HÖLDER REGULARITY OF THE INTEGRATED DENSITY OF STATES FOR QUASI-PERIODIC LONG-RANGE OPERATORS ON $\ell^2(\mathbb{Z}^d)$

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Abstract. We prove the Hölder continuity of the integrated density of states for a class of quasi-periodic long-range operators on $\ell^2(\mathbb{Z}^d)$ with large trigonometric polynomial potentials and Diophantine frequencies. Moreover, we give the Hölder exponent in terms of the cardinality of the level sets of the potentials, which improves, in the perturbative regime, the result obtained by Goldstein and Schlag [31]. Our approach is a combination of Aubry duality, generalized Thouless formula and the regularity of the Lyapunov exponents of analytic quasi-periodic $GL(m,\mathbb{C})$ cocycles which is proved by quantitative almost reducibility method.

1. Introduction

In this paper, we consider the following quasi-periodic long-range operator on $\ell^2(\mathbb{Z}^d)$ with trigonometric polynomial potentials:

$$(L^{\lambda W}_{V,\alpha,\theta}u)_n = \sum_{k\in\mathbb{Z}^d}\widehat{V}_k u_{n-k} + \lambda W(\theta+\langle n,\alpha\rangle)u_n, \quad n\in\mathbb{Z}^d, \tag{1.1}$$

where $W(\theta)=\sum_{k=-m}^{m}\widehat{W}(k)e^{2\pi ik\theta}$ is a real trigonometric polynomial and $V(x)=\sum_{k\in\mathbb{Z}^d}\widehat{V}(k)e^{2\pi i\langle k,x\rangle}$ is a real analytic function. W, $\theta\in\mathbb{T}$, $\alpha\in\mathbb{R}^d$ are called the potential, the phase and the frequency respectively. We always assume that $\{1,\alpha_1,\cdots,\alpha_d\}$ are independent over $\mathbb{Q}$. (1.1) includes and relates to several interesting quasi-periodic models.

Example 1. Taking $V(x)=\sum_{i=1}^{d}2\cos 2\pi x_i$, it is the quasi-periodic Schrödinger operator on $\ell^2(\mathbb{Z}^d)$,

$$H=\Delta+\lambda W(\theta+\langle n,\alpha\rangle)\delta_{nn'}, \tag{1.2}$$

where Δ is the usual Laplacian on $\mathbb{Z}^d$ lattice.

Example 2. Taking $d=1$, $V(x)=2\cos 2\pi x$, it is the one-frequency quasi-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$,

$$(H_{\lambda W,\alpha,\theta}u)(n)=u_{n+1}+u_{n-1}+\lambda W(\theta+n\alpha)u(n). \tag{1.3}$$

Example 3. Taking $d=m=1$, $V(x)=W(x)=2\cos 2\pi x$, it is the famous almost Mathieu operator,

$$(H_{\lambda,\alpha,\theta}u)(n)=u_{n+1}+u_{n-1}+2\lambda\cos 2\pi(\theta+n\alpha)u(n). \tag{1.4}$$

Example 4. The Aubry dual of (1.1) is the following multi-frequency quasi-periodic finite-range operator on $\ell^2(\mathbb{Z})$,

$$(L^{\lambda^{-1}V}_{W,\alpha,x}u)(n)=\sum_{k=-m}^{m}\widehat{W}_k u_{n-k}+\lambda^{-1}V(x+n\alpha)u(n). \tag{1.5}$$

Example 5. The special case of (1.5) is the following quasi-periodic Schrödinger operator with d frequencies,

$$(H_{\lambda^{-1}V,\alpha,x}u)(n)=u_{n+1}+u_{n-1}+\lambda^{-1}V(x+n\alpha)u(n). \tag{1.6}$$

The Integrated Density of States (IDS) is a quantity of fundamental importance for models in condensed matter physics. It is defined in a uniform way for all quasi-periodic family self-adjoint operators $(L_x)_{x\in\mathbb{T}^d}$ by

$$N(E)=\int_{\mathbb{T}^d}\mu_x(-\infty,E]dx,$$

where μ_x is the spectral measure associated with L_x and δ_0. Roughly speaking, the density of states measure $N([E_1,E_2])$ gives the "number of states per unit volume" with energy between E_1 and E_2.

The regularity of IDS is a fascinating subject in the spectral theory of quasi-periodic operators, especially for the absolute continuity [3, 5, 6] and the Hölder regularity [1, 6, 30, 31]. The regularity of IDS is also closely related to many other topics in the spectral theory of quasi-periodic operators. For example, the absolute continuity of IDS is closely related to purely absolutely continuous spectrum in the regime of zero Lyapunov exponent [1] [20, 43]. Hölder continuity of IDS is closely related to homogeneity of the spectrum [21, 22, 44].

1.1. **Regularity of IDS.** We first review the previous results on Hölder regularity of IDS for one dimensional quasi-periodic Schrödinger operators. For (1.6) with real analytic potential V (equivalently for its dual operator (1.1) with $W=2\cos 2\pi\theta$), $\frac{1}{2}$-Hölder continuity of IDS was obtained by Amor [1] for sufficiently large λ and Diophantine α [2]. For the one-frequency case, i.e., operator (1.3) with W being a real analytic potential, Amor's result was extended by Avila-Jitomirskaya [6] to the non-perturbative regime (actually to the almost reducible regime). The Hölder continuity of IDS in the positive Lyapunov exponent regime was proved by Goldstein and Schlag [30] assuming that α is strong Diophantine. In view of Avila's global theory [2], for one-frequency analytic quasi-periodic Schrödinger operators, the energy can be divided into three regimes: the subcritical regime (almost reducible regime), the supercritical regime (positive Lyapunov exponent regime) and the critical regime (otherwise), and typically, there is no critical energy [2]. Thus the Hölder continuity of IDS for one-frequency quasi-periodic Schrödinger operators is clear: it is Hölder continuous in both subcritical regime and supercritical regime, while one should not expect any modulus of continuity of IDS at the critical energies [12]. We remark that for small potentials, [1, 6] got the optimal result, i.e., $\frac{1}{2}$-Hölder continuity, while for big potentials, [30] did not give any information on the Hölder exponent. So it is natural to ask: what is the modulus of continuity of IDS for big potentials? For the almost Mathieu operator, Bourgain [10] proved that for Diophantine α and large enough λ, the Hölder exponent is larger than $\frac{1}{2}-\varepsilon$ for any $\varepsilon>0$. The Hölder exponent was further optimized as $\frac{1}{2}$ by Avila and Jitomirskaya [6] for non-critical almost Mathieu operator with Diophantine frequency. For general trigonometric polynomial of degree m, Goldstein and Schlag [31] proved that the Hölder exponent is larger than $\frac{1}{2m}-\varepsilon$ for all $\varepsilon>0$ in the positive Lyapunov exponent regime, which is a generalization of Bourgain's result [10]. Their methods are the Large Deviation Theorem (LDT) and Avalanch Principle (AP), which were initially developed by Bourgain-Goldstein [14] and Goldstein-Schlag [30], further developed by Goldstein-Schlag [31]. LDT and AP were later recognized as powerful tools to study the regularity of IDS for quasi-periodic Schrödinger operators in the positive Lyapunov exponent regime. Based on LDT and AP, various Hölder continuity results of IDS for quasi-periodic Schrödinger operators were obtained, here is a partial list [10, 30, 31, 33, 45, 50, 51, 53]. Initially, LDT and AP were developed for Schrödinger cocycles, which works only for $SL(2,\mathbb{R})$ cocycles. More recently, LDT and AP were

[1]See section 2.1 for the definition of Lyapunov exponent of Schrödinger cocycles.

[2] Recall that $\alpha\in\mathbb{R}^d$ is called *Diophantine* if there are $\kappa>0$ and $\tau>d-1$ such that $\alpha\in\mathrm{DC}_d(\kappa,\tau)$, where

$$\mathrm{DC}_d(\kappa,\tau):=\left\{\alpha\in\mathbb{R}^d:\inf_{j\in\mathbb{Z}}|\langle n,\alpha\rangle-j|>\frac{\kappa}{|n|^\tau},\quad\forall\, n\in\mathbb{Z}^d\backslash\{0\}\right\},\tag{1.7}$$

and $\mathrm{DC}_d:=\bigcup_{\kappa>0,\tau>d-1}\mathrm{DC}_d(\kappa,\tau)$ is a full Lebesgue measure set.

further developed to higher dimensional linear cocycles [23, 24, 48] for proving the continuity of the Lyapunov exponents. We remark that the characterization of the Hölder exponent is much more difficult in the supercritical regime, which is closely related to the "multiple resonances" [3] and developing techniques to deal with such "multiple resonances" is very difficult ([31] is an 114 pages paper). Different from the small potentials, $\frac{1}{2}$-Hölder continuity of IDS usually should not be expected for large potentials. It seems that the Hölder exponent is closely related to the profile of the potential. In [31], Goldstein and Schlag linked the Hölder exponent to the degree of the trigonometric polynomial potential.

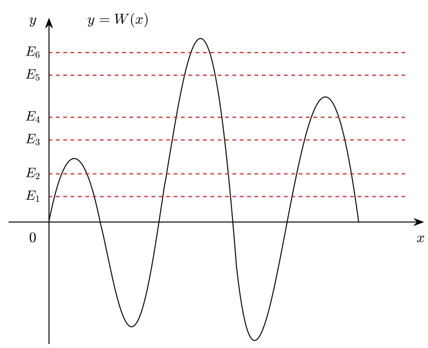
In this paper, different from [31], we use the Quantitative Almost Reducibility Theorem (QART) and Aubry duality to study this problem. Our approach has some advantages: it does not sensitive to d and m ([31] treated the case $m = d = 1$), thus it works for more general classes of operators (1.1) and (1.5). More importantly, it brings a more delicate estimate on the Hölder exponent. More precisely, for large trigonometric polynomial potentials, the Hölder exponent of IDS is controlled by the cardinality of the level sets $\#\{\theta : W(\theta) = E\}$ instead of the degree of W. Our main result is

Theorem 1.1. *Assume that $\alpha \in \mathrm{DC}_d$, $\varepsilon > 0$ and $\#\{\theta : W(\theta) = E\}$, i.e., the number of zeros (counting multiplicities) of $W(\theta) - E$ on $\mathbb{T}$, is no more than $2m_0$ for all $E \in \mathbb{R}$. Then there exist $\lambda_0 = \lambda_0(\alpha, W, V, \varepsilon) > 0$ and $C = C(\alpha, W, V) > 0$ such that if $\lambda \geq \lambda_0$, for operators (1.1) and (1.5), we have*

$$|N(E) - N(E')| \leq C|E - E'|^{\frac{1}{2m_0} - \varepsilon},$$

for all $E, E' \in \mathbb{R}$.

Remark 1.1. We indeed characterize the Hölder exponent in the neighborhood of E by the cardinality of level sets $\#\{\theta : W(\theta) = E\}$. For W in the following figure



we have

(1) For $E \in [E_1, E_2]$, the Hölder exponent is larger than $\frac{1}{6} - \varepsilon$,
(2) For $E \in [E_3, E_4]$, the Hölder exponent is larger than $\frac{1}{4} - \varepsilon$,
(3) For $E \in [E_5, E_6]$, the Hölder exponent is larger than $\frac{1}{2} - \varepsilon$.

Remark 1.2. If $d = 1$ and $V(x) = 2\cos 2\pi x$, it is possible to extend the result to operator (1.1) with any real analytic potential W and large λ. We guess that the Hölder exponent of IDS is closely related to the acceleration [4] in the positive Lyapunov exponent regime.

[3] Roughly speaking, on the almost reducibility side, resonance means the eigenvalues of the constant matrix come very close to each other. On the large deviation side, resonance means the eigenvalues of the operators restricted to a box come very close to each other.

[4] See section 2.4 for the definition.

Remark 1.3. In the perturbative regime, if $m = d = 1$, our result is a refinement of Goldstein and Schlag's result [31].

Remark 1.4. To the best of our knowledge, Theorem 1.1 is the first Hölder regularity result of IDS for quasi-periodic long-range operators (1.1) on $\ell^2(\mathbb{Z}^d)$ with potentials beyond the cosine function. It is also the first Hölder regularity result of IDS for multi-frequency finite difference analytic quasi-periodic operators (1.5).

Remark 1.5. It is possible to improve the Hölder exponent to $\frac{1}{2m_0}$ in Theorem 1.1 by refining estimates of QART for $GL(m,\mathbb{C})$ cocycles. We do not go further in the present paper.

Theorem 1.1 is a generalization of [31] from one dimensional Schrödinger operators to Schrödinger operators on $\mathbb{Z}^d$ lattice in the perturbative regime. In fact, since $m_0 \le m$, we immediately get the following corollary which was proved by Goldstein and Schlag in [31] for the case $d = 1$ and $V(x) = 2\cos 2\pi x$.

Corollary 1.1. *Assume $\alpha \in \mathrm{DC}_d$ and $\varepsilon > 0$, there exist $\lambda_0 = \lambda_0(\alpha, W, V, \varepsilon)$ and $C_0 = C(\alpha, W, V)$ such that if $\lambda \ge \lambda_0$, for operators (1.1) and (1.5), we have*

$$|N(E) - N(E')| \le C|E - E'|^{\frac{1}{2m} - \varepsilon},$$

for all $E, E' \in \mathbb{R}$.

We briefly review some other related regularity results on IDS. The Hölder continuity of IDS for one-frequency analytic quasi-periodic Schrödinger operators with Liouvillean frequencies was obtained by You-Zhang [53] and Han-Zhang [33]. Weak Hölder continuity of IDS for quasi-periodic Schrödinger operators on $\ell^2(\mathbb{Z}^2)$ with Diophantine frequencies and large analytic potentials was proved by Schlag [47]. For the lower regularity case, Klein [42] proved that for Schrödinger operators with potentials in a Gevrey class, the IDS is weak Hölder continuous on any compact interval of the energy provided that the coupling constant is large enough, the frequency is Diophantine and the potential satisfies some transversality condition. Wang-Zhang [50] obtained the weak Hölder continuity of the IDS as function of energies, for a class of C^2 quasi-periodic potentials and for any Diophantine frequency. Subsequently it was improved to be Hölder continuous by Liang-Wang-You [45]. Recently, the Hölder exponent was proved to be 1/2 by Xu-Ge-Wang [51] which is optimal. More recently, Cai-Chavaudret-You-Zhou [16] proved 1/2-Hölder continuity of IDS for quasi-periodic Schrödinger operator with small finitely differential potentials and Diophantine frequencies. Jitomirskaya-Kachkovskiy [37] proved Lipschitz continuity of IDS for quasi-periodic Schrödinger operators with bounded monotonic potentials. Their result [37] was recently extended by Kachkovskiy [41] to unbounded monotonic potentials.

1.2. **Regularity of the Lyapunov exponents for quasi-periodic $GL(m,\mathbb{C})$ cocycles.** We denote by $GL(m,\mathbb{C})$ the set of all $m \times m$ invertible matrices. Given $A \in C^\omega(\mathbb{T}^d, GL(m,\mathbb{C}))$ and rational independent $\alpha \in \mathbb{R}^d$, we define the *quasi-periodic $GL(m,\mathbb{C})$ cocycle* (α, A):

$$(\alpha, A)\colon \begin{cases} \mathbb{T}^d \times \mathbb{C}^m & \to \quad \mathbb{T}^d \times \mathbb{C}^m \\ \quad (x, v) & \mapsto \quad (x+\alpha, A(x)\cdot v) \end{cases}.$$

The iterates of (α, A) are of the form $(\alpha, A)^n = (n\alpha, A_n)$, where

$$A_n(x) := \begin{cases} A(x+(n-1)\alpha)\cdots A(x+\alpha)A(x), & n \ge 0 \\ A^{-1}(x+n\alpha)A^{-1}(x+(n+1)\alpha)\cdots A^{-1}(x-\alpha), & n < 0 \end{cases}.$$

We denote by $L_1(\alpha, A) \ge L_2(\alpha, A) \ge ... \ge L_m(\alpha, A)$ the Lyapunov exponents of (α, A) repeatedly according to their multiplicities, i.e.,

$$L_k(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}^d} \ln \sigma_k(A_n(x))dx,$$

where for a matrix $B \in GL(m,\mathbb{C})$, we denote by $\sigma_1(B) \geq ... \geq \sigma_m(B)$ its singular values (eigenvalues of $\sqrt{B^*B}$).

The continuity of the Lyapunov exponents of linear cocycles has been extensively studied. It was proved by Bourgain-Jitomirskaya in [15] that LE is joint continuous for $SL(2,\mathbb{R})$ cocycles, in frequency and cocycle map, at any irrational frequencies. Jitomirskaya-Koslover-Schulteis [38] got the continuity of LE with respect to potentials for a class of analytic quasi-periodic $M(2,\mathbb{C})$ cocycles. Bourgain [13] extended the results in [15] to multi-frequency case. Jitomirskaya-Marx [39] extended the results in [15] to all (including singular) $M(2,\mathbb{C})$ cocycles. More recently, continuity of the Lyapunov exponents for one-frequency analytic $M(m,\mathbb{C})$ cocycles was given by Avila-Jitomirskaya-Sadel [7]. Weak Hölder continuity of the Lyapunov exponents for multi-frequency $GL(m,\mathbb{C})$ cocycles, $m \geq 2$, was recently obtained by Schlag [48] and Duarte-Klein [24].

In this paper, we show that if $A \in C^\omega(\mathbb{T}^d, GL(m,\mathbb{C}))$, the Lyapunov exponents can still be Hölder continuous, provided that the cocycle is almost reducible. Our main result is the following:

Theorem 1.2. *Let $\alpha \in DC_d(\kappa,\tau)$ and $\varepsilon > 0$, if (α, A) is C_h^ω almost reducible, then for any $\widetilde{A} \in C^\omega(\mathbb{T}^d, GL(m,\mathbb{C}))$, we have*

$$|L_i(\alpha, A) - L_i(\alpha, \widetilde{A})| \leq C_0 |\widetilde{A} - A|_0^{\frac{1}{2n_i} - \varepsilon}, \quad 1 \leq i \leq m,$$

where C_0 is a constant depending on $A, d, \kappa, \tau, m, \varepsilon$ and n_i is the multiplicity of $L_i(\alpha, A)$.

Remark 1.6. Almost reducible cocycles include a large class of cocycles. For example, consider $(\alpha, A_0 e^{f_0(\cdot)})$ where $A_0 \in GL(m,\mathbb{C})$ and $f_0 \in C_h^\omega(\mathbb{T}^d, gl(m,\mathbb{C}))$, then $(\alpha, A_0 e^{f_0(\cdot)})$ is almost reducible provided $|f_0|_h$ is sufficiently small. In one-frequency case, all subcritical cocycles are almost reducible [3, 4].

1.3. **Outline of the proofs.** The Quantitative Almost Reducibility Theorem (QART), was initially developed by Eliasson [25], further developed by Hou-You [35], Avila-Jitomirskaya [6] and Avila [3, 4]. It has been proved to be a powerful tool in studying various spectral problems of quasi-periodic operators in the almost reducible regime which is the "dual regime" of the positive Lyapunov exponent regime [2] [5].

We will prove our main results by establishing the QART for higher dimensional linear cocycles. More precisely, we establish a quantitative version of the almost reducibility theorem in the higher dimension emphasizing the quantitative estimates on the Lyapunov exponents at each KAM iteration. The advantage of QART is that it works for higher dimensional cocycles with multiple frequencies, which seems to be highly nontrivial by LDT and AP. Let us explain why QART is a powerful tool for investigating the regularity of the Lyapunov exponents of analytic quasi-periodic $GL(m,\mathbb{C})$ cocycles.

Definition 1.1. $(\alpha, A) \in C^\omega(\mathbb{T}^d, GL(m,\mathbb{C}))$ is said to be C_h^ω almost reducible if there exist $B_j \in C_h^\omega(\mathbb{T}^d, GL(m,\mathbb{C}))$, $A_j \in GL(m,\mathbb{C})$ and $f_j \in C_h^\omega(\mathbb{T}^d, gl(m,\mathbb{C}))$ such that

$$B_j(x+\alpha)A(x)B_j^{-1}(x) = A_j e^{f_j(x)},$$

with $|f_j|_h \to 0$ and $A_j \to A_\infty \in GL(m,\mathbb{C})$.

The followings are three basic facts of the Lyapunov exponents:

(1) The Lyapunov exponents are invariant under continuous conjugation, i.e.,

$$L_i(\alpha, A) = L_i(\alpha, B_j(\cdot+\alpha)A(\cdot)B_j^{-1}(\cdot)) = L_i(\alpha, A_j e^{f_j(\cdot)}),\ 1 \leq i \leq m.$$

[5]For example, for the almost Mathieu operator, the positive Lyapunov exponent regime corresponds to $|\lambda| > 1$ and the almost reducible regime corresponds to $|\lambda| < 1$.

Furthermore, if the base frequency is Diophantine, all Lyapunov exponents are continuous in A in C_h^ω topology [23, 48]. It follows that

$$L_i(\alpha, A_j e^{f_j(\cdot)}) \to L_i(\alpha, A_\infty),\ 1 \le i \le m.$$

Thus,

$$L_i(\alpha, A) = L_i(\alpha, A_\infty),\ 1 \le i \le m.$$

(2) Almost reducibility is open [6]. Thus if $|\widetilde{A} - A|_h$ is sufficiently small, $(\alpha, \widetilde{A})$ is also almost reducible to $(\alpha, \widetilde{A}_\infty)$ with

$$L_i(\alpha, \widetilde{A}) = L_i(\alpha, \widetilde{A}_\infty),\ 1 \le i \le m.$$

(3) The Lyapunov exponents of constant cocycles are computable. If we denote the eigenvalues of A_∞ ($\widetilde{A}_\infty$) by $\{e^{-2\pi i\rho_i}\}_{i=1}^m$ ($\{e^{-2\pi i\tilde{\rho}_i}\}_{i=1}^m$) respectively, then

$$\{L_i(\alpha, A)\}_{i=1}^m = \{2\pi\Im\rho_i\}_{i=1}^m, \quad \{L_i(\alpha, \widetilde{A})\}_{i=1}^m = \{2\pi\Im\tilde{\rho}_i\}_{i=1}^m.$$

QART can give us very precise estimates on the differences $|L_i(\alpha, A_\infty) - L_i(\alpha, \widetilde{A}_\infty)|$, from which one can find Hölder continuity easily. The Hölder continuity of IDS is then a consequence in view of the generalized Thouless formula [18, 19].

Finally, we introduce our motivations. During the past ten years, QART for $SL(2,\mathbb{R})$ cocycles along with Aubry duality [32] has been proved to be a powerful tool to solve various central problems in the field of spectral theory of quasi-periodic operators, see [6, 8, 9, 27–29, 36] for some recent progresses. While all the results are essentially restricted to the quasi-periodic long-range operator on $\ell^2(\mathbb{Z}^d)$ with a cosine potential or the almost Mathieu operator. When the potential goes beyond cosine, things become dramatically complicated, and the so-called "multiple resonances" becomes a real issue. KAM theory provides us a way to deal with "multiple resonances" [52] and it may have wider applications in other aspects of the spectral theory for quasi-periodic Schrödinger operators with potentials beyond the cosine function.

2. Preliminaries

For a bounded analytic (possibly matrix valued) function F defined on $\{x||\Im x| < h\}$, let $|F|_h = \sup_{|\Im x|<h} |F(x)|$ and denote by $C_h^\omega(\mathbb{T}^d, *)$ the set of all these $*$-valued functions ($*$ will usually denote $\mathbb{R}$, $\mathbb{C}$, $gl(m,\mathbb{C})$ and $GL(m,\mathbb{C})$). Also we denote $C^\omega(\mathbb{T}^d, *) = \cup_{h>0} C_h^\omega(\mathbb{T}^d, *)$, and $|F|_0 = \sup_{x\in\mathbb{T}^d} |F(x)|$.

2.1. **Quasi-periodic finite-range cocycles.** In this subsection, we review some basic concepts for quasi-periodic finite-range operators, especially for quasi-periodic Schrödinger operators. An important example of quasi-periodic $GL(2m,\mathbb{C})$ cocycle is the finite-range cocycle $(\alpha, L_{E,W}^{\lambda^{-1}V})$ where

$$L_{E,W}^{\lambda^{-1}V}(x) = \frac{1}{\widehat{W}_m} \begin{pmatrix} -\widehat{W}_{m-1} & \cdots & -\widehat{W}_1 & E-\lambda^{-1}V(x)-\widehat{W}_0 & -\widehat{W}_{-1} & \cdots & -\widehat{W}_{-m+1} & -\widehat{W}_{-m} \\ \widehat{W}_m & & & & & & & \\ & & & \ddots & & & & \\ & & & & & & \widehat{W}_m & \end{pmatrix}.$$

The finite-range cocycles are equivalent to the eigenvalue equations of operators (1.5), i.e., $L_{W,\alpha,x}^{\lambda^{-1}V} u = Eu$. Note that $(\alpha, L_{E,W}^{\lambda^{-1}V})$ is more special than general $GL(2m,\mathbb{C})$ cocycles in the following senses,

[6]It follows from the results in [17, 26].

(1) The m-th iteration of $(\alpha, L_{E,W}^{\lambda^{-1}V})$, denoted by $(m\alpha, (L_{E,W}^{\lambda^{-1}V})_m)$, is a symplectic cocycle [34]. As a corollary, the Lyapunov exponents of $(\alpha, L_{E,W}^{\lambda^{-1}V})$ come into pairs $\pm L_i(\alpha, L_{E,W}^{\lambda^{-1}V})$ $(1 \leq i \leq m)$.

(2) We denote $L_i(E) = L_i(\alpha, L_{E,W}^{\lambda^{-1}V}) \geq 0$ $(1 \leq i \leq m)$ and the IDS of $(L_{W,\alpha,x}^{\lambda^{-1}V})_{x\in\mathbb{T}^d}$ by $N(E)$, then the sum of the nonnegative Lyapunov exponents and the IDS are linked by the famous generalized Thouless formula [18, 19, 34],

$$\sum_{i=1}^{m} L_i(E) + \ln|\widehat{W}_m| = \int \ln|E - E'| dN(E'). \tag{2.1}$$

By the Hilbert transform and the theory of singular integral operators, the Hölder continuity passes from $\sum\limits_{i=1}^{m} L_i(E)$ to $N(E)$ and vice versa (see [30] for details). Notice that the Aubry duals of $(L_{W,\alpha,x}^{\lambda^{-1}V})_{x\in\mathbb{T}^d}$ are $(L_{\lambda^{-1}V,\alpha,\theta}^{W})_{\theta\in\mathbb{T}}$, we denote the IDS of $(L_{\lambda^{-1}V,\alpha,\theta}^{W})_{\theta\in\mathbb{T}}$ by $\widehat{N}(E)$. The IDS is invariant under Aubry dual, i.e.,

Proposition 2.1 ([34, 40])**.** *$N(E) = \widehat{N}(E)$.*

2.2. **Some basic properties of the Lyapunov exponents.** In the introduction, we have given some basic facts on the Lyapunov exponents of linear cocycles. In this subsection, we give an elementary proof of them.

1. Lyapunov exponents are invariant under continuous conjugations.

Proposition 2.2. *Assume $(\alpha, A) \in \mathbb{T}^d \times C^0(\mathbb{T}^d, GL(m,\mathbb{C}))$, $B \in C^0(\mathbb{T}^d, GL(m,\mathbb{C}))$, and $\widetilde{A}(x) = B(x+\alpha)A(x)B^{-1}(x)$, we have*

$$L_i(\alpha, \widetilde{A}) = L_i(\alpha, A), \quad 1 \leq i \leq m.$$

Proof. For any $1 \leq i \leq m$, we have

$$\Lambda^i \widetilde{A}_n(x) = \Lambda^i B(x+n\alpha)A_n(x)B^{-1}(x),$$

where $\Lambda^i M$ is the i-th wedge of M. Thus

$$|B|_0^{-m}|B^{-1}|_0^{-m}|\Lambda^i A_n(x)| \leq |\Lambda^i \widetilde{A}_n(x)| \leq |B|_0^{m}|B^{-1}|_0^{m}|\Lambda^i A_n(x)|,$$

which implies that

$$\frac{1}{n}\int_{\mathbb{T}^d} \ln|\Lambda^i \widetilde{A}_n(x)|dx = \frac{1}{n}\int_{\mathbb{T}^d} \ln|\Lambda^i A_n(x)|dx + o(1).$$

Hence

$$L_i(\alpha, \widetilde{A}) = L_i(\alpha, A), \quad 1 \leq i \leq m.$$

□

2. The Lyapunov exponents of constant cocycles are computable.

Proposition 2.3. *If we denote the eigenvalues of $A \in GL(m,\mathbb{C})$ by $\{e^{-2\pi i\rho_j}\}_{j=1}^m$, then*

$$\{L_j(\alpha, A)\}_{j=1}^m = \{2\pi\Im\rho_j\}_{j=1}^m.$$

Proof. We only give the proof of $L_1(\alpha, A) = \max_{1\leq j\leq m}\{2\pi\Im\rho_j\}$ and the other proofs are similar. By Proposition 2.2, we can always assume that A is in Jordan form. Then one can directly compute that

$$L_1(\alpha, A) = \lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}^d} \ln\sigma_1(A^n)dx = \lim_{n\to\infty}\frac{1}{n}\ln\sigma_1(A^n) = \max_{1\leq j\leq m}\{2\pi\Im\rho_j\}.$$

□

2.3. **Perturbation theory of constant matrices.** In this subsection, we briefly introduce the perturbation theory of constant matrices which will be used in Section 3.

Definition 2.1. Let A be an $m \times m$ matrix, denote

$$|A|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} |a_{ij}|^2}.$$

We call A normal if $AA^T = A^T A$ where A^T is the transpose conjugation of A.

Lemma 2.1 ([49]). *Let A and B be two $m \times m$ matrices, where A is normal and B is nonnormal, with spectrum $\lambda(A) = \{\lambda_1, \cdots, \lambda_m\}$ and $\lambda(B) = \{\mu_1, \cdots, \mu_m\}$, then there exists a permutation π of $\{1, 2, \cdots, m\}$ such that*

$$\sqrt{\sum_{j=1}^{m} \left|\mu_{\pi(j)} - \lambda_j\right|^2} \leq \sqrt{m}|B - A|_F.$$

The following proposition follows immediately.

Proposition 2.4. *Let A and B be two $m \times m$ matrices, with spectrum $\lambda(A) = \{\lambda_1, \cdots, \lambda_m\}$ and $\lambda(B) = \{\mu_1, \cdots, \mu_m\}$, then there exists a permutation π of $\{1, 2, \cdots, m\}$ such that*

$$\sqrt{\sum_{j=1}^{m} \left|\mu_{\pi(j)} - \lambda_j\right|^2} \leq C(m, A)|B - A|^{\frac{1}{m}}.$$

Proof. Let $\epsilon = |B - A|$. There exists a unitary matrix $U \in GL(m, \mathbb{C})$ such that

$$U^{-1}AU = \begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_m \end{pmatrix}.$$

If we denote $T = diag\{1, \epsilon^{\frac{1}{m}}, \cdots, \epsilon^{\frac{m-1}{m}}\}$, then

$$\begin{aligned}(UT)^{-1}BUT &= (UT)^{-1}AUT + (UT)^{-1}(B - A)UT \\ &= diag\{\lambda_1, \cdots, \lambda_m\} + F,\end{aligned}$$

with $|F| \leq 2|A|\epsilon^{\frac{1}{m}}$. By Lemma 2.1, there exists a permutation π of $\{1, 2, \cdots, m\}$ such that

$$\sqrt{\sum_{j=1}^{m}(\mu_{\pi(j)} - \lambda_j)^2} \leq \sqrt{m}|F|_F \leq C(m, A)|B - A|^{\frac{1}{m}}.$$

□

2.4. **Global theory for analytic one-frequency quasi-periodic Schrödinger operators.** In 2015, Avila [2] gave a qualitative spectral picture for one-frequency quasi-periodic Schrödinger operators. To explain more, we denote the associated cocycle of the eigenequation $H_{W(\cdot+i\varepsilon),\alpha,\theta}u = Eu$ by $(\alpha, S_E^W(\cdot + i\varepsilon))$ and the associated nonnegative Lyapunov exponent by $L_\varepsilon(E)$. The *acceleration* is defined in [2] by

$$\omega(E) = \lim_{\varepsilon \to 0^+} \frac{L_\varepsilon(E) - L(E)}{2\pi\varepsilon}.$$

The global theory discovers that the spectral set of the Schrödinger operator can be divided into three regimes based on the Lyapunov exponent and acceleration:

(1) The subcritical regime: $L(E) = 0$ and $\omega(E) = 0$.
(2) The critical regime: $L(E) = 0$ and $\omega(E) > 0$.

(3) The supercritical regime: $L(E) > 0$ and $\omega(E) > 0$.

Moreover, the subcritical regime is equivalent to the almost reducible regime, announced by Avila [3, 4]. Usually, the almost reducible regime can be viewed as the dual regime of the positive Lyapunov exponent regime.

3. A quantitative almost reducibility theorem for $GL(m,\mathbb{C})$ cocycles

Consider the local quasi-periodic $GL(m,\mathbb{C})$ cocycle

$$(\alpha, A_0 e^{f_0(\cdot)}) \colon \left\{ \begin{array}{ccc} \mathbb{T}^d \times \mathbb{C}^m & \to & \mathbb{T}^d \times \mathbb{C}^m \\ (x,v) & \mapsto & (x+\alpha, A_0 e^{f_0(x)} \cdot v) \end{array} \right. ,$$

where $A_0 \in GL(m,\mathbb{C})$ is a constant matrix, $f_0 \in C_h^\omega(\mathbb{T}^d, gl(m,\mathbb{C}))$ is an analytic perturbation. $\alpha \in DC_d(\kappa,\tau)$ for some $\kappa > 0$, $\tau > d-1$. In this section, we prove the following quantitative almost reducibility result. For our purpose, we specially pay attention to shift of the norms of the eigenvalues during the KAM iteration.

Theorem 3.1. *For any given $0 < \sigma < \frac{1}{500m^3}$ and $0 < h' < h$, there exists $\epsilon_0 = \epsilon(\alpha, h, h', m, |A_0|, \sigma)$ such that if $|f_0|_h \le \epsilon_0$, then $\left(\alpha, A_0 e^{f_0(\cdot)}\right)$ is $C_{h'}^\omega$ almost reducible, i.e., there exist $B_j \in C_{h'}^\omega(\mathbb{T}^d, GL(m,\mathbb{C}))$, $A_j \in GL(m,\mathbb{C})$ and $f_j \in C_{h'}^\omega(\mathbb{T}^d, gl(m,\mathbb{C}))$, such that*

$$B_j(x+\alpha)(A_0 e^{f_0(x)}) B_j^{-1}(x) = A_j e^{f_j(x)}.$$

Moreover, we have the following estimates

$$|f_j|_{h'} \le \epsilon_j, \quad |B_j|_0 \le \epsilon_{j-1}^{-200m^2\sigma}, \tag{3.1}$$

$$\left| \sum_{\ell=1}^m \Im \rho_\ell^j - \sum_{\ell=1}^m \Im \rho_\ell^{j-1} \right| \le C(m, A_0)\epsilon_{j-1}^{\frac{1}{2}}, \tag{3.2}$$

$$\left| \Im \rho_\ell^j - \Im \rho_\ell^{j-1} \right| \le C(m, A_0)\epsilon_{j-1}^{\frac{1}{m} - 200m^2\sigma}, \quad 1 \le \ell \le m, \tag{3.3}$$

where $\left\{ e^{-2\pi i \rho_\ell^j} \right\}_{\ell=1}^m$ are the eigenvalues of A_j and $\epsilon_j = \epsilon_0^{2^j}$.

Proof. Suppose that

$$|f_0|_h \le \epsilon_0 \le \frac{C}{|A_0|^{\frac{500dm^3\tau}{\sigma}}} (h-h')^{\frac{500dm^3\tau}{\sigma}}, \tag{3.4}$$

where $C = \min\left\{ 2^{-\frac{2\tau}{\sigma}} (4m+1)^{-\frac{2}{\sigma}} \kappa^{\frac{2}{\sigma}}, C_0^{-\frac{500dm^3\tau}{\sigma}} \right\}$ [7]. Then we define the following sequences inductively,

$$\epsilon_{j+1} = \epsilon_j^2, \quad h_j - h_{j+1} = \frac{h-h'}{4^{j+1}}, \quad N_j = \frac{2|\ln \epsilon_j|}{h_j - h_{j+1}}.$$

By our selection of ϵ_0, it's easy to check that

$$\epsilon_j \le \frac{C}{|A_j|^{\frac{500dm^3\tau}{\sigma}}} (h_j - h_{j+1})^{\frac{500dm^3\tau}{\sigma}}. \tag{3.5}$$

Indeed, ϵ_j on the left side of the inequality decays at least super-exponentially with j, while $(h_j - h_{j+1})^{\frac{500dm^3\tau}{\sigma}}$ on the right side decays exponentially with j.

[7] C_0 is an absolute constant only depending on m.

Assume after j steps of iteration, we are at the $(j+1)^{th}$ KAM step. That is, we have already constructed $B_j \in C^{\omega}_{h_j}(\mathbb{T}^d, GL(m,\mathbb{C}))$ such that

$$B_j(x+\alpha)A_0 e^{f_0(x)} B_j^{-1}(x) = A_j e^{f_j(x)}, \tag{3.6}$$

where $A_j \in GL(m,\mathbb{C})$ with eigenvalues $\{e^{-2\pi i \rho_\ell^j}\}_{\ell=1}^m$ and

$$|f_j|_{h'} \le \epsilon_j, \quad |B_j|_0 \le \epsilon_{j-1}^{-200m^2\sigma}, \tag{3.7}$$

$$\left|\sum_{\ell=1}^m \Im\rho_\ell^j - \sum_{\ell=1}^m \Im\rho_\ell^{j-1}\right| \le C(m,A_0)\epsilon_{j-1}^{\frac{1}{2}},$$

$$\left|\Im\rho_\ell^j - \Im\rho_\ell^{j-1}\right| \le C(m,A_0)\epsilon_{j-1}^{\frac{1}{m}-200m^2\sigma}, \quad 1 \le \ell \le m.$$

We will construct

$$\bar{B}_j \in C^{\omega}_{h_{j+1}}(\mathbb{T}^d, GL(m,\mathbb{C})), \quad A_{j+1} \in GL(m,\mathbb{C}), \quad f_{j+1} \in C_{h_{j+1}}(\mathbb{T}^d, gl(m,\mathbb{C}))$$

such that

$$\bar{B}_j(x+\alpha)A_j e^{f_j(x)} \bar{B}_j^{-1}(x) = A_{j+1} e^{f_{j+1}(x)},$$

with desired estimates. The proof is divided into the following four steps. We denote $A_j, f_j, \epsilon_j, h_j, \epsilon_{j+1}, h_{j+1}$ by $A, f, \epsilon, h, \epsilon_+, h_+$ for simplicity.

Step 1: Block diagonalizing A. We denote $\{e^{-2\pi i\rho_\ell}\}_{1\le\ell\le m}$ the eigenvalues of A.

Lemma 3.1 (Block diagonalization). *$\{\rho_\ell\}_{1\le\ell\le m}$ can be grouped as $\bigcup_{\ell=1}^r E_\ell$ with $\#E_\ell = n_\ell$, such that*

$$\left|\rho - \rho'\right| \le m\epsilon^{\sigma}, \quad \rho, \rho' \in E_\ell, \tag{3.8}$$

$$\left|\rho - \rho'\right| \ge \epsilon^{\sigma}, \quad \rho \in E_k, \rho' \in E_\ell, \quad k \ne \ell. \tag{3.9}$$

Moreover, there exist $C_0 \ge 1$ depending only on m, $P \in GL(m,\mathbb{C})$ with estimate

$$|P^{-1}|, \quad |P| \le C_0(2|A|\epsilon^{-\sigma})^{m(m+1)}, \tag{3.10}$$

and upper triangular matrices $\{A_\ell\}_{\ell=1}^r$ with $spec(A_\ell) = \{e^{-2\pi i\rho} | \rho \in E_\ell\}$, such that

$$PAP^{-1} = diag\{A_1 \cdots A_r\}.$$

Proof. (3.8) and (3.9) follow from a simple observation. (3.10) follows from Lemma A^1 in [26]. □

Thus we can conjugate $\left(\alpha, Ae^{f(\cdot)}\right)$ to a new cocycle,

$$PAe^{f(x)}P^{-1} = diag\{A_1 \cdots A_r\} e^{Pf(x)P^{-1}} := \widetilde{A} e^{\tilde{f}(x)}. \tag{3.11}$$

Since $C \le C_0^{-\frac{500dm^3\tau}{\sigma}}$, together with the assumption that $\sigma < \frac{1}{500m^3}$, we have

$$|\tilde{f}|_h \le |P||f|_h|P^{-1}| \le \epsilon C_0^2(2|A|\epsilon^{-\sigma})^{2m(m+1)} \le \min\left\{\epsilon^{1-200m^2\sigma}, \epsilon^{\frac{9}{10}}\right\}. \tag{3.12}$$

Step 2: Eliminating the non-resonant terms. For any given $\alpha \in \mathbb{R}^d$ and $A \in GL(m,\mathbb{C})$, we decompose $\mathcal{B}_h = C_h^{\omega}(\mathbb{T}^d, gl(m,\mathbb{C})) = \mathcal{B}_h^{nre}(\epsilon^{\frac{2}{5}}) \bigoplus \mathcal{B}_h^{re}(\epsilon^{\frac{2}{5}})$ in such a way that for any $Y \in \mathcal{B}_h^{nre}(\epsilon^{\frac{2}{5}})$,

$$A^{-1}Y(\cdot+\alpha)A \in \mathcal{B}_h^{nre}(\epsilon^{\frac{2}{5}}), \qquad |A^{-1}Y(\cdot+\alpha)A - Y(\cdot)|_h \ge \epsilon^{\frac{2}{5}}|Y|_h. \tag{3.13}$$

Now we define

$$\Lambda_{k,\ell}(\epsilon^\sigma) = \left\{ n \in \mathbb{Z}^d : \|E_k - E_\ell - \langle n, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \epsilon^\sigma \right\}, \quad 1 \leq k, \ell \leq r.$$

where $\|E_k - E_\ell - \langle n, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} = \min_{\rho\in E_k, \rho' \in E_\ell} \|\rho - \rho' - \langle n, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}}$.

$$\Lambda_N = \left\{ f \in C_h^\omega(\mathbb{T}^d, gl(m, \mathbb{C})) \mid f(x) = \sum_{1\leq k,\ell\leq r} \sum_{n\in\Lambda_{k,\ell}(\epsilon^\sigma)} \hat{f}_{k,\ell}(n) e^{2\pi i\langle n,x\rangle} \right\},$$

where $\hat{f}_{k,\ell}(n) = P_{[\sum_{j=1}^{k-1} n_j+1, \sum_{j=1}^{k} n_j]} \hat{f}(n) P_{[\sum_{j=1}^{\ell-1} n_j+1, \sum_{j=1}^{\ell} n_j]}$ and $P_I : [1, m] \to I$ is a projection. The following lemma gives a characterization of the non-resonant space.

Lemma 3.2. *For any $Y \in \Lambda_N$, we have*

$$\left| \widetilde{A}^{-1} Y(\cdot + \alpha) \widetilde{A} - Y(\cdot) \right|_h \geq \epsilon^{\frac{2}{5}} |Y|_h.$$

Proof. Notice that $\widetilde{A} = diag\{A_1, \cdots, A_r\}$. Thus for any $Y \in \Lambda_N$, we have

$$Y(x + \alpha)\widetilde{A} - \widetilde{A}Y(x) = (Y_{k,\ell}(x + \alpha)A_\ell - A_k Y_{k,\ell}(x))_{1\leq k,\ell\leq r},$$

where $Y_{k,\ell}(x) = P_{[\sum_{j=1}^{k-1} n_j+1, \sum_{j=1}^{k} n_j]} Y(x) P_{[\sum_{j=1}^{\ell-1} n_j+1, \sum_{j=1}^{\ell} n_j]}$. For any (k, ℓ) block, we apply Lemma 5.1 by taking $A = A_\ell$, $B = A_k$, $\Lambda = \Lambda_{k,\ell}(\epsilon^\sigma)$ and $\eta = \epsilon^\sigma$. By the definition of Λ_N and (3.4), it's easy to verify that condition (5.1) in Lemma 5.1 is satisfied. Thus

$$\left| \widetilde{A}^{-1} Y(\cdot + \alpha) \widetilde{A} - Y(\cdot) \right|_h \geq \sum_{1\leq k,\ell\leq r} \epsilon^\sigma \left(1 + m\left(|A_k| + |A_\ell|\right)\epsilon^{-\sigma}\right)^{-2m} |Y_{k,\ell}|_h \geq \epsilon^{\frac{2}{5}} |Y|_h,$$

the last inequality holds because of $\sigma < \frac{1}{500m^3}$ and (3.4). □

This implies that $\Lambda_N \subset \mathcal{B}_h^{nre}(\epsilon^{\frac{2}{5}})$. Since $\epsilon^{\frac{2}{5}} \geq 20|A|^2 \epsilon^{\frac{9}{20}}$, we can apply Lemma 5.2 to remove all the non-resonant terms of $\tilde{f}$, which means there exist $Y \in \mathcal{B}_h$ and $\tilde{f}^{re} \in \mathcal{B}_h^{re}(\epsilon^{\frac{2}{5}})$ such that

$$e^{Y(x+\alpha)} \widetilde{A} e^{\tilde{f}(x)} e^{-Y(x)} = \widetilde{A} e^{\tilde{f}^{re}(x)}, \tag{3.14}$$

with $|Y|_h \leq \epsilon^{\frac{1}{3}}$ and $|\tilde{f}^{re}|_h \leq 2\min\left\{\epsilon^{1-200m^2\sigma}, \epsilon^{\frac{9}{10}}\right\}$.

Step 3: Structure of the resonant terms $\tilde{f}^{re}$. By Lemma 3.1, we assume that

$$\tilde{A} = diag\{A_1 \cdots A_r\}$$

is a block diagonal matrix with $spec(A_\ell) = \{e^{-2\pi i\rho} | \rho \in E_\ell\}$ $(\ell = 1, \cdots, r)$, satisfying (3.8) and (3.9). We say A_k and A_ℓ are N-resonant if there are $\rho \in E_k$, $\rho' \in E_\ell$, $n_{k,\ell} \in \mathbb{Z}^d$ with $0 < |n_{k,\ell}| \leq N$ such that

$$\|\rho - \rho' - \langle n_{k,\ell}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < \epsilon^\sigma.$$

By (3.8), for any $\rho \in E_k$ and $\rho' \in E_\ell$, we have

$$\|\rho - \rho' - \langle n_{k,\ell}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < 4m\epsilon^\sigma.$$

Let $\mathcal{R}_N \subset \{(k, \ell) | 1 \leq k < \ell \leq r\}$ be the set of all indexes such that A_k and A_ℓ are N-resonant.

Lemma 3.3 (Uniqueness of N-resonance)**.** *$n_{k,\ell}$ is the unique resonant site within length $\epsilon^{-\frac{\sigma}{2\tau}} \gg N$.*

Proof. Indeed, if there exists $n'_{k,\ell} \neq n_{k,\ell}$ satisfying $\|\rho - \rho' - \langle n'_{k,\ell}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < 4m\epsilon^\sigma$, then by the Diophantine condition of α, we have

$$\frac{\kappa}{|n'_{k,\ell} - n_{k,\ell}|^\tau} \leq \|\langle n'_{k,\ell} - n_{k,\ell}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} < 8m\epsilon^\sigma,$$

which implies that

$$|n'_{k,\ell}| > (8m)^{-\frac{1}{\tau}}\kappa^{\frac{1}{\tau}}\epsilon^{-\frac{\sigma}{\tau}} - \epsilon^{-\frac{\sigma}{2\tau}} \geq \epsilon^{-\frac{\sigma}{2\tau}}.$$

The last inequality holds since $\epsilon \leq 2^{-\frac{2\tau}{\sigma}}(4m+1)^{-\frac{2}{\sigma}}\kappa^{\frac{2}{\sigma}}$. □

Now we are in the position to characterize $\tilde{f}^{re}(x)$.

Lemma 3.4 (Structure of resonances)**.** *There exists $1 \leq i_0 \leq m^2$ such that*

$$\tilde{f}^{re}(x) = \tilde{f}^{re}_0 + \tilde{f}^{re}_1(x) + \tilde{f}^{re}_2(x),$$

$$\tilde{f}^{re}_0 = \sum_{\ell=1}^{r}\widehat{\tilde{f}^{re}_{\ell,\ell}}(0), \quad \tilde{f}^{re}_2(x) = \sum_{|n|\geq N^{i_0+1}}\widehat{\tilde{f}^{re}}(n)e^{2\pi i\langle n,x\rangle},$$

$$\tilde{f}^{re}_1(x) = \sum_{(k,\ell)\in\mathcal{R}_{N^{i_0}}}\left(\widehat{\tilde{f}^{re}_{k,\ell}}(n_{k,\ell})e^{2\pi i\langle n_{k,\ell},x\rangle} + \widehat{\tilde{f}^{re}_{\ell,k}}(-n_{k,\ell})e^{-2\pi i\langle n_{k,\ell},x\rangle}\right).$$

Proof. Since $\Lambda_N \in \mathcal{B}^{nre}_h(\epsilon^{\frac{2}{5}})$, we have

$$\tilde{f}^{re}(x) = \sum_{1\leq k,\ell\leq r}\sum_{n\in\mathbb{Z}^d\backslash\Lambda_{k,\ell}(\epsilon^\sigma)}\widehat{\tilde{f}^{re}_{k,\ell}}(n)e^{2\pi i\langle n,x\rangle}.$$

Let $\mathcal{N}_i := [N^i, N^{i+1})$ for $1 \leq i \leq m^2$. There are at most $m(m-1)$ different $n_{k,\ell}$'s, and m^2 different $\mathcal{N}_i$'s, thus by pigeonhole principle, there exists $1 \leq i_0 \leq m^2$ such that $n_{k,\ell} \notin \mathcal{N}_{i_0}$. I.e.,

$$\|E_k - E_\ell - \langle n,\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \epsilon^\sigma, \quad N^{i_0} \leq n < N^{i_0+1}, \quad \forall 1 \leq k,\ell \leq r,$$

which means there is no resonance in a large scale. We define

$$\tilde{f}^{re}_2(x) = \sum_{|n|\geq N^{i_0+1}}\widehat{\tilde{f}^{re}}(n)e^{2\pi i\langle n,x\rangle}, \tag{3.15}$$

then it follows

$$\tilde{f}^{re}(x) - \tilde{f}^{re}_2(x) = \sum_{1\leq k,\ell\leq r}\sum_{n\in\{\mathbb{Z}^d\backslash\Lambda_{k,\ell}(\epsilon^\sigma)\}\cap\{n||n|\leq N^{i_0}\}}\widehat{\tilde{f}^{re}_{k,\ell}}(n)e^{2\pi i\langle n,x\rangle}. \tag{3.16}$$

By the Diophantine condition on the frequency α and (3.8), for any $\rho, \rho' \in E_\ell$ $(1 \leq \ell \leq r)$ and $0 < n \leq N^{i_0}$, we have

$$\|\rho - \rho' - \langle n,\alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\kappa}{|n|^\tau} - 4m\epsilon^\sigma \geq \kappa N^{-m^2\tau} - 4m\epsilon^\sigma \geq \epsilon^\sigma, \tag{3.17}$$

the last inequality holds since $\epsilon \leq \frac{2^{-\frac{2\tau}{\sigma}}(4m+1)^{-\frac{2}{\sigma}}\kappa^{\frac{2}{\sigma}}}{|A|^{\frac{500dm^3\tau}{\sigma}}}(h-h')^{\frac{500dm^3\tau}{\sigma}}$. It follows from (3.17) that

$$\left\{\mathbb{Z}^d\backslash\Lambda_{\ell,\ell}(\epsilon^\sigma)\right\}\cap\{n\in\mathbb{Z}^d : |n| \leq N^{i_0}\} = \{0\}, \quad \ell = 1,\cdots,r. \tag{3.18}$$

On the other hand, by Lemma 3.3, we have

$$\left\{\mathbb{Z}^d\backslash\Lambda_{k,\ell}(\epsilon^\sigma)\right\}\cap\{n\in\mathbb{Z}^d : |n| \leq N^{i_0}\} = \{\pm n_{k,\ell}\}, \quad k \neq \ell, \tag{3.19}$$

since $N^{m^2} \leq \epsilon^{-\frac{\sigma}{2\tau}}$ by (3.4).

(3.15), (3.16),(3.18) and (3.19) finish the proof. □

Step 4. Eliminating the lower order resonant terms.

Lemma 3.5. *There exists a family* $m_1, \cdots, m_r$ *with* $\max_{1\le\ell\le r} |m_\ell| \le rN^{i_0}$ *such that*

$$m_k - m_\ell = -n_{k,\ell}, \quad (k,\ell) \in \mathcal{R}_{N^{i_0}}.$$

Proof. We prove this by induction, assume for $\mathcal{R}_{N^{i_0}} \cap \{(k,\ell)|1 \le k < \ell \le r-1\}$, the above lemma holds, which means there exists a family $m_1 \cdots m_{r-1}$ with $\max_{1\le\ell\le r} |m_\ell| \le (r-1)N^{i_0}$ such that

$$m_k - m_\ell = -n_{k,\ell}, \quad (k,\ell) \in \mathcal{R}_{N^{i_0}} \cap \{(k,\ell)|1 \le k < \ell \le r-1\}.$$

We consider $\mathcal{R}_{N^{i_0}}$. There are two possible cases.
Case I: There exists $1 \le \ell \le r-1$, such that $(\ell, r) \in \mathcal{R}_{N^{i_0}}$. Thus there exists $0 < |n_{\ell,r}| \le N^{i_0}$ such that

$$\|\rho - \rho' - \langle n_{\ell,r}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \le \epsilon^\sigma,$$

for some $\rho \in E_\ell$ and $\rho' \in E_r$. On the other hand, if there exists $1 \le \ell' \le r-1$, such that $(\ell', r) \in \mathcal{R}_{N^{i_0}}$, then there exists $0 < |n_{\ell',r}| \le N^{i_0}$ such that

$$\|\tilde\rho - \tilde\rho' - \langle n_{\ell',r}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \le \epsilon^\sigma,$$

for some $\tilde\rho \in E_{\ell'}$ and $\tilde\rho' \in E_r$. This implies that

$$\|\rho - \tilde\rho - \langle n_{\ell,r} - n_{\ell',r}, \alpha\rangle\|_{\mathbb{R}/\mathbb{Z}} \le 4m\epsilon^\sigma.$$

Thus

$$n_{\ell,r} - n_{\ell',r} = n_{\ell,\ell'}.$$

Let $m_r = m_\ell + n_{\ell,r}$, then

$$m_{\ell'} - m_r = m_{\ell'} - m_\ell - n_{\ell,r} = n_{\ell,\ell'} - n_{\ell,r} = -n_{\ell',r},$$

for $\ell' \ne r$, with the estimate

$$|m_r| \le |m_\ell| + |n_{\ell,r}| \le (r-1)N^{i_0} + N^{i_0} = rN^{i_0}.$$

Case II: $(\ell, r) \notin \mathcal{R}_{N^{i_0}}$ for any $1 \le \ell \le r-1$, let $m_r = 0$, we get the result. □

Define the $\mathbb{Z}^d$-periodic rotation $Q(x)$ as below:

$$Q(x) = diag\{e^{-2\pi i\langle m_1, x\rangle} I_{n_1}, \cdots, e^{-2\pi i\langle m_r, x\rangle} I_{n_r}\}.$$

So we have

$$|Q|_{h_+} \le e^{rN^{i_0}h_+} \le e^{mN^{i_0}h_+}. \tag{3.20}$$

One can also show that

$$Q(x+\alpha)\widetilde{A}e^{\tilde f^{re}(x)}Q^{-1}(x) = \widehat{A}e^{\hat f(x)}, \tag{3.21}$$

where

$$\widehat{A} = Q(x+\alpha)diag\{A_1, \cdots, A_r\}Q^{-1}(x) = diag\{A_1 e^{-2\pi i\langle m_1,\alpha\rangle}, \cdots, A_r e^{-2\pi i\langle m_r,\alpha\rangle}\},$$

and

$$\hat f(x) = Q(x)\tilde f^{re}(x)Q^{-1}(x) = Q(x)\tilde f_0^{re}Q^{-1}(x) + Q(x)\tilde f_1^{re}(x)Q^{-1}(x) + Q(x)\tilde f_2^{re}(x)Q^{-1}(x).$$

Moreover,

$$Q(x)\tilde f_0^{re}Q^{-1}(x) = \tilde f_0^{re} \in gl(m,\mathbb{C}),$$

$$Q(x)\tilde f_1^{re}(x)Q^{-1}(x) = \sum_{(k,\ell)\in\mathcal{R}_{N^{i_0}}} \left(\widehat{\tilde f^{re}_{k,\ell}}(n_{k,\ell}) + \widehat{\tilde f^{re}_{\ell,k}}(-n_{k,\ell})\right) \in gl(m,\mathbb{C}).$$

Denote

$$L=Q(x)\tilde{f}_0^{re}Q^{-1}(x)+Q(x)f_1^{re}(x)Q^{-1}(x),\quad F(x)=Q(x)f_2^{re}(x)Q^{-1}(x),\quad \bar{B}(x)=Q(x)e^{Y(x)}P.$$

By (3.11), (3.14) and (3.21), we have

$$\bar{B}(x+\alpha)Ae^{f(x)}\bar{B}^{-1}(x)=\widehat{A}e^{\hat{f}(x)}=\widehat{A}e^{L+F(x)}. \tag{3.22}$$

Moreover, we have the following estimates:

$$|\bar{B}|_0\le |e^Y|_{h_+}|P|\le 2C_0(2|A|\epsilon^{-\sigma})^{m(m+1)}\le \epsilon^{-50m^2\sigma}, \tag{3.23}$$

$$|F|_{h_+}\le |Qf_2^{re}Q^{-1}|_{h_+}\le 2\epsilon^{\frac{9}{10}}e^{-N^{i_0+1}(h-h_+)}e^{2mN^{i_0}h_+}\le 2\epsilon^{\frac{9}{10}}e^{-N^{i_0}(N(h-h_+)-2mh_+)}\le\epsilon^3.$$

$$|L|\le |\tilde{f}_0^{re}|+\sum_{(k,\ell)\in\mathcal{R}_{N^{i_0}}}\left(|\widehat{\tilde{f}^{re}_{k,\ell}}(n_{k,\ell})|+|\widehat{\tilde{f}^{re}_{\ell,k}}(-n_{k,\ell})|\right)\le 2m^2|\tilde{f}^{re}|_h\le 2m^2\epsilon^{1-200m^2\sigma}. \tag{3.24}$$

Direct computation shows that

$$e^{L+F(x)}=e^L+\mathcal{O}(F(x))=e^L(Id+e^{-L}\mathcal{O}(F(x)))=e^Le^{f_+(x)}. \tag{3.25}$$

It immediately implies that

$$|f_+|_{h'}\le 2|F|_{h_+}\le\epsilon_+.$$

Thus we can rewrite (3.22) as

$$\bar{B}(x+\alpha)(Ae^{f(x)})\bar{B}^{-1}(x)=A_+e^{f_+(x)}, \tag{3.26}$$

with

$$A_+=diag\{A_1e^{-2\pi i\langle m_1,\alpha\rangle},\cdots,A_re^{-2\pi i\langle m_r,\alpha\rangle}\}e^L. \tag{3.27}$$

Combining the above four steps, let $B_{j+1}(x)=\bar{B}(x)B_j(x)$, $A_{j+1}=A_+$ and $f_{j+1}=f_+$, then $B_{j+1}\in C^\omega_{h_{j+1}}(\mathbb{T}^d,GL(m,\mathbb{C}))$, by (3.6) and (3.26), we have

$$B_{j+1}(x+\alpha)A_0e^{f_0(x)}B_{j+1}^{-1}(x)=A_{j+1}e^{f_{j+1}(x)}. \tag{3.28}$$

By (3.7) and (3.23), we have

$$|B_{j+1}|_0\le |B_j|_0|\bar{B}|_0\le \epsilon_{j-1}^{-200m^2\sigma}\epsilon_j^{-50m^2\sigma}\le\epsilon_j^{-200m^2\sigma}. \tag{3.29}$$

By (3.27), Proposition 2.4 and (3.24), we can permute the eigenvalues of A_{j+1} as $\{e^{-2\pi i\rho_\ell^{j+1}}\}_{\ell=1}^m$, such that

$$\left|\Im\rho_\ell^{j+1}-\Im\rho_\ell^j\right|\le C(m,A_j)(2m^2\epsilon_j^{(1-200m^2\sigma)})^{\frac{1}{m}}\le C(m,A_0)\epsilon_j^{\frac{1}{m}-200m^2\sigma},\quad 1\le\ell\le m. \tag{3.30}$$

On the other hand, it is obvious that

$$\left|\sum_{\ell=1}^m\Im\rho_\ell^{j+1}-\sum_{\ell=1}^m\Im\rho_\ell^j\right|\le C(m,A_0)\epsilon_j^{\frac12}. \tag{3.31}$$

(3.28), (3.29), (3.30) and (3.31) finish the proof. □

Corollary 3.1. *Assume (α,A) is C_h^ω almost reducible, for any given $0<\sigma<\frac{1}{500m^3}$ and $0<h'<h$, there exist $B_j\in C^\omega_{h'}(\mathbb{T}^d,GL(m,\mathbb{C}))$, $A_j\in GL(m,\mathbb{C})$ and $f_j\in C^\omega_{h'}(\mathbb{T}^d,gl(m,\mathbb{C}))$, such that*

$$B_j(x+\alpha)A(x)B_j^{-1}(x)=A_je^{f_j(x)}.$$

Moreover, we have the following estimates

$$|f_j|_{h'}\le\epsilon_j,\quad |B_j|_0\le C(\alpha,A)\epsilon_{j-1}^{-200m^2\sigma},$$

$$\left|\sum_{\ell=1}^{m}\Im\rho_\ell^j-\sum_{\ell=1}^{m}\Im\rho_\ell^{j-1}\right|\leq C(m,A)\epsilon_{j-1}^{\frac{1}{2}},$$

$$\left|\Im\rho_\ell^j-\Im\rho_\ell^{j-1}\right|\leq C(m,A)\epsilon_{j-1}^{\frac{1}{m}-200m^2\sigma},\quad 1\leq\ell\leq m,$$

where $\left\{e^{-2\pi i\rho_\ell^j}\right\}_{\ell=1}^m$ *are the eigenvalues of* A_j *and* $\epsilon_j=\epsilon_0^{2^j}$.

Proof. By the assumption, (α,A) is C_h^ω almost reducible, there exist $B\in C_h^\omega(\mathbb{T}^d,\ GL(m,\mathbb{C}))$, $A_0\in GL(m,\mathbb{C})$, $f_0\in C_h^\omega(\mathbb{T}^d,gl(m,\mathbb{C}))$, such that

$$B(x+\alpha)A(x)B^{-1}(x)=A_0e^{f_0(x)},$$

with $|f_0|_h\leq\epsilon_0\leq\epsilon(\alpha,h,h',m,|A|,\sigma)$ which is defined in Theorem 3.1. Applying Theorem 3.1 to $(\alpha,A_0e^{f_0(\cdot)})$, there exist $\bar{B}_j\in C_{h'}^\omega(\mathbb{T}^d,\ GL(m,\mathbb{C}))$, $A_j\in GL(m,\mathbb{C})$, $f_j\in C_{h'}^\omega(\mathbb{T}^d,gl(m,\mathbb{C}))$, such that

$$\bar{B}_j(x+\alpha)(A_0e^{f_0(x)})\bar{B}_j^{-1}(x)=A_je^{f_j(x)}.$$

Moreover, we have the following estimates

$$|f_j|_{h'}\leq\epsilon_j,\quad|\bar{B}_j|_0\leq\epsilon_{j-1}^{-200m^2\sigma},$$

$$\left|\sum_{\ell=1}^{m}\Im\rho_\ell^j-\sum_{\ell=1}^{m}\Im\rho_\ell^{j-1}\right|\leq C(m,A_0)\epsilon_{j-1}^{\frac{1}{2}},$$

$$\left|\Im\rho_\ell^j-\Im\rho_\ell^{j-1}\right|\leq C(m,A_0)\epsilon_{j-1}^{\frac{1}{m}-200m^2\sigma},\quad 1\leq\ell\leq m,$$

where $\left\{e^{-2\pi i\rho_\ell^j}\right\}_{\ell=1}^m$ are the eigenvalues of A_j and $\epsilon_j=\epsilon_0^{2^j}$.

The desired results follow if we denote $B_j(x)=\bar{B}_j(x)B(x)$. □

4. Proof of the main Theorems

In this section, we give the proof of the main theorems. Our main ideas are based on the following basic facts.

Proposition 4.1. *Assume that* $(\alpha,A)/(\alpha,\widetilde{A})$ *is* C_h^ω *almost reducible to* $(\alpha,A_\infty)/(\alpha,\widetilde{A}_\infty)$. *Denote the eigenvalues of* $A_\infty/\widetilde{A}_\infty$ *by* $\{e^{-2\pi i\rho_j^\infty}\}_{j=1}^m/\{e^{-2\pi i\tilde{\rho}_j^\infty}\}_{j=1}^m$ *respectively. If*

$$|\Im\rho_j^\infty-\Im\tilde{\rho}_j^\infty|\leq\epsilon,\quad 1\leq j\leq m,\tag{4.1}$$

then

$$|L_j(\alpha,A)-L_j(\alpha,\widetilde{A})|\leq 100\epsilon,\quad 1\leq j\leq m.\tag{4.2}$$

Proof. We prove (4.2) inductively based on (4.1). For $m=1$, by Proposition 2.3, it is obvious. Now we assume (4.2) holds for $m\leq k$. We consider the case $m=k+1$, again by Proposition 2.3, we can assume

$$L_1(\alpha,A)=\max_j 2\pi\Im\rho_j^\infty=2\pi\Im\rho_{j_0}^\infty,\quad L_1(\alpha,\widetilde{A})=\max_j 2\pi\Im\tilde{\rho}_j^\infty=2\pi\Im\tilde{\rho}_{j_1}^\infty.$$

We distinguish into two cases,
Case I: $j_0=j_1$, by (4.1),

$$|L_1(\alpha,A)-L_1(\alpha,\widetilde{A})|=2\pi|\Im\rho_{j_0}^\infty-\Im\tilde{\rho}_{j_1}^\infty|\leq 2\pi\epsilon.$$

By induction, we have

$$|L_j(\alpha,A)-L_j(\alpha,\widetilde{A})|\leq 100\epsilon,\quad 2\leq j\leq k+1.$$

Case II: $j_0 \neq j_1$, by (4.1),

$$|\Im\rho_{j_0}^{\infty} - \Im\tilde{\rho}_{j_0}^{\infty}| \leq \epsilon, \quad |\Im\rho_{j_1}^{\infty} - \Im\tilde{\rho}_{j_1}^{\infty}| \leq \epsilon.$$

On the other hand,

$$L_1(\alpha, A) = \max_j 2\pi\Im\rho_j^{\infty} = 2\pi\Im\rho_{j_0}^{\infty}, \quad L_1(\alpha, \widetilde{A}) = \max_j 2\pi\Im\tilde{\rho}_j^{\infty} = 2\pi\Im\tilde{\rho}_{j_1}^{\infty}.$$

We must have

$$|\Im\rho_{j_0}^{\infty} - \Im\rho_{j_1}^{\infty}| \leq 50\epsilon.$$

Thus

$$|L_1(\alpha, A) - L_1(\alpha, \widetilde{A})|| = |\Im\rho_{j_0}^{\infty} - \Im\tilde{\rho}_{j_1}^{\infty}| \leq |\Im\rho_{j_0}^{\infty} - \Im\rho_{j_1}^{\infty}| + |\Im\rho_{j_1}^{\infty} - \Im\tilde{\rho}_{j_1}^{\infty}| \leq 100\epsilon.$$

By induction, we have

$$|L_j(\alpha, A) - L_j(\alpha, \widetilde{A})| \leq 100\epsilon, \quad 2 \leq j \leq k+1.$$

Thus we finish the proof. □

For any given $A_0 \in GL(m, \mathbb{C})$ with eigenvalues $\{e^{-2\pi i \rho_\ell^0}\}_{\ell=1}^m$, we assume that $\{\rho_\ell^0\}_{1\leq \ell\leq m}$ can be grouped as $\bigcup_{k=1}^{r} E_k$ with $\#E_k = n_k$, such that

$$|\rho - \rho'| \geq \delta > 0, \quad \rho \in E_k, \rho' \in E_\ell, \quad \forall k \neq \ell. \tag{4.3}$$

Then we have the following refinements of Theorem 3.1.

Proposition 4.2. *Assume $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, $0 < \sigma < \frac{1}{500m^3}$ and $0 < h' < h$, there exists $\epsilon_0 = \epsilon(\alpha, h, h', m, |A_0|, \sigma, \delta)$ such that if $|f_0|_h \leq \epsilon_0$, then $(\alpha, A_0 e^{f_0(\cdot)})$ is $C_{h'}^{\omega}$ almost reducible to (α, A_∞). Moreover, $\{e^{-2\pi i\rho_\ell^\infty}\}_{\ell=1}^m$, the eigenvalues of A_∞ satisfy*

$$\left|\Im\rho_\ell^{\infty} - \Im\rho_\ell^0\right| \leq C(m, \delta, A_0)\epsilon_0^{\frac{1}{n_k} - 200m^2\sigma}, \quad \sum_{j=1}^{k-1} n_j + 1 \leq \ell \leq \sum_{j=1}^{k} n_j, \quad 1 \leq k \leq r. \tag{4.4}$$

$$\left|\sum_{\ell=1}^{n_k}\Im\rho_\ell^{\infty} - \sum_{\ell=1}^{n_k}\Im\rho_\ell^0\right| \leq C(m, \delta, A_0)\epsilon_0^{\frac{1}{2}}, \quad 1 \leq k \leq r. \tag{4.5}$$

Proof. By Lemma A^1 in [26], there exists $P \in GL(m, \mathbb{C})$ such that

$$PA_0P^{-1} = \widetilde{A}_0 := diag\{A_0^1, A_0^2, \cdots, A_0^r\},$$

with $\|P\| \leq C(\delta, m, |A_0|)$ and $spec(A_0^k) = \{e^{-2\pi i\rho} | \rho \in E_k\} (k = 1, \cdots, r)$.

Thus we can conjugate $\left(\alpha, A_0 e^{f_0(\cdot)}\right)$ to a new cocycle,

$$PA_0 e^{f_0(x)}P^{-1} = diag\{A_0^1 \cdots A_0^r\} e^{Pf_0(x)P^{-1}} := \widetilde{A}_0 e^{\tilde{f}_0(x)},$$

with

$$|\tilde{f}|_h \leq |P||f_0|_h|P^{-1}| \leq C(m, \delta, |A_0|)|f_0|_h.$$

Now we define

$$\Lambda = \left\{ f \in C_h^{\omega}(\mathbb{T}^d, gl(m, \mathbb{C})) \mid f(x) = \sum_{1\leq k\neq \ell\leq r} \hat{f}_{k,\ell}(n) e^{2\pi i\langle n, x\rangle} \right\},$$

where $\hat{f}_{k,\ell}(n) = P_{[\sum_{j=1}^{k-1} n_j+1, \sum_{j=1}^{k} n_j]}\hat{f}(n)P_{[\sum_{j=1}^{\ell-1} n_j+1, \sum_{j=1}^{\ell} n_j]}$ and $P_I : [1, m] \to I$ is a projection. Similar to Lemma 3.2, one can verify that for any $Y \in \Lambda$,

$$|\widetilde{A}_0^{-1}Y(\cdot + \alpha)\widetilde{A}_0 - Y(\cdot)|_h \geq c(m)\delta\left(1 + m\left(|\widetilde{A}_0| + |\widetilde{A}_0|\right)\delta^{-1}\right)^{-2m}|Y|_h \geq c(m, A_0, \delta)|Y|_h.$$

Thus $\Lambda \subset \mathcal{B}_h^{nre}(c)$. Choose ϵ_0 sufficiently small such that $c \geq 13|\widetilde{A}_0|^2 C^2 \epsilon_0^2$, by Lemma 5.2, there exists $B \in C^\omega(\mathbb{T}^d, GL(m,\mathbb{C}))$ such that

$$B(x+\alpha)\widetilde{A}_0 e^{\tilde{f}_0(x)} B^{-1}(x) = \widetilde{A}_0 e^{\tilde{f}_0^{re}(x)} := diag\left\{A_0^1 e^{f_0^1(x)}, A_0^2 e^{f_0^2(x)}, \cdots, A_0^r e^{f_0^r(x)}\right\},$$

with estimates

$$|f_0^k|_h \leq C(m, A_0, \delta)|f_0|_h \leq C(m, A_0, \delta)\epsilon_0, \quad k = 1, 2, \cdots, r, \quad |B|_0 \leq 2. \tag{4.6}$$

Assume $\epsilon_0 \leq \epsilon(\alpha, h, h', m, |A_0|, \sigma)\delta^{100}$ where $\epsilon(\alpha, h, h', m, |A_0|, \sigma)$ is defined in Theorem 3.1. Applying Theorem 3.1 to $(\alpha, A_0^k e^{f_0^k(x)})$ $(k = 1, \cdots, r)$, there exist $B_j^k \in C_{h'}^\omega(\mathbb{T}^d, GL(n_k, \mathbb{C}))$, $A_j^k \in GL(n_k, \mathbb{C})$, $f_j^k \in C_{h'}^\omega(\mathbb{T}^d, gl(n_k, \mathbb{C}))$, such that

$$B_j^k(x+\alpha)(A_0^k e^{f_0^k(x)})(B_j^k)^{-1}(x) = A_j^k e^{f_j^k(x)}.$$

Moreover, we have the following estimates

$$\left| \sum_{\ell=\sum_{j=1}^{k-1} n_j + 1}^{\sum_{j=1}^{k} n_j} \Im\rho_\ell^j - \sum_{\ell=\sum_{j=1}^{k-1} n_j + 1}^{\sum_{j=1}^{k} n_j} \Im\rho_\ell^{j-1} \right| \leq C(m, A_0)\epsilon_{j-1}^{\frac{1}{2}},$$

$$\left|\Im\rho_\ell^j - \Im\rho_\ell^{j-1}\right| \leq C(m, A_0)\epsilon_{j-1}^{\frac{1}{n_k} - 200m^2\sigma}, \quad \sum_{j=1}^{k-1} n_j + 1 \leq \ell \leq \sum_{j=1}^{k} n_j,$$

where $\left\{e^{-2\pi i \rho_\ell^j}\right\}_{\ell=\sum_{j=1}^{k-1} n_j + 1}^{\sum_{j=1}^{k} n_j}$ are the eigenvalues of A_j^k and $\epsilon_j = \epsilon_0^{2^j}$.

It follows that

$$\left| \sum_{\ell=\sum_{j=1}^{k-1} n_j + 1}^{\sum_{j=1}^{k} n_j} \Im\rho_\ell^\infty - \sum_{\ell=\sum_{j=1}^{k-1} n_j + 1}^{\sum_{j=1}^{k} n_j} \Im\rho_\ell^0 \right| \leq \sum_{j=1}^{\infty} \left| \sum_{\ell=\sum_{j=1}^{k-1} n_j + 1}^{\sum_{j=1}^{k} n_j} \Im\rho_\ell^j - \sum_{\ell=\sum_{j=1}^{k-1} n_j + 1}^{\sum_{j=1}^{k} n_j} \Im\rho_\ell^{j-1} \right| \leq C(m, \delta, A_0)\epsilon_0^{\frac{1}{2}},$$

$$\begin{aligned} \left|\Im\rho_\ell^\infty - \Im\rho_\ell^0\right| &\leq \sum_{j=1}^{\infty} \left|\Im\rho_\ell^j - \Im\rho_\ell^{j-1}\right| \leq \sum_{j=1}^{\infty} C(m, \delta, A_0)\epsilon_{j-1}^{\frac{1}{n_k} - 200m^2\sigma} \\ &\leq C(m, \delta, A_0)\epsilon_0^{\frac{1}{n_k} - 200m^2\sigma}, \quad \sum_{j=1}^{k-1} n_j + 1 \leq \ell \leq \sum_{j=1}^{k} n_j. \end{aligned}$$

Thus we finish the whole proof. □

If we further assume the eigenvalues of A_0 satisfy

$$\rho - \rho' \geq \delta > 0, \quad \rho \in E_k, \rho' \in E_\ell, \quad \forall k < \ell. \tag{4.7}$$

We immediately have the following corollary.

Corollary 4.1. *Assume $\alpha \in \mathrm{DC}_d(\kappa, \tau)$ and $0 < \sigma < \frac{1}{500m^3}$. There exists $\epsilon_0 = \epsilon(\alpha, h, m, |A_0|, \sigma, \delta)$ such that if $|f_0|_h \leq \epsilon_0$, then*

$$\left|L_\ell(\alpha, A_0 e^{f_0(\cdot)}) - \Im\rho_\ell^0\right| \leq C(m, \delta, A_0)\epsilon_0^{\frac{1}{n_k} - 200m^2\sigma}, \quad \sum_{j=1}^{k-1} n_j + 1 \leq \ell \leq \sum_{j=1}^{k} n_j, \quad 1 \leq k \leq r.$$

$$\left|\sum_{\ell=1}^{n_k} L_\ell(\alpha, A_0 e^{f_0(\cdot)}) - \sum_{\ell=1}^{n_k} \Im\rho_\ell^0\right| \le C(m,\delta,A_0)\epsilon_0^{\frac{1}{2}}, \quad 1\le k\le r.$$

Proof. Take $h' = \frac{h}{2}$ and $\epsilon_0 = \epsilon(\alpha, h, h/2, m, |A_0|, \sigma, \delta)$ which is defined in Proposition 4.2, then $(\alpha, A_0 e^{f_0(\cdot)})$ is $C^\omega_{h'}$ almost reducible to (α, A_∞). Moreover, $\{e^{-2\pi i\rho_\ell^\infty}\}_{\ell=1}^m$, the eigenvalues of A_∞ satisfy

$$\left|\Im\rho_\ell^\infty - \Im\rho_\ell^0\right| \le C(m,\delta,A_0)\epsilon_0^{\frac{1}{n_k}-200m^2\sigma}, \quad \sum_{j=1}^{k-1} n_j + 1 \le \ell \le \sum_{j=1}^{k} n_j, \quad 1\le k\le r. \tag{4.8}$$

$$\left|\sum_{\ell=1}^{n_k} \Im\rho_\ell^\infty - \sum_{\ell=1}^{n_k} \Im\rho_\ell^0\right| \le C(m,\delta,A_0)\epsilon_0^{\frac{1}{2}}, \quad 1\le k\le r. \tag{4.9}$$

By (4.7) and (4.8), $\{\rho_\ell^\infty\}_{\ell=1}^m$ can be grouped as $\bigcup_{k=1}^r E_k^\infty$ with $\#E_k^\infty = n_k$, such that

$$\rho-\rho' \ge \frac{\delta}{2}, \quad \rho\in E_k^\infty, \rho'\in E_\ell^\infty, \quad \forall k<\ell. \tag{4.10}$$

By (4.10) and Proposition 2.3, we have

$$\left\{L_\ell(\alpha, A_0 e^{f_0(\cdot)})\right\}_{\ell=\sum_{j=1}^{k-1} n_j+1}^{\sum_{j=1}^{k} n_j} = E_k^\infty, \quad 1\le k\le r. \tag{4.11}$$

The desired result follows from (4.8), (4.9), (4.11) and Proposition 4.1. □

4.1. **Proof of Theorem 1.2.** We assume $(\alpha, A)/(\alpha, \widetilde{A})$ is almost reducible to $(\alpha, A_\infty)/(\alpha, \widetilde{A}_\infty)$ respectively. We divide the proof into the following three steps.

Step 1. Group the eigenvalues. Since (α, A) is C^ω_h almost reducible, by Corollary 3.1, for any $\varepsilon>0$, there exist $B_j \in C^\omega_{h'}(\mathbb{T}^d, GL(m,\mathbb{C}))$, $A_j\in GL(m,\mathbb{C})$ and $f_j\in C^\omega_{h'}(\mathbb{T}^d, gl(m,\mathbb{C}))$, such that

$$B_j(x+\alpha)A(x)B_j^{-1}(x) = A_j e^{f_j(x)}.$$

Moreover, we have the following estimates

$$|f_j|_{h'} \le \epsilon_j, \quad |B_j|_0 \le C(\alpha, A)\epsilon_{j-1}^{-200m^2\varepsilon}, \tag{4.12}$$

$$\left|\Im\rho_\ell^j - \Im\rho_\ell^{j-1}\right| \le C(m,A)\epsilon_{j-1}^{\frac{1}{m}-200m^2\varepsilon}, \quad 1\le \ell\le m, \tag{4.13}$$

where $\left\{e^{-2\pi i\rho_\ell^j}\right\}_{\ell=1}^m$ are the eigenvalues of A_j and $\epsilon_j = \epsilon_0^{2^j}$.

We assume there are r distinct Lyapunov exponents of (α, A). By Proposition 2.3, there are r distinct $\Im\rho_\ell^\infty$'s. Thus $\{\rho_\ell^\infty\}_{\ell=1}^m$ can be grouped as $\bigcup_{k=1}^r E_k^\infty$ with $\#E_k^\infty = n_k$, such that

$$\rho-\rho' \ge 2\eta > 0, \quad \rho\in E_k^\infty, \rho'\in E_\ell^\infty, \quad k<\ell.$$

By (4.13), for j sufficiently large depending on α, A, η, we can group $\{\rho_\ell^{j+1}\}_{\ell=1}^m$ as $\bigcup_{k=1}^r E_k^{j+1}$ with $\#E_k^{j+1} = n_k$, such that

$$\rho-\rho' \ge \eta, \quad \rho\in E_k^{j+1}, \rho'\in E_\ell^{j+1}, \quad k<\ell.$$

Step 2. Compare (α, A) and $(\alpha, \widetilde{A})$. Denote $\delta = |B - A|_h$, we only need to consider the case that δ is sufficiently small. Assume that $\epsilon_0 \le \epsilon(\alpha, h, h', |A|, m, \varepsilon, \eta)$ which is defined in Proposition 4.2 and $\epsilon_{j+1} \le \delta \le \epsilon_j$. It is obvious that

$$\begin{aligned} B_{j+1}(x+\alpha)\widetilde{A}(x)B_{j+1}^{-1}(x) &= B_{j+1}(x+\alpha)(\widetilde{A}(x) - A(x))B_{j+1}^{-1}(x) + B_{j+1}(x+\alpha)A(x)B_{j+1}^{-1}(x) \\ &= B_{j+1}(x+\alpha)(B(x) - A(x))B_{j+1}^{-1}(x) + A_{j+1}e^{f_{j+1}(x)} \\ &:= A_{j+1}e^{\tilde{f}_{j+1}(x)}. \end{aligned}$$

By (4.12), we have

$$|\tilde{f}_{j+1}|_h \le 100\delta |B_{j+1}|_0^2 + \epsilon_{j+1} \le C(\alpha, A)\delta\epsilon_j^{-800m^2\varepsilon} \le C(\alpha, A)\delta^{1-800m^2\varepsilon}.$$

Step 3. Hölder continuity of the Lyapunov exponents. By Corollary 4.1, we have

(1) $\left|L_\ell\left(\alpha, A_{j+1}e^{f_{j+1}(\cdot)}\right) - \Im\rho_\ell^{j+1}\right| \le C(m, \delta, A)\delta^{\frac{1}{n_k} - 200m^2\varepsilon}, \quad \sum_{j=1}^{k-1} n_j + 1 \le \ell \le \sum_{j=1}^{k} n_j, \ \ 1 \le k \le r.$

(2) $\left|L_\ell\left(\alpha, A_{j+1}e^{\tilde{f}_{j+1}(\cdot)}\right) - \Im\rho_\ell^{j+1}\right| \le C(m, \delta, A)\delta^{\frac{1}{n_k} - 800m^2\varepsilon}, \quad \sum_{i=1}^{k-1} n_j + 1 \le \ell \le \sum_{j=1}^{k} n_j, \ \ 1 \le k \le r.$

Thus

$$\left|L_\ell\left(\alpha, A_{j+1}e^{f_{j+1}(\cdot)}\right) - L_\ell\left(\alpha, A_{j+1}e^{\tilde{f}_{j+1}(\cdot)}\right)\right| \le C(m, \delta, A)\delta^{\frac{1}{n_k} - 800m^2\varepsilon}, \quad \sum_{j=1}^{k-1} n_j + 1 \le \ell \le \sum_{j=1}^{k} n_j, \ \ 1 \le k \le r.$$

By Proposition 2.2, we have

$$|L_\ell(\alpha, A) - L_\ell(\alpha, \widetilde{A})| \le C\delta^{\frac{1}{n_k}(1 - 800m^2\varepsilon)}, \quad \sum_{j=1}^{k-1} n_j + 1 \le \ell \le \sum_{j=1}^{k} n_j, \quad 1 \le k \le r.$$

4.2. **Proof of Theorem 1.1.** We only need to consider the case that E is located in a bounded set $\mathcal{B}$, otherwise $(\alpha, L_{E,W}^{\lambda^{-1}V})$ is uniformly hyperbolic and the IDS is smooth. For any $E \in \mathcal{B}$, assume $\lambda^{-1} \le \lambda_0(m, V, W)$, we can rewrite $(\alpha, L_{E,W}^{\lambda^{-1}V})$ as $(\alpha, L_{E,W}^0 e^{f(x)})$ with $|f|_h \le \lambda^{-\frac{1}{2}}$. We divide the proof into the following four steps.

Step 1. Group the eigenvalues. We denote by the eigenvalues of $L_{E,W}^0$ by $\left\{e^{-2\pi i \rho_\ell^0}\right\}_{\ell=1}^{2m}$. By Proposition 5.1, for any $E \in \mathcal{B}$, there exists $\eta(V) > 0$ and $1 \le i_0(E) \le 2m$, such that we can group $\{\rho_\ell^0\}_{\ell=1}^{2m}$ as

$$E_1 = \left\{\rho | \rho > \frac{i_0 + 1}{4m}\eta\right\}, \quad E_3 = \left\{\rho | \rho < -\frac{i_0 + 1}{4m}\eta\right\},$$
$$E_2 = \left\{\rho | |\rho| \le \frac{i_0}{4m}\eta\right\}, \quad \#E_2 \le 2m_0,$$
$$n_1 = \#E_1 = \#E_3 = n_3.$$

Thus

$$\rho > \rho' \ge \frac{\eta}{4m}, \quad \rho \in E_k, \rho' \in E_\ell, \quad \forall k < \ell.$$

For any given $\varepsilon > 0$ and $0 < h' < h$, assume $\lambda^{-\frac{1}{2}} \le \epsilon(\alpha, h, h', m, |L_{E,W}^0|, \varepsilon, \eta/4m)$ which is defined in Proposition 4.2. Thus by Proposition 4.2, there exist $B_j \in C_{h'}^\omega(\mathbb{T}^d, GL(m, \mathbb{C}))$, $A_j \in GL(m, \mathbb{C})$ and $f_j \in C_{h'}^\omega(\mathbb{T}^d, gl(m, \mathbb{C}))$, such that

$$B_j(x+\alpha)L_{E_0,W}^{\lambda^{-1}V}(x)B_j^{-1}(x) = A_j e^{f_j(x)}.$$

Moreover, we have the following estimates

$$|f_j|_{h'} \le \epsilon_j, \quad |B_j|_0 \le C(\alpha, A)\epsilon_{j-1}^{-200m^2\sigma}, \tag{4.14}$$

$$\left|\Im\rho_\ell^j - \Im\rho_\ell^{j-1}\right| \le C(m,A)\epsilon_{j-1}^{\frac{1}{m}-200m^2\sigma}, \quad 1 \le \ell \le m, \tag{4.15}$$

where $\left\{e^{-2\pi i\rho_\ell^j}\right\}_{\ell=1}^m$ are the eigenvalues of A_j and $\epsilon_j = \epsilon_0^{2^j}$.

By (4.15), for j sufficiently large, we can group the eigenvalues of A_{j+1} as $\bigcup_{k=1}^{3} E_k^{j+1}$ with $\#E_k^{j+1} = n_k$, such that

$$\rho - \rho' \ge \frac{\eta}{8m}, \quad \rho \in E_k^{j+1}, \rho' \in E_\ell^{j+1}, \quad k < \ell.$$

Step 2. Compare $(\alpha, L_{E,W}^{\lambda^{-1}V})$ and $(\alpha, L_{E',W}^{\lambda^{-1}V})$. Assume $\delta = |E-E'|$ satisfies $\epsilon_{j+1} \le \delta \le \epsilon_j$. Then

$$\begin{aligned} B_{j+1}(x+\alpha)L_{E',W}^{\lambda^{-1}V}B_{j+1}^{-1}(x) &= A_{j+1}e^{f_{j+1}(x)} + B_j(x+\alpha)(L_{E,W}^{\lambda^{-1}V} - L_{E',W}^{\lambda^{-1}V})B_j^{-1}(x) \\ &:= A_{j+1}e^{\tilde f_{j+1}}. \end{aligned}$$

By (4.14), we have

$$|\tilde f_{j+1}|_h \le 100\delta|B_{j+1}|_0^2 + \epsilon_{j+1} \le 100\delta\epsilon_j^{-800m^2\varepsilon} \le 100\delta^{1-800m^2\varepsilon}.$$

Step 3. Hölder continuity of the Lyapunov exponents. Recall that $\{L_\ell(E)\}_{\ell=1}^m$ are the nonnegative Lyapunov exponents of $(\alpha, L_{E,W}^{\lambda^{-1}V})$. By Corollary 4.1, we have

$$\left|\sum_{\ell=1}^{n_1} L_\ell(E) - \sum_{\ell=1}^{n_1}\Im\rho_\ell^{j+1}\right| \le C(\eta,V)\delta^{\frac12},$$

$$\left|L_\ell(E) - \Im\rho_\ell^{j+1}\right| \le C(\eta,V)\delta^{\frac{1}{n_2}-800m^2\varepsilon}, \quad n_1+1 \le \ell \le n_1+n_2,$$

$$\left|\sum_{\ell=1}^{n_1} L_\ell(E') - \sum_{\ell=1}^{n_1}\Im\rho_\ell^{j+1}\right| \le C(\eta,V)\delta^{\frac12},$$

$$\left|L_\ell(E') - \Im\rho_\ell^{j+1}\right| \le C(\eta,V)\delta^{\frac{1}{n_2}-800m^2\varepsilon}, \quad n_1+1 \le \ell \le n_1+n_2.$$

It follows

$$\left|\sum_{\ell=1}^{n_1} L_\ell(E) - \sum_{\ell=1}^{n_1} L_\ell(E')\right| \le C(\eta,V)\delta^{\frac12}, \tag{4.16}$$

$$\left|L_\ell(E') - L_\ell(E)\right| \le C(\eta,V)\delta^{\frac{1}{n_2}-800m^2\varepsilon}, \quad n_1+1 \le \ell \le n_1+n_2. \tag{4.17}$$

(4.16) and (4.17) imply

$$\begin{aligned} \left|\sum_{\ell=1}^{m} L_\ell(E) - \sum_{\ell=1}^{m} L_\ell(E')\right| &\le \left|\sum_{\ell=1}^{n_1} L_\ell(E) - \sum_{\ell=1}^{n_1} L_\ell(E')\right| + \left|\sum_{\ell=n_1+1}^{m} L_\ell(E) - \sum_{\ell=n_1+1}^{m} L_\ell(E')\right| \\ &\le C(\eta,V)\delta^{\frac12} + 2mC(\eta,V)\delta^{\frac{1}{2m_0}-800m^2\varepsilon} \\ &\le 4mC(\eta,V)|E-E'|^{\frac{1}{2m_0}-800m^2\varepsilon}. \end{aligned}$$

Step 4. Hölder continuity of the IDS. By Thouless formula

$$\sum_{\ell=1}^{m} L_\ell(E) + \ln|W_m| = \int \ln|E-E'|dN(E'),$$

we have

$$L(E+i\epsilon)-L(E)=\frac{1}{2}\int \ln(1+\frac{\epsilon^2}{(E-E')^2})dN(E')$$
$$\geq c'|N(E+\epsilon)-N(E-\epsilon)|.$$

Thus

$$|N(E+\epsilon)-N(E-\epsilon)|\leq C(\alpha,V,W)\epsilon^{\frac{1}{2m_0}-800m^2\varepsilon}.$$

5. Appendix

In the appendix, we give the some useful lemmas.

Lemma 5.1. *Assume*

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn}\end{pmatrix},\quad B=\begin{pmatrix} b_{11} & \cdots & b_{1m} \\ & \ddots & \vdots \\ & & b_{mm}\end{pmatrix},$$

$$|a_{ii}e^{\pm 2\pi i\langle k,\alpha\rangle}-b_{jj}|\geq\eta,\quad 1\leq i\leq n,\quad 1\leq j\leq m,\quad \forall k\in\Lambda. \tag{5.1}$$

Then for any $Y\in\{f|f(x)=\sum_{k\in\Lambda}\hat{f}(k)e^{2\pi i\langle k,x\rangle}\}$*, we have*

$$|Y(\cdot+\alpha)A-BY(\cdot)|_h\geq\eta\left(1+\max\{m,n\}(|A|+|B)|\eta^{-1}\right)^{-(m+n)}|Y|_h.$$

Proof. Without loss of generality, we assume that $n\leq m$. Let $F(x)=Y(x+\alpha)A-BY(x)$, we inductively prove for $k\in\Lambda$,

$$|\widehat{Y}_{i,1}(k)|\leq\eta^{-1}\left(1+m(|A|+|B|)\eta^{-1}\right)^{m-i}|\widehat{F}(k)|. \tag{5.2}$$

$$|\widehat{Y}_{m,j}(k)|\leq\eta^{-1}\left(1+m(|A|+|B|)\eta^{-1}\right)^{j-1}|\widehat{F}(k)|. \tag{5.3}$$

$$|\widehat{Y}_{i,j}(k)|\leq\eta^{-1}\left(1+m(|A|+|B)|\eta^{-1}\right)^{m-i+j-1}|\widehat{F}(k)|. \tag{5.4}$$

We first prove (5.2), for $i=m$, we have

$$F_{m,1}(x)=a_{11}Y_{m,1}(x+\alpha)-b_{mm}Y_{m,1}(x),$$

thus

$$\widehat{F}_{m,1}(k)=a_{11}e^{2\pi i\langle k,\alpha\rangle}\widehat{Y}_{m,1}(k)-b_{mm}\widehat{Y}_{m,1}(k),$$

by (5.1), we have

$$|\widehat{Y}_{m,1}(k)|\leq\eta^{-1}|\widehat{F}_{m,1}(k)|\leq\eta^{-1}|\widehat{F}(k)|.$$

Assume for $i_0\leq i\leq m$, we already have

$$|\widehat{Y}_{i,1}(k)|\leq\eta^{-1}\left(1+m(|A|+|B|)\eta^{-1}\right)^{m-i}|\widehat{F}(k)|.$$

Then

$$F_{i_0-1,1}(x)=a_{11}Y_{i_0-1,1}(x+\alpha)-b_{i_0-1,i_0-1}Y_{i_0-1,1}(x)+\sum_{\ell=i_0}^{m}b_{i_0-1,\ell}Y_{\ell,1}(x).$$

By (5.1), we have

$$|\widehat{F}_{i_0-1,1}(k)|\geq\eta^{-1}|\hat{Y}_{i_0-1,1}(k)|-|B|\sum_{\ell=i_0}^{m}|\widehat{Y}_{\ell,1}(k)|,$$

this implies that

$$\begin{aligned}
|\widehat{Y}_{i_0-1,1}(k)| &\leq \eta^{-1}\left(|\widehat{F}_{i_0-1,1}(k)| + |B|\sum_{\ell=i_0}^{m}|\widehat{Y}_{\ell,1}(k)|\right)\\
&\leq \eta^{-1}\left(1+|B|\sum_{\ell=i_0}^{m}\eta^{-1}(1+m(|A|+|B|)\eta^{-1})^{m-\ell}\right)|\hat{F}(k)|\\
&\leq \eta^{-1}\left(1+m|B|\eta^{-1}(1+m(|A|+|B|)\eta^{-1})^{m-i_0}\right)|\widehat{F}(k)|\\
&\leq \eta^{-1}\left(1+m(|A|+|B|)\eta^{-1}\right)^{m-i_0+1}|\widehat{F}(k)|.
\end{aligned}$$

The proof of (5.3) is exactly the same.

Now, we inductively prove (5.4), assume for $(i,j)\in\mathcal{B}_{i_0,j_0}$ where

$$\mathcal{B}_{i_0,j_0}=\{(i,j)|i_0\leq i\leq n, 1\leq j\leq m\}\cup\{(i,j)|1\leq i\leq m, 1\leq j\leq j_0\},$$

we already have

$$|\widehat{Y}_{i,j}(k)|\leq \eta^{-1}\left(1+m(|A|+|B|)\eta^{-1}\right)^{m-i+j-1}|\widehat{F}(k)|.$$

Then

$$\begin{aligned}
F_{i_0-1,j_0+1}(x) = &-b_{i_0-1,i_0-1}Y_{i_0-1,j_0+1}(x)+a_{j_0+1,j_0+1}Y_{i_0-1,j_0+1}(x+\alpha)\\
&-\sum_{\ell=i_0}^{m}b_{i_0-1,\ell}Y_{\ell,j_0+1}(x)+\sum_{\ell=1}^{j_0}a_{\ell,j_0+1}Y_{i_0-1,\ell}(x+\alpha).
\end{aligned}$$

By (5.1), we have

$$|\widehat{F}_{i_0-1,j_0+1}(k)|\geq \eta|\widehat{Y}_{i_0-1,j_0+1}(k)|-|B|\sum_{\ell=i_0}^{m}|\widehat{Y}_{\ell,j_0+1}(k)|-|A|\sum_{\ell=1}^{j_0}|\widehat{Y}_{i_0-1,\ell}(k)|,$$

this implies that

$$\begin{aligned}
|\widehat{Y}_{i_0-1,j_0+1}(k)| &\leq \eta^{-1}\left(|\widehat{F}_{i_0-1,j_0+1}(k)|+|B|\sum_{\ell=i_0}^{m}|\widehat{Y}_{\ell,j_0+1}(k)|+|A|\sum_{\ell=1}^{j_0}|\hat{Y}_{i_0-1,\ell}(k)|\right)\\
&\leq \eta^{-1}\left(1+|B|\sum_{\ell=i_0}^{m}\eta^{-1}(1+m(|A|+|B|)\eta^{-1})^{m-\ell+j_0}\right)|F(k)|\\
&\quad+\left(|A|\sum_{\ell=1}^{j_0}\eta^{-1}(1+m(|A|+|B|)\eta^{-1})^{m-i_0+\ell}\right)|F(k)|\\
&\leq \eta^{-1}\left(1+m(|A|+|B|)\eta^{-1}(1+m(|A|+|B|)\eta^{-1})^{m-i_0+j_0}\right)|F(k)|\\
&\leq \eta^{-1}\left(1+m(|A|+|B|)\eta^{-1}\right)^{m-i_0+1+j_0}|F(k)|.
\end{aligned}$$

Similarly, we can prove that for any $(i,j)\in\mathcal{B}_{i_0-1,j_0+1}$,

$$|\widehat{Y}_{i,j}(k)|\leq \eta^{-1}\left(1+m(|A|+|B|)\eta^{-1}\right)^{m-i+j-1}|\hat{F}(k)|.$$

Thus we finish the proof of (5.4). By the definition of analytic norm, we have

$$|Y(\cdot+\alpha)A-BY(\cdot)|_h\geq \eta^{-1}\left(1+m(|A|+|B)|\eta^{-1}\right)^{-(m+n)}|Y|_h.$$

□

For any given $\eta > 0$, $\alpha \in \mathbb{R}^d$ and $A \in GL(m,\mathbb{C})$, we decompose $\mathcal{B}_h = C_h^\omega(\mathbb{T}^d, gl(m,\mathbb{C})) = \mathcal{B}_h^{nre}(\eta) \bigoplus \mathcal{B}_h^{re}(\eta)$ in such a way that for any $Y \in \mathcal{B}_h^{nre}(\eta)$,

$$A^{-1}Y(\theta+\alpha)A \in \mathcal{B}_h^{nre}(\eta), \qquad |A^{-1}Y(\theta+\alpha)A - Y(\theta)|_h \geq \eta |Y(\theta)|_h. \tag{5.5}$$

Moreover, let $\mathbb{P}_{nre}$ and $\mathbb{P}_{re}$ be the standard projections from $\mathcal{B}_h$ onto $\mathcal{B}_h^{nre}(\eta)$ and $\mathcal{B}_h^{re}(\eta)$ respectively.

Lemma 5.2 (Lemma 3.1 of [16])**.** *Assume that $\epsilon \leq (4|A|)^{-4}$ and $\eta \geq 13|A|^2\epsilon^{\frac{1}{2}}$. For any $g \in \mathcal{B}_h$ with $|g|_h \leq \epsilon$, there exist $Y \in \mathcal{B}_h$ and $g^{re} \in \mathcal{B}_h^{re}(\eta)$ such that*

$$e^{Y(x+\alpha)}(Ae^{g(x)})e^{-Y(x)} = Ae^{g^{re}(x)},$$

with $|Y|_h \leq \epsilon^{\frac{1}{2}}$ and $|g^{re}|_h \leq 2\epsilon$.

Remark 5.1. Although in Lemma 3.1 in [16], the authors assume that $A \in SU(1,1)$, the proof only uses implicit function theorem and essentially works for any $A \in GL(m,\mathbb{C})$. See the continuous version of Lemma 5.2 in [35].

Proposition 5.1. *Assume the zeros (counting multiplicity) of $W(\theta) - E$ on $\mathbb{T}$ are no more than $2m_0$ for all $E \in \mathbb{R}$, let $\mathcal{Z}(E) = \{z \in \mathbb{C} | W(z) - E = 0\}$. There exist $\eta(V) > 0$ and $1 \leq i_0(E) \leq 2m$, such that $\mathcal{Z}(E) = \mathcal{Z}^+(E) \cup \mathcal{Z}^-(E) \cup \mathcal{Z}_0(E)$ where*

$$\mathcal{Z}^+(E) = \left\{ z \in \mathcal{Z}(E) | \ln|z| > \frac{i_0+1}{4m}\eta \right\}, \quad \mathcal{Z}^-(E) = \left\{ z \in \mathcal{Z}(E) | \ln|z| < -\frac{i_0+1}{4m}\eta \right\},$$

$$\mathcal{Z}_0(E) = \left\{ z \in \mathcal{Z}(E) || \ln|z|| \leq \frac{i_0}{4m}\eta \right\}, \quad |\mathcal{Z}_0(E)| \leq 2m_0,$$

$$|\mathcal{Z}^+(E)| = |\mathcal{Z}^-(E)|.$$

Proof. Let $\mathcal{Z}(E) = \{z_i(E)\}_{i=1}^{2m}$ satisfying $|z_1(E)| \geq |z_2(E)| \geq \cdots \geq |z_{2m}(E)|$. Since the zeros (counting multiplicity) of $W(\theta) - E$ on $\mathbb{T}$ are no more than $2m_0$ for all $E \in \mathbb{R}$, we have $|z_{m-m_0}(E)| > 1$ for all $E \in \mathbb{R}$. On the other hand, the zeros the polynomial depend continuously on E. Thus there exists $\eta(V) > 0$, such that $|z_{m-m_0}(E)| \geq 1 + \eta$ for all $E \in \mathbb{R}$. Now we fix E, by pigeonhole principle, there exists $1 \leq i_0 \leq 2m$ such that $\{|\ln|z|| : z \in \mathcal{Z}_0\} \cap [\frac{i_0}{3m}\eta, \frac{i_0+1}{3m}\eta] = \emptyset$. We denote by

$$\mathcal{Z}^+(E) = \left\{ z \in \mathcal{Z}(E) | \ln|z| > \frac{i_0+1}{3m}\eta \right\}, \quad \mathcal{Z}^-(E) = \left\{ z \in \mathcal{Z}(E) | \ln|z| < -\frac{i_0+1}{3m}\eta \right\},$$

$$\mathcal{Z}_0(E) = \left\{ z \in \mathcal{Z}(E) || \ln|z|| \leq \frac{i_0}{3m}\eta \right\}, \quad |\mathcal{Z}_0(E)| \leq 2m_0,$$

then $\mathcal{Z}(E) = \mathcal{Z}^+(E) \cup \mathcal{Z}^-(E) \cup \mathcal{Z}_0(E)$. ∎

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