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GROWTH OF THE WANG-CASATI-PROSEN COUNTER IN AN INTEGRABLE BILLIARD

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Abstract. This work is motivated by an article by Wang, Casati, and Prosen [5] devoted to a study of ergodicity in two-dimensional irrational right-triangular billiards. Numerical results presented there suggest that the irrational contribution to the velocity orientation remains localized as the system evolves, hence pointing to an absence of ergodicity.

We study an analogue of the Wang-Casati-Prosen counter for a 45°: 45°: 90° billiard—a manifestly non-ergodic system with only eight velocity orientations for any generic trajectory—and find no localization *even* there. What we found instead is an extremely slow, logarithmic growth that could be hard to detect numerically.

1. Introduction

This work is motivated by an article by Wang, Casati, and Prosen [5] devoted to a study of ergodicity in two-dimensional irrational right-triangular billiards. Authors introduce a particular integer-valued counter (see Section 6 for details) that monitors the value of an irrational contribution to the orientation of the particle's velocity. Numerical results presented in [5] suggest a localization of the counter, which in turn indicates an absence of ergodicity.

In our article, we study an analogue for the Wang-Casati-Prosen for a 45°:45°:90° billiard—a manifestly non-ergodic system with only eight velocity orientations for any generic trajectory—and find no localization *even* there. What we found instead is an extremely slow, logarithmic growth along at least an exponentially sparse subsequence that could be hard to detect numerically.

2. The object of study

Given $\alpha : 0 \leq \alpha < 1$, we consider the α-rotational trajectory emerging at $x_0 \in [0, 1)$,

$$x_{j+1}^{(\alpha)} = (x_j^{(\alpha)} + \alpha) \mod 1 \ , \ j \in \mathbb{N}_0 \ . \tag{1}$$

$$x_0^{(\alpha)} = x_0 \ , \tag{2}$$

where, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For fixed $\beta : 0 \leq \beta < 1$, introduce an "observable" $f^{(\beta)} : [0, 1) \to [0, 1)$,

$$f^{(\beta)}(x) = \begin{cases} +1 & , \text{for} \quad x \in \mathcal{I}_{\mathrm{I}} := [0, \beta) \ , \\ -1 & , \text{for} \quad x \in \mathcal{I}_{\mathrm{II}} := [\beta, 1) \ . \end{cases} \tag{3}$$

and a corresponding sequence

$$f_j^{(\alpha,\,\beta)} := f^{(\beta)}(x_j^{(\alpha)}) \ , \ j \in \mathbb{N}_0 \ . \tag{4}$$

Finally, for each $j \in \mathbb{N}_0$, consider an "increment"

$$\epsilon_j^{(\alpha,\,\beta)} := \chi(f_j^{(\alpha,\,\beta)}) \prod_{j'=0}^{j} f_{j'}^{(\alpha,\,\beta)} \ , \tag{5}$$

with

$$\chi(f) := \left\{ \begin{array}{ll} 2 & ,\text{ for} \quad f = +1 \\ 1 & ,\text{ for} \quad f = -1 \end{array} \right. . \tag{6}$$

Our primary object of interest is the following "counter:"

$$S_j^{(\alpha,\,\beta)} = \sum_{j'=0}^{j-1} \epsilon_{j'}^{(\alpha,\,\beta)} \ , \ j \in \mathbb{N}_0 \ . \tag{7}$$

A similar object has been considered in [4]. There, $\mathcal{I}_{\mathrm{II}}$ was any connected nonempty subset of $[0, 1)$ and $\chi(f) = 1$.

Our main result, Theorem 5.1, considers the "diagonal" case

$$\alpha = \beta =: \sigma \ ,$$

for which we prove:

Theorem 2.1. *There exists a non-empty set of irrationals* $\Sigma \subseteq (0, 1)$ *such that for each* $\sigma \in \Sigma$ *there is an associated full measure set of initial conditions* $\Omega(\sigma)$ *such that for all* $x_0 \in \Omega(\sigma)$*, one has*

$$\limsup_{j\to\infty} |S_j| = \infty \ . \tag{8}$$

The set Σ is described explicitly in terms of continued fraction expansion of its elements, see Theorem 5.1 of Section 5. The description of the full measure set of initial conditions $\Omega(\sigma)$ is given in Remark 5.2 of Section 5. It remains an interesting open question whether the conditions on σ and x_0 are indeed necessary; see also our remarks at the end of Section 5.

3. The method of study: auxiliary rational rotations

For a given irrational $\sigma : 0 < \sigma < 1$, consider its continued fraction expansion

$$\sigma = [0;\, a_1,\, a_2,\, \ldots] \ ,$$

and the corresponding convergents,

$$\sigma_n := [0;\, a_1,\, a_2,\, \ldots,\, a_n] = \frac{p_n}{q_n} \ , \ n \in \mathbb{N} \ ,$$

where p_n and q_n are mutually prime.

For $n \in \mathbb{N}$, consider an auxiliary trajectory $(x_j^{(\sigma_n)})_{j\in\mathbb{N}_0}$. Observe that this trajectory is q_n-periodic, i.e.

$$x_{j+q_n}^{(\sigma_n)} = x_j^{(\sigma_n)} \ , \text{for all } j \in \mathbb{N}_0 \ , \tag{9}$$

whence so is the sequence of observables $(f_j^{(\sigma_n,\,\sigma_n)})_{j\in\mathbb{N}_0}$. The definition in (5) thus implies that $(\epsilon^{(\sigma_n,\sigma_n)})_{j\in\mathbb{N}_0}$ is either periodic or anti-periodic with period q_n,

$$\epsilon_{j+q_n}^{(\sigma_n,\,\sigma_n)} = \eta_n^{(\sigma_n,\,\sigma_n)} \epsilon_j^{(\sigma_n,\,\sigma_n)} \ ,$$

where

$$\eta_n^{(\alpha,\,\beta)} := \operatorname{sign}(\epsilon^{(\alpha,\,\beta)}_{q_n-1}) = \prod_{j=0}^{q_n-1} f_j^{(\alpha,\,\beta)} \ .$$

As a result, the sequence of corresponding counters either grows indefinitely, period after period, or remains trapped around zero:

$$S^{(\sigma_n,\,\sigma_n)}_{Nq_n} = \left\{ \begin{array}{cc} N & \text{, for} \quad \eta_n^{(\sigma_n,\,\sigma_n)} = +1 \\ \left\{ \begin{array}{ll} 0 & \text{, for} \quad N = \text{even} \\ 1 & \text{, for} \quad N = \text{odd} \end{array} \right\} & \text{, for} \quad \eta_n^{(\sigma_n,\,\sigma_n)} = -1 \end{array} \right\} \cdot S^{(\sigma_n,\,\sigma_n)}_{q_n} \ ,$$

for any $N \in \mathbb{N}$. In particular, if $\eta_n^{(\sigma_n,\,\sigma_n)} = +1$ and $S^{(\sigma_n,\,\sigma_n)}_{q_n} \neq 0$, the counter $S^{(\sigma_n,\,\sigma_n)}_j$ is, obviously, unbounded, since it is unbounded on the subsequence with $j = Nq_n$, where it grows as

$$S^{(\sigma_n,\,\sigma_n)}_{Nq_n} \overset{\eta_n^{(\sigma_n,\,\sigma_n)}=+1}{=} N S^{(\sigma_n,\,\sigma_n)}_{q_n} \ . \tag{10}$$

This observation is the cornerstone of the proof of unboundedness of the $S_j^{(\sigma,\,\sigma)}$ for appropriately chosen irrational rotations σ and initial conditions x_0, which will be constructed in Section 5 (Theorem 5.1).

4. GROWTH FOR (RATIONAL) AUXILIARY ROTATIONS

Fix $n \in \mathbb{N}$ and consider the q_n-periodic auxiliary trajectory $(x_j^{(\sigma_n)})_{j\in\mathbb{N}_0}$. The goal of this section is to explore the structure of the trajectory to quantify the growth of the counters $S_j^{(\sigma,\,\sigma)}$ over one period $0 \leq j \leq q_n - 1$. Specifically, we will establish the following:

Proposition 4.1. *Suppose that* both p_n and q_n *are odd. Then, one has the lower bound*

$$\left| S^{(\sigma_n,\,\sigma_n)}_{q_n} \right| \geq 2 \ . \tag{11}$$

To prove Proposition 4.1, we start with recalling some basic facts about rational rotations. First observe that since $\gcd(p_n, q_n) = 1$, the finite trajectory $\{x_j^{(\sigma_n)} \ , \ 0 \leq j \leq q_n - 1\}$ consists of q_n *distinct* points, each of which is visited only once. Moreover, identifying $\mathbb{R}/\mathbb{Z}$ with the unit circle $S^1 \subseteq \mathbb{C}$ via the bijection $t \mapsto \mathrm{e}^{2\pi i t}$ and noticing that

$$\left(\mathrm{e}^{2\pi i (x_0 - x_j^{(\sigma_n)})} \right)^{q_n} = 1 \ , \tag{12}$$

shows that the points $\mathrm{e}^{2\pi i (x_0 - x_j^{(\sigma_n)})}$ are merely a permutation of the q_n-th roots of unity, i.e.

$$\{x_j^{(\sigma_n)} \ , \, 0 \leq j \leq q_n - 1\} = \{x_0 + \frac{k}{q_n} \ , \, 0 \leq k \leq q_n - 1\} \ . \tag{13}$$

In particular, writing

$$x_0 = \left(x_0 \mod \frac{1}{q_n} \right) + \frac{\lfloor x_0 q_n \rfloor}{q_n} =: x_{0,n} + k_{0,n} \frac{1}{q_n} \ , \tag{14}$$

we may represent the elements of the finite trajectory $\{x_j^{(\sigma_n)} \ , \ 0 \leq j \leq q_n - 1\}$ in the form

$$x_j^{(\sigma_n)} = x_{0,n} + \frac{k_{j,n}}{q_n} \ , \, 0 \leq j \leq q_n - 1 \ , \tag{15}$$

with

$$k_{j,n} := (k_{0,n} + j p_n) \mod q_n \ . \tag{16}$$

The main merit of the representation of the auxiliary trajectory in (14)-(16) is that it allows to keep track of the value value of the observable since

$$f_j^{(\sigma_n, \sigma_n)} = \begin{cases} +1 & , \text{for} \quad k_{j,n} = 0, 1, \ldots, p_n - 1 \\ -1 & , \text{for} \quad k_{j,n} = p_n, \ldots, q_n - 1 \end{cases} . \tag{17}$$

In particular, (17) immediately yields:

Lemma 4.2.

$$\eta_n^{(\sigma_n, \sigma_n)} = +1 \quad , \text{for} \quad q_n - p_n = \text{even} .$$

Proof. Since for $0 \leq j \leq q_n - 1$, $k_{j,n}$ will visit each of the q_n points $0, 1, \ldots, q_n - 1$ precisely one time, (17) implies that

$$\eta_n^{(\sigma_n, \sigma_n)} = \prod_{j=0}^{q_n-1} f_j^{(\sigma_n, \sigma_n)}$$

is product of p_n factors $+1$ and $q_n - p_n$ factors -1. Thus, if $q_n - p_n$ is even, we conclude that $\eta_n^{(\sigma_n, \sigma_n)} = +1$. □

We are now ready to prove the main result of this section:

Proof of Proposition 4.1. According to (17), we have

$$\epsilon_j^{(\sigma_n, \sigma_n)} = \left\{ \begin{array}{lll} +2 & , \text{for} & k_{j,n} = 0, 1 \ldots, p_n - 1 \\ -1 & , \text{for} & k_{j,n} = p_n, p_n, \ldots, q_n - 1 \end{array} \right\} \cdot \operatorname{sign}(\epsilon_{j-1}^{(\sigma_n, \sigma_n)}) . \tag{18}$$

Using that, for $0 \leq j \leq q_n - 1$, $k_{j,n}$ will visit each of the q_n points $0, 1, \ldots, q_n - 1$ precisely once, we may consider the subsequence j_l consisting of all the instances j for which $f_j^{(\sigma_n, \sigma_n)} = -1$; in particular, along this subsequence one has

$$\epsilon_{j_l}^{(\sigma_n, \sigma_n)} = -\epsilon_{j_{l-1}}^{(\sigma_n, \sigma_n)} .$$

The sequence $\epsilon_{j_l}^{(\sigma_n, \sigma_n)}$ has an even number of terms (i.e. $q_n - p_n$), and thus does not contribute to the sum $S_{q_n}^{(\sigma_n, \sigma_n)} = \sum_{j'=0}^{q_n-1} \epsilon_j^{(\sigma_n, \sigma_n)}$. By (18), the remaining summands are all ± 2, and there is an odd number of them (i.e. p_n). In summary, we conclude $|S_{q_n}^{(\sigma_n, \sigma_n)}| \geq 2$, as claimed. □

5. Constructing an unbounded subsequence for an irrational rotation

We are now in a position to formulate and prove our main result:

Theorem 5.1. *Let $\sigma = [0; a_1, a_2, \ldots] \in (0, 1)$ be irrational such that its continued fraction expansion has the following properties: there exists a subsequence $(n_m)_{m \in \mathbb{N}}$ of $2\mathbb{N}$ such that*

(a) *$(a_{n_m})_{m \in \mathbb{N}}$ is unbounded*
(b) *for all $m \in \mathbb{N}$, both p_{n_m} and q_{n_m} are odd (and thus the conditions of the Lemma 4.1 are satisfied, for $n = n_m$);*

Then, there exists a full measure set of initial conditions $\Omega(\sigma) \subseteq [0, 1)$ such that for each $x_0 \in \Omega(\sigma)$, one has

$$\limsup_{j \to \infty} |S_j| = \infty . \tag{19}$$

Remark 5.2. Our proof of Theorem 5.1 implies the following explicit description of the set $\Omega(\sigma)$, letting

$$\Omega(\sigma) := \bigcup_{Q\in\mathbb{N}} \left\{ x_0 \in [0,1) \ : \ \{x_0 q_{n_m}\} < 1 - \frac{1}{Q} \text{ , for infinitely many } m \in \mathbb{N} \right\} \ . \tag{20}$$

Here, as common, $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of $x \in [0,1)$.

Observe that $\Omega(\sigma)$ is a set of *full* Lebesgue measure in $[0,1)$. Indeed, considering its complement

$$\begin{aligned} [0,1) \setminus \Omega(\sigma) &= \{x_0 \in [0,1) \ : \ \{x_0 q_{n_m}\} \to 1\} \qquad (21) \\ &\subseteq \{x_0 \in [0,1) \ : \ |||x_0 q_{n_m}||| \to 0\} \ , \qquad (22) \end{aligned}$$

where $|||x||| := \inf_{n\in\mathbb{Z}} |x-n|$ is the usual norm in $\mathbb{R}/\mathbb{Z}$, shows that the set on the right-hand side of (22) is a proper subgroup of $\mathbb{R}/\mathbb{Z}$. Thus, a well known fact from harmonic analysis (see e.g. problem 14 in Sec. 1 of Katznelson [2]) implies that (22), and hence also $[0,1) \setminus \Omega(\sigma)$, has zero Lebesgue measure.

Before turning to the proof of Theorem 5.1, we comment on the existence of irrationals σ described in Theorem 5.1. To construct explicit examples of such σ, we first recall that the continued fraction expansion of $\sigma = [0; a_1, a_2, \dots] \in (0,1)$ satisfies the recursion relations (see e.g. [3]):

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \ , \qquad (23) \\ q_n &= a_n q_{n-1} + q_{n-2} \text{ , for all } n \in \mathbb{N} \ , \qquad (24) \end{aligned}$$

with initial conditions

$$\begin{aligned} p_0 &= 0 \ , \ p_{-1} = 1 \ , \\ q_0 &= 1 \ , \ p_{-1} = 0 \ . \qquad (25) \end{aligned}$$

Suppose that (a_n) is a sequence in $(2\mathbb{N}-1)$. Using induction, the recursion relations (23)-(24) together with the initial conditions (25), imply that

$$p_n = \begin{cases} \mathit{odd} & \text{, if } n \equiv 1,2 \mod 3 \ , \\ \mathit{even} & \text{, if } n \equiv 0 \mod 3 \ , \end{cases} \tag{26}$$

and

$$q_n = \begin{cases} \mathit{odd} & \text{, if } n \equiv 0,1 \mod 3 \ , \\ \mathit{even} & \text{, if } n \equiv 2 \mod 3 \ . \end{cases} \tag{27}$$

The conditions of Theorem 5.1 are thus satisfied for ($n \equiv 1 \mod 3$), specifically by letting

$$n_m = 3(2m-1) + 1 \ , \ m \in \mathbb{N} \ . \tag{28}$$

In summary, we have shown that all irrationals $\sigma = [0; a_1, a_2, \dots] \in (0,1)$ for which the sequence of elements (a_n) has *odd parity* satisfy the hypotheses of Theorem 5.1.

Proof of Theorem 5.1. Let $\sigma \in (0,1)$ as in Theorem 5.1. Observe that since both conditions (a) and (b) are assumed to only hold along some *sub*sequence $(n_m)_{m\in\mathbb{N}}$ of $2\mathbb{N}$, possibly passing to a sub-subsequence one may replace hypothesis (a) with

$$\lim_{m\to\infty} a_{n_m} = \infty \ . \tag{29}$$

Fix an initial condition $x_0 \in \Omega(\sigma)$, where $\Omega(\sigma)$ is described in (20); in particular, there exists $Q \in \mathbb{N}$ such that

$$\{x_0 q_{n_m}\} < 1 - \frac{1}{Q} \text{ , for infinitely many } m \in \mathbb{N} \text{ .} \tag{30}$$

Again, possibly passing to an appropriate subsequence, we may simply assume that

$$\{x_0 q_{n_m}\} < 1 - \frac{1}{Q} \text{ , for all } m \in \mathbb{N} \text{ .} \tag{31}$$

We recall two useful properties of continued fractions (see e.g. [3]): For all $n \in \mathbb{N}$, one has

$$\vert \sigma_n - \sigma \vert < \frac{1}{q_n q_{n+1}} \text{ ;} \tag{32}$$

morevoer, since by hypothesis $n_m \in 2\mathbb{N}$, one has

$$\sigma_{n_m} < \sigma \text{ , for all } m \in \mathbb{N} \text{ .} \tag{33}$$

Now fix $m \in \mathbb{N}$, and let

$$j_m^{(Q)} := \lfloor \frac{a_{n_m+1}}{Q} \rfloor q_{n_m} \text{ .} \tag{34}$$

Then, for $1 \leq j < j_m^{(Q)}$, we estimate, using (32), (34), and the recursion relation in (24):

$$\vert x_j^{(\sigma)} - x_j^{(\sigma_{n_m})} \vert < j_m^{(Q)} \vert \sigma - \sigma_m \vert \leq \frac{1}{Q q_{n_m}} \text{ .} \tag{35}$$

Moreover, notice that the representation of the elements of the rational auxiliary trajectory in (14) - (15) implies

$$\begin{aligned} x_j^{(\sigma_{n_m})} &= \left(x_0 \mod \frac{1}{q_{n_m}} \right) + \frac{k_{j,n_m}}{q_{n_m}} \\ &= \frac{\{x_0 q_{n_m}\}}{q_{n_m}} + \frac{k_{j,n_m}}{q_{n_m}} < \frac{1 - \frac{1}{Q}}{q_{n_m}} + \frac{k_{j,n_m}}{q_{n_m}} \qquad (36) \\ &\leq \frac{1 - \frac{1}{Q}}{q_{n_m}} + \frac{q_{n_m} - 1}{q_{n_m}} \leq 1 - \frac{1}{Q q_{n_m}} \text{ , for } 0 \leq j \leq q_{n_m} - 1 \text{ .} \qquad (37) \end{aligned}$$

Thus, for $0 \leq j < j_m^{(Q)}$, we conclude from (35) and (37) that

$$\begin{aligned} x_j^{(\sigma)} &= \left(\underbrace{j(\sigma - \sigma_{n_m})}_{< \frac{1}{Q q_{n_m}}} + \underbrace{x_j^{(\sigma_{n_m})}}_{\leq 1 - \frac{1}{Q q_{n_m}}} \right) \mod 1 \\ &= j(\sigma - \sigma_{n_m}) + x_j^{(\sigma_{n_m})} \text{ ,} \end{aligned}$$

whence,

$$x_j^{(\sigma)} < x_j^{(\sigma_{n_m})} + \frac{1}{Q q_{n_m}} \text{ , } 0 \leq j < j_m^{(Q)} \text{ ,} \tag{38}$$

$$x_j^{(\sigma)} \geq x_j^{(\sigma_{n_m})} + (\sigma - \sigma_{n_m}) \text{ , } 1 \leq j < j_m^{(Q)} \text{ .} \tag{39}$$

Let us now prove that

$$f_j^{(\sigma,\sigma)} = f_j^{(\sigma_{n_m},\sigma_{n_m})} \text{ , for } 0 \leq j < j_m^{(Q)} \text{ .} \tag{40}$$

Using (38), one has for $j < j_m^{(Q)}$

$$\begin{aligned} f_j^{(\sigma_{n_m},\,\sigma_{n_m})} = +1 \quad &\Leftrightarrow \quad x_j^{(\sigma_{n_m})} < \sigma_{n_m} \ \text{ and } k_{j,n_m} \leq p_{n_m} - 1 \\ &\Rightarrow \quad x_j^{(\sigma_{n_m})} + \tfrac{1}{Qq_{n_m}} \overset{(36)}{\leq} \sigma_{n_m} \\ &\overset{(38)}{\Rightarrow} \quad x_j^{(\sigma)} \leq \sigma_{n_m} \overset{(33)}{<} \sigma \\ &\Rightarrow \quad f_j^{(\sigma,\,\sigma)} = +1 \end{aligned} \tag{41}$$

Similarly, for $1 \leq j < j_m^{(Q)}$, (39) yields that

$$\begin{aligned} f_j^{(\sigma_{n_m},\,\sigma_{n_m})} = -1 \quad &\Leftrightarrow \quad x_j^{(\sigma_{n_m})} \geq \sigma_{n_m} \\ &\Leftrightarrow \quad x_j^{(\sigma_{n_m})} + (\sigma - \sigma_{n_m}) \geq \sigma \\ &\overset{(39)}{\Rightarrow} \quad x_j^{(\sigma)} \geq \sigma \\ &\Leftrightarrow \quad f_j^{(\sigma,\,\sigma)} = -1 \end{aligned} \tag{42}$$

Finally, since $x_0^{(\sigma_{n_m})} = x_0^{(\sigma)} = x_0$,

$$f_j^{(\sigma_{n_m},\,\sigma_{n_m})} \overset{j=0}{=} f_j^{(\sigma,\,\sigma)} \ . \tag{43}$$

In summary, (41), (42), and (43) validate (40).

Therefore, for $j < j_m^{(Q)}$, any conclusion about the auxiliary sequence $f_j^{(\sigma_{n_m},\,\sigma_{n_m})}$ will be valid for the sequence $f_j^{(\sigma,\,\sigma)}$. By the definition of the increments $\epsilon_j^{(\alpha,\,\beta)}$ in (5), (40) implies

$$\epsilon_j^{(\sigma,\,\sigma)} = \epsilon_j^{(\sigma_{n_m},\,\sigma_{n_m})} \ , \text{for } 0 \leq j < j_m^{(Q)} \ . \tag{44}$$

Finally, since the counter $S_j^{(\alpha,\,\beta)}$ is only sensitive to the values of $\epsilon_{j'}^{(\alpha,\,\beta)}$, for $j' : 0 \leq j' \leq j-1$ (see (7)), we obtain

$$S_j^{(\sigma,\,\sigma)} = S_j^{(\sigma_{n_m},\,\sigma_{n_m})} \ , \text{for } j \leq j_m^{(Q)} \ . \tag{45}$$

In particular, the property (10), will be valid for $S_{Nq_{n_m}}^{(\sigma,\,\sigma)}$, with $N = \lfloor a_{n_m+1}/Q \rfloor$: combining (45), condition (b) of Theorem 5.1, and Proposition 4.1, we thus get

$$\left| S_{j_m^{(Q)}}^{(\sigma,\,\sigma)} \right| = \left| S_{j_m^{(Q)}}^{(\sigma_{n_m},\,\sigma_{n_m})} \right| = \left| S_{\lfloor \frac{a_{n_m+1}}{Q} \rfloor q_{n_m}}^{(\sigma_{n_m},\,\sigma_{n_m})} \right| = \lfloor \frac{a_{n_m+1}}{Q} \rfloor \left| S_{q_{n_m}}^{(\sigma_{n_m},\,\sigma_{n_m})} \right| \geq 2 \lfloor \frac{a_{n_m+1}}{Q} \rfloor \ . \tag{46}$$

which by hypothesis (a) of Theorem 5.1 implies the claim in (19). □

What follows from the Theorem 5.1, is that overall, the sequence $S_j^{(\sigma,\,\sigma)}$ is unbounded, in a seeming contradiction to the observation [5].

Remark 5.3. Inequality (46) implies, using (34) and properties of continued fractions [3], that the growth along the subsequence $j_m^{(Q)}$ is at least logarithmic. Note that the subsequence itself is at least exponentially sparse. However, choosing a very fast growing sequence a_{n_m+1} one can also ensure any faster sublinear growth rate along $j_m^{(Q)}$, although it will come at the expense of making the subsequence of growth even sparser.

We mention that it remains an open question whether the condition on the rotations σ in Theorem 5.1, or the respective condition on the initial conditions in (20) are indeed necessary. In particular, an interesting question for future research may be to examine the case of σ of bounded type (e.g. take σ equal to the golden mean), which however requires development of a different proof strategy.

6. Motivation: triangular billiards

In [5], authors consider a single two-dimensional point-like particle moving in a right-triangular billiard (Fig. 1) of an acute angle $\tilde{\alpha}$ (α in the original). The physical model behind the right-triangular billiards is two one-dimensional hard-core point-like particles between two walls. The map is described in [5] and elsewhere. According to the map, the vertical cathetus, horizontal cathetus, and hypothenuse correspond to the left particles colliding with a wall (L), right particle colliding with a wall (R), and a particle-particle collision (C) respectively, hence the naming of the sides at Fig. 1 and throughout the text.

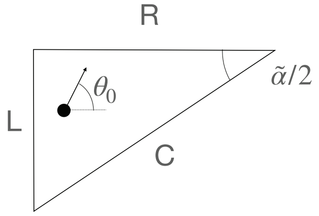

Figure 1. The right-triangular billiard considered in [5]. θ_0 gives orientation of the initial velocity. $\tilde{\alpha}/2$ is one of the angles. See text for the rational behind naming the sides of the triangle.

Particle's trajectory starts at some point inside the billiard, with a velocity vector at an angle θ_0 to the horizontal axis. Authors of [5] observe that at any instant of time, the angle θ between the velocity and the horizontal has a form

$$\theta = K\tilde{\alpha} + \left\{ \begin{array}{l} +\theta_0 \\ \text{or} \\ -\theta_0 \\ \text{or} \\ +(\pi - \theta_0) \\ \text{or} \\ -(\pi - \theta_0) \end{array} \right\},$$

with the *counter* K being an integer. The paper studies the growth of the magnitude of K over time. The numerical evidence presented suggests that K is localized within a $\sim \pm 20$ range around the origin, suggesting absence of ergodicity. In our paper we study an analogue of the counter K for the case $\tilde{\alpha} = 90^\circ$, and find no localization.

To proceed, observe that the velocity orientation, and, thus, the counter value does not change between the particle wall collisions; hence, the counter K is a function of the number i of particle-wall collisions prior. As such, the temporal index i labels the time intervals between two successive partcle-wall collisions. Accordingly, from now on, we will denote K as K_i. It is easy to show (see [5]) that the rule for updating the counter K_i is as follows:

$$\begin{aligned} & K_{i+1} = \\ & \qquad \left\{ \begin{array}{ll} -K_i + 1 & \text{, if the event that separates the } i+1\text{'st and } i\text{'th time interval is } C \\ -K_i & \text{, if the event that separates the } i+1\text{'st and } i\text{'th time interval is } L \text{ or } R \end{array} \right\}. \\ & K_0 = 0 \end{aligned} \tag{47}$$

Without loss of generality, assume that $i = 0$ labels an interval between an L and an R event, the former preceded by C (see Fig. 1). Name the two events preceding the interval $i = 0$ as $C^\star$ and L^*, for future reference. Introduce a counter

$$P_i := (-1)^{i+1} K_i$$

a temporal index

$$\bar{m}(i) := \#\ C\text{-events between the } C^\star\text{-event and the } i\text{'th interval, excluding } C^\star\ ,$$

and another temporal index

$$i'(m) := \inf\{i : \bar{m}(i) = m\}\ .$$

Observe that the counter P_i is a function of $\bar{m}(i)$ alone as it does not change after L and R events. Accordingly, introduce a counter

$$Q_m := P_{i'(m)}\ .$$

Recall that, by construction,

$$|Q_m| = |K_{i'(m)}|\ ,$$

so that is Q_m is found to be unbounded, then K_i will be unbounded as well.

Note also that m labels a temporal interval between two successive C events. A crucial observation is that the dynamics of the counter Q_m is governed by

$$\begin{aligned} &Q_{m+1} = Q_m + \varepsilon_m \\ &Q_0 = 0 \end{aligned}\ , \tag{48}$$

with

$$\varepsilon_m := (-1)^{\#\ CLRC\text{- or } CRLC\text{-intervals between the } C^\star\text{-event inclusive and the } m\text{'th interval inclusive}}\ . \tag{49}$$

So far, our discussion concerned a generic right triangle of Fig. 1. From now on, let us assume a 45°: 45°: 90° billiard, i.e. that

$$\tilde{\alpha} = \frac{\pi}{2}\ . \tag{50}$$

Also assume, without loss of generality that

$$\frac{\pi}{4} \leq \theta_0 < \frac{\pi}{2}\ . \tag{51}$$

Thanks to the method of images—described in a cation to Fig. 2—particle's trajectory becomes fully predictable, even for long propagation times, with no sensitivity to the initial conditions. The evolution of the counter Q (see (48)-(49)) remains sensitive to the initial conditions. During a sequence of $CLRC$ and $CRLC$ fragments, the counter Q lingers around a particular value, with no substantial evolution. Such sequences are interrupted an occasional (isolated, given (50)-(51)) $CRCLC$ fragment that changes the value of the counter by ±2. Whether the change keeps the sign of the previous ±2 jump or flips it depends on the parity of the number of the $CLRC$ and $CRLC$ fragment in between: this parity, in turn, can be altered by a small change in the initial position, leading to large deviations at long propagation times.

Let us introduce another temporal index,

$$\bar{\jmath}(m) := \\ \#\ CLRC,\ CRLC,\text{ and } CRCLC \text{ fragments between the } C^\star\text{-event and the } m\text{'th interval}\ .$$

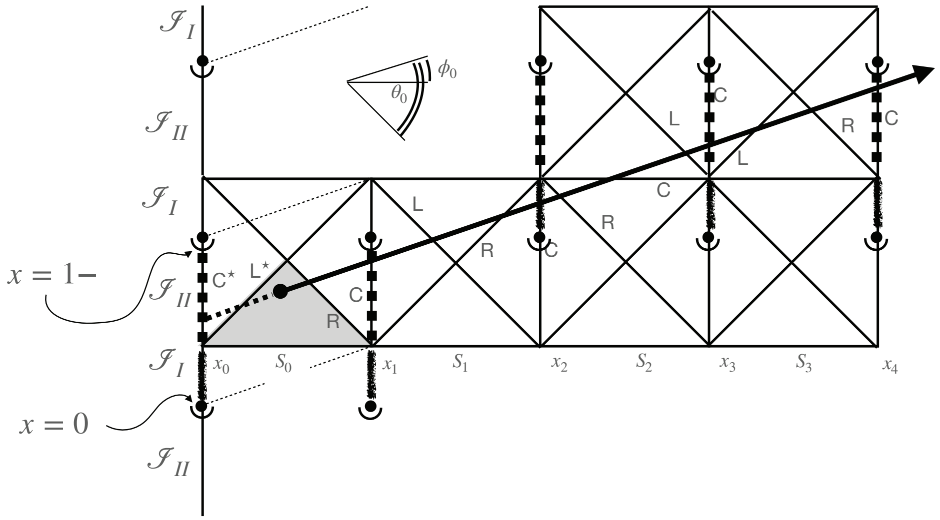

FIGURE 2. The relationship between dynamics in a 45°: 45°: 90° billiard and irrational rotations. The grey triangle corresponds to the billiard in question. The white triangles are images of the original billiard obtained via a mirror reflection with respect to the wall that the particle hits. The subsequent fragment of particle's trajectory is reflected as well; as a result the trajectory becomes a straight line traveling through a plane tiled by the copies of the original billiard. The large black circle corresponds to the initial position of the particle. Vertical dotted and "hand-drawn" lines correspond to two intervals in the related irrational rotation model. x_j is the phase space variable in the irrational rotations. See the text for a definition of the counter S_j. The time j counts the vertical C-lines crossed by the trajectory.

Note that $\bar{j}(m)$ counts the number of "vertical" C-events between $C^\star$, exclusive, and the m'th interval (see Fig. 2).

Accordingly, introduce

$$m'(j) := \inf\{m : \bar{j}(m) = j\} \ .$$

Observe now that the counter Q_m is a function of $\bar{j}(m)$ alone. Finally, introduce a counter

$$S_j := Q_{m'(j)} \ .$$

Again, by construction,

$$|S_j| = |K_{i'(m'(j))}| \ ,$$

and the growth of the artificially constructed counter S_j would indicate a growth of the physical counter of interest, K_i.

Recall that j labels the temporal intervals between successive "vertical" C events. Let it also label the left "vertical" C event for a given j'th interval. Without loss of generality assume that the hypothenuse C has unit length:

$$\text{length}(C) = 1 \ .$$

Introduce

$$x_j := \text{position of the particle-wall } C \text{ collision },$$

in turns of the tiling depicted at Fig. 2. It is easy to see that x_j undergoes a sequence of irrational rotations, with a shift

$$\sigma = \tan(\phi_0) ,$$

with

$$\begin{aligned} &\phi_0 := \theta_0 - \frac{\pi}{4} \\ &0 \leq \phi_0 < \frac{\pi}{4} . \end{aligned}$$

Now, divide each of the "vertical" C-walls onto two areas

$$\begin{aligned} &\mathcal{I}_{\mathrm{I}} := [0,\, \sigma[\\ &\text{and} \\ &\mathcal{I}_{\mathrm{II}} := [\sigma,\, 1[\; . \end{aligned}$$

Observe that if x_j gets to the $\mathcal{I}_{\mathrm{I}}$ area, a $CRCLC$ fragment will follow and the counter S will change by ± 2, with *the same* sign as the preceding increment. (Recall that (50)-(51) dictate that $CLCRC$ fragments are impossible, and that each $CRCLC$ fragment is isolated, i.e. it is surrounded by either CLC or CRC fragments.) Likewise, when x_j is in $\mathcal{I}_{\mathrm{II}}$, a CLC or CRC fragment follows, and the counter S changes by ± 1, *reversing* the sign of the previous change. All in all, it is easy to show that the counter S_j follows the dynamics described by the equations (1), (3), (4), (5), (6), and (7).

7. Conclusions: implications for triangular billiards

The result of Theorem 2.1 suggest that the counter K_i (see (47)) of [5] shows absence of localization of the counter, for a $45^\circ : 45^\circ : 90^\circ$ billiard, at least for one particular initial condition, and, if proven, for all rational initial conditions. It seems unlikely localization can reemerge in the generic $\tilde{\alpha}/2 : (90^\circ - \tilde{\alpha}/2) : 90^\circ$ case of [5]. Numerical results of [5] do however suggest localization. A potential resolution for this contradiction can be offered by a probable very slow growth of the counter. In what follows, we will provide an explicit example that confirms this slowness.

Consider

$$\begin{aligned} \sigma &= [0;\, 2,\, 3,\, \ldots,\, n+1,\, \ldots] \\ &\approx 0.433127 \; . \end{aligned} \tag{52}$$

Using the recurrence relations (23)-(24) with the initial conditions (25) one can easily show that the subsequence of the rational approximants with

$$n_m = 4m + 2$$

will satisfy all conditions of Theorem 5.1. Thus, for $x_0 = 0$, the proof of Theorem 5.1 (use (34) with $Q = 1$) shows that S_{j_m} is unbounded for

$$j_m = a_{4m+3}\, q_{4m+2}$$

The first three temporal instances showing a provable growth are

$$\begin{array}{rcccc} m = & 0 & 1 & 2 & \dots \\ j = & 28 & 55688 & 695991252 & \dots \\ S_j = & +8 & +16 & +24 & \dots \\ |S_j| \geq 2\, a_{4m+3} = & 8 & 16 & 24 & \dots \end{array}$$

(see Fig. 3). The last line gives the lower bound (46). Notice that $|S_j|$ stays at its lowest value allowed, something that we can neither prove nor disprove at the moment.

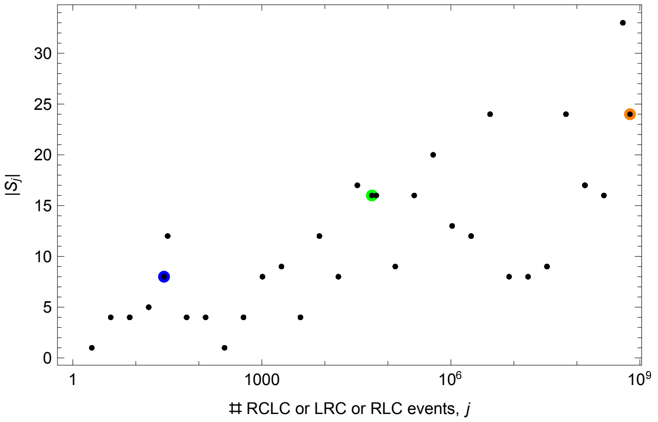

FIGURE 3. Counter S_j as a function of "time" j. We show 200 randomly chosen instances. In addition, we show S_j at $j_{m=0,\,1,\,2}$ (see (34) for $Q = 1$ and $x_0 = 0$), along with the lower bound (46). Blue, green, and orange dots correspond to $m = 0$, 1, 2 respectively.

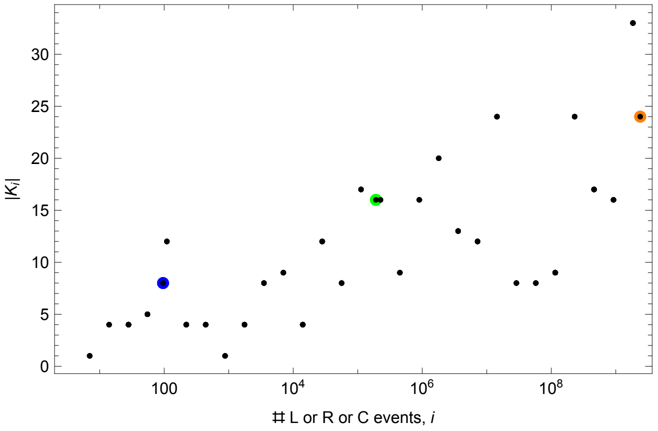

FIGURE 4. Counter K_i of [5] as a function of "time" i. In terms of [5], The rest is the same as at Fig. 3

The growth of the counter, while present, appears to be slow. We suggest the following order of magnitude estimate for the growth of the counter. From (24), we get

$$\ln(q_n) \sim \ln(a_n!) \approx a_n \ln(a_n) \Rightarrow \ln(a_n) \sim \frac{\ln(q_n)}{\ln(\ln(q_n))} \Rightarrow S_j \gtrsim 2\frac{\ln(j)}{\ln(\ln(j)) - 1} \ .$$

The corresponding values of the "physical" counter in question, [5], can also be computed (see Fig. 4).:

$$\begin{array}{rcccc} m = & 0 & 1 & 2 & \dots \\ i = & 96 & 191184 & 2389426656 & \dots \\ K_i = & -8 & -16 & -24 & \dots \\ |K_i| \geq 2\, a_{4m+3} = & 8 & 16 & 24 & \dots \end{array} \ .$$

Recall that $S_j = (-1)^{i(j)+1} K_{i(j)}$. Also, $i(j)$ can be estimated as

$$i(j) \approx (\sigma \times 4 + (1-\sigma) \times 3) \times j \approx 3.43313 \times j \ ,$$

producing $j = 96.1, 191184.0, 2389426656.0, \dots$ in the second line above. The estimate is using (a) the ergodicity of the irrational rotations, leading to $\mathrm{Prob}\,(x \in \mathcal{I}_{\mathrm{I}} \equiv [0,\, \sigma[\,) = \sigma$ and $\mathrm{Prob}\,(x \in \mathcal{I}_{\mathrm{II}} \equiv [\sigma,\, 1[\,) = 1 - \sigma$ and (b) the fact that $x \in \mathcal{I}_{\mathrm{I}}$ corresponds to a four-physical-events RCLC fragment, while $x \in \mathcal{I}_{\mathrm{II}}$ corresponds to either LRC or RLC three-physical-events fragments.

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