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On the spectrum of critical almost Mathieu operators in the rational case

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Abstract

We derive a new Chambers-type formula and prove sharper upper bounds on the measure of the spectrum of critical almost Mathieu operators with rational frequencies.

1 Introduction

The Harper operator, a.k.a. the discrete magnetic Laplacian[1], is a tight-binding model of an electron confined to a 2D square lattice in a uniform magnetic field orthogonal to the lattice plane and with flux $2\pi\alpha$ through an elementary cell. It acts on $\ell^2(\mathbb{Z}^2)$ and is usually given in the Landau gauge representation

$$(H(\alpha)\psi)_{m,n} = \psi_{m,n-1} + \psi_{m,n+1} + e^{-i2\pi\alpha n}\psi_{m-1,n} + e^{i2\pi\alpha n}\psi_{m+1,n}, \tag{1}$$

first considered by Peierls [17], who noticed that it makes the Hamiltonian separable and turns it into the direct integral in θ of operators on $\ell^2(\mathbb{Z})$ given by:

$$(H_{\alpha,\theta}\varphi)(n) = \varphi(n-1) + \varphi(n+1) + 2\cos 2\pi(\alpha n+\theta)\varphi(n), \qquad \alpha,\theta \in [0,1). \tag{2}$$

In physics literature, it also appears under the names Harper's or the Azbel-Hofstadter model, with both names used also for the discrete magnetic Laplacian $H(\alpha)$. In mathematics, it is universally called the critical almost Mathieu operator.[2] In addition to importance in physics, this model is of special interest, being at the boundary of two reasonably well understood regimes: (almost) localization and (almost) reducibility, and not being amenable to methods of either side. Recently, there has been some progress in the study of the fine structure of its spectrum [6, 7, 9, 13, 15].

Denote the spectrum of an operator H, as a set, by $\sigma(H)$. An important object is the union of $\sigma(H_{\alpha,\theta})$ over θ, which coincides with the spectrum of $H(\alpha)$. We denote it $S(\alpha) := \sigma(H(\alpha)) = \cup_{\theta\in[0,1)}\sigma(H_{\alpha,\theta})$. Note that by the general theory of ergodic operators, if α is irrational, $\sigma(H_{\alpha,\theta})$ is independent of θ. We denote the Lebesgue measure of a set A by $|A|$.

For irrational α, the Lebesgue measure $|S(\alpha)| = 0$, and $S(\alpha)$ is a set of Hausdorff dimension no greater than $1/2$ [14, 2, 8]. The proof of the Hausdorff dimension result in [8]

[1] The name "discrete magnetic Laplacian" was first introduced by M. Shubin in [18].

[2] This name was originally introduced by Barry Simon [19].

(which was a conjecture of D. J. Thouless) is based on upper bounds of the measure of the spectrum for $\alpha \in \mathbb{Q}$ and a strong continuity. For rational $\alpha = \frac{p_0}{q_0}$, where p_0, q_0 are coprime positive integers, Last obtained the bounds [14, Lemma 1]:

$$\frac{2(\sqrt{5}+1)}{q_0} < \left| S\left(\frac{p_0}{q_0}\right) \right| < \frac{8e}{q_0}, \tag{3}$$

where $e = \exp(1) = 2.71\ldots$. While the upper bound in (3) was sufficient for the argument of [8], the measure of the spectrum is subject to another conjecture of Thouless [20, 21]: that in the limit $p_n/q_n \to \alpha$, we have $q_n|S(p_n/q_n)| \to c$, where $c = 32C_c/\pi = 9.32\ldots$, $C_c = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2}$ being the Catalan constant. Thouless provided a partly heuristic argument in the case $p_n = 1$, $q_n \to \infty$. A rigorous proof for $\alpha = 0$ and $p_n = 1$ or $p_n = 2$, q_n odd, was given in [5].

The purpose of this note is to present a sharper upper bound, for all $\alpha \in \mathbb{Q}$:

Theorem 1. *For all positive coprime integers p_0 and q_0,*

$$\left| S\left(\frac{p_0}{q_0}\right) \right| \leq \frac{4\pi}{q_0}.$$

Thus, the upper bound is reduced from $8e = 21.74\ldots$ to $4\pi = 12.56\ldots$. The way we prove Theorem 1 is very different from that of [14]; we use the chiral gauge representation [8] and Lidskii's inequalities. The chiral gauge representation of the almost Mathieu operator also leads to a new type of Chambers' relation (equations (14), (15) below).

2 Proof of Theorem 1

Consider the following operator on $\ell^2(\mathbb{Z})$:

$$(\widetilde{H}_{\alpha,\theta}\varphi)(n) = 2\sin 2\pi(\alpha(n-1)+\theta)\varphi(n-1) + 2\sin 2\pi(\alpha n+\theta)\varphi(n+1), \qquad \alpha, \theta \in [0, 1), \tag{4}$$

and define $\widetilde{S}(\alpha) := \cup_{\theta\in[0,1)}\sigma(\widetilde{H}_{\alpha,\theta})$. It was shown in [8, Theorem 3.1] that the operators $M_{2\alpha} := \oplus_{\theta\in[0,1)}H_{2\alpha,\theta}$ and $\widetilde{M}_\alpha := \oplus_{\theta\in[0,1)}\widetilde{H}_{\alpha,\theta}$ are unitarily equivalent, so that $S(\alpha) = \widetilde{S}(\alpha/2)$. (Note that $\sigma(H_{2\alpha,\theta}) \neq \sigma(\widetilde{H}_{\alpha,\theta})$, in general.) See also related partly non-rigorous considerations in [16, 10, 22, 11, 12], and an application of the rational case in [13]. Operator (4) corresponds to the chiral gauge representation of the Harper operator.

From now on, we always consider the case of rational α. Furthemore, the analysis below for $q_0 = 1$, $q_0 = 2$ becomes especially elementary, and gives $|S(1)| = 8$, $|S(1/2)| = 4\sqrt{2}$, so that Theorem 1 obviously holds in these cases. From now on, we assume $q_0 \geq 3$.

If p_0 is even, define $p := \frac{p_0}{2}$ and $q := q_0$ (note that q is necessarily odd in this case). This corresponds to case I below. If p_0 is odd, define $p := p_0$ and $q := 2q_0$. This corresponds to case II below. We note that in either case p and q are coprime and $S(p_0/q_0) = \widetilde{S}(p/q)$.

Let $b(x) := 2\sin(2\pi x)$, and further identify $b_n(\theta) := b((p/q)n+\theta)$. For the operator $\widetilde{H}_{\frac{p}{q},\theta}$, Floquet theory states that $E \in \sigma(\widetilde{H}_{\frac{p}{q},\theta})$ if and only if the equation $(\widetilde{H}_{\frac{p}{q},\theta}\varphi)(n) = E\varphi(n)$ has a solution $\{\varphi(n)\}_{n\in\mathbb{Z}}$ satisfying $\varphi(n+q) = e^{ikq}\varphi(n)$ for all n, and for some real k. Therefore, for a fixed k, there exist q values of E satisfying the eigenvalue equation

$$B_{\theta,k,\ell}\begin{pmatrix}\varphi(\ell)\\ \vdots\\ \varphi(\ell+q-1)\end{pmatrix} = E\begin{pmatrix}\varphi(\ell)\\ \vdots\\ \varphi(\ell+q-1)\end{pmatrix} \tag{5}$$

for any ℓ, where

$$B_{\theta,k,\ell} := \begin{pmatrix} 0 & b_\ell & 0 & 0 & \cdots & 0 & 0 & e^{-ikq}b_{\ell+q-1}\\ b_\ell & 0 & b_{\ell+1} & 0 & \cdots & 0 & 0 & 0\\ 0 & b_{\ell+1} & 0 & b_{\ell+2} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & b_{\ell+q-3} & 0 & b_{\ell+q-2}\\ e^{ikq}b_{\ell+q-1} & 0 & 0 & 0 & \cdots & 0 & b_{\ell+q-2} & 0 \end{pmatrix}. \tag{6}$$

Thus, the eigenvalues of $B_{\theta,k,\ell}$ are independent of ℓ.

2.1 Chambers-type formula

The celebrated Chambers' formula presents the dependence of the determinant of the almost Mathieu operator with $\alpha = p_0/q_0$ restricted to the period q_0 with Floquet boundary conditions, on the phase θ and quasimomentum k. In the critical case it is given by (see, e.g., [14])

$$\det(A_{\theta,k,\ell} - E) = \Delta(E) - 2(-1)^{q_0}(\cos(2\pi q_0\theta) + \cos(kq_0)), \tag{7}$$

where

$$A_{\theta,k,\ell} := \begin{pmatrix} a_\ell & 1 & 0 & 0 & \cdots & 0 & 0 & e^{-ikq}\\ 1 & a_{\ell+1} & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 1 & a_{\ell+2} & 1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & a_{\ell+q-2} & 1\\ e^{ikq} & 0 & 0 & 0 & \cdots & 0 & 1 & a_{\ell+q-1} \end{pmatrix}, \quad \ell\in\mathbb{Z}, \tag{8}$$

$$a(x) := 2\cos(2\pi x), \qquad a_n(\theta) := a((p_0/q_0)n + \theta), \tag{9}$$

and Δ, the discriminant[3], is independent of θ and k. An immediate corollary of this formula is that $S\left(\frac{p_0}{q_0}\right) = \Delta^{-1}([-4,4])$, e.g., [14].

Here we obtain a formula of this type for $\det(B_{\theta,k,\ell} - E)$. Indeed, as usual, separating the terms containing k in the determinant, we obtain, for the characteristic polynomial $D_{\theta,k}(E) := \det(B_{\theta,k,\ell} - E)$:

$$D_{\theta,k}(E) = D_\theta^{(0)}(E) - (-1)^q b_0\cdots b_{q-1}\cdot 2\cos(kq), \tag{10}$$

where $D_\theta^{(0)}(E)$ is independent of k and equal therefore to $D_{\theta,k=\frac{\pi}{2q}}(E)$.

For the product of b_j's we have:

[3] In [14], the discriminant differs from $\Delta(E)$ by the factor $(-1)^{q_0}$.

Lemma 1.

$$\begin{aligned} b_0 \cdots b_{q-1} &= \prod_{j=0}^{q-1} 2\sin 2\pi\left(\frac{p}{q}j+\theta\right) \\ &= 4\sin(\pi q\theta)\sin\pi q(\theta+1/2) = 2(\cos(\pi q/2) - \cos\pi q(2\theta+1/2)). \end{aligned} \tag{11}$$

Proof. To evaluate the product of b_j's, we expand sine in terms of exponentials and use the formula $1 - z^{-q} = \prod_{j=0}^{q-1}(1 - z^{-1}e^{2\pi i \frac{p}{q}j})$. An alternative derivation can go along the lines of the proof of Lemma 9.6 in [1]. ∎

Substituting (11) into (10), we have

$$D_{\theta,k}(E) = D_\theta^{(0)}(E) - 8(-1)^q \sin(\pi q\theta)\sin\pi q(\theta+1/2)\cos(kq). \tag{12}$$

We can further obtain the dependence of $D_\theta^{(0)}(E)$ on θ:

Lemma 2.
$$D_\theta^{(0)}(E) = \widetilde{\Delta}(E) + \begin{cases} 0, & q \ \textit{odd} \\ 4(\cos(2\pi q\theta) - 1), & q \ \textit{even}, \end{cases}$$

where the discriminant $\widetilde{\Delta}(E) := D_{\theta=0}^{(0)}(E)$ is independent of θ.

Proof. Since $D_{\theta,k}(E)$ is independent of ℓ, it is $1/q$ periodic in θ, i.e., $D_{\theta,k}(E) = D_{\theta+1/q,k}(E)$, and by (10) so is $D_\theta^{(0)}(E)$. Therefore, since, clearly, $D_\theta^{(0)}(E) = \sum_{n=-q}^{q} c_n(E)e^{2\pi i\theta n}$, the terms c_k other than $k = mq$ vanish, and $D_\theta^{(0)}(E)$ has the following Fourier expansion:

$$D_\theta^{(0)}(E) = c_0(E) + c_q e^{2\pi i q\theta} + c_{-q}e^{-2\pi i q\theta}.$$

It is easily seen that the c_q and c_{-q} can be obtained from the expansion of the determinant and that, moreover, they do not depend on E. Expanding $D_\theta^{(0)}(E)$ with $E = 0$ in rows and columns (cf. [13]), we obtain

$$D_\theta^{(0)}(0) = D_{\theta,k=\frac{\pi}{2q}}(0) = \begin{cases} 0, & q \text{ odd} \\ (-1)^{q/2}(b_0^2 b_2^2 \cdots b_{q-2}^2 + b_1^2 b_3^2 \cdots b_{q-1}^2), & q \text{ even.} \end{cases} \tag{13}$$

This gives $c_q = c_{-q} = 0$ for q odd, and $c_q = \prod_{j=0}^{\frac{q-2}{2}} e^{8\pi i \frac{p}{q}j} + \prod_{j=0}^{\frac{q-2}{2}} e^{4\pi i \frac{p}{q}(2j+1)} = 2 = c_{-q}$, for q even. It remains to denote $\widetilde{\Delta}(E) = c_0(E)$ for q odd, and $\widetilde{\Delta}(E) = c_0(E) + 4$ for q even, and the proof is complete. ∎

We therefore have, by (12) and Lemma 2:

Lemma 3 (Chambers-type formula)**.**

$$D_{\theta,k}(E) = \widetilde{\Delta}(E) + 4(-1)^{(q-1)/2}\sin(2\pi q\theta)\cos(kq), \qquad q \ \textit{odd}. \tag{14}$$

$$D_{\theta,k}(E) = \widetilde{\Delta}(E) - 4(1 - \cos(2\pi q\theta))(1 + (-1)^{q/2}\cos(kq)), \qquad q \ \textit{even}. \tag{15}$$

Note that $\widetilde{\Delta}(E)$ is a polynomial of degree q independent of $k \in \mathbb{R}$ and $\theta \in [0,1)$. By Floquet theory, the spectrum $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ is the union of the eigenvalues of $B_{\theta,k,\ell}$ over k, a collection of q intervals.

We make the following observations.

Case I: q is odd.

By (14), $D_{\theta,k}(E) \equiv \det(B_{\theta,k,\ell} - E) = 0$ if and only if $\widetilde{\Delta}(E) = 4(-1)^{(q+1)/2}\sin(2\pi q\theta)\cos(kq)$. Thus, $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ is the preimage of $[-4|\sin(2\pi q\theta)|, 4|\sin(2\pi q\theta)|]$ under the mapping $\widetilde{\Delta}(E)$. If $\theta = m/(2q)$, $m \in \mathbb{Z}$, $\sigma(\widetilde{H}_{\frac{p}{q},\frac{m}{2q}})$ is a collection of q points where $\widetilde{\Delta}(E) = 0$. (In this case, $b_0(m/(2q)) = 0$, so that $\widetilde{H}$ splits into the direct sum of an infinite number of copies of a q-dimensional matrix.) We note that the spectra $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ for different θ are nested in one another as θ grows from 0 to $1/(4q)$; in particular, for each $\theta \in [0,1)$,

$$\sigma(\widetilde{H}_{\frac{p}{q},\theta}) = \widetilde{\Delta}^{-1}([-4|\sin(2\pi q\theta)|, 4|\sin(2\pi q\theta)|]) \subseteq \sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{4q}}) = \widetilde{\Delta}^{-1}([-4,4]). \tag{16}$$

This implies that all the maxima of $\widetilde{\Delta}(E)$ are no less than 4, and all the minima are no greater than -4. Moreover, taking the union over all $\theta \in [0,1)$ gives:

$$\widetilde{S}\left(\frac{p}{q}\right) = \sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{4q}}) = \widetilde{\Delta}^{-1}([-4,4]). \tag{17}$$

Clearly, it is sufficient to consider only $\theta \in [0, 1/(4q)]$.

Case II: q is even. This case is similar to case I, so we omit some details for brevity. By (15), $D_{\theta,k}(E) = 0$ if and only if $\widetilde{\Delta}(E) = 4(1-\cos(2\pi q\theta))(1+(-1)^{q/2}\cos(kq))$. Considering the cases $k = 0, \frac{\pi}{q}$, it is easy to see that $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ is the preimage of $[0, 8-8\cos(2\pi q\theta)]$ under the mapping $\widetilde{\Delta}(E)$. If $\theta = m/q$, $m \in \mathbb{Z}$, $\sigma(\widetilde{H}_{\frac{p}{q},\frac{m}{q}})$ is a collection of q points where $\widetilde{\Delta}(E) = 0$. We note that the spectra $\sigma(\widetilde{H}_{\frac{p}{q},\theta})$ for different θ are nested in one another as θ grows from 0 to $1/(2q)$; in particular, for each $\theta \in [0,1)$,

$$\sigma(\widetilde{H}_{\frac{p}{q},\theta}) = \widetilde{\Delta}^{-1}([0, 8-8\cos(2\pi q\theta)]) \subseteq \sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{2q}}) = \widetilde{\Delta}^{-1}([0,16]). \tag{18}$$

This implies that all the maxima of $\widetilde{\Delta}(E)$ are no less than 16, and all the minima are no greater than 0. Moreover, taking the union over all $\theta \in [0,1)$ gives:

$$\widetilde{S}\left(\frac{p}{q}\right) = \sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{2q}}) = \widetilde{\Delta}^{-1}([0,16]). \tag{19}$$

Clearly, it is sufficient to consider only $\theta \in [0, 1/(2q)]$.

In this case of even q we can say more about the form of $\widetilde{\Delta}(E)$. Note that $b_0(0) = b_{q/2}(0) = 0$ and $b_k(0) = b_{-k}(0)$. Recall that by Floquet theory, $D_{\theta,k}(E) = \det(B_{\theta,k,\ell} - E)$ is independent of the choice of ℓ. For convenience, choose $\ell = -q/2+1$. It is easily seen that $B_{\theta=0,k,\ell=-q/2+1}$ decomposes into a direct sum, and moreover $\widetilde{\Delta}(E) = D_{\theta=0,k}(E) = (-1)^{q/2}P_{q/2}(-E)P_{q/2}(E)$, where $P_{q/2}(E)$ is a polynomial of degree $q/2$, odd if $q/2$ is odd, and even if $q/2$ is even (as it is a characteristic polynomial of a tridiagonal matrix with zero main diagonal). Thus $\widetilde{\Delta}(E) = P_{q/2}(E)^2$ is a square.

The discriminants $\widetilde{\Delta}(E) \equiv \widetilde{\Delta}_{p/q}(E)$ and $\Delta(E) \equiv \Delta_{p_0/q_0}(E)$ are related in the following way:

Lemma 4. *For q odd,*

$$\widetilde{\Delta}_{p/q}(E) = \Delta_{p_0/q_0}(E), \qquad p_0 = 2p, \quad q_0 = q. \tag{20}$$

For q even,

$$\widetilde{\Delta}_{p/q}(E) = \Delta^2_{p_0/q_0}(E), \qquad p_0 = p, \quad q_0 = q/2. \tag{21}$$

Proof. Case I: q is odd. Here, by our definitions at the start of the section, $p_0 = 2p$ and $q_0 = q$. $\widetilde{\Delta}_{p/q}(E)$ and $\Delta_{p_0/q_0}(E)$ are polynomials in E of degree q with the same coefficient -1 of E^q. Since $\widetilde{\Delta}(E) = \Delta(E) = \pm 4$ at the $2q \geq q+1$ distinct edges of the bands (cf. [4, 3.3]), these polynomials coincide: $\widetilde{\Delta}(E) = \Delta(E)$ for each E.

Case II: q is even. Here, $p_0 = p$ and $q_0 = q/2$. $\widetilde{S}\left(\frac{p}{q}\right) = S\left(\frac{p_0}{q_0}\right)$ is the preimage of $[0,16]$ under $\widetilde{\Delta}_{p/q}$ and of $[-4,4]$ under Δ_{p_0/q_0}, hence also of $[0,16]$ under $\Delta^2_{p_0/q_0}$. On the other hand, we have seen above that $\widetilde{\Delta}(E) = P^2_{q/2}(E)$ for some polynomial $P_{q/2}(E)$ of degree $q/2 = q_0$. Thus, $P^2_{q/2}(E)$ and $\Delta^2(E)$ coincide at the $2q_0 \geq q_0 + 1$ (for q_0 odd) and $2q_0 - 1 \geq q_0 + 1$ (for q_0 even) distinct edges of the bands (cf. [4, 3.3]; the central bands merge for q_0 even), so these polynomials of degree q are equal: $\widetilde{\Delta}(E) = \Delta^2(E)$ for each E. ■

2.2 Measure of the spectrum

The rest of the proof follows the argument of [3], namely it uses Lidskii's inequalities to bound $|\widetilde{S}(\frac{p}{q})|$. The key observation is that choosing ℓ appropriately, we can make the corner elements of the matrix $B_{\theta,k,\ell}$ very small, of order $1/q$ when q is large. This is not possible to do in the standard representation for the almost Mathieu operator. Here are the details.

Case I: q is odd. Assume without loss that $(-1)^{(q+1)/2} > 0$, $\theta \in (0, 1/(4q)]$. (If $(-1)^{(q+1)/2} < 0$, the analysis is similar.) Then the eigenvalues $\{\lambda_i(\theta)\}_{i=1}^q$ of $B_{\theta,k=0,\ell}$ labelled in decreasing order are the edges of the spectral bands where $\widetilde{\Delta}(E)$ reaches its maximum $4\sin(2\pi q\theta)$ on the band; and the eigenvalues $\{\widehat{\lambda}_i(\theta)\}_{i=1}^q$ of $B_{\theta,k=\pi/q,\ell}$ labelled in decreasing order are the edges of the spectral bands where $\widetilde{\Delta}(E)$ reaches its minimum $-4\sin(2\pi q\theta)$ on the band. Then

$$|\sigma(\widetilde{H}_{\frac{p}{q},\theta})| = \sum_{j=1}^{q} (-1)^{q-j}(\widehat{\lambda}_j(\theta) - \lambda_j(\theta)) = \sum_{j=1}^{(q+1)/2} (\widehat{\lambda}_{2j-1}(\theta) - \lambda_{2j-1}(\theta)) + \sum_{j=1}^{(q-1)/2} (\lambda_{2j}(\theta) - \widehat{\lambda}_{2j}(\theta));$$
$$\widehat{\lambda}_j(\theta) - \lambda_j(\theta) > 0, \quad \text{if } j \text{ is odd}; \qquad \widehat{\lambda}_j(\theta) - \lambda_j(\theta) < 0, \quad \text{if } j \text{ is even}. \tag{22}$$

Now we view $B_{\theta,k=\pi/q,\ell}$ as $B_{\theta,k=0,\ell}$ with the added perturbation

$$B_{\theta,k=\pi/q,\ell} - B_{\theta,k=0,\ell} = \begin{pmatrix} & & -2b_{\ell+q-1} \\ & & \\ -2b_{\ell+q-1} & & \end{pmatrix},$$

which has the eigenvalues $\{E_i(\theta)\}_{i=1}^{q}$ given by:

$$E_q(\theta) = -2|b_{\ell+q-1}(\theta)| < 0 = E_{q-1}(\theta) = \cdots = E_2(\theta) = 0 < 2|b_{\ell+q-1}(\theta)| = E_1(\theta).$$

The Lidskii inequalities (e.g., (2.51) in [3]) are:

Theorem 2. *For any $q \times q$ self-adjoint matrix M, we denote is eigenvalues by $E_1(M) \geq E_2(M) \geq \cdots \geq E_q(M)$. For $q \times q$ self-adjoint matrices A and B, we have:*

$$E_{i_1}(A+B) + \cdots + E_{i_m}(A+B) \leq E_{i_1}(A) + \cdots + E_{i_m}(A) + E_1(B) + \cdots + E_m(B);$$
$$E_{i_1}(A+B) + \cdots + E_{i_m}(A+B) \geq E_{i_1}(A) + \cdots + E_{i_m}(A) + E_{q-m+1}(B) + \cdots + E_q(B),$$

for any $1 \leq i_1 < \cdots < i_m \leq q$.

Applying these inequalities with $A = B_{\theta,k=0,\ell}$, $B = B_{\theta,k=\pi/q,\ell} - B_{\theta,k=0,\ell}$ gives:

$$\sum_{j=1}^{(q+1)/2} (\widehat{\lambda}_{2j-1}(\theta) - \lambda_{2j-1}(\theta)) \leq \sum_{j=1}^{(q+1)/2} E_j(\theta) = E_1(\theta);$$
$$\sum_{j=1}^{(q-1)/2} (\lambda_{2j}(\theta) - \widehat{\lambda}_{2j}(\theta)) \leq -\sum_{j=(q-1)/2}^{q} E_j(\theta) = -E_q(\theta).$$

Substituting these into (22), we obtain:

$$|\sigma(\widetilde{H}_{\frac{p}{q},\theta})| \leq E_1(\theta) - E_q(\theta) = 4|b_{\ell+q-1}(\theta)|. \tag{23}$$

Moreover, by the invariance of $D_{\theta,k}(E)$ under the mapping $b_n \mapsto b_{n+m}$, for $n = 0, 1, \ldots, q-1$ and any m, we can choose any ℓ in (23), so that

$$|\sigma(\widetilde{H}_{\frac{p}{q},\theta})| \leq 4 \min_{\ell} |b_{\ell+q-1}(\theta)|. \tag{24}$$

In particular,

$$\left|\widetilde{S}\left(\frac{p}{q}\right)\right| = |\sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{4q}})| \leq 4 \min_{\ell} \left|b_{\ell+q-1}\left(\frac{1}{4q}\right)\right| = 4 \cdot 2 \left|\sin 2\pi \left(\frac{1}{4q}\right)\right| \leq \frac{4\pi}{q}. \tag{25}$$

Therefore, $|S(\frac{p_0}{q_0})| = |\widetilde{S}(\frac{p}{q})| \leq \frac{4\pi}{q} = \frac{4\pi}{q_0}$, as required.

Case II: q is even. This case is similar to case I, so we omit some details for brevity. This time, the Lidskii equations of Theorem 2 show that $|\widetilde{S}(\frac{p}{q})| \leq \frac{8\pi}{q}$. Indeed, as in (24), we have (note the doubling of the eigenvalues for $\widetilde{\Delta}(E) = 0$)

$$|\sigma(\widetilde{H}_{\frac{p}{q},\theta})| \leq 4 \min_{\ell} |b_{\ell+q-1}(\theta)|. \tag{26}$$

In particular,

$$\left|\widetilde{S}\left(\frac{p}{q}\right)\right| = |\sigma(\widetilde{H}_{\frac{p}{q},\theta=\frac{1}{2q}})| \leq 4\min_{\ell}\left|b_{\ell+q-1}\left(\frac{1}{2q}\right)\right| = 4\cdot 2\left|\sin 2\pi\left(\frac{1}{2q}\right)\right| \leq \frac{8\pi}{q}. \tag{27}$$

Therefore, $|S(\frac{p_0}{q_0})| = |\widetilde{S}(\frac{p}{q})| \leq \frac{8\pi}{q} = \frac{4\pi}{q_0}$, as required.

This completes the proof of Theorem 1.

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