Fractal dimension of spectral measures of rank one perturbations of probability measures

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Abstract

In this paper, we consider the packing dimension of a family of measures consisting of a probability measure, μ, and the spectral measures of its rank one perturbations, μ_λ. Past results have determined that the Hausdorff dimension of spectral measures of almost every rank one perturbation can be determined if the limit inferior of a ratio involving μ is constant on a Lebesgue typical set. This ratio is sometimes called the pointwise dimension of μ and is related to the upper derivative of μ. Work has been done to make a similar argument for the packing dimension, but with little success. Using the theory of rank one perturbations and Borel transforms, also known as Weyl functions, we introduce the concept of Lebesgue exact dimension for μ, which allows us to determine the packing dimension of spectral measures of almost every rank one perturbation μ_λ. As an interesting corollary, we find that this limit condition implies a stronger result: the Hausdorff and packing dimensions are equal for spectral measures of almost every rank one perturbation.

1. Introduction

In this paper we consider the packing dimension of a family of measures consisting of a probability measure, μ, and the spectral measures of its rank one perturbations, μ_λ. Previous results have established that the Hausdorff dimension of μ and μ_λ can be determined for almost every λ by the limit

$$\liminf_{\delta\to 0} \frac{\log \mu(x-\delta, x+\delta)}{\log \delta} \tag{1.1}$$

on a Lebesgue typical set [2]. In [2], it was found that if this lim inf is equal to α on a Lebesgue typical set, then the Hausdorff dimension of the spectral measures of almost every rank one perturbation is $2-\alpha$.

It has also been established that the packing dimensions of μ can be determined by the limit

$$\limsup_{\delta\to 0} \frac{\log \mu(x-\delta, x+\delta)}{\log \delta}, \tag{1.2}$$

on a Lebesgue typical set, but there has been no result relating this limit to the packing dimension of the spectral measures of the rank one perturbations, μ_λ.

An obvious relation arising from the definitions is that if (1.1) and (1.2) coincide, that is if the limit exists, on a μ-typical set, then the Hausdorff and packing dimensions for μ are equal. When this happens, μ is said to have *exact dimension* equal to the limit. When the limit exists on a Lebesgue typical set, we will say μ has *Lebesgue exact dimension* equal to the limit. We will make these two definitions precise in Section 3. It is known that Borel transforms provide a way to relate spectral measures of rank one perturbations back to the original measure. Using Borel transforms, we prove that, if we replace lim sup with lim in the above limit, and if the limit exists and is equal to some $1 < \alpha < 2$ on a Lebesgue typical set, then the packing dimension of spectral measures of almost every rank one perturbation is given by $2 - \alpha$.

We use concepts from the study of rank one perturbations and Borel transforms in our paper and thus our results can be applied to situations where spectral measures are the starting point, rather than pure probability measures. Indeed, Borel transforms were used to study spectra of operators under rank one perturbations in [3]. The existence of mixed spectral types was also studied in [4] with regards to the real parameter.

The theory of rank one perturbations and Borel transforms is outlined in detail in [7]. For completeness, we list the definitions and results needed for this paper.

Given a probability measure μ we define the Borel transform as

$$F_\mu(z) = \int \frac{d\mu(x)}{x-z}$$

for $\operatorname{Im} z > 0$. We note that

$$F_\mu : \{z : \operatorname{Im}(z) > 0\} \to \{z : \operatorname{Im}(z) > 0\}$$

is an analytic function. In this paper, we usually call F the Borel transform of μ.

Let μ be a probability measure, A a positive self-adjoint operator on $L^2(\mathbb{R}, d\mu)$, and φ be a unit vector in $L^2(\mathbb{R}, d\mu)$. We define the rank one perturbations of A, associated to the real parameter λ, as

$$A_\lambda = A + \lambda(\varphi, \cdot)\varphi.$$

If $d\mu_\lambda$ is the spectral measure for φ and the operator A_λ, then we also define

$$F_\lambda(z) = \int \frac{d\mu_\lambda(x)}{x - z}$$

and denote $F(z)$ for $F_0(z)$. These objects are well studied and there are some well-known properties that arise from the definitions:

$$F_\lambda(z) = \frac{F(z)}{1 + \lambda F(z)}, \tag{1.3}$$

$$\text{Im} F_\lambda(z) = \frac{\text{Im} F(z)}{|1 + \lambda F(z)|^2}. \tag{1.4}$$

With this we can define a family of probability measures by looking at a probability measure and the associated spectral measures, $d\mu_\lambda$, of the rank one perturbations. See Section 7 for an example.

The study of the fractal dimension of these measures and the spectral measures of their rank one perturbations has many varied applications. The Lebesgue Decomposition Theorem allows us to decompose any measure μ into three parts:

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$$

where μ_{ac} is the absolutely continuous part of the measure, μ_{sc} is the singular continuous part of the measure, and μ_{pp} is the pure point part of the measure. It is interesting to note that absolutely continuous measures have Hausdorff and packing dimension 1, while pure point measures have Hausdorff dimension 0. The interesting case is the singular continuous measures; these measures can have Hasudorff and packing dimension between 0 and 1. See Section 7 for an example of such a measure. Physicists tend to study the singular continuous spectrum, and so an important property of these measures is their Hausdorff and packing dimensions. In this paper, we will consider only singular continuous measures.

The relation between Borel transforms and the Hausdorff dimension of a measure has been studied in [2]. Roughly speaking, the Hausdorff dimension of measure is related to the Power law property of the imaginary part of its Borel transform: μ has Hausdorff dimension α if $\text{Im} F(E + i\epsilon) \sim \epsilon^\alpha$.

However, a similar argument cannot be generalized to a packing dimension result[1]. Jitomirskaya, Liu and Tcheremchantsev show that a weaker result still holds for packing dimension in [5]. The concept of spectral dimension was introduced in [6] to establish a lower bound on the packing dimension. We briefly discuss applications of our result to Spectral dimension in Section 6.

In this paper, we consider the Hilbert Space $L^2(\mathbb{R}, d\mu)$. The unit vector $\phi \in L^2(\mathbb{R}, d\mu)$ and the positive self-adjoint operator A will be fixed for the purposes of examining rank one perturbations. We prove

Theorem 1.1. *If* $\lim_{\epsilon\to 0} \frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} = \alpha$, *for Lebesgue a.e.* x *and some* $1 < \alpha < 2$, *and if we let* μ_λ *denote the spectral measure of the rank one perturbation with parameter* λ, *then*

1. $\lim_{\epsilon\to 0} \frac{\log \mu_\lambda(x-\epsilon, x+\epsilon)}{\log \epsilon} = 2 - \alpha$ *for Lebesgue a.e.* λ *and* μ_λ*-a.e.* x.
2. *The packing dimension of the spectral measure of the rank one perturbation,* μ_λ, *is* $2 - \alpha$ *for Lebesgue a.e.* λ.

See Sections 2 and 3 for the relevant packing dimension definitions.

We can then use this theorem to prove a result relating packing dimension to Spectral dimension.

Theorem 1.2. *Under the conditions of Theorem 1.1, we have*

1. $\dim_s^-(\mu_\lambda) = \dim_s^+(\mu_\lambda)$ *and*
2. $\dim_s^+(\mu_\lambda) = \dim_P(\mu_\lambda)$

for Lebesgue a.e. λ.

Spectral dimension was introduced in [6], and we review the relevant definitions and results in Section 6.

In Section 4, we study what happens when this limit just holds on a set E, without any assumptions on the measure of E. The assumption that E has full Lebesgue measure is both reasonable, see Section 7, and useful, see the start of Section 5. Regardless, by studying the set, E, where this limit holds, we can provide information about the packing dimension of the spectral measure of a rank one perturbation, see the end of Section 4.

Our paper is organized in the following way. Section 2 defines the various fractal dimensions that we consider in this paper. Section 3 gives the broad

[1] It was pointed out in [6] that a lower bound on packing dimension can be recovered from the limit behavior of the Borel transform.

definition for dimension of a measure and we then introduce well-known results that simplify this definition for Hausdorff and packing dimension. We include proofs to keep our paper self contained. In sections 4 and 5 we prove that, if a measure obeys a simple limit statement, then the spectral measure for almost every rank one perturbation has packing dimension equal to two minus this limit. We then study spectral dimension under the same assumptions in Section 6. Section 7 analyzes a particular family of measures and we apply our theorem to this family. Much of the work done in finding the limit value in section 7 was completed in [2].

2. Fractal Dimension of Sets

Given a Borel set $E \subset [0,1]$, we can define the dimension of E in many different ways. We are, in particular, interested in two different but related definitions.

First, we consider the Hausdorff dimension. Fix $E \subset [0,1]$ and $\alpha \in [0,1]$. We define

$$H_{\alpha,\delta}(E) = \inf \left\{ \sum_{j=1}^{\infty} |B_j|^{\alpha} : |B_j| < \delta; E \subset \bigcup_{j=1}^{\infty} B_j \right\},$$

where the inf is over all δ-covers by intervals B_j of diameter at most δ. We see that as δ decreases, H increases, so the limit

$$h^{\alpha}(E) = \lim_{\delta \to 0^+} H_{\alpha,\delta}(E)$$

exists. We define h^{α} to be the *α-dimensional Hausdorff measure*. This is a non-sigma-finite measure on the Borel sets with some interesting properties as we vary α. Notice that if $\beta < \alpha < \gamma$, then

$$\delta^{\alpha-\gamma} H_{\gamma,\delta}(E) \leq H_{\alpha,\delta}(E) \leq \delta^{\alpha-\beta} H_{\beta,\delta}(E).$$

Thus if $h^{\alpha}(E) < \infty$, then $h^{\gamma}(E) = 0$, and if $h^{\alpha}(E) > 0$, then $h^{\beta}(E) = \infty$. We then see that for every E there is some α_0 such that, for every $\beta < \alpha_0 < \gamma$ we have $h^{\beta}(E) = \infty$ and $h^{\gamma}(E) = 0$. In other words,

$$\alpha_0 = \inf\{\alpha : h^{\alpha}(E) = 0\}$$

and

$$\alpha_0 = \sup\{\alpha : h^{\alpha}(E) = \infty\}.$$

We say that α_0 is the *Hausdorff dimension* of E.

Next, we consider the packing dimension. We begin by defining

$$P_{\alpha,\delta}(F) = \sup\left\{\sum_{j=1}^{\infty} |B_j|^{\alpha} : |B_j| < \delta; E \subset \bigcup_{j=1}^{\infty} B_j\right\},$$

where the supremum is over all disjoint δ-covers of F by balls B_j of radius at most δ and centers in F.

We can see that $P_{\alpha,\delta}(F)$ is decreasing with δ, so the limit

$$\lim_{\delta\to 0} P_{\alpha,\delta}(F) = P_{\alpha,0}(F)$$

exists and may be ∞.

This is not quite a measure. Indeed, if we consider any countable dense set Q, then $P_{\alpha,0}(Q) = \infty$, yet given any point $q \in Q$ we have $P_{\alpha,0}(\{q\}) = 0$. This indicates that $P_{\alpha,0}$ is not countably additive, and thus not a measure. In order to construct a measure from $P_{\alpha,0}$, we decompose F into a countable collection of sets and define

$$p^{\alpha}(F) = \inf\left\{\sum_{i=1}^{\infty} P_{\alpha,0}(F_i) : F \subset \bigcup_{i=1}^{\infty} F_i\right\}.$$

We call p^{α} the α-dimensional packing measure. Just as we did for Hausdorff dimension, we now define the packing dimension of F as

$$P(F) = \sup\{\alpha \geq 0 : p^{\alpha}(F) = \infty\}$$

or

$$P(F) = \inf\{\alpha \geq 0 : p^{\alpha}(F) = 0\}.$$

3. Fractal Dimension of Measures

Given a probability measure μ we are interested in defining the dimension of μ.

Definition 3.1. If μ is a probability measure and if $\dim(E)$ gives the dimension of a set E, then the upper and lower dimensions of μ are, respectively,

$$\dim^{+}(\mu) = \inf\{\dim(E) : \mu(E) = 1\} \tag{3.1}$$

$$\dim^{-}(\mu) = \inf\{\dim(E) : \mu(E) > 0\}. \tag{3.2}$$

While Definition 3.1 is concise and works for any definition of dimension that we could consider, it is not the easiest to apply to problems, so we look at a way to simplify the work necessary to do this calculation for Hausdorff and packing dimensions.

Definition 3.2. Fix $x \in \mathbb{R}$. The upper and lower local dimensions are

$$d_\mu^+(x) = \limsup_{\delta \to 0} \frac{\log \mu(x-\delta, x+\delta)}{\log \delta} \tag{3.3}$$

$$d_\mu^-(x) = \liminf_{\delta \to 0} \frac{\log \mu(x-\delta, x+\delta)}{\log \delta}. \tag{3.4}$$

Rogers–Taylor and others have done work to show that Definition 3.2 can be used to define the Hausdorff and packing dimension of a measure.

Theorem 3.1. *The upper and lower Hausdorff Dimension of μ is*

$$\dim_H^+(\mu) = \mu - ess \sup d_\mu^-(x) \tag{3.5}$$

$$\dim_H^-(\mu) = \mu - ess \inf d_\mu^-(x). \tag{3.6}$$

Theorem 3.2. *The upper and lower packing dimension of μ is*

$$\dim_P^+(\mu) = \mu - ess \sup d_\mu^+(x) \tag{3.7}$$

$$\dim_P^-(\mu) = \mu - ess \inf d_\mu^+(x). \tag{3.8}$$

Proof of Theorem 3.1

Proof. For the first equality, suppose $a = \mu - ess \sup d_\mu^-(x)$. Then μ is supported on a set of dimension a by [2] and so $a \geq \dim_H^+(\mu)$. It follows that $a \geq \dim_H^+(\mu)$.

Suppose $a < \mu - ess \sup d_\mu^-(x)$ and let S be any set supporting μ. Then for some set $B \subset S$ of positive measure, we know $d_\mu^-(x) > a$ for $x \in B$. Consider χ_B, the characteristic function of B, and the measure $d\mu_B = \chi_B d\mu$. Clearly $d\mu_B$ is supported on B. Also notice that $d\mu_B$ is continuous with respect to h^a, by [2]. Furthermore, this measure satisfies $d_{\mu_B}^-(x) \geq d_\mu^-(x) > a$ for μ_B-a.e. x. Now if E is a set of Hausdorff dimension less than a, then $\mu_B(E) = 0$ by the continuity of $d\mu_B$. That is, $H(B) \geq a$. Since $B \subset S$, we also have $H(S) \geq a$. Since this holds for any supporting set S, we can conclude that $\dim_H^+(\mu) \geq a$. This establishes the first equality.

To establish the second equality, suppose $b = \mu - ess \inf d_\mu^-(x)$. First we want to show $b \leq \dim_H^-(\mu)$. If $b = 0$, then it is trivially true that $\dim_H^-(\mu) \geq b$, so suppose $b > 0$. Let $0 < \beta < b$. Then $d_\mu^-(x) > \beta$ for μ-a.e. x, so $\mu(E) = 0$

for any set E of Hausdorff dimension $< \beta$ by [2]. Since $\beta < b$ was chosen arbitrarily, this means $\dim_H^-(\mu) \geq b$.

Now we want to show $\dim_H^-(\mu) \leq b + 3\epsilon$. Recall that, by the definition of b, for every $\epsilon > 0$ there exists a set $F, \mu(F) > 0$ such that

$$\liminf_{\delta \to 0} \frac{\log \mu(x - \delta, x + \delta)}{\log \delta} < b + \epsilon$$

for every $x \in F$. Consider this set F. Either F is bounded or it can be expressed as a countable union of bounded sets of positive measure. We want to show that $H(F) \leq b + 3\epsilon$, so it suffices to show that one of these bounded subsets of positive measure has a Hausdorff dimension less than $b + 3\epsilon$. WLOG suppose F is bounded. Then for every $\delta > 0$ we can find a finite set of points $x_i \in F$ such that $B_i(\delta) = (x_i - \delta, x_i + \delta)$ is a cover of F. By the Besicovitch Covering Theorem, we can also assume that each point of F appears in at most two of the B_i independent of δ. That is,

$$\sum \mu(B_i(\delta)) \leq 2\mu(\mathbb{R}) = 2.$$

Furthermore, if we denote by N_δ the number of sets $B_i(\delta)$ in our cover of F, then

$$\sum |B_i|^{b+3\epsilon} = N_\delta (2\delta)^{b+3\epsilon}.$$

By our choice of F, we have

$$\mu(x_i - \delta, x_i + \delta) \geq \delta^{b+2\epsilon}$$

for every $\delta < \delta_0$, so

$$N_\delta \delta^{b+2\epsilon} \leq \sum \mu(B_i(\delta)) \leq 2.$$

In particular, we see

$$\sum |B_i(\delta)|^{b+3\epsilon} = N_\delta (2\delta)^{b+3\epsilon} = N_\delta \delta^{b+2\epsilon} 2^{b+3\epsilon} \delta^\epsilon \to 0$$

as $\delta \to 0$. This implies $h^{b+3\epsilon}(F) = 0$, so $H(F) \leq b + 3\epsilon$ which in turn allows us to conclude $\dim_H^-(\mu) \leq b + 3\epsilon$. This establishes the second equality. □

To prove the result for the packing dimension, the proof is similar to the above arguments, with the only technical difficulty being in the proof that $\dim_P^-(\mu) \leq \mu - ess \inf d_\mu^+(x) + 3\epsilon$. We provide a proof of this below.

Proof that $\dim_P^-(\mu) \leq \mu - ess\inf d_\mu^+(x) + 3\epsilon$

Proof. Let $a = \mu - ess\inf d_\mu^+(x)$. By definition of $\mu - ess\inf d_\mu(x)$, we know that for every $\epsilon > 0$, there exists a set B of positive measure such that $d_\mu^+(x) < a + \epsilon$ for every $x \in E$. In particular, this means that for every $x \in B$ there exists $\delta > 0$ such that

$$\mu(x_i - \delta, x_i + \delta) \geq \delta^{a+2\epsilon}.$$

Consider the sets

$$B_n = \left\{ x \in B : \delta^* = \sup\{\delta : \mu(x-\delta, x+\delta) > \delta^{a+2\epsilon}\} < \frac{1}{n} \right\}$$

for $n \geq 1$. Denote by B_0 all points $x \in B$ such that $x \notin B_n$ for any n. Observe that every $x \in B_0$ obeys the inequality

$$\mu(x-\delta, x+\delta) \geq \delta^{a+2\epsilon}$$

for every $\delta \leq 1$. The B_n form a decomposition of B.

Recall that we chose B so that $\mu(B) > 0$, so there exists N such that $\mu(B_N) > 0$. If $N \neq 0$ then consider any countable disjoint cover of B_N by balls of radius less than $\delta = \frac{1}{N+1}$ with centers in B_N, and denote the cover $\{C_n\}$. Let x_n be the center of each C_n. We know that there exists a sequence $\{\delta_n\}$ such that $C_n = (x_n - \delta_n, x_n + \delta_n)$ or some variation where the endpoints are included. Note that $\delta_n < \frac{1}{N+1}$. Since $x_n \in B_N$ and $\delta_n < \frac{1}{N+1}$, we know

$$\mu(x_n - \delta_n, x_n + \delta_n) \geq \delta_n^{a+2\epsilon}.$$

This inequality holds for every $\delta_n < \delta = \frac{1}{N+1}$. We now have

$$\begin{aligned}
\sum |C_n|^{a+3\epsilon} &\leq \sum (2\delta_n)^{a+3\epsilon} \\
&\leq \sum 2^{a+3\epsilon} \delta_n^\epsilon \mu(x_n - \delta_n, x_n + \delta_n) \\
&= 2^{a+3\epsilon} \sum \delta_n^\epsilon \mu(x_n - \delta, x_n + \delta) \\
&\leq 2^{a+3\epsilon} \delta^\epsilon.
\end{aligned}$$

This last inequality follows by noting, first, that $\delta_n < \delta$ and secondly that $(x_n - \delta_n, x_n + \delta_n)$ and $(x_m - \delta_m, x_m + \delta_m)$ are disjoint for $n \neq m$, and so $\sum \mu(x_n - \delta, x_n + \delta) \leq 1$.

Now we let $\delta \to 0$ and see $\sum |C_n|^{a+3\epsilon} \to 0$. This holds for any disjoint cover, $\{C_n\}$ of B_N, so $p^{a+3\epsilon}(B_N) = 0$. Hence, $P(B_N) \leq a + 3\epsilon$. Since $\mu(B_N) > 0$, it follows that $\dim_P^-(\mu) \leq a + 3\epsilon$.

If the set of positive measure is B_0, then we can repeat the above procedure and take our initial δ to be less than 1. This completes our proof.

□

Now we introduce the two definitions of exact dimension briefly discussed in Section 1.

Definition 3.3. A measure μ has exact dimension α if

$$\lim_{\epsilon \to 0} \frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} = \alpha$$

for μ-a.e. x.

The concept of exact dimension is important for many applications, but the concept central to our main result is Lebesgue exact dimension.

Definition 3.4. A measure μ has Lebesgue exact dimension α if

$$\lim_{\epsilon \to 0} \frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} = \alpha$$

for Lebesgue a.e. x.

There is a final characterization of packing dimension that will be very useful in Section 6.

Proposition 3.1. *If*

$$\dim_P^+ = \dim_P^- = \dim_P,$$

then

$$\dim_P(\mu) = \sup\left\{\gamma : \liminf_{\epsilon \to 0} \frac{\mu(x-\epsilon, x+\epsilon)}{\epsilon^\gamma} < \infty\right\}.$$

Proof. Suppose $\liminf_{\epsilon \to 0} \frac{\log \mu(x-\epsilon,x+\epsilon)}{\log \epsilon} = \alpha$ and fix $\delta > 0$. Then there exists $\epsilon^* > 0$ such that for every $\epsilon < \epsilon^*$ we have

$$\mu(x-\epsilon, x+\epsilon) \geq \epsilon^{\alpha+\epsilon}.$$

In particular, this gives us

$$\frac{\mu(x-\epsilon, x+\epsilon)}{\epsilon^{\alpha+2\delta}} \geq \epsilon^{-\delta} \to \infty \text{ as } \delta \to 0.$$

Thus $\alpha \geq \sup\left\{\gamma : \liminf_{\epsilon \to 0} \frac{\mu(x-\epsilon,x+\epsilon)}{\epsilon^\gamma} < \infty\right\}$.

Similarly, there exists a sequence of $\epsilon_n \to 0$ such that

$$\frac{\mu(x-\epsilon_n, x+\epsilon_n)}{\epsilon_n^{\alpha-2\delta}} \leq \epsilon_n^{\delta} \to 0 \text{ as } \delta \to 0.$$

Thus $\alpha \leq \sup\left\{\gamma : \liminf_{\epsilon\to 0} \frac{\mu(x-\epsilon,x+\epsilon)}{\epsilon^\gamma} < \infty\right\}$. This establishes the desired result. □

4. Packing Dimension Results

We would like to study the packing dimension of a measure μ subject to a certain limit condition. We know that if

$$\lim_{\epsilon\to 0} \frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} = \alpha \tag{4.1}$$

for μ-a.e. x, then μ has packing dimension equal to α. We will use this as our starting point to evaluate the packing dimension of the spectral measures of rank one perturbations of μ. First we consider what happens when (4.1) holds for every x in some set A.

Lemma 4.1. *Suppose $\delta > 0$ and $1 < \alpha < 2$. If*

$$\lim_{\epsilon\to 0} \frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} = \alpha$$

for $x \in A \subset \mathbb{R}$, then for sufficiently small ϵ dependent on x and δ, we have the bounds $\epsilon^{\alpha+\delta} \leq \mu(x-\epsilon, x+\epsilon) \leq \epsilon^{\alpha-\delta}$.

Proof. Since

$$\lim_{\epsilon\to 0} \frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} = \alpha$$

for $x \in A$, we see that for every $\delta > 0$, there is $\epsilon^* > 0$ such that for every $\epsilon < \epsilon^*$, we have

$$\alpha - \delta < \frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} < \alpha + \delta.$$

For ϵ small, $\log \epsilon < 0$, so after multiplying everything by $\log \epsilon$ and exponentiating, we have

$$\epsilon^{\alpha+\delta} < \mu(x-\epsilon, x+\epsilon) < \epsilon^{\alpha-\delta}.$$

□

In this section, from this point onwards, unless otherwise stated, we assume

$$\lim_{\epsilon\to 0} \frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} = \alpha$$

for all x in some set A.

Knowing a pair of bounds for μ, we can study bounds on objects that are related to μ, like the Borel transform F. We do this so that we may translate a bound on μ to a bound on F and then to a bound for F_λ which will in turn give us a bound for μ_λ.

Lemma 4.2. *Suppose $\delta > 0$ and $1 < \alpha < 2$. Then for sufficiently small ϵ dependent on x and δ, we have the bound*

$$ImF(x+i\epsilon) \geq \frac{1}{2}\epsilon^{\alpha+\delta-1}$$

for $x \in A \subset \mathbb{R}$.

Proof. First observe that the definition of $F(x+i\epsilon)$ gives us

$$\begin{aligned} \mathrm{Im}F(x+i\epsilon) &= \mathrm{Im}\int \frac{d\mu(y)}{y-x+i\epsilon} && (4.2)\\ &= \epsilon \int \frac{d\mu(y)}{(y-x)^2+\epsilon^2}. && (4.3) \end{aligned}$$

Using the montonicity of the integral, and the fact that the integrand is nonnegative, we have

$$\geq \epsilon \int_{x-\epsilon}^{x+\epsilon} \frac{d\mu(y)}{(y-x)^2+\epsilon^2}.$$

We can now use our bound on $\mu(x-\epsilon, x+\epsilon)$ to get a final bound.

$$\geq \frac{1}{2\epsilon}\epsilon^{\alpha+\delta} = \frac{1}{2}\epsilon^{\alpha+\delta-1}.$$

Observe that the second inequality only holds for $x \in A$ and follows from the established lower bound on $\mu(x-\epsilon, x+\epsilon)$, which is why our bound on $\mathrm{Im}F$ only holds on A. □

It turns out that, for our applications, all we need is a lower bound on $\mathrm{Im}F(x+i\epsilon)$. Now that we have such a bound for every x in a set A, we want to study the conditions on A that allow us to talk about μ_λ. We could consider the case when A supports μ_λ for some λ, but that will only allow us to talk about one particular λ, so instead we consider a much stronger case that is easier to apply.

5. Proof of Theorem 1.1

It is an interesting fact that there is a deep relationship between μ_λ and Lebesgue measure. Indeed, the next Theorem, given as Theorem I.8 in [7], will allow us to take our results from the previous section and use them to find bounds on $\mathrm{Im}F_\lambda(x+i\epsilon)$ for μ_λ a.e. x and Lebesgue a.e. λ.

Theorem 5.1. *Let $\int_{-\infty}^{\infty}\mu_\lambda(E)d\lambda = \eta(E)$. Then η is equivalent to Lebesgue measure.*

A consequence of this equality is that any set of full Lebesgue measure is also of full μ_λ measure for Lebesgue a.e. λ.

Observe: if

$$0 = \eta(A) = \int \mu_\lambda(A)d\lambda$$

then

$$\mu_\lambda(A) = 0$$

for η-a.e.λ. By Theorem 5.1, η is equivalent to Lebesgue measure, so $\mu_\lambda(A) = 0$ for Lebesgue a.e. λ. Hence, A^c is full Lebesgue measure and full μ_λ measure for Lebesgue a.e. λ.

We now proceed to bound $\mathrm{Im}F_\lambda$ using the assumption that A has full Lebesgue measure. This is not a completely arbitrary assumption, as can be seen from the lemma above, but it is not clear that many measures, if any, possess this limit property for Lebesgue a.e. x. In Section 7, we discuss a concrete example of a measure with this limit property.

Lemma 5.1. *Suppose $\delta > 0$ and $1 < \alpha < 2$ and μ has Lebesgue exact dimension α. Then for Lebesgue a.e. λ, μ_λ-a.e. x and sufficiently small ϵ dependent on x and δ, there exists $C(\lambda) > 0$ such that*

$$Im\, F_\lambda(x+i\epsilon) \leq C\epsilon^{1-\alpha-\delta}.$$

Proof. Recall that μ having Lebesgue exact dimension α is equivalent to

$$\lim_{\epsilon\to 0}\frac{\log \mu(x-\epsilon, x+\epsilon)}{\log \epsilon} = \alpha$$

for every x in a set A of full Lebesgue measure. Let $x \in A$. By Theorem 5.1, we know that A is a set of full μ_λ measure for Lebesgue a.e. x. Recall, from (1.4), that

$$\mathrm{Im}F_\lambda(x+i\epsilon) = \frac{\mathrm{Im}F(x+i\epsilon)}{|1+\lambda F(x+i\epsilon)|^2}$$

By bounding the denominator by $|\lambda \text{Im} F(x+i\epsilon)|^2$, we have the bound

$$\text{Im} F_\lambda(x+i\epsilon) \le \frac{1}{|\lambda|^2 \text{Im} F(x+i\epsilon)}.$$

From Lemma 4.2,

$$\text{Im} F_\lambda(x+i\epsilon) \le \frac{2}{|\lambda|^2 \epsilon^{\alpha+\delta-1}} = \frac{1}{|\lambda|^2}\epsilon^{1-\alpha-\delta}.$$

Setting $C = \frac{1}{|\lambda|^2}$ yields the desired inequality.

□

Lemma 5.2. *Suppose $\delta > 0$, $1 < \alpha < 2$, and μ has Lebesgue exact dimension α. Then for Lebesgue a.e. λ, μ_λ-a.e. x and sufficiently small ϵ dependent on x and δ, there exists $C > 0$ dependent on x, α, δ, and λ such that*

$$Im\, F_\lambda(x+i\epsilon) \ge C\epsilon^{1-\alpha+\delta}.$$

Proof. Let A be the set of full Lebesgue measure where the Lebesgue exact dimension condition holds and let $x \in A$. Theorem 4.2 from [2] shows

$$|1+\lambda F(x+i\epsilon)| \le C\epsilon^{\alpha-1}$$

for $C > 0$ dependent on x, α, and δ, so (1.4) gives us

$$\text{Im} F_\lambda(x+i\epsilon) \ge \frac{\text{Im} F(x+i\epsilon)}{C_1\epsilon^{2(\alpha-1)}}.$$

Then Lemma 4.2 yields

$$\begin{aligned} &\ge C\frac{\epsilon^{\alpha+\delta-1}}{\epsilon^{2(\alpha-1)}} \\ &= C\epsilon^{1-\alpha+\delta}. \end{aligned}$$

□

We can now use the definition of F_λ to place bounds on the measure μ_λ.

Theorem 5.2. *Suppose $\delta > 0$, $1 < \alpha < 2$, and μ has Lebesgue exact dimension α. Then for Lebesgue a.e. λ, μ_λ-a.e. x and sufficiently small ϵ dependent on x and δ, there exists $C > 0$ dependent only on λ such that*

$$\mu_\lambda(x-\epsilon, x+\epsilon) \le C\epsilon^{2-\alpha-\delta}.$$

Proof. Let A be the set of full Lebesgue measure where the Lebesgue exact dimension condition holds and let $x \in A$. By definition of F_λ, we have

$$\mathrm{Im} F_\lambda(x+i\epsilon) = \int \frac{\epsilon d\mu_\lambda}{(y-x)^2+\epsilon^2}.$$

Since the integrand is nonnegative, we can use monotonicity to see

$$\geq \epsilon \int_{x-\epsilon}^{x+\epsilon} \frac{d\mu_\lambda}{2\epsilon^2}$$

and evaluating this integral yields

$$= \frac{1}{2\epsilon}\mu_\lambda(x-\epsilon, x+\epsilon).$$

Since $\mathrm{Im} F_\lambda(x+i\epsilon) \leq C\epsilon^{1-\alpha-\delta}$, we have $\mu_\lambda(x-\epsilon, x+\epsilon) \leq 2C\epsilon^{2-\alpha-\delta}$. □

Theorem 5.3. *Suppose $\delta > 0$, $1 < \alpha < 2$, and μ has Lebesgue exact dimension α. Then for Lebesgue a.e. λ, μ_λ-a.e. x and sufficiently small ϵ dependent on x and δ, there exists $C > 0$ dependent only on λ such that $\mu_\lambda(x-\epsilon, x+\epsilon) \geq C\epsilon^{2-\alpha+\delta}$ for Lebesgue a.e. λ and μ_λ-a.e. x.*

Proof. Let A be the set of full Lebesgue measure where the Lebesgue exact dimension condition holds and let $x \in A$. From (4.3), we have $\mathrm{Im} F_\lambda(x+i\epsilon) = \epsilon \int \frac{d\mu_\lambda(y)}{(y-x)^2+\epsilon^2}$, so

$$\mathrm{Im} F_\lambda(x+i\epsilon) \leq \int_{\mathbb{R}} \frac{d\mu_\lambda(y)}{((y-x)^2+\epsilon^2)^{1/2}}.$$

Fix $C_1 > 0$. By partitioning $\mathbb{R}$ into

$$(x - C_1\epsilon, x + C_1\epsilon) \text{ and } \bigcup_{n=0}^{\infty} \{2^n C_1 \epsilon \leq |x-y| < 2^{n+1} C_1 \epsilon\},$$

and setting $E_n = \{2^n C_1 \epsilon \leq |x-y| < 2^{n+1} C_1 \epsilon\}$, we have

$$\mathrm{Im} F_\lambda(x+i\epsilon) \leq \int_{x-C_1\epsilon}^{x+C_1\epsilon} \frac{d\mu_\lambda(y)}{((y-x)^2+\epsilon^2)^{1/2}} + \sum_{n=0}^{\infty} \int_{E_n} \frac{d\mu_\lambda(y)}{((y-x)^2+\epsilon^2)^{1/2}}.$$

The first integral is bounded above by $\frac{1}{\sqrt{2}\epsilon}\mu_\lambda(x - C_1\epsilon, x + C_1\epsilon)$ and the second integrand is bounded above by $\frac{d\mu_\lambda}{(2^{2n}C_1^2\epsilon^2+\epsilon^2)^{1/2}}$ so we have

$$\begin{aligned}\text{Im}F_\lambda(x + i\epsilon) \leq &\frac{1}{\sqrt{2}\epsilon}\mu_\lambda(x - C_1\epsilon, x + C_1\epsilon)\\ &+\sum_{n=0}^{\infty}\int_{E_n}\frac{d\mu_\lambda}{(2^{2n}C_1^2\epsilon^2 + \epsilon^2)^{1/2}}.\end{aligned}$$

Observe that

$$\int_{E_n}\frac{d\mu_\lambda}{(2^{2n}C_1^2\epsilon^2 + \epsilon^2)^{1/2}} \leq \frac{1}{C_1\epsilon}2^{-n}\mu_\lambda(x - 2^{n+1}C_1\epsilon, x + 2^{n+1}C_1\epsilon),$$

hence,

$$\sum_{n=0}^{\infty}\int_{E_n}\frac{d\mu_\lambda}{(2^{2n}C_1^2\epsilon^2 + \epsilon^2)^{1/2}} \leq \frac{1}{C_1\epsilon}\sum_{n=0}^{\infty}2^{-n}\mu_\lambda(x - 2^{n+1}C_1\epsilon, x + 2^{n+1}C_1\epsilon).$$

Theorem 5.2 gives us $\mu_\lambda(x-2^{n+1}C_1\epsilon, x+2^{n+1}C_1\epsilon) \leq C_2(2^{n+1}C_1\epsilon)^{2-\alpha-\delta}$. Thus,

$$\sum_{n=0}^{\infty}\int_{E_n}\frac{d\mu_\lambda}{(2^{2n}C_1^2\epsilon^2 + \epsilon^2)^{1/2}} \leq \frac{1}{C_1\epsilon}\sum_{n=0}^{\infty}2^{-n}C_2(2^{n+1}C_1\epsilon)^{2-\alpha-\delta}.$$

Since $2^{-n}C_2(2^{n+1})^{2-\alpha-\delta}C_1^{1-\alpha-\delta}$ is summable, by our assumptions on α and δ, we have

$$\sum_{n=0}^{\infty}\int_{E_n}\frac{d\mu_\lambda}{(2^{2n}C_1^2\epsilon^2 + \epsilon^2)^{1/2}} = C_3\epsilon^{1-\alpha-\delta}.$$

Therefore, we have

$$\text{Im}F_\lambda(x + i\epsilon) \leq \frac{1}{\epsilon\sqrt{2}}\mu_\lambda(x - C_1\epsilon, x + C_1\epsilon) + C_3\epsilon^{1-\alpha-\delta}.$$

This C_3 can be made as small as we wish by taking C_1 sufficiently large. Since $\text{Im}F_\lambda(x + i\epsilon) \geq C_4\epsilon^{1-\alpha+\delta}$, we have

$$C_4\epsilon^{2-\alpha+\delta} - C_3\epsilon^{2-\alpha-\delta} = \epsilon^{2-\alpha+\delta}(C_4 - C_3\epsilon^{-2\delta}) \leq \mu_\lambda(x - C_1\epsilon, x + C_1\epsilon).$$

We can make $C_4 - C_3\epsilon^{-2\delta} > 0$ for sufficiently large fixed C_1, hence we have $\mu_\lambda(x - C_1\epsilon, x + C_1\epsilon) \geq C_5\epsilon^{2-\alpha+\delta}$ with $C_5 > 0$. This holds for all sufficiently small ϵ, so in particular it will hold for all ϵ/C_1 for sufficiently small ϵ. Setting $C = C_5/C_1^{2-\alpha-\delta}$ will give us the desired inequality. □

With these bounds on μ_λ, we may now turn our full attention to Theorem 1.1.

Proof of Theorem 1.1

Proof. First, we prove part one of the theorem. Consider

$$\lim_{\epsilon\to 0} \frac{\log \mu_\lambda(x-\epsilon, x+\epsilon)}{\log \epsilon}.$$

We know this is bounded by

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{(2-\alpha+\delta)\log\epsilon + \log C_1}{\log\epsilon} &\geq \lim_{\epsilon\to 0} \frac{\log \mu_\lambda(x-\epsilon, x+\epsilon)}{\log\epsilon} \\ &\geq \lim_{\epsilon\to 0} \frac{(2-\alpha-\delta)\log\epsilon + \log C_2}{\log\epsilon}. \end{aligned} \tag{5.1}$$

The term on the left is equal to $2-\alpha+\delta$ and the term on the right is equal to $2-\alpha-\delta$ so by sending $\delta \to 0$ we have the desired equality.

The second part follows immediately from (5.1) and Theorem 3.2. □

This theorem allows us to connect our packing dimension results to the Hausdorff dimension results from [2].

Corollary (to Theorem 1.1)**.** *Under the conditions of Theorem 1.1, the Hausdorff and packing dimensions for the spectral measure of almost every rank one perturbation are equal to* $2-\alpha$*.*

Proof. The first part of Theorem 1.1 shows

$$\lim_{\epsilon\to 0} \frac{\log \mu_\lambda(x-\epsilon, x+\epsilon)}{\log\epsilon} = 2-\alpha$$

on a μ_λ-typical set for a.e. λ. In particular, the $\liminf$ is equal to $2-\alpha$. Using Definition 3.1, we conclude $\dim_H(\mu_\lambda) = 2-\alpha$ for a.e. λ. This is also the packing dimension established in the second part of Theorem 1.1, so we are done. □

6. Applications to Spectral Dimension

The concept of spectral dimension was introduced in [6], and we can use our results to illustrate a relation between the spectral dimension and the packing dimension of measures under the limit condition from Theorem 1.1.

Fix $\gamma \in (0,1)$. A measure μ is said to be *upper γ-spectral continuous* if

$$\liminf_{\epsilon \to 0} \epsilon^{1-\gamma} |F(x+i\epsilon)| < \infty$$

for μ-a.e. x. Replacing $\liminf$ with $\limsup$ gives us the definition for *lower spectral continuous.*

Definition 6.1. We say a measure μ has lower spectral dimension α if

$$\alpha = \sup\{\gamma \in (0,1) : \mu \text{ is lower } \gamma\text{-spectral continuous}\}.$$

Similarly, we can define the upper spectral dimension

Definition 6.2. We say a measure μ has upper spectral dimension α if

$$\alpha = \sup\{\gamma \in (0,1) : \mu \text{ is upper } \gamma\text{-spectral continuous}\}.$$

We will denote upper spectral dimension $\dim_s^+(\mu)$ and lower spectral dimension $\dim_s^-(\mu)$.

In [2], it was shown that lower spectral dimension corresponds to Hausdorff dimension. It was commented in [6] that the upper spectral dimension provides a lower bound for packing dimension.

Proposition 6.1. $\liminf_{\epsilon \to 0} \frac{\mu(x-\epsilon,x+\epsilon)}{\epsilon^\gamma} \leq 2 \liminf_{\epsilon \to 0} \epsilon^{1-\gamma}|F(x+i\epsilon)|$

Proof. Recall that

$$\text{Im}F(x+i\epsilon) = \int \frac{\epsilon d\mu}{(y-x)^2+\epsilon^2}.$$

Furthermore, $|F(x+i\epsilon)| \geq \text{Im}F(x+i\epsilon)$. We can also see

$$\begin{aligned} |\text{Im}F(x+i\epsilon)| &= \epsilon \int_{x-\epsilon}^{x+\epsilon} \frac{d\mu}{(x-y)^2+\epsilon^2} \\ &\geq \frac{1}{2\epsilon}\mu(x-\epsilon, x+\epsilon). \end{aligned}$$

Thus, $\epsilon^{1-\gamma}|F(x+i\epsilon)| \geq \frac{1}{2}\epsilon^{-\gamma}\mu(x-\epsilon,x+\epsilon)$. Taking a $\liminf$ of both sides proves our result. □

Proposition 6.2. $\dim_P(\mu) \geq \dim_s^+(\mu)$.

Proof. We know

$$\liminf_{\epsilon\to 0} \frac{\mu(x-\epsilon, x+\epsilon)}{\epsilon^\gamma} \leq \liminf_{\epsilon\to 0} \epsilon^{1-\gamma}|F(x+i\epsilon)|,$$

so in particular, we can see that

$$\sup\left\{\gamma : \liminf_{\epsilon\to 0} \frac{\mu(x-\epsilon, x+\epsilon)}{\epsilon^\gamma} < \infty\right\} \geq \sup\left\{\gamma : \liminf_{\epsilon\to 0} \epsilon^{1-\gamma}|F(x+i\epsilon)| < \infty\right\}.$$

Appealing to Proposition 3.1 establishes our result. □

We can now turn our attention to Theorem 1.2.

Proof of Theorem 1.2

Proof. From Theorem 1.1, we know that Hausdorff and packing dimension are equal for spectral measures of a.e. rank one perturbation. Furthermore, we know $\dim_H(\mu_\lambda) = \dim_s^-(\mu_\lambda)$ and $\dim_s^+(\mu_\lambda) \leq \dim_P(\mu_\lambda)$ for every λ. Hence, we have

$$\dim_H(\mu_\lambda) = \dim_s^-(\mu_\lambda) \leq \dim_s^+(\mu_\lambda) \leq \dim_P(\mu_\lambda) = \dim_H(\mu_\lambda)$$

for Lebesgue a.e. λ. That is, we have equality everywhere for a.e. λ. □

7. A Family of Probability Measures

We would now like to use Theorem 1.1 to study a particular family of measures. In particular, we wish to show that the assumption that

$$\lim_{\delta\to 0} \frac{\log \mu(x-\delta, x+\delta)}{\log \delta}$$

exists on a Lebesgue typical set is reasonable.

Consider the binary expansion of $x \in [0,1]$ such that for $x \neq 1$, the representation eventually ends in 0s. We make this choice of representation to resolve the non-uniqueness of the binary expansions of certain $x \in [0,1]$. Let $f : [0,1] \to \{0,1\}^{\mathbb{N}}$ be the function associated with this representation. That is, if

$$x = \sum_{n=1}^{\infty} \frac{a_n(x)}{2^n}$$

is the unique binary representation for x that we consider, then $f(x) = \{a_n(x)\}_{n=1}^{\infty}$. We can see that this function is a bijection from $[0,1]$ to some proper subset $K \subset \{0,1\}^{\mathbb{N}}$.

Fix $p \in [0,1]$ and let P be the product measure on K with each factor giving weight p to 0 and weight $1-p$ to 1. More precisely, we denote P by

$$\prod_{n=1}^{\infty} \left(pd\delta_0 + (1-p)d\delta_1\right).$$

From P we can obtain a probability measure, μ_p, on $[0,1]$ using the relation

$$\mu_p(E) = P(f(E)).$$

The measure μ_p is studied in detail with respect to Hausdorff dimension in [2].

Now to build our family of measures we consider the measures μ_{p_λ} such that $\mu_{p_0} = \mu_p$ and μ_{p_λ} is the spectral measure of the rank one perturbation of μ_p.

From now on, unless otherwise specified, we denote μ_p as μ and μ_{p_λ} as μ_λ.

It was shown in [2] that

$$\lim_{\delta\to 0} \frac{\log \mu(x-\delta, x+\delta)}{\log 2\delta} = -\frac{q\log p + (1-q)\log(1-p)}{\log 2}$$

for a.e. x w.r.t. μ_q. It is well known that $\mu_{1/2}$ is equivalent to Lebesgue measure, so in particular, if we take $q = \frac{1}{2}$, then we have

$$\lim_{\delta\to 0} \frac{\log \mu(x-\delta, x+\delta)}{\log 2\delta} = -\frac{\log p(1-p)}{2\log 2} = \alpha.$$

Now Theorem 1.1 implies that $\dim_P(\mu_\lambda) = 2-\alpha$ for Lebesgue a.e. λ.

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