# SCH'NOL'S THEOREM AND THE SPECTRUM OF LONG RANGE OPERATORS

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ABSTRACT. We extend some basic results known for finite range operators to long range operators with off-diagonal decay. Namely, we prove an analogy of Sch'nol's theorem. We also establish the connection between the almost sure spectrum of long range random operators and the spectra of deterministic periodic operators.

## 1. INTRODUCTION

Long range operators on $l^2(\mathbb{Z}^d)$ arise naturally from the discrete Laplacian on half-space $\mathbb{Z}_+^{d+1}$ after dimension reduction (see e.g. [9, 6, 7]). However, compared to Schrödinger operators or finite range operators, little has been studied. Our goal is to extend some basic results known for finite range operators to long range setting with off-diagonal decay. The first part of this note concerns a generalization of Sch'nol's theorem to the case of a long range normal operator. The second part provides a connection between the almost sure spectrum of long range self-adjoint random operators and the spectra of deterministic periodic operators. We hope these could serve as ready-to-use tools in the future for people who study discrete operators.

The classical Sch'nol's theorem for Schrödinger operators is well-known, see [15, 2, 16, 4] for the continuum case, from which the discrete version could be derived. It asserts that any spectral measure gives full weight to the set of energies with generalized eigenfunctions, moreover, the spectrum is the closure of this set. It has many applications, for example, relating the spectrum to non-uniform hyperbolicity of the corresponding cocycle, as well as providing a priori estimate which turns out to be crucial in the proofs of the almost-localization and localization. So far there have been various generalizations of Sch'nol's theorem (see e.g. [13, 14, 3, 5]). In this note, we generalize this result to the long range case.

Let us consider long range normal operators with polynomially decaying off-diagonal terms.

$$(Hu)(n) = \sum_{j\in\mathbb{Z}^d\setminus\{0\}} a_j^n u(n-j) + a_0^n u(n), \tag{1.1}$$

with

$$|a_j^m| \leqslant C(1+\|j\|)^{-r} \text{ for any } j, m \in \mathbb{Z}^d, \tag{1.2}$$

where $\|j\|$ is the Euclidean norm of $j$ and $C > 0$ is a constant. We assume

$$\sum_{j\in\mathbb{Z}^d} a_j^n \overline{a_{j-m}^{n-m}} = \sum_{j\in\mathbb{Z}^d} \overline{a_{-j}^{n-j}} a_{m-j}^{n-j} \ \text{ for any } m, n \in \mathbb{Z}^d, \tag{1.3}$$

to ensure $H$ is a normal operator.

We introduce generalized eigenfunctions.

**Definition 1.1.** (*$\epsilon$-generalized eigenvalue/eigenfunction*) An energy $z$ of $H$ is called an $\epsilon$-generalized eigenvalue if there is a formal solution to the equation $H\phi_z = z\phi_z$, with $\phi_z(0) = 1$ and $|\phi_z(n)| \leqslant C'(1+\|n\|)^{\frac{d}{2}+\epsilon}$ for some constant $C'$.

We denote the set of $\epsilon$-generalized eigenvalues by $G_\epsilon$. The spectral measures of $H$ are defined by $\mu_{n,m}(B) = (e_n, \chi_B(H)e_m)$, for any Borel sets $B \subset \mathbb{C}$ and $m, n \in \mathbb{Z}^d$. We denote $\mu_{n,n} := \mu_n$. Let $\mu = \sum_{n\in\mathbb{Z}^d} \lambda_n \mu_n$, where $\lambda_n = (1 + \|n\|)^{-d-2\epsilon}/(\sum_{n\in\mathbb{Z}^d}(1 + \|n\|)^{-d-2\epsilon})$. Clearly, any spectral measure is absolutely continuous with respect to $\mu$.

Our main theorem is:

**Theorem 1.1.** *Let $H$ be a normal operator defined as in (1.1) with $r > 2d$. Then for any $0 < \epsilon < r - 2d$, we have the following:*
*(a). $G_\epsilon \subseteq \sigma(H)$,*
*(b). $\mu(\sigma(H)\backslash G_\epsilon) = 0$,*
*(c). $\overline{G_\epsilon} = \sigma(H)$.*

The proof of part (a) relies on Lemma 2.1. We will discuss both in Section 2. The proofs of (b), (c) are standard. We will include them in the appendix for completeness.

Switching our attention to the long range self-adjoint case, we are able to cover operators with unbounded potentials $a_0^n$. Let $H$ be a self-adjoint operator defined as follows.

$$(Hu)(n) = \sum_{j\in\mathbb{Z}^d\backslash\{0\}} a_j^n u(n-j) + a_0^n u(n), \tag{1.4}$$

with

$$|a_j^m| \leqslant C(1 + \|j\|)^{-r} \text{ for any } j \neq 0, m \in \mathbb{Z}^d, \quad \text{and} \quad a_j^n = \overline{a_{-j}^{n-j}}. \tag{1.5}$$

Note that the polynomial decay condition (1.5) does not involve $j = 0$, allowing us to take unbounded potentials into account. Following exactly the same line of the proof of Theorem 1.1, we have Sch'nol's theorem for long range self-adjoint operators (with unbounded potentials).

**Theorem 1.2.** *Let $H$ be a self-adjoint operator defined as in (1.4) with $r > 2d$. Then for any $0 < \epsilon < r - 2d$, the conclusions of Theorem 1.1 hold.*

The localization property of long range self-adjoint operators with polynomially decaying off-diagonal terms has been studied in the random potential case, when the potentials $a_0^n = V_\omega(n)$ are i.i.d. random variables. Some known results are pure point spectrum in the large disorder and high energy regime when $r > d$ [1], purely singular spectrum for typical $\omega$ when $d = 1$ and $r > 4$ [17], pure point spectrum for typical $\omega$ when $d = 1$ and $r > 8$ (conjectured to be $r > 2$) under some conditions on the density of distribution [8]. We hope Theorem 1.2 could serve as a step towards the proof of the conjecture, as well as establishing localization in more general setting.

The second part of this note is devoted to extending some basic results which were previously known for random Schrödinger operators to long range self-adjoint cases.

Let $\Gamma$ be a subset of $\mathbb{Z}^d$ such that $\Gamma$ and $-\Gamma$ form a partition of $\mathbb{Z}^d\backslash\{0\}$ in the sense that $\Gamma \bigcap(-\Gamma) = \varnothing$ and $\Gamma \bigcup(-\Gamma) = \mathbb{Z}^d\backslash\{0\}$. Denote $\Gamma_0 = \{0\} \bigcup \Gamma$. Let $\{\gamma^{(i)}\}_{i\in\Gamma}$ be a sequence of compactly supported probability measures on $\mathbb{C}$ and $\gamma^{(0)}$ be a probability measure on $\mathbb{R}$. Similar to Theorem 1.2 we do not make compact support assumption on $\gamma^{(0)}$. This enables us to cover unbounded operators. Let $\mathrm{d}\kappa = \bigtimes_{i\in\Gamma_0}(\mathrm{d}\gamma^{(i)})^{\mathbb{Z}^d}$ and $\Omega = \bigtimes_{i\in\Gamma_0}(\operatorname{supp}\gamma^{(i)})^{\mathbb{Z}^d} = \{(\omega^{(i)})_{i\in\Gamma_0} \mid \omega^{(i)} = (\omega_j^{(i)})_{j\in\mathbb{Z}^d},\ \omega_j^{(i)} \in \operatorname{supp}(\gamma^{(i)})\}$. For any $n \in \mathbb{Z}^d$, we define the translations $\tilde{T}^n$ on $\Omega$ as $\tilde{T}^n(\omega^{(i)}) = (T^n\omega^{(i)})$, where $(T^n\omega^{(i)})_j = \omega_{j+n}^{(i)}$.

Consider long range self-adjoint operator $H_{(\omega^{(i)})}$ on $l^2(\mathbb{Z}^d)$:

$$(H_{(\omega^{(i)})}u)(n) = \sum_{j\in\mathbb{Z}^d} a_{n-j}(\tilde{T}^n(\omega^{(i)}))u(j), \tag{1.6}$$

where $a_k((\omega^{(i)})) = \omega_0^{(k)}$ for $k \in \Gamma_0$, and $a_k((\omega^{(i)})) = \overline{\omega_{-k}^{(-k)}}$ for $k \in (-\Gamma_0)$. We will further assume $\gamma^{(k)}$ have shrinking supports, namely,

$$\operatorname{supp} \gamma^{(k)} \subseteq B(C(1 + \|k\|)^{-r}) \quad \text{for} \quad k \in \Gamma, \tag{1.7}$$

for some constant $C > 0$ and $r > \frac{d}{2}$, with $B(M)$ being the ball in $\mathbb{C}$ centered at 0 with radius $M$.

Following a standard argument, one can show that there exists a non-random $\Sigma$, such that $\sigma(H_{(\omega^{(i)})}) = \Sigma$ for $\kappa-$almost every $(\omega^{(i)}) \in \Omega$ [12].

We say a set $\Omega_1$ is a dense subset of $\Omega$, if for any $(\omega^{(i)}) \in \Omega$, $0 < \xi \in \mathbb{R}$ and any finite sets $\bigwedge_1 \subset \Gamma_0$ and $\bigwedge_2 \subset \mathbb{Z}^d$, there exists $(\tilde{\omega}^{(i)}) \in \Omega_1$ such that $|\tilde{\omega}_j^{(i)} - \omega_j^{(i)}| < \xi$ for any $i \in \bigwedge_1$ and $j \in \bigwedge_2$. It is clear that any subset $\Omega_0 \subseteq \Omega$ with $\kappa(\Omega_0) = 1$ is dense in $\Omega$.

The following result is similar to Theorem 3 [11].

**Theorem 1.3.** *Let $\Omega_1$ be a dense subset of $\Omega$, then*

$$\Sigma = \overline{\bigcup_{(\omega^{(i)}) \in \Omega_1} \sigma(H_{(\omega^{(i)})})}$$

This theorem has a direct corollary which implies the almost sure spectrum is determined by the spectra of periodic operators. We say $(\omega^{(i)})$ is $p-$periodic if $\omega_j^{(i)} = \omega_{j+p}^{(i)}$ for any $i, j \in \mathbb{Z}^d$.

**Corollary 1.4.**

$$\Sigma = \overline{\bigcup_{p \in \mathbb{Z}_+} \bigcup_{p-periodic\ (\omega^{(i)})} \sigma(H_{(\omega^{(i)})})}$$

This can be viewed as an extension of Theorem 3.9 [10] for Schrödinger operators. We also point out that in [5], the authors established a similar result for the extended CMV matrices.

If we focus on one-dimensional random Jacobi matrices, it turns out we could get a better result than Corollary 1.4. Let $\gamma$ be a compactly supported measure on $\mathbb{R}$. Let $d\tilde{\kappa} = (\mathrm{d}\gamma)^{\mathbb{Z}}$ and $\tilde{\Omega} = (\operatorname{supp} \gamma)^{\mathbb{Z}}$. For any $\omega \in \tilde{\Omega}$, we consider

$$(H_\omega u)(n) = a(T^n \omega) u(n+1) + a(T^{n-1}\omega) u(n-1) + b(T^n \omega) u(n) \tag{1.8}$$

where $a(\omega) = \omega_1$, $b(\omega) = \omega_0$ and $T\omega_n = \omega_{n+2}$, which ensures that all $a(T^i\omega)$ and $b(T^j\omega)$ are independent. We have

**Theorem 1.5.** *Let $H_\omega$ be a random Jacobi matrice defined as in (1.8). We have*

$$\Sigma = \bigcup_{2-periodic\ \{\alpha_j\}} \sigma(H_{\{\alpha_j\}}).$$

*Remark* 1.1. Unlike the random Schrödinger case, for Jacobi matrices, $\Sigma = \bigcup_{\text{constant}\{\alpha\}} \sigma(H_{\text{constant}\{\alpha\}})$ is in general not true. For example, take $\text{supp}\gamma = \{0, 1\}$, then $\bigcup_{\text{constant}\{\alpha\}} \sigma(H_{\text{constant}\{\alpha\}}) = [-1, 3]$, however $\Sigma = [-2, 3]$.

We organize the note as follows: a key lemma and proof of Theorem 1.1 will be presented in Section 2, proofs of Theorems 1.3 and 1.5 will be given in Section 3, the proof of our key lemma will be included in Section 4.

## 2. Key lemma and the proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemma.

**Lemma 2.1.** *Let $H$ be defined as in (1.1) with $r > (2d+\epsilon)\frac{q}{q-1}$ for some $\epsilon > 0$ and $q \in \mathbb{Z}^+$. Let $z \in G_\epsilon$ and $\phi$ be a corresponding generalized eigenfunction. Let*

$$\phi_N(j) = \begin{cases} \phi(j) & -N \leqslant \|j\| \leqslant N, \\ 0 & otherwise. \end{cases} \tag{2.1}$$

*and $\Phi_N = (H-z)\phi_N$. We have*

$$\liminf_{L\to\infty} \frac{\|\Phi_{L^q}\|_{l^2}}{\|\phi_{L^q}\|_{l^2}} = 0. \tag{2.2}$$

**Proof of Theorem 1.1.** We only prove part (a) here, the proofs of part (b) and (c) are standard, we include them in the appendix for reader's convenience.

Since $r > 2d+\epsilon$, we can choose $q \in \mathbb{Z}^+$ large enough such that $r > (2d+\epsilon)\frac{q}{q-1}$. Then Lemma 2.1 directly implies $\text{dist}\,(z, \sigma(H)) = 0$. □

## 3. Proofs of Theorems 1.3 and 1.5

3.1. **Proof of Theorem 1.3.** Let $\Omega_0$ be a shift invariant set with $\kappa(\Omega_0) = 1$ such that $\sigma(H_{(\omega^{(i)})}) = \Sigma$ for every $(\omega^{(i)}) \in \Omega_0$. Then given any $(\omega^{(i)}) \in \Omega$, $\xi > 0$, and any finite sets $\bigwedge_1 \subset \Gamma_0$, $\bigwedge_2 \subset \mathbb{Z}^d$, there exists $(\tilde{\omega}^{(i)}) \in \Omega_0$ such that $|\omega_n^{(j)} - \tilde{\omega}_n^{(j)}| < \xi$ for any $j \in \bigwedge_1$ and $n \in \bigwedge_2$.

**The " $\supseteq$ " direction.** Take $(\omega^{(i)}) \in \Omega_1$ and $E \in \sigma(H_{(\omega^{(i)})})$, by Weyl's criterion, for any $L > 0$ there exists $\phi^{(L)} \in l^2(\mathbb{Z}^d)$ such that $\|(H_{(\omega^{(i)})} - E)\phi^{(L)}\| < \frac{1}{L}\|\phi^{(L)}\|$.

First, we show such $\phi^{(L)}$ can be taken with compact support. This is standard if $H_{(\omega^{(i)})}$ is a bounded operator, but since $H_{(\omega^{(i)})}$ could be unbounded here, we will work out the detail.

Let us consider a cut-off function $\phi_k^{(L)}(n) = \chi_{\|j\|\leqslant k}(n)\phi^{(L)}(n)$. We split $H_{(\omega^{(i)})}$ into two parts $H^0_{(\omega^{(i)})} + H^1_{(\omega^{(i)})}$, where $H^0_{(\omega^{(i)})}$ contains only the diagonal multiplication, namely $(H^0_{(\omega^{(i)})}u)(n) = a_0(\tilde{T}^n(\omega^{(i)}))u(n)$, and $H^1_{(\omega^{(i)})}$ is the off-diagonal part. Clearly, $H^1_{(\omega^{(i)})}$ is a bounded operator while $H^0_{(\omega^{(i)})}$ could be unbounded. Now let's consider

$$(H_{(\omega^{(i)})} - E)(\phi^{(L)} - \phi_k^{(L)}) = (H^1_{(\omega^{(i)})} - E)(\phi^{(L)} - \phi_k^{(L)}) + H^0_{(\omega^{(i)})}(\phi^{(L)} - \phi_k^{(L)}).$$

We know $\|(H^1_{(\omega^{(i)})} - E)(\phi^{(L)} - \phi_k^{(L)})\|_{l^2} \leqslant \|(H^1_{(\omega^{(i)})} - E)\|\;\|(\phi^{(L)} - \phi_k^{(L)})\|_{l^2} \to 0$ as $k \to \infty$. Also since $\|H^0_{(\omega^{(i)})}\phi^{(L)}\| \leqslant \|(H^1_{(\omega^{(i)})} - E)\phi^{(L)}\| + \frac{1}{L}\|\phi^{(L)}\| < \infty$, we have $\|H^0_{(\omega^{(i)})}(\phi^{(L)} - \phi_k^{(L)})\|_{l^2} \to 0$, as $k \to \infty$. Thus $\|(H_{(\omega^{(i)})} - E)(\phi^{(L)} - \phi_k^{(L)})\| \to 0$ as $k \to \infty$. Taking $k_L$ large enough, we may assume

$$\|(H_{(\omega^{(i)})} - E)\phi_{k_L}^{(L)}\| < \frac{2}{L}\|\phi_{k_L}^{(L)}\|. \tag{3.1}$$

For simplicity, we still denote $\phi_{k_L}^{(L)}$ by $\phi^{(L)}$ and we assume it is supported on a compact set $\{\|n\| \leqslant K\}$.

Now, we take an integer $m$ large enough such that $(m-1)^{2r-d} > L^2K^{2d-2r}$, this choice is guaranteed by $r > d/2$. Let $\bigwedge_1 = \Gamma_0 \bigcap \{\|j\| \leqslant (m+1)K\}$ and $\bigwedge_2 = \{\|n\| \leqslant mK\}$. There exists

$(\tilde{\omega}^{(i)}) \in \Omega_0$ such that $|\omega_n^{(j)} - \tilde{\omega}_n^{(j)}| < \frac{1}{LK^d m^{d/2}}$ for any $j \in \bigwedge_1$ and $n \in \bigwedge_2$. Now let us consider

$$\begin{aligned}
&\|(H_{\tilde{\omega}^{(i)}} - H_{\omega^{(i)}})\phi^{(L)}\|_{l^2}^2 \\
=&\sum_{n\in\mathbb{Z}^d} |\sum_{\|j\|\leqslant K} (a_{n-j}(\tilde{T}^n(\tilde{\omega}^{(i)})) - a_{n-j}(\tilde{T}^n(\omega^{(i)})))\phi^{(L)}(j)|^2 \\
\leqslant&\|\phi^{(L)}\|_{l^2}^2 \sum_{n\in\mathbb{Z}^d} \sum_{\|j\|\leqslant K} |a_{n-j}(\tilde{T}^n(\tilde{\omega}^{(i)})) - a_{n-j}(\tilde{T}^n(\omega^{(i)}))|^2 \\
=&\|\phi^{(L)}\|_{l^2}^2 \sum_{\|j\|\leqslant K} (\sum_{\|n\|>mK} + \sum_{\|n\|\leqslant mK, n-j\in\Gamma_0} + \sum_{\|n\|\leqslant mK, n-j\in -\Gamma}) |a_{n-j}(\tilde{T}^n(\tilde{\omega}^{(i)})) - a_{n-j}(\tilde{T}^n(\omega^{(i)}))|^2 \\
\leqslant&\|\phi^{(L)}\|_{l^2}^2 \sum_{\|j\|\leqslant K} (\sum_{\|n\|>mK} 2(1+\|n-j\|)^{-2r} + \sum_{\|n\|\leqslant mK, n-j\in\Gamma_0} |\tilde{\omega}_n^{(n-j)} - \omega_n^{(n-j)}|^2 \\
&\qquad + \sum_{\|n\|\leqslant mK, n-j\in -\Gamma} |\tilde{\omega}_n^{(j-n)} - \omega_n^{(j-n)}|^2) \\
\leqslant&C\|\phi^{(L)}\|_{l^2}^2 (\sum_{\|n\|\geqslant mK} \sum_{\|j\|\leqslant K} \|n-j\|^{-2r} + K^{2d} m^d \frac{1}{L^2 K^{2d} m^d}) \\
\leqslant&C\|\phi^{(L)}\|_{l^2}^2 ((m-1)^{d-2r} K^{2d-2r} + \frac{1}{L^2}) \\
\leqslant&\frac{C}{L^2}\|\phi^{(L)}\|_{l^2}^2
\end{aligned}$$

Thus $\|(H_{\tilde{\omega}^{(i)}} - E)\phi^{(L)}\| \leqslant \frac{\sqrt{C}+1}{L}\|\phi^{(L)}\|$. Taking $L \to \infty$, we get $E \in \sigma(H_{(\tilde{\omega}^{(i)})}) = \Sigma$.

**The " $\subseteq$ " direction.** Note that since $\Omega_1$ is dense in $\Omega$, the proof follows from that of " $\supseteq$ " by interchanging the roles of $\Omega_0$ and $\Omega_1$. □

3.2. **Proof of Theorem 1.5.** Let $H_\omega = H_\omega^1 + H_\omega^2$, where $(H_\omega^1 u)(n) = a(T^n\omega)u(n+1) + a(T^{n-1}\omega)u(n-1)$ and $(H_\omega^2 u)(n) = b(T^n\omega)u(n)$. Let $\Sigma_1 = \sigma(H_\omega^1)$ a.s., $\Sigma_2 = \sigma(H_\omega^2)$ a.s. and $M = \sup\{|\alpha|, \alpha \in \operatorname{supp}\gamma\}$. Clearly,

$$\bigcup_{2-periodic\ \{\alpha_j\}} \sigma(H_{\{\alpha_j\}}) = [-2M, 2M] + \operatorname{supp}\gamma. \tag{3.2}$$

**The " $\subseteq$ " direction.** $\|H_\omega^1\| \leqslant 2M$, we have $\Sigma_1 \subset [-2M, 2M]$. This immediately implies $\Sigma \subseteq \Sigma_1 + \Sigma_2 \subseteq [-2M, 2M] + \operatorname{supp}\gamma$.

**The " $\supseteq$ " direction.** Let $\Omega_0$ be the full $\tilde{\kappa}$-measure set so that $\sigma(H_\omega) = \Sigma$ for any $\omega \in \Omega_0$. Since $M = \sup\{|\alpha|, \alpha \in \operatorname{supp}\gamma\}$, for any $\beta \in (-2M, 2M)$, there exists a set $\mathcal{F}$ with $\gamma(\mathcal{F}) > 0$ such that $\frac{1}{2}|\beta| < \inf\{|\alpha|, \alpha \in \mathcal{F}\}$. Then taking any $\xi \in \operatorname{supp}\gamma$, there exists a sequence $(\alpha_n, \xi_n) \in \Omega_0$ such that $\frac{1}{2}|\beta| < \alpha_n$ and $\xi_n \to \xi$ as $n \to \infty$. Clearly, $\beta + \xi \in \bigcup_n \sigma(H_{\{\ldots\alpha_n\xi_n\alpha_n\xi_n\alpha_n\xi_n\ldots\}}) = \Sigma$. Thus we have $(-2M, 2M) + \operatorname{supp}\gamma \subseteq \Sigma$, which implies the desired result after taking closure. □

## 4. Proof of Lemma 2.1

For simplicity, we denote $a_0^j - z$ by $a_0^j$. Then $\Phi_N(n) = \sum_{\|j\|\leqslant N} a_{n-j}^j \phi(j)$. Clearly,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^d} |\Phi_{L^q}(n)|^2 &= \sum_{n\in\mathbb{Z}^d} |\sum_{\|j\|\leqslant L^q} a_{n-j}^j \phi(j)|^2 \\
&= \sum_{\|n\|>L^q} |\sum_{\|j\|\leqslant L^q, \|j-n\|>L^{q-1}} a_{n-j}^j \phi(j) + \sum_{\|j\|\leqslant L^q, \|j-n\|\leqslant L^{q-1}} a_{n-j}^j \phi(j)|^2 \\
&\quad + \sum_{\|n\|\leqslant L^q} |\sum_{\|j\|>L^q, \|j-n\|>L^{q-1}} a_{n-j}^j \phi(j) + \sum_{\|j\|>L^q, \|j-n\|\leqslant L^{q-1}} a_{n-j}^j \phi(j)|^2 \\
&\leqslant 2 \sum_{\|n\|>L^q} \left( |\sum_{\|j\|\leqslant L^q, \|j-n\|>L^{q-1}} a_{n-j}^j \phi(j)|^2 + |\sum_{\|j\|\leqslant L^q, \|j-n\|\leqslant L^{q-1}} a_{n-j}^j \phi(j)|^2 \right) \\
&\quad + 2 \sum_{\|n\|\leqslant L^q} \left( |\sum_{\|j\|>L^q, \|j-n\|>L^{q-1}} a_{n-j}^j \phi(j)|^2 + |\sum_{\|j\|>L^q, \|j-n\|\leqslant L^{q-1}} a_{n-j}^j \phi(j)|^2 \right) \\
&:= 2(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4). \qquad (4.1)
\end{aligned}$$

**Estimate of $\Sigma_1$.**

$$\begin{aligned}
&\sum_{\|n\|>L^q} |\sum_{\|j\|<L^q, \|j-n\|>L^{q-1}} a_{n-j}^j \phi(j)|^2 \\
\leqslant &\|\phi_{L^q}\|_{l^2}^2 \sum_{\|n\|>L^q} \sum_{\|j\|<L^q, \|j-n\|>L^{q-1}} |a_{n-j}^j|^2 \\
\leqslant &\|\phi_{L^q}\|_{l^2}^2 \sum_{\|n\|>L^q} \sum_{\|j\|<L^q, \|j-n\|>L^{q-1}} (1+\|n-j\|)^{-2r} \\
\leqslant &C\|\phi_{L^q}\|_{l^2}^2 L^{q(2d-2r)+2r-1}. \qquad (4.2)
\end{aligned}$$

The proof of (4.2) will be included in the appendix.

**Estimate of $\Sigma_2$.**

$$\begin{aligned}
&\sum_{\|n\|>L^q} |\sum_{\|j\|\leqslant L^q, \|j-n\|\leqslant L^{q-1}} a_{n-j}^j \phi(j)|^2 \\
\leqslant &\sum_{L^q+L^{q-1}\geqslant\|n\|>L^q} \left( \sum_{\|m\|\leqslant L^{q-1}} \chi_{\|n-m\|\leqslant L^q} \ |a_m^{n-m}\phi(n-m)| \right)^2 \\
\leqslant &\left\{ \sum_{\|m\|\leqslant L^{q-1}} (1+\|m\|)^{-r} \left( \sum_{L^q+L^{q-1}\geqslant\|n\|>L^q} \chi_{\|n-m\|\leqslant L^q} |\phi(n-m)|^2 \right)^{\frac{1}{2}} \right\}^2 \\
\leqslant &(\|\phi_{(L+1)^q}\|_{l^2}^2 - \|\phi_{(L-1)^q}\|_{l^2}^2)(\sum_{m\in\mathbb{Z}^d} (1+\|m\|)^{-r})^2. \qquad (4.3)
\end{aligned}$$

**Estimate of $\Sigma_3$.**

$$
\begin{aligned}
&\sum_{\|n\|\leqslant L^q} \Big| \sum_{\|j\|>L^q, \|j-n\|>L^{q-1}} a^j_{n-j}\phi(j)\Big|^2 \\
\leqslant &\sum_{\|n\|\leqslant L^q} \Big( \sum_{\|j\|>L^q, \|j-n\|>L^{q-1}} |a^j_{n-j}\phi(j)|\Big)^2 \\
\leqslant &\left\{ \sum_{\|j\|>L^q} |\phi(j)| \left( \sum_{\|n\|\leqslant L^q} \chi_{\|j-n\|>L^{q-1}} \, |a^j_{n-j}|^2 \right)^{\frac{1}{2}} \right\}^2 \\
\leqslant &\left\{ \sum_{L^q+L^{q-1}\geqslant \|j\|>L^q} \|j\|^{\frac{d}{2}+\epsilon} \left( \sum_{\|n\|\leqslant L^q} \chi_{\|j-n\|>L^{q-1}} \, (1+\|n-j\|)^{-2r} \right)^{\frac{1}{2}} \right. \\
&\left. + \sum_{\|j\|>L^q+L^{q-1}} \|j\|^{\frac{d}{2}+\epsilon} \left( \sum_{\|n\|\leqslant L^q} (1+\|n-j\|)^{-2r} \right)^{\frac{1}{2}} \right\}^2 \\
\leqslant &\left\{ CL^{q(2d-r)+r+\epsilon q} + \sum_{\|j\|>L^q+L^{q-1}} \|j\|^{\frac{d}{2}+\epsilon} \left( \sum_{\|n\|\leqslant L^q} (1+\|n-j\|)^{-2r} \right)^{\frac{1}{2}} \right\}^2 \\
\leqslant &(CL^{q(2d-r)+r+\epsilon q})^{\frac{1}{2}}.
\end{aligned} \tag{4.4}
$$

The proof of (4.4) will be included in the appendix.

**Estimate of $\Sigma_4$.**

$$
\begin{aligned}
&\sum_{\|n\|\leqslant L^q} \Big| \sum_{\|j\|>L^q, \|j-n\|\leqslant L^{q-1}} a^j_{n-j}\phi(j)\Big|^2 \\
\leqslant &\sum_{\|n\|\leqslant L^q} \left( \sum_{\|m\|\leqslant L^{q-1}} \chi_{\|n-m\|>L^q} |a^{n-m}_m \phi(n-m)| \right)^2 \\
\leqslant &\left\{ \sum_{\|m\|\leqslant L^{q-1}} (1+\|m\|)^{-r} \left( \sum_{\|n\|\leqslant L^q} \chi_{\|n-m\|>L^q} |\phi(n-m)|^2 \right)^{\frac{1}{2}} \right\}^2 \\
\leqslant &(\|\phi_{(L+1)^q}\|^2_{l^2} - \|\phi_{(L-1)^q}\|^2_{l^2})(\sum_{m\in\mathbb{Z}^d} (1+\|m\|)^{-r})^2.
\end{aligned} \tag{4.5}
$$

Eventually combining the estimates (4.2), (4.3), (4.4), (4.5) with our choice of $r$, we conclude that

$$
\liminf_{l\to\infty} \frac{\|\Phi_{L^q}\|^2_{l^2}}{\|\phi_{L^q}\|^2_{l^2}} \leqslant C \liminf_{L\to\infty} \frac{\|\phi_{(L+1)^q}\|^2_{l^2} - \|\phi_{(L-1)^q}\|^2_{l^2}}{\|\phi_{L^q}\|^2_{l^2}} \quad \text{as} \quad L\to\infty. \tag{4.6}
$$

Then $\liminf_{L\to\infty} \frac{\|\Phi_{L^q}\|^2_{l^2}}{\|\phi_{L^q}\|^2_{l^2}} > \kappa > 0$ would imply that $\|\phi_{(L+1)^q}\|^2_{l^2} \geqslant (1+\frac{\kappa}{C})\|\phi_{(L-1)^q}\|^2_{l^2}$, which leads to exponential growth of $\phi$, contradiction. □

## Appendix

The proofs of (4.2) and (4.4) are in the same spirit.

**Proof of (4.2).**

$$
\begin{aligned}
&\sum_{\|n\|>L^q}\ \sum_{\|j\|<L^q,\|j-n\|>L^{q-1}} (1+\|j-n\|)^{-2r}\\
=&\sum_{L^q+L^{q-1}\geqslant\|n\|>L^q}\ \sum_{\|j\|<L^q,\|j-n\|>L^{q-1}} (1+\|j-n\|)^{-2r}+\sum_{\|n\|>L^q+L^{q-1}}\ \sum_{\|j\|<L^q}(1+\|j-n\|)^{-2r}\\
\leqslant& CL^{q(2d-2r)+2r-1}+CL^{qd}\int_{L^q+L^{q-1}}^{\infty}\frac{x^{d-1}}{(x-L^q)^{2r}}\ \mathrm{d}x\\
\leqslant& CL^{q(2d-2r)+2r-1}.
\end{aligned}
$$

□

**Proof of (4.4).**

$$
\begin{aligned}
&\sum_{\|j\|>L^q+L^{q-1}}(1+\|j\|)^{\frac{d}{2}+\epsilon}\left(\sum_{\|n\|<L^q}(1+\|n-j\|)^{-2r}\right)^{\frac{1}{2}}\\
\leqslant& L^{\frac{dq}{2}}\sum_{\|j\|>L^q+L^{q-1}}(1+\|j\|)^{\frac{d}{2}+\epsilon}\ (\|j\|-L^q)^{-r}\\
\sim& L^{\frac{dq}{2}}\int_{L^q+L^{q-1}}^{\infty}\frac{x^{\frac{3d}{2}+\epsilon-1}}{(x-L^q)^{r}}\ \mathrm{d}x\\
\leqslant& C^{L^{q(2d-r)+r+\epsilon q}}.
\end{aligned}
$$

□

**Part (b).** The proof is standard, we present it here for readers' convenience and completeness.

Since $\mu_{n,m}\ll\mu$, there exists a $F_{n,m}(E)=\frac{d\mu_{n,m}}{d\mu}(E)$ defined for $\mu$-a.e. $E$. We will show that for any fixed $n$, for $\mu$-a.e. $E$, $F_{n,m}(E)$ is an $\epsilon$-generalized eigenfunction, namely

(1) $F_{n,m}(E)$ is a solution to $Hu=Eu$,
(2) $|F_{n,m}(E)|\leqslant(1+\|m\|)^{\frac{d}{2}+\epsilon}$.

**Proof.**

(1).$(H-E)F_{n,m}(E)=0$, $\mu$-a.e. $E$, is equivalent to

$$\int_{\sigma(H)}(HF_{n,m})(E)g(E)\ d\mu(E)=\int_{\sigma(H)}EF_{n,m}(E)g(E)\ d\mu(E)$$

for any compactly supported continuous function $g$ on $\sigma(H)$. For simplicity, we denote $a_0^j-E$ by $a_0^j$. Then for any such $g$, $g(H)$ is a bounded operator, moreover,

$$
\begin{aligned}
&\textstyle\int_{\sigma(H)}EF_{n,m}(E)g(E)\ d\mu(E)\\
=&(e_n,Hg(H)e_m)\\
=&(g(H)^*e_n,He_m)\\
=&\textstyle(g(H)^*e_n,\sum_{k\in\mathbb{Z}^d}a_{k-m}^m e_k)\\
=&\textstyle(e_n,\sum_{k\in\mathbb{Z}^d}a_{k-m}^m g(H)e_k)\\
=&\textstyle\int_{\sigma(H)}\sum_{k\in\mathbb{Z}^d}a_{k-m}^m F_{n,k}(E)g(E)\ d\mu(E)
\end{aligned}
$$

$= \int_{\sigma(H)} (HF_{n,m})(E)g(E)\ d\mu(E)$.

(2). For any Borel set $B \subset \mathbb{C}$,

$$|\int_B F_{n,m}(E)\ d\mu(E)| = |\mu_{n,m}(B)| \leqslant \sqrt{\mu_n(B)\mu_m(B)}| \leqslant \frac{\mu(B)}{\sqrt{\lambda_n \lambda_m}} \leqslant C\mu(B)(1+\|n\|)^{\frac{d}{2}+\epsilon}(1+\|m\|)^{\frac{d}{2}+\epsilon}$$

Thus $|F_{n,m}(E)| \leqslant C(1+\|m\|)^{\frac{d}{2}+\epsilon}$ for some constant $C$ and $\mu$-a.e. $E$. □

**Part (c).** Part (c) follows directly from (a), (b) and the fact that spectrum of $H$ is the smallest closed set which supports every spectral measure of $H$. □

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