CONTINUITY OF MEASURE OF THE SPECTRUM FOR SCHRÖDINGER OPERATORS WITH POTENTIALS DRIVEN BY SHIFTS AND SKEW-SHIFTS ON TORI

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Abstract. We study discrete Schrödinger operators on $l^2(\mathbb{Z})$ with γ-Lipschitz potentials defined on higher dimensional torus $(\mathbb{T}^d, T)$, where T is a shift or skew-shift with frequency α. We show that under the positive Lyapunov exponent condition, measure of the spectrum at irrational frequency is the limit of measures of spectra at rational approximations.

1. introduction

Consider Schrödinger operators acting on $l^2(\mathbb{Z})$:

$$H_{f,T,\theta}u(n) = u(n+1) + u(n-1) + f(T^n\theta)u(n). \tag{1.1}$$

where f is the potential, $\theta \in \mathbb{T}^d$ is the phase and T is shift or skew-shift with frequency α on the torus $\mathbb{T}^d$. We study continuity of the spectra in frequency α. In particular, since the spectrum at rational frequencies can be obtained numerically and are easier to study, continuity in frequency allows us to study the spectrum at irrational frequencies via rational approximation. While many recent significant advances in discrete Schrödinger operators, see e.g. [6, 9, 2], require one dimensional torus shift and analytic potentials, our results reveal that continuity of the spectrum is a much more general phenomenon: it holds for both shift and skew-shift on higher dimensional torus and also Hölder continuous potentials. Our results can be viewed as a generalization of [12], where a similar result was obtained for $d = 1$ and T is a rotation of the circle.

Let $T_{s,\alpha} : \theta \to \theta + \alpha$ be the shift and $T_{ss,\alpha} : (\theta_1, \theta_2, ..., \theta_d) \to (\theta_1 + \alpha, \theta_2 + \theta_1, ..., \theta_d + \theta_{d-1})$ be the skew-shift. For a fixed $f, T_{*,\alpha}$, let us denote the spectrum of $H_{f,T_{*,\alpha},\theta}$ by $S(\alpha, \theta)$. Let $S(\alpha) = \cup_{\theta\in\mathbb{T}^d} S(\alpha, \theta)$. It is known that if α is irrational, $S(\alpha) = S(\alpha, \theta)$ for any $\theta \in \mathbb{T}^d$, while if α is rational, $S(\alpha, \theta)$ depends on θ and $S(\frac{\vec{p}_n}{q_n})$ is a union of at most q_n bands. We would like to establish that $\lim_{n\to\infty} S(\frac{\vec{p}_n}{q_n}) = S(\alpha)$ in the sense that $\lim_{n\to\infty} \chi_{S(\frac{\vec{p}_n}{q_n})}(E) = \chi_{S(\alpha)}(E)$ for a.e. $E \in \mathbb{R}$.

This question first arose from the Aubry-Andre conjecture [1] on the measure of the spectrum of the almost Mathieu operator ($d = 1$, $T = T_{s,\alpha}$ and $f(\theta) = 2\lambda\cos 2\pi\theta$) to be $4|1 - |\lambda||$. This conjecture has been proved for all irrational α, with partial results obtained in [5, 16, 17, 7, 14] and the extension to all irrational α was made in [10, 4] [1] (see e.g. [12] for a complete history). The proof of the Aubry-Andre conjecture contains two important ingredients: one is to obtain estimates about the rational frequencies [5, 17]: $|S(\frac{p_n}{q_n})| \to 4|1 - |\lambda||$; the other is to prove continuity of measure of the spectrum in frequency at irrationals. While the first ingredient clearly specializes to the almost Mathieu operator, the second ingredient, related to quantitative estimates on the Hausdorff continuity of the spectrum, have been studied for much more general potentials.

When $d = 1$ and $T = T_{s,\alpha}$, it was proved [10] that for any analytic f in the regime of positive Lyapunov exponent, $|S(\frac{p_n}{q_n})| \to |S(\alpha)|$ for every Diophantine α and its continued fraction approximants.

[1] The argument of [4], applies to the critical value $\lambda = 1$, did not involve continuity in frequency

Later, it was shown [11] that positivity of the Lyapunov exponent is not need for this result, in particular, $S(\frac{p_n}{q_n}) \to S(\alpha)$ for any analytic f and all irrational α. More recently, it has been proved [12] that under the condition of positive Lyapunov exponent, the regularity of f can be relaxed to Hölder continuity.

One of the key ingredients of the proof of [12] is strongly (weakly) M-dense property of the irrational rotation of the circle defined in the abstract form in Section 2.3. We say a dynamical system is strongly M-dense if any point will enter a ball with radius r within r^{-M} steps under the map as long as r is small, while the weak version requires only a sequence of lengths $r_k \to 0$. The strongly M-dense property for the irrational rotation of the circle is guaranteed by the Diophantine condition on α and proved using continued fraction expansion. For the higher dimensional shift and skew-shift, strongly M-dense properties have been studied using different methods in [3, 8], and some results on weakly M-dense property were obtained in [8]. These properties are important in our generalization of the results of [12] to both $(\mathbb{T}^d, T_{s,\alpha})$ and $(\mathbb{T}^d, T_{ss,\alpha})$ cases.

Let $L(\alpha, E)$ be the Lyapunov exponent of the operator $H_{f,T_{*,\alpha},\theta}$ at energy E (see (2.1)). Let $L_+(\alpha) = \{E : L(\alpha, E) > 0\}$ and $L_{\epsilon+}(\alpha) = \{E : L(\alpha, E) > \epsilon\}$.

With the Diophantine conditions defined in section 2.2, our main results are:

Theorem 1.1. *Let $T_{s,\alpha}$ be an irrational shift on $\mathbb{T}^d$. Let $1 \geq \gamma > \frac{d}{d+1}$ be a constant. Then if $\alpha \notin WDC(\frac{1}{\gamma})$ or $\alpha \in DC(\tau)$ for some $\tau > 1$, there exists a sequence of rationals $\frac{\vec{p}_n}{q_n} = (\frac{p_{1,n}}{q_n}, ..., \frac{p_{d,n}}{q_n}) \to \alpha$ such that for any $f \in C^\gamma(\mathbb{T}^d)$,*

$$\lim_{n\to\infty} S(\frac{\vec{p}_n}{q_n}) \cap L_+(\alpha) = S(\alpha) \cap L_+(\alpha).$$

Remark 1.1. The sequence of rationals can be taken as the full sequence of best simultaneous approximation, of α (see section 2.2.2) when $\alpha \in DC(\tau)$, and a proper subsequence when $\alpha \notin WDC(\frac{1}{\gamma})$.

A direct corollary is:

Corollary 1.2. *Let $\frac{\vec{p}_n}{q_n}$ be the chosen sequence of rationals as in Theorem 1.1, we have,*

$$\lim_{n\to\infty} |S(\frac{\vec{p}_n}{q_n}) \cap L_+(\alpha)| = |S(\alpha) \cap L_+(\alpha)|.$$

Theorem 1.3. *Let $T_{ss,\alpha}$ be a skew-shift on $\mathbb{T}^d$. For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. There exists a sequence of rationals $\frac{p_n}{q_n} \to \alpha$ such that for any $f \in C^\gamma(\mathbb{T}^d)$ with $1 \geq \gamma > \frac{1}{2}$,*

$$\lim_{n\to\infty} S(\frac{p_n}{q_n}) \cap L_+(\alpha) = S(\alpha) \cap L_+(\alpha).$$

Remark 1.2. The sequence of rationals will the be full sequence of continued fraction approximants if $\alpha \in DC(\tau)$ for some $\tau > 1$, and a proper subsequence otherwise.

A direct corollary is:

Corollary 1.4. *Let $\frac{p_n}{q_n}$ be the chosen sequence of rationals as in Theorem 1.3, we have,*

$$\lim_{n\to\infty} |S(\frac{p_n}{q_n}) \cap L_+(\alpha)| = |S(\alpha) \cap L_+(\alpha)|.$$

For shifts on two dimensional torus, as for the skew-shifts, we are able to cover all frequencies.

Theorem 1.5. *Let $T_{s,\alpha}$ be an irrational shift on $\mathbb{T}^2$. Let $1 \geq \gamma > \frac{2}{3}$ be a constant. Then for any $\epsilon_0 > 0$ and for any irrational α, there exists a sequence of rationals $\frac{\vec{p}_n}{q_n} \to \alpha$ (depending on ϵ_0) such* *that for any* $f \in C^\gamma(\mathbb{T}^2)$,

$$\lim_{n\to\infty} S(\frac{\vec{p}_n}{q_n}) \cap L_{\epsilon_0+}(\alpha) = S(\alpha) \cap L_{\epsilon_0+}(\alpha).$$

Similarly, we have

Corollary 1.6. *Let* $\frac{\vec{p}_n}{q_n}$ *be the chosen sequence of rationals as in Theorem 1.5, we have,*

$$\lim_{n\to\infty} |S(\frac{\vec{p}_n}{q_n}) \cap L_+(\alpha)| = |S(\alpha) \cap L_+(\alpha)|.$$

We organize this paper as follows: some preliminaries are presented in section 2, then the two key lemmas proved in section 3 prepare us for the proofs of main theorems in section 4.

2. Preparation

For $x \in \mathbb{R}^d$, let $\|x\|_{\mathbb{T}^d} = \operatorname{dist}(x, \mathbb{Z}^d)$. For a Borel set $U \subseteq \mathbb{R}^d$, let $|U|$ be its Lebesgue measure. Let d_0 be the dimension of the frequency α and $d_1 = d - d_0 + 1$, hence we have $d_0 = d$ and $d_1 = 1$ when $T_{*,\alpha} = T_{s,\alpha}$, while $d_0 = 1$ and $d_1 = d$ when $T_{*,\alpha} = T_{ss,\alpha}$. Let $D_r(x) \subset \mathbb{T}^d$ be the Euclidean ball centered at x with radius r.

2.1. Cocycles and Lyapunov exponent.

For a given $z \in \mathbb{C}$, a formal solution u of $H_{f,T_{*,\alpha},\theta} u = zu$ can be reconstructed using the transfer matrix

$$A(\theta, z) = \begin{pmatrix} z - f(\theta) & -1 \\ 1 & 0 \end{pmatrix}$$

via the equation

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A(T^n_{*,\alpha}\theta, z) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}$$

Indeed, let $A_k(\theta, z)$ be the product of consecutive transfer matrices:

$$A_k(\theta, z) = A(T^{k-1}_{*,\alpha}\theta, z) \cdots A(T_{*,\alpha}\theta, z) A(\theta, z) \quad \text{for } k > 0, \qquad A_0(\alpha, \theta, z) = I \quad \text{and}$$
$$A_k(\theta, z) = (A_{-k}(T^k_{*,\alpha}\theta, z))^{-1} \quad \text{for } k < 0.$$

Then for any $k \in \mathbb{Z}$ we have the following relation

$$\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = A_k(\theta, z) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}.$$

We define the Lyapunov exponent

$$L(\alpha, z) = \lim_k \frac{1}{k} \int_{\mathbb{T}^d} \ln \|A_k(\theta, z)\| \, \mathrm{d}\theta = \inf_k \frac{1}{k} \int_{\mathbb{T}^d} \ln \|A_k(\theta, z)\| \, \mathrm{d}\theta. \tag{2.1}$$

Furthermore, $L(\alpha, z) = \lim_k \frac{1}{k} \ln \|A_k(\theta, z)\|$ for a.e. $\theta \in \mathbb{T}^d$.

2.2. Rational approximation.

Let us introduce the Diophantine condition on $\mathbb{T}^d$:

$$DC(\tau) = \cup_{c>0} DC(c, \tau) = \cup_{c>0} \{(\alpha_1, ..., \alpha_d) | \|\langle \vec{h}, \alpha \rangle\|_{\mathbb{T}} \geq \frac{c}{r(\vec{h})^\tau} \text{ for any } \vec{0} \neq \vec{h} \in \mathbb{Z}^d\}$$

where $r(\vec{h}) = \prod_{i=1}^d \max(|h_i|, 1)$. It is well-known that when $\tau > 1$, $DC(\tau)$ is a full measure set.

We also introduce the weak Diophantine condition:

$$WDC(\tau) = \cup_{c>0} WDC(c, \tau) = \cup_{c>0} \{(\alpha_1, ..., \alpha_d) | \max\{\|h\alpha_i\|_{\mathbb{T}}\} \geq \frac{c}{|h|^\tau} \text{ for any } 0 \neq h \in \mathbb{Z}\}.$$

It is well-known that when $\tau > \frac{1}{d}$, $WDC(\tau)$ is a full measure set.

Clearly, in general $DC(\tau) \subseteq WDC(\tau)$, while in the single frequency case $DC(\tau) = WDC(\tau)$.

2.2.1. *Single frequency.* Let α be an irrational number and let $\{\frac{p_n}{q_n}\}$ be its continued fraction approximants. The following properties (see e.g.[15]) are well-known:

$$\frac{1}{2q_{n+1}} \leq \|q_n\alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}}. \tag{2.2}$$

$$\|k\alpha\|_{\mathbb{T}} > \|q_n\alpha\|_{\mathbb{T}} \text{ for } q_n < |k| < q_{n+1}. \tag{2.3}$$

If $\alpha \in DC(c,\tau)$ for some $c > 0$, we have

$$\|k\alpha\|_{\mathbb{T}} \geq \frac{c}{|k|^\tau} \text{ for any } k \neq 0. \tag{2.4}$$

In particular, combining (2.2) with (2.4) we have

$$cq_{n+1} \leq q_n^\tau. \tag{2.5}$$

2.2.2. *Multiple frequencies.* Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ be a set of irrational frequencies. Let $\{\frac{\vec{p}_n}{q_n}\}$ be its best simultaneous approximation with respect to the Euclidean norm on $\mathbb{T}^d$, namely,

$$\sum_{j=1}^d \|q_n\alpha_j\|_{\mathbb{T}}^2 < \sum_{j=1}^d \|k\alpha_j\|_{\mathbb{T}}^2 \text{ for any } 0 < |k| < q_n.$$

Clearly, by the pigeonhole principle, we have

$$\sqrt{\sum_{j=1}^d \|q_n\alpha_j\|_{\mathbb{T}}{}^2} \leq \frac{2\Gamma(\frac{d}{2}+1)^{\frac{1}{d}}}{\sqrt{\pi} q_{n+1}^{\frac{1}{d}}}. \tag{2.6}$$

By the definition of Diophantine and weak-Diophantine condition.

(1) If $\alpha \in DC(c,\tau)$, then

$$\|\langle \vec{k}, \alpha\rangle\|_{\mathbb{T}} \geq \frac{c}{r(\vec{k})^\tau} \text{ for any } \vec{k} \in \mathbb{Z}^d \backslash \{\vec{0}\}. \tag{2.7}$$

(2) If $\alpha \in WDC(c,\tau)$, then

$$\max_{1\leq j\leq d} \|k\alpha_j\|_{\mathbb{T}} \geq \frac{c}{|k|^\tau} \text{ for any } k \in \mathbb{Z}\backslash\{\vec{0}\}. \tag{2.8}$$

In particular combining (2.6) with (2.8), we have for $\alpha \in WDC(\tau)$,

$$c' q_{n+1}^{\frac{1}{d}} \leq q_n^\tau \text{ for some constant } c'. \tag{2.9}$$

(3) If $\alpha \notin WDC(\tau)$, there exists a subsequence of the best simultaneous Diophantine approximation $\{\frac{\vec{p}_{n_k}}{q_{n_k}}\}$ so that

$$\lim_{k\to\infty} q_{n_k}^\tau \max_{1\leq j\leq d} \|q_{n_k}\alpha_j\|_{\mathbb{T}} = 0. \tag{2.10}$$

2.3. **Covering $\mathbb{T}^d$ with the orbit of a ball.** We say a point x in $\mathbb{T}^d$ is *(T,r,M)-dense* for some $r>0$, $M\geq 1$, if $\cup_{j=0}^{r^{-M}} D_r(T^j x)=\mathbb{T}^d$. This means, the ball $D_r(x)$ with radius r will cover the whole $\mathbb{T}^d$ in r^{-M} steps under the map T. We say $(\mathbb{T}^d,T)$ is *strongly M-dense* if there exists $r_0>0$ such that any point in $\mathbb{T}^d$ is (T,r,M)-dense. We say $(\mathbb{T}^d,T)$ is *weakly M-dense* if there exists a sequence $r_k\to 0$ as $k\to\infty$ such that any point in $\mathbb{T}^d$ is (T,r_k,M)-dense.

The following lemmas are extracted from section 3 of [8].

Lemma 2.1. *Let T_s be an irrational shift on $\mathbb{T}^d$ and T_{ss} be a skew-shift. We have,*

- *if $\alpha\in DC(\tau)\subset\mathbb{T}^d$, then $(\mathbb{T}^d,T_s)$ is strongly M-dense for some $M\geq 1$.*
- *if $\alpha\in DC(\tau)\subset\mathbb{T}$, then $(\mathbb{T}^d,T_{ss})$ is strongly M-dense for some $M\geq 1$.*
- *if $\alpha\notin DC(d)\subset\mathbb{T}$, then $(\mathbb{T}^d,T_{ss})$ is weakly M-dense for some $M\geq 1$.*
- *if $\alpha\in WDC(\tau)\subset\mathbb{T}^2$, then $(\mathbb{T}^2,T_s)$ is weakly M-dense for some $M\geq 1$.*

2.4. **Upper and lower bounds on transfer matrices.** The following lemma on the uniform upper bound of transfer matrix is essentially from [13], we have adapted it into the following form for convenience.

Lemma 2.2. [13] *Let f be a function whose discontinuity set has Lebesgue measure 0 and T be a uniquely ergodic map on $\mathbb{T}^d$. Let $L(E)$ be positive on a Borel set U and μ be a measure such that $\mu(U)>0$. Then for any $\zeta,\epsilon>0$ there exists a number $D_\zeta>0$, a set $B_{\zeta,\epsilon}$ with $0<\mu(B_{\zeta,\epsilon})<\zeta$, and an integer $N_{\zeta,\epsilon}$ such that for any $E\in U\backslash B_{\zeta,\epsilon}$:*

- *$L(E)\geq D_\zeta$,*
- *for $n>N_{\zeta,\epsilon}$, $|z-E|<e^{-4\epsilon n}$ and $\theta\in\mathbb{T}^d$, we have $\frac{1}{n}\ln\|A_n(\theta,z)\|<L(E)+\epsilon$.*

We also have the following lemma on the lower bound of transfer matrix.

Lemma 2.3. [8] *Let $f\in C^\gamma(\mathbb{T}^d)$ with $1\geq\gamma>0$ and $T_{*,\alpha}=T_{s,\alpha}$ or $T_{ss,\alpha}$. Let $L(E)$ be positive on a Borel set U and a measure μ with $\mu(U)>0$. For any ζ,ϵ, let $D_\zeta, B_{\zeta,\epsilon}$ and $N_{\zeta,\epsilon}$ be defined as in Lemma 2.2. Then*

(1) *if $(\mathbb{T}^d,T_{*,\alpha})$ is strongly M-dense for some $M>0$, then for $n>N'_{\zeta,\epsilon}$, any $E\in U\backslash B_{\zeta,\epsilon}$, $|z-E|<e^{-4\epsilon n}$ and $\theta\in\mathbb{T}^d$ we have*
$$\min_{\iota\in\{-1,1\}}\max_{\iota j=0,\ldots,e^{\frac{5M\epsilon}{\gamma}n}}\|A_n(T_{*,\alpha}^j\theta,z)\|\geq e^{n(L(E)-3\epsilon)}.$$

(2) *if $(\mathbb{T}^d,T_{*,\alpha})$ is weakly M-dense for some $M>0$, then there exists a sequence $\{n_k(\epsilon)\}$ such that for any $k>k_{\zeta,\epsilon}$, any $E\in U\backslash B_{\zeta,\epsilon}$, $|z-E|<e^{-4\epsilon n_k}$ and $\theta\in\mathbb{T}^d$ we have*
$$\min_{\iota\in\{-1,1\}}\max_{\iota j=0,\ldots,e^{\frac{5M\epsilon}{\gamma}n_k}}\|A_{n_k}(T_{*,\alpha}^j\theta,z)\|\geq e^{n_k(L(E)-3\epsilon)}.$$

2.5. **Continuity of the spectrum for well approximated frequencies.** The following lemma enables us to establish the continuity of the spectrum at frequencies that are well approximated by the rationals, it is an extension of the $(\mathbb{T},\mathbb{T}_{s,\alpha})$ case in [5, 12].

Lemma 2.4. *Let $f\in C^\gamma(\mathbb{T}^d)$ with $1\geq\gamma>0$ and $T_{*,\alpha}=T_{s,\alpha}$ or $T_{ss,\alpha}$. Then for each $E\in S(\alpha)$, for $\|\alpha'-\alpha\|_{\mathbb{T}^{d_0}}$ small enough, there exists $E'\in S(\alpha')$ such that*

$$|E-E'|<C_f\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^{\frac{\gamma}{1+d_1\gamma}}. \tag{2.11}$$

Two direct corollaries of Lemma 2.4 are:

Lemma 2.5. *Let $T = T_{s,\alpha}$. If $\alpha \notin WDC(\frac{1}{\gamma})$, then there exists a proper subsequence of the best simultaneous approximation $\{\frac{\vec{p}_{n_k}}{q_{n_k}}\}$ of α, such that for any $f \in C^\gamma(\mathbb{T}^d)$, we have*

$$S(\alpha) \subseteq \liminf_{k\to\infty} S(\frac{\vec{p}_{n_k}}{q_{n_k}}). \tag{2.12}$$

Lemma 2.6. *Let $T = T_{ss,\alpha}$. If $\alpha \notin DC(d-1+\frac{1}{\gamma})$, then there exists a proper subsequence of the continued fraction approximants $\{\frac{p_{n_k}}{q_{n_k}}\}$ of α, such that for any $f \in C^\gamma(\mathbb{T}^d)$, we have*

$$S(\alpha) \subseteq \liminf_{k\to\infty} S(\frac{p_{n_k}}{q_{n_k}}). \tag{2.13}$$

The proofs of Lemmas 2.4, 2.5, 2.6 will be included in the appendix.
In the next sections, we therefore focus on the Diophantine α.

3. Key Lemmas

Lemma 3.1. *Let $f \in C^\gamma(\mathbb{T}^d)$ with $1 \geq \gamma > 0$ and $T_{*,\alpha} = T_{s,\alpha}$ or $T_{ss,\alpha}$. Recall that $d_0 = d, d_1 = 1$ for $T_{ss,\alpha}$ and $d_0 = 1, d_1 = d$ for $T_{ss,\alpha}$. Then*

1. *for any $\zeta, \epsilon > 0$, let $D_\zeta, B_{\zeta,\epsilon}$ and $N_{\zeta,\epsilon}$ be defined as in Lemma 2.2. If $(\mathbb{T}^d, T_{*,\alpha})$ is strongly M-dense, then for $n > N'_{\zeta,\epsilon}$, where $N'_{\zeta,\epsilon}$ is defined as in Lemma 2.3, $E \in S(\alpha) \cap L_+(\alpha) \setminus B_{\zeta,\epsilon}$ and $\|\alpha' - \alpha\|_{\mathbb{T}^{d_0}}$ small enough, there exists $E' \in S(\alpha')$ so that*
$$|E - E'| \leq Ce^{-n(\frac{D_\zeta}{4} - \frac{5M\epsilon}{\gamma})} + C_f \|\alpha - \alpha'\|^\gamma_{\mathbb{T}^{d_0}} e^{5M\epsilon d_1 n}, \tag{3.1}$$
where C is an absolute constant.
2. *for any $\zeta, \epsilon > 0$, let $B_{\zeta,\epsilon}$ and $N_{\zeta,\epsilon}$ be defined as in Lemma 2.2. If $(\mathbb{T}^d, T_{*,\alpha})$ is weakly M-dense, then for $k > k_{\zeta,\epsilon}$, where $\{n_k(\epsilon)\}$ and $k_{\zeta,\epsilon}$ are defined as in Lemma 2.3, $E \in S(\alpha) \cap L_{\epsilon_0+}(\alpha) \setminus B_{\zeta,\epsilon}$ and $\|\alpha' - \alpha\|_{\mathbb{T}^{d_0}}$ small enough, there exists $E' \in S(\alpha')$ so that*
$$|E - E'| \leq Ce^{-n_k(\frac{\epsilon_0}{4} - \frac{5M\epsilon}{\gamma})} + C_f \|\alpha - \alpha'\|^\gamma_{\mathbb{T}^{d_0}} e^{5M\epsilon d_1 n_k}, \tag{3.2}$$
where C is an absolute constant.

Proof of Lemma 3.1. We will prove part (2). Part (1) will be discussed briefly at the end of the proof. For $E \in S(\alpha) \cap L_{\epsilon_0+} \setminus B_{\zeta,\epsilon}$, by Lemma 2.2, for $n > N_{\zeta,\epsilon}$ and $|z - E| < e^{-4\epsilon n}$ we have

$$\|A_n(\theta, z)\| \leq e^{n(L(E)+\epsilon)}. \tag{3.3}$$

By Lemma 2.3, for $k > k_{\zeta,\epsilon}$, $|z - E| < e^{-4\epsilon n_k}$ and any $\theta \in \mathbb{T}^d$ we have

$$\min_{\iota\in\{-1,1\}} \max_{\iota j=0,\ldots,e^{\frac{5M\epsilon}{\gamma}n_k}} \|A_{n_k}(T^j_{*,\alpha}\theta, z)\| \geq e^{n_k(L(E)-3\epsilon)}. \tag{3.4}$$

Let E_0 be a generalized eigenvalue of $H_{f,T_{*,\alpha},\theta}$ such that $|E - E_0| < e^{-n_k(L(E)+4\epsilon)}$, with generalized eigenvector ψ satisfying $|\psi(x)| = o((1+|x|)^{1/2+\epsilon})$. Then there exists x_m so that

$$\frac{|\psi(x_m)|}{1+|x_m|} = \max_x \frac{|\psi(x)|}{1+|x|}. \tag{3.5}$$

Let ψ be normalized so that

$$\frac{|\psi(x_m)|}{1+|x_m|} = 1. \tag{3.6}$$

For $k > k_{\zeta,\epsilon}$, let $Q_{n_k} = e^{\frac{5M\epsilon}{\gamma} n_k}$. There exists an x_1' with $x_m - Q_{n_k} - n_k \le x_1' \le x_m - n_k$ such that $\|A_{n_k}(T_{*,\alpha}^{x_1'}\theta, E_0)\| > e^{n_k(L(E)-3\epsilon)}$. Similarly there exists an x_3' with $x_m \le x_3' \le x_m + Q_{n_k}$ such that $\|A_{n_k}(T_{*,\alpha}^{x_3'}\theta, E_0)\| > e^{n_k(L(E)-3\epsilon)}$. In general, we have

$$A_n(T_{*,\alpha}^l \theta, z) = \begin{pmatrix} P_n(T_{*,\alpha}^l\theta, z) & -P_{n-1}(T_{*,\alpha}^{l-1}\theta, z) \\ P_{n-1}(T_{*,\alpha}^{l-1}\theta, z) & -P_{n-2}(T_{*,\alpha}^{l-2}\theta, z) \end{pmatrix}.$$

This implies for $x_1 = x_1'$ or $x_1' - 1$ and $k_l = n_k, n_k - 1$ or $n_k - 2$, we have

$$|P_{k_l}(T_{*,\alpha}^{x_1}\theta, E_0)| > \frac{1}{4} e^{k_l(L(E)-3\epsilon)}. \tag{3.7}$$

Similarly, for $x_3 = x_3'$ or $x_3' - 1$ and $k_r = n_k, n_k - 1$ or $n_k - 2$, we have

$$|P_{k_r}(T_{*,\alpha}^{x_3}\theta, E_0)| > \frac{1}{4} e^{k_r(L(E)-3\epsilon)}. \tag{3.8}$$

Let

$$x_l = x_1 + \left[\frac{k_l}{2}\right]; \quad x_r = x_3 + \left[\frac{k_r}{2}\right]. \tag{3.9}$$

Also set $x_2 = x_1 + k_l - 1$ and $x_4 = x_3 + k_r - 1$. By Cramer's rule and (3.3), (3.7),

$$|G_{[x_1,x_2]}^{E_0}(x_l, x_1)| = \left|\frac{P_{x_2-x_l}(T_{*,\alpha}^{x_l+1}\theta, E_0)}{P_{k_l}(T_{*,\alpha}^{x_1}\theta, E_0)}\right| \le \frac{e^{\frac{k_l}{2}(L(E)+\epsilon)}}{\frac{1}{4}e^{k_l(L(E)-3\epsilon)}} < e^{-\frac{n_k}{4}L(E)}. \tag{3.10}$$

Similarly

$$|G_{[x_3,x_4]}^{E_0}(x_r, x_3)| < e^{-\frac{n_k}{4}L(E)}. \tag{3.11}$$

For similar reasons, (3.10) holds if we replace (x_l, x_1) with (x_l, x_2), $(x_l - 1, x_1)$ or $(x_l - 1, x_2)$; (3.11) holds if we replace (x_r, x_3) with (x_r, x_4), $(x_r + 1, x_3)$ or $(x_r + 1, x_4)$. Let $\Lambda = [x_l, x_r]$, we have $|\Lambda| < 3Q_{n_k} = 3e^{\frac{5M\epsilon}{\gamma}n_k}$. Let ψ_Λ be the truncation of ψ to Λ. For $x = x_i \pm 1$, $i = 1, 2, 3, 4$, by (3.5) and (3.6),

$$\frac{|\psi(x)|}{1+|x_m|} = \frac{|\psi(x)|}{1+|x|} \cdot \frac{1+|x|}{1+|x_m|} \le \frac{1+|x_m|+|x_m-x|}{1+|x_m|} \le 2e^{\frac{5M\epsilon}{\gamma}n_k}. \tag{3.12}$$

For $x_1 \le x \le x_2$,

$$\psi(x) = -G_{[x_1,x_2]}^{E_0}(x, x_1)\psi(x_1 - 1) - G_{[x_1,x_2]}^{E_0}(x, x_2)\psi(x_2 + 1). \tag{3.13}$$

Thus by (3.10) and (3.12),

$$|\psi(x_l)| \le 4(1+|x_m|)e^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})}.$$

Similarly

$$|\psi(x_r)| \le 4(1+|x_m|)e^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})}.$$

Hence the cut-off function satisfies

$$\|(H_{f,T_{*,\alpha},\theta} - E_0)\psi_\Lambda\| \le C(1+|x_m|)e^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})}.$$

Let $\phi_\Lambda = \frac{\psi_\Lambda}{\|\psi_\Lambda\|}$. Then by (3.6),

$$\|(H_{f,T_{*,\alpha},\theta} - E_0)\phi_\Lambda\| \le Ce^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})}. \tag{3.14}$$

For $T_{*,\alpha'}$, set $\theta' = T_{*,\alpha'}^{-\frac{x_l+x_r}{2}} T_{*,\alpha}^{\frac{x_l+x_r}{2}}\theta$. Then $T_{*,\alpha'}^{\frac{x_l+x_r}{2}}\theta' = T_{*,\alpha}^{\frac{x_l+x_r}{2}}\theta$, furthermore for $|k| \le \frac{x_r-x_l}{2}$,

$$\|T_{*,\alpha'}^{k+\frac{x_l+x_r}{2}}\theta' - T_{*,\alpha}^{k+\frac{x_l+x_r}{2}}\theta\| = \|T_{*,\alpha'}^{k} T_{*,\alpha}^{\frac{x_l+x_r}{2}}\theta - T_{*,\alpha}^{k} T_{*,\alpha}^{\frac{x_l+x_r}{2}}\theta\| \le C|k|^{d_1}\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}. \tag{3.15}$$

Thus since $f \in C^\gamma(\mathbb{T}^d)$,

$$\begin{aligned} \|(H_{f,T_{*,\alpha},\theta} - H_{f,T_{*,\alpha'},\theta'})\phi_\Lambda\| &\le \max_{|k|\le (x_r-x_l)/2} |f(T_{*,\alpha}^{k+\frac{x_l+x_r}{2}}\theta) - f(T_{*,\alpha'}^{k+\frac{x_l+x_r}{2}}\theta')| \\ &\le C_f(|\Lambda|^{d_1}\|\alpha-\alpha'\|_{d_0})^\gamma \\ &= C_f\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^\gamma e^{5M\epsilon d_1 n_k}. \end{aligned} \tag{3.16}$$

Then by the choice of E_0 and (3.14), (3.16),

$$\begin{aligned} \|(E - H_{f,T_{*,\alpha'},\theta'})\phi_\Lambda\| &\le |E-E_0| + \|(E_0 - H_{f,T_{*,\alpha},\theta})\phi_\Lambda\| + \|(H_{f,T_{*,\alpha},\theta} - H_{f,T_{*,\alpha'},\theta'})\phi_\Lambda\| \\ &\le Ce^{-n_k(\frac{L(E)}{4} - \frac{5M\epsilon}{\gamma})} + C_f\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^\gamma e^{5M\epsilon d_1 n_k}. \end{aligned} \tag{3.17}$$

This implies there exists $E' \in S(\alpha')$ so that

$$|E-E'| \le Ce^{-n_k(\frac{\epsilon_0}{4} - \frac{5M\epsilon}{\gamma})} + C_f\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^\gamma e^{5M\epsilon d_1 n_k}.$$

Remark 3.1. Part (1) can be proved by considering $S(\alpha) \cap L_+(\alpha)$ instead of $S(\alpha) \cap L_{\epsilon_0+}(\alpha)$ and without taking a subsequence $\{n_k(\epsilon)\}$. □

Lemma 3.2. *Let $f \in C^\gamma(\mathbb{T}^d)$ with $1 \ge \gamma > 0$ and $T_{*,\alpha} = T_{s,\alpha}$ or $T_{ss,\alpha}$.*

1. *If $(\mathbb{T}^d, T_{*,\alpha})$ is strongly M-dense for some $M > 1$, then for any $\zeta > 0$ and $\gamma > \beta > 0$ there exists a set B_ζ^β with $0 < |B_\zeta^\beta| < \zeta$ such that for any $E \in S(\alpha) \cap L_+(\alpha) \backslash B_\zeta^\beta$ and $\|\alpha'-\alpha\|_{\mathbb{T}^{d_0}}$ small enough, there exists $E' \in S(\alpha')$ satisfying*
$$|E-E'| < C_f\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^\beta.$$
2. *Let $d = 2$, $T = T_{s,\alpha}$ and $\alpha \in WDC(\frac{1}{\gamma})$. Then for any $\epsilon_0 > 0$ and $\gamma > \beta > 0$ there exists a sequence $\frac{\vec{p}_{m_k}}{q_{m_k}} \to \alpha$, with the property that for any $\zeta > 0$ there exists a set $B_\zeta^{\beta,\epsilon_0}$ with $0 < |B_\zeta^{\beta,\epsilon_0}| < \zeta$ such that for any $E \in S(\alpha) \cap L_{\epsilon_0+}(\alpha) \backslash B_\zeta^{\beta,\epsilon_0}$ there exists $E' \in S(\frac{\vec{p}_{m_k}}{q_{m_k}})$ satisfying*
$$|E-E'| < C_f\|\alpha - \frac{\vec{p}_{m_k}}{q_{m_k}}\|_{\mathbb{T}^2}^\beta.$$

Proof of Lemma 3.2.

Part (1). Given $\zeta > 0$, let $D_\zeta > 0$ be from Lemma 2.2. Fix $\epsilon = \epsilon(\zeta,\beta) = \frac{\gamma(\gamma-\beta)D_\zeta}{20M(\gamma-\beta+4d_1\gamma\beta)} < \frac{D_\zeta}{4}$. Let $B_\zeta^\beta := B_{\zeta,\epsilon(\zeta,\beta)}, N_\zeta^\beta := N_{\zeta,\epsilon(\zeta,\beta)}$ with $B_{\zeta,\epsilon}, N_{\zeta,\epsilon}$ as in Lemma 2.2. Let $\tilde{N}_\zeta^\beta := N'_{\zeta,\epsilon(\zeta,\beta)}$ be defined as in Lemma 2.3. By Lemma 3.1, for any $n > \tilde{N}_\zeta^\beta$, $E \in S(\alpha) \cap L_+(\alpha) \setminus B_\zeta^\beta$ and $\|\alpha'-\alpha\|_{\mathbb{T}^{d_0}}$ small enough, there exists $E' \in S(\alpha')$ so that E' is close to E, namely,

$$|E-E'| \le Ce^{-n(\frac{D_\zeta}{4} - \frac{5M\epsilon}{\gamma})} + C_f\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^\gamma e^{5M\epsilon d_1 n}. \tag{3.18}$$

There exists a small constant $\varrho_{\zeta,\beta} > 0$ so that when $\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}} < \varrho_{\zeta,\beta}$ we have

$$N'_{\zeta,\beta} < \frac{\gamma-\beta+2d_1\gamma\beta}{d_1\gamma D_\zeta}(-\ln\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}).$$

Then we could take $n > N'_{\zeta,\beta}$ satisfying

$$\frac{\gamma-\beta+4d_1\gamma\beta}{d_1\gamma D_\zeta}(-\ln\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}) \leq n \leq \frac{4(\gamma-\beta+4d_1\gamma\beta)}{d_1\gamma D_\zeta}(-\ln\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}),$$

so that by (3.18) there exists $E' \in S(\alpha')$ with

$$|E-E'| < C_f\|\alpha-\alpha'\|^{\beta}_{\mathbb{T}^{d_0}}. \tag{3.19}$$

□

Part (2). For $\epsilon_0 > 0$, fix a constant $\epsilon = \epsilon(\beta,\epsilon_0) = \frac{\gamma(\gamma-\beta)\epsilon_0}{20M(\gamma-\beta+4\beta)} < \frac{\epsilon_0}{4}$. For any $\zeta > 0$, let $B^{\beta,\epsilon_0}_\zeta := B_{\zeta,\epsilon(\beta,\epsilon_0)}$ and $N^{\beta,\epsilon_0}_\zeta := N_{\zeta,\epsilon(\beta,\epsilon_0)}$ be as in Lemma 2.2. Let $\{n_k(\beta,\epsilon_0)\} := \{n_k(\epsilon(\beta,\epsilon_0))\}$ and $k^{\beta,\epsilon_0}_\zeta := k_{\zeta,\epsilon(\beta,\epsilon_0)}$ be as in Lemma 2.3. By Lemma 3.1, for any $k > k^{\beta,\epsilon_0}_\zeta$, $E \in S(\alpha)\cap L_{\epsilon_0+}(\alpha)\setminus B^{\beta,\epsilon_0}_\zeta$ and $\|\alpha'-\alpha\|_{\mathbb{T}^2}$ small enough, there exists $E' \in S(\alpha')$ so that

$$|E-E'| \leq Ce^{-n_k(\frac{\epsilon_0}{4}-\frac{5M\epsilon}{\gamma})} + C_f\|\alpha-\alpha'\|^{\gamma}_{\mathbb{T}^2}e^{5M\epsilon n_k}. \tag{3.20}$$

$\alpha \in WDC(c,\frac{1}{\gamma})$ for some $c > 0$. Take the sequence of best simultaneous approximation $\{\frac{\vec{p}_m}{q_m}\}$. By (2.9) we have $q_m \geq c^\gamma q_{m+1}^{\frac{\gamma}{2}}$. Combining this with (2.8) and (2.6), we have

$$\|\alpha-\frac{\vec{p}_{m+1}}{q_{m+1}}\|_{\mathbb{T}^2} \geq \frac{c}{q_{m+1}^{1+\frac{1}{\gamma}}} \geq c(\frac{1}{q_m}\frac{1}{\sqrt{q_{m+1}}})^{\frac{2}{\gamma}} \geq c\|\alpha-\frac{\vec{p}_m}{q_m}\|^{\frac{2}{\gamma}}_{\mathbb{T}^2}.$$

Which implies

$$-\ln\|\alpha-\frac{\vec{p}_m}{q_m}\|_{\mathbb{T}^2} < -\ln\|\alpha-\frac{\vec{p}_{m+1}}{q_{m+1}}\|_{\mathbb{T}^2} \lesssim -\frac{2}{\gamma}\ln\|\alpha-\frac{\vec{p}_m}{q_m}\|_{\mathbb{T}^2}.$$

Therefore for each $n_k(\beta,\epsilon_0)$ there must be a corresponding $m_k(\beta,\epsilon_0)$ such that that

$$\frac{\gamma\epsilon_0}{4(\gamma-\beta+4\beta)}n_k \leq -\ln\|\alpha-\frac{\vec{p}_{m_k}}{q_{m_k}}\|_{\mathbb{T}^2} \leq \frac{\epsilon_0}{\gamma-\beta+4\beta}n_k.$$

By (3.20) and the choice of m_k, there exists $E' \in S(\frac{\vec{p}_{m_k}}{q_{m_k}})$ so that

$$|E-E'| \leq C_f\|\alpha-\frac{\vec{p}_{m_k}}{q_{m_k}}\|^{\beta}_{\mathbb{T}^2}. \tag{3.21}$$

□

4. Proof of Theorems 1.1, 1.3 and 1.5

First of all, the continuity of $S(\alpha)$ in the Hausdorff metric implies that for any sequence $\frac{\vec{p}_n}{q_n} \to \alpha$,

$$\limsup_{n\to\infty} S(\frac{\vec{p}_n}{q_n}) \subseteq S(\alpha). \tag{4.1}$$

By (4.1) and Lemmas 2.5, 2.6, the proofs are all reduced to proving a statement of the following type

$$S(\alpha)\cap L_+(\alpha) \subseteq \liminf_{k\to\infty} S(\frac{\vec{p}_{n_k}}{q_{n_k}}).$$

Since the proofs for $(\mathbb{T}^d, T_{s,\alpha})$, $(\mathbb{T}^d, T_{ss,\alpha})$ and $(\mathbb{T}^2, T_{s,\alpha})$ (weakly M-dense) relying on Lemma 3.2 are quite similar, we will only give the proof for $(\mathbb{T}^d, T_{ss,\alpha})$ in detail. The other two proofs will be discussed briefly at the end of this section.

Proof of Theorem 1.3. Let $\frac{p_n}{q_n}$ be the full sequence of continued fraction approximants of α. Since $\gamma > \frac{1}{2}$, we could fix $\frac{1}{2} < \beta < \gamma$. By Lemma 3.2, for any $\zeta > 0$ there exists $B_\zeta := B_\zeta^\beta$, $0 < |B_\zeta| < \zeta$, such that for n large enough we have

$$S(\alpha) \cap L_+(\alpha) \setminus B_\zeta \subset \cup_{i=1}^{q_n'} [a_{n,i} - C_f \|\alpha - \frac{p_n}{q_n}\|_{\mathbb{T}}^\beta,\ b_{n,i} + C_f \|\alpha - \frac{p_n}{q_n}\|_{\mathbb{T}}^\beta] := S(\frac{p_n}{q_n}) \cup F_n,$$

where $q_n' \leq q_n$ and

$$S(\frac{p_n}{q_n}) = \cup_{i=1}^{q_n'} [a_{n,i}, b_{n,i}].$$

This implies

$$S(\alpha) \cap L_+(\alpha) \setminus B_\zeta \subset \liminf_{n\to\infty} S(\frac{p_n}{q_n}) \cup F_n,$$

furthermore,

$$|S(\alpha) \cap L_+(\alpha) \setminus (\liminf_{n\to\infty} S(\frac{p_n}{q_n}) \cup F_n)| < \zeta. \tag{4.2}$$

By (2.2),

$$|F_n| \leq 2C_f q_n \|\alpha - \frac{p_n}{q_n}\|_{\mathbb{T}}^\beta \leq 2C_f q_{n+1}^{1-2\beta}, \tag{4.3}$$

which implies $\sum_n |F_n| < \infty$, thus $|\limsup_{n\to\infty} F_n| = 0$. This implies

$$|\liminf_{n\to\infty} S(\frac{p_n}{q_n}) \cup F_n| = |\liminf_{n\to\infty} S(\frac{p_n}{q_n})|. \tag{4.4}$$

Combining (4.2) with (4.4), we have

$$|S(\alpha) \cap L_+(\alpha) \setminus \liminf_{n\to\infty} S(\frac{p_n}{q_n})| < \zeta$$

for any $\zeta > 0$. Thus

$$S(\alpha) \cap L_+(\alpha) \subseteq \liminf_{n\to\infty} S(\frac{p_n}{q_n}). \tag{4.5}$$

□

Theorem 1.1 could be proved by taking $\frac{\vec{p}_n}{q_n}$ to be the full sequence of best simultaneous approximation. One needs to apply (2.7) to obtain the following (similar to (4.3))

$$|F_n| \leq 2C_f q_{n+1}^{1-\frac{d+1}{d}\beta}. \tag{4.6}$$

Theorem 1.5 could be proved by applying part (2) of Lemma 3.2. □

Appendix A. Proofs of Lemmas 2.4, 2.5, 2.6

A.1. **Lemma 2.4.** The proof is very similar to that of [5, 12]. Given $\epsilon > 0$ and $E \in S(\alpha)$, there exists an approximate eigenfunction $\phi_\epsilon \in l^2(\mathbb{Z})$ such that $\|(H_{T_{*,\alpha},\theta} - E)\phi_\epsilon\| < \epsilon\|\phi_\epsilon\|$. Set $g_{j,L}(n) = \max(1 - \frac{|j-n|}{L}, 0)$. Avron-van Mouche-Simon [5] proved that for sufficiently large L, for any bounded $f : \mathbb{T}^d \to \mathbb{R}$ there exists j such that $g_{j,L}\phi_\epsilon \neq 0$ and for any $\epsilon > 0$,

$$\|(H_{T_{*,\alpha},\theta} - E)g_{j,L}\phi_\epsilon\|^2 \leq C(\epsilon^2 + L^{-2})\|g_{j,L}\phi_\epsilon\|^2, \tag{A.1}$$

where C is universal. Now let $\theta' = T_{*,\alpha'}^{-j} T_{*,\alpha}^j \theta$. By the Hölder assumption on f and $j - L \leq n \leq j + L$, we have

$$|f(T_{*,\alpha'}^n \theta') - f(T_{*,\alpha}^n \theta)| \leq C_f (L^{d_1} \|\alpha' - \alpha\|_{\mathbb{T}^{d_0}})^\gamma.$$

Thus,

$$\|(H_{T_{*,\alpha'},\theta'}-E)g_{j,L}\phi_\epsilon\| \le \|(H_{T_{*,\alpha'},\theta'}-H_{T_{*,\alpha},\theta})g_{j,L}\phi_\epsilon\| + \|(H_{T_{*,\alpha},\theta}-E)g_{j,L}\phi_\epsilon\| \tag{A.2}$$

$$\le (C_f(L^{d_1}\|\alpha'-\alpha\|_{\mathbb{T}^{d_0}})^\gamma + C(\epsilon^2+L^{-2})^{\frac{1}{2}})\|g_{j,L}\phi_\epsilon\|. \tag{A.3}$$

Choosing $\epsilon = L^{-1} = C_f\|\alpha-\alpha'\|_{\mathbb{T}^{d_0}}^{-\frac{\gamma}{1+d_1\gamma}}$, we obtain the statement of Lemma 2.4. □

A.2. **Lemma 2.5.** Assume $\alpha \notin WDC(\frac{1}{\gamma})$. Then by (2.10), there exists a subsequence of the best simultaneous Diophantine approximation $\{\frac{\vec{p}_{n_k}}{q_{n_k}}\}$ so that

$$\lim_{k\to\infty} q_{n_k}^{\frac{1}{1+\gamma}} \max_{1\le j\le d} \|q_{n_k}\alpha_j\|_{\mathbb{T}}^{\frac{\gamma}{1+\gamma}} = 0. \tag{A.4}$$

By Lemma 2.4, we have

$$S(\alpha) \subset \cup_{i=1}^{q'_{n_k}} [a_{n_k,i} - C_f\|\alpha - \frac{\vec{p}_{n_k}}{q_{n_k}}\|_{\mathbb{T}^d}^{\frac{\gamma}{1+\gamma}},\ b_{n_k,i} + C_f\|\alpha - \frac{\vec{p}_{n_k}}{q_{n_k}}\|_{\mathbb{T}^d}^{\frac{\gamma}{1+\gamma}}] := S(\frac{\vec{p}_{n_k}}{q_{n_k}}) \cup F_{n_k},$$

where $q'_{n_k} \le q_{n_k}$ and

$$S(\frac{\vec{p}_{n_k}}{q_{n_k}}) = \cup_{i=1}^{q'_{n_k}} [a_{n_k,i}, b_{n_k,i}].$$

Thus, by (A.4),

$$S(\alpha) \subseteq \liminf_{k\to\infty} S(\frac{\vec{p}_{n_k}}{q_{n_k}}).$$

□

A.3. **Lemma 2.6.** Assume $\alpha \notin DC(d-1+\frac{1}{\gamma})$. Then by (2.10), there exists a subsequence of the continued fraction approximants $\frac{p_{n_k}}{q_{n_k}}$ so that

$$\lim_{k\to\infty} q_{n_k}^{\frac{1+(d-1)\gamma}{1+d\gamma}} \|q_{n_k}\alpha\|_{\mathbb{T}}^{\frac{\gamma}{1+d\gamma}} = 0 \tag{A.5}$$

By Lemma 2.4, we have

$$S(\alpha) \subset \cup_{i=1}^{q'_{n_k}} [a_{n_k,i} - C_f\|\alpha - \frac{p_{n_k}}{q_{n_k}}\|_{\mathbb{T}}^{\frac{\gamma}{1+d\gamma}},\ b_{n_k,i} + C_f\|\alpha - \frac{p_{n_k}}{q_{n_k}}\|_{\mathbb{T}}^{\frac{\gamma}{1+d\gamma}}] := S(\frac{p_{n_k}}{q_{n_k}}) \cup F_{n_k},$$

where $q'_{n_k} \le q_{n_k}$ and

$$S(\frac{p_{n_k}}{q_{n_k}}) = \cup_{i=1}^{q'_{n_k}} [a_{n_k,i}, b_{n_k,i}].$$

Thus, by (A.5),

$$S(\alpha) \subseteq \liminf_{k\to\infty} S(\frac{p_{n_k}}{q_{n_k}}).$$

□

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