

LOWER BOUND TO THE ENTANGLEMENT ENTROPY OF THE XXZ SPIN RING

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ABSTRACT. We study the free XXZ quantum spin model defined on a ring of size L and show that the bipartite entanglement entropy of eigenstates belonging to the first energy band above the vacuum ground state satisfies a logarithmically corrected area law. Along the way, we show a Combes-Thomas estimate for fiber operators which can also be applied to discrete many-particle Schrödinger operators on more general translation-invariant graphs.

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1. INTRODUCTION

Considered to be one of the indicators of many-body-localization (MBL), area laws for the entanglement entropy have attracted significant interest by the physics [ECP10, Laf16] as well as the mathematics community [PS14, PS18a, EPS17, Has07, BW18, ARFS20]. In contrast, delocalization induced by long-range correlations leads to increase stronger than a mere area law. However, only a few examples, where such phenomena were observed, have been

rigorously studied as of yet [LSS14, LSS17, PS18b, Wol06, MPS20, MS20]. It is noteworthy that almost all of these examples are found within the non-interacting setting. In particular, logarithmic corrections of area laws seem to be a common occurrence in physical systems that are known to be delocalized.

An interacting system that exhibits MBL phenomena is the disordered XXZ spin chain. Recently, localization phenomena for this model have been rigorously studied in [EKS18b, EKS18a, BW17]; see also [Sto20] for a survey of the newest developments. An area law of the entanglement entropy for low energy states in such a system has been proven in [BW18]. However, the situation for the infinite XXZ spin chain without disorder is fundamentally different: with the help of the Bethe ansatz, it can be shown that the lowest spectral band is purely absolutely continuous with delocalized generalized eigenfunctions [NSS06, FS14]. It is thus reasonable to also expect a different scaling behavior for the entanglement entropy. In this paper, we therefore consider finite XXZ spin chains of arbitrary size with periodic boundary conditions and constant magnetization density.

In [BW18], Beaud and Warzel already showed for low energy states in the finite XXZ chain with droplet boundary conditions, that there is a logarithmic upper bound of the entanglement entropy, independent of non-negative background potentials. The explicit results in [BW18, Prop. 1.2] and [ARFS20, Thm. 1.2] show that the log-term is optimal in the Ising model. By developing a suitable perturbational approach, we extend this to the XXZ model in the Ising phase.

It is well-known that the XXZ Hamiltonian preserves the total magnetization and is equivalent to a direct sum of discrete many-particle Schrödinger operators of hard-core bosons. In the Ising phase, the associated potential energetically favors clustered configurations [NSS06, NS01, FS14, FS18], commonly referred to as “droplets”. Thus, the mass of low energy states is mainly concentrated around these droplet configurations. Our perturbative result will rely on the fact that in the Ising limit, droplet configurations and low energy states coincide. This follows from a suitable Combes-Thomas bound which we will show in the first part of this paper.

While the main result of the paper contributes to the question whether the logarithmic upper bound is optimal, we also believe its perturbative approach to be of more general interest. In particular, for the disordered Ising model, it was shown in [ARFS20] that it is possible to find states in the next-highest energy band whose entanglement entropy exhibits a logarithmic lower bound with arbitrary high probability. This suggests delocalization phenomena for higher energy states in the XXZ model despite disorder.

We will proceed as follows:

In Section 2, we introduce the XXZ model on the ring and review some of its basic properties. Exploiting its preservation of total magnetization, we decompose the Hamiltonian into a direct sum of operators acting on subspaces of fixed total magnetization. We will also state our main results (Theorems 2.1 and 2.3).

Section 3 is dedicated to obtaining estimates on eigenfunctions of the XXZ Hamiltonian. To this end, we firstly exploit the ring’s translational symmetry and define a suitable Fourier transform. We then introduce an equivalent formulation of the XXZ Hamiltonian using Schrödinger operators, which we will use to show an appropriate Combes-Thomas estimate 2.3. At the end of this section, we consider the Ising model, for which we pick a suitable

low-energy state which exhibits the desired logarithmic lower bound. The purpose of the remainder of the paper will therefore be to show that if one does not move too far away from the Ising limit, this logarithmic lower bound persists.

To this end, we then focus in Section 4 on showing that the reduced state of droplet eigenstates in the Ising phase is exponentially close to the reduced state coming from the Ising limit (with respect to a suitable distance function). The underlying geometry of the ring poses some technical difficulties which are overcome by suitable estimates, basically allowing us to treat the model with methods developed for the chain.

After this, in Section 5, we estimate the eigenvalues of the difference of these reduced states, which allows us to find suitable bounds of its Schatten-quasinorms. Using a result by Combes, Hislop and Nakamura [CHN01], this allows us to estimate the L^p -norms of the associated Kreĭn's spectral shift function which we then use to show Theorem 2.1.

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2. MODEL AND MAIN RESULTS

For any $L \in \mathbb{N}$, consider the XXZ model on a discrete ring of size L . We start by describing the ring using the graph $\mathcal{G}_L := (\mathcal{V}_L, \mathcal{E}_L)$ with vertex set $\mathcal{V}_L := \{0, 1, \dots, L-1\}$ and edge set $\mathcal{E}_L := \{\{j, (j+1) \bmod L\} : j \in \mathcal{V}_L\}$.

The underlying 2^L -dimensional Fock space \mathbb{H}_L is given by $\mathbb{H}_L = \bigotimes_{j \in \mathcal{V}_L} \mathbb{C}^2$. Let $|\uparrow\rangle := (10)^t$ and $|\downarrow\rangle := (01)^t$ denote the canonical basis of \mathbb{C}^2 . To construct a basis for the Fock-space we define the spin lowering operator

$$(2.1) \quad S^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For any set \mathcal{A} let $\mathcal{P}(\mathcal{A})$ denote its power set. We now introduce a canonical basis $\{|\delta_x^L\rangle\}_{x \in \mathcal{P}(\mathcal{V}_L)}$ of \mathbb{H}_L by $|\delta_\emptyset^L\rangle := |\uparrow\rangle^{\otimes L}$ and for any other $x \in \mathcal{P}(\mathcal{V}_L)$ by

$$(2.2) \quad |\delta_x^L\rangle := \prod_{j \in x} S_j^- |\delta_\emptyset^L\rangle.$$

Here, and in the following, for any $A \in \mathbb{C}^{2 \times 2}$ the notation A_j refers to a spin operator acting as A on the site $j \in \mathcal{V}_L$ and as the identity else.

The XXZ-Hamiltonian $H_L : \mathbb{H}_L \rightarrow \mathbb{H}_L$ is given by

$$(2.3) \quad H_L \equiv H_L(\Delta) := \sum_{\{j,k\} \in \mathcal{E}_L} h_{jk}(\Delta),$$

where the two-site operator h_{jk} describes an interaction between two spins located at the two sites $\{j, k\} \in \mathcal{E}_L$. It is given by

$$(2.4) \quad h_{jk} \equiv h_{jk}(\Delta) := \left(\frac{1}{4} - S_j^3 S_k^3 \right) - \frac{1}{\Delta} (S_j^1 S_k^1 + S_j^2 S_k^2),$$

with S^1 , S^2 and S^3 being the standard spin-1/2 matrices

$$(2.5) \quad S^1 := \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad S^2 := \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} \quad \text{and} \quad S^3 := \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

In the following, we will assume that the *anisotropy parameter* Δ satisfies $\Delta > 1$, which ensures that for each $\{j, k\} \in \mathcal{E}_L$, we have $h_{jk} \geq 0$ and consequently $H_L \geq 0$. The case $\Delta > 1$ is commonly referred to as the ‘‘Ising-phase’’ of the XXZ model. The two-dimensional ground-state space of H_L corresponding to the ground state energy $E_0 = 0$ is the linear span of the two vectors $|\delta_{\emptyset}^L\rangle$ (‘‘all spins-up’’) and $|\delta_{\mathcal{V}_L}^L\rangle$ (‘‘all spins-down’’). The operator H_L preserves the total magnetization (for more details see [FS18]). We therefore treat each down-spin as a particle. For all $N \in \{0, \dots, L\}$, let us define the N -particle subspace by

$$(2.6) \quad \mathbb{H}_L^N := \text{span}\{|\delta_x^L\rangle : x \in \mathcal{P}(\mathcal{V}_L), |x| = N\}.$$

Since H_L is particle number preserving, each \mathbb{H}_L^N reduces the operator H_L . Hence, we express it as the direct sum

$$(2.7) \quad H_L = \bigoplus_{N=0}^L H_L^N,$$

where $H_L^N := H_L \upharpoonright_{\mathbb{H}_L^N}$ for all $N \in \{0, 1, \dots, L\}$. The operators H_L^L and H_L^0 are identical to the zero operator on $\mathbb{H}_L^L = \text{span}\{|\delta_{\mathcal{V}_L}^L\rangle\}$ and $\mathbb{H}_L^0 = \text{span}\{|\delta_{\emptyset}^L\rangle\}$ respectively.

Now, let $\Lambda \subset \mathcal{V}_L$ and $\mathbb{H}_\Lambda := \bigotimes_{j \in \Lambda} \mathbb{C}^2$. Given any normalized state $|\psi\rangle \in \mathbb{H}_L$, we consider its entanglement entropy with respect to the spatial decomposition $\mathbb{H}_L = \mathbb{H}_\Lambda \otimes \mathbb{H}_{\Lambda^c}$. As before, we denote by $\{|\delta_x^\Lambda\rangle\}_{x \in \mathcal{P}(\Lambda)}$ the canonical basis of \mathbb{H}_Λ . Analogously, we define the N -particle subspace by

$$(2.8) \quad \mathbb{H}_\Lambda^N := \text{span}\{|\delta_x^\Lambda\rangle : x \in \mathcal{P}(\Lambda), |x| = N\}.$$

for all $N \in \{0, \dots, |\Lambda|\}$. This choice allows the convenient identification

$$(2.9) \quad |\delta_{x \cup y}^L\rangle = |\delta_x^\Lambda\rangle \otimes |\delta_y^{\Lambda^c}\rangle \in \mathbb{H}_L$$

for any $x \in \mathcal{P}(\Lambda)$ and $y \in \mathcal{P}(\Lambda^c)$. Let $\rho(\psi) := |\psi\rangle\langle\psi|$ denote the density operator corresponding to $|\psi\rangle$ and moreover let $\rho_\Lambda(\psi) := \text{tr}_{\Lambda^c}\{\rho(\psi)\} \in L(\mathbb{H}_\Lambda)$ denote the respective partial trace over \mathbb{H}_{Λ^c} . The entanglement entropy of ψ is given by

$$(2.10) \quad S(\psi, \Lambda) := \text{tr } s(\rho_\Lambda(\psi)),$$

where $s : [0, 1] \rightarrow \mathbb{R}$, $x \mapsto -x \ln x$.

Our main result concerns the entanglement entropy of low energy states whose eigenenergy belongs to the interval $I_1 \equiv I_1(\Delta) := [1 - \frac{1}{\Delta}, 2(1 - \frac{1}{\Delta})]$.

Theorem 2.1. *Let $\epsilon \in (0, 1/16)$, let $\theta \in (\epsilon, 1/16)$. For $L \in \mathbb{N}$, let $N \equiv N(L) := \lfloor \epsilon L \rfloor$ and $\Lambda_L := \{0, \dots, 2\lfloor \theta L \rfloor\} \subset \mathcal{V}_L$. Then there exists $\Delta_0 \equiv \Delta_0(\epsilon) > 3$ such that for all $\Delta \geq \Delta_0$, $L \in \mathbb{N}$ and $E \in \sigma(H_L^N) \cap I_1$ there exists a corresponding eigenstate $|\varphi_L^N(\Delta, E)\rangle \in \mathbb{H}_L^N$ such that*

$$(2.11) \quad \liminf_{L \rightarrow \infty} \frac{S(\varphi_L^N(\Delta, E), \Lambda_L)}{\ln L} \geq \frac{\epsilon}{2}.$$

Remark 2.2. (i) *While we have made the particular choice for Λ_L to scale proportionally to the ring size L and not independently of it, our result nevertheless shows that an area law could not possibly be true in the generic case. Similar choices are often considered in the physics literature [ISL12, PY14, VLRK03].*

(ii) *While we were not able to find any reference in the literature, we expect that for almost every $\Delta > 3$, the multiplicity of an eigenvalue within the droplet band is at most two. This would imply that the result of Theorem 2.1 holds for every eigenfunction in the droplet band.*

An important ingredient for the proof of Theorem 2.1 is the following estimate showing that low-energy eigenfunctions are mainly concentrated around droplet configurations, which are given by

$$(2.12) \quad \mathcal{V}_{L,1}^N := \{\{j, (j+1) \bmod L, \dots, (j+N-1) \bmod L\} : j \in \mathcal{V}_L\}.$$

This result follows from a Combes-Thomas estimate similar to those shown in [EKS18b, ARFS20]. The main new feature of our estimate here is that the ring's symmetry is taken into account which allows to obtain an additional factor of $L^{-1/2}$.

Theorem 2.3. *Let $L, N \in \mathbb{N}$ with $N < L$ and $\Delta > 3$. For any $E \in \sigma(H_L^N) \cap I_1$ there exists a corresponding eigenstate $|\varphi_L^N\rangle \equiv |\varphi_L^N(\Delta, E)\rangle \in \mathbb{H}_L^N$ such that*

$$(2.13) \quad |\langle \delta_x^L, \varphi_L^N \rangle| \leq \frac{2^4}{\sqrt{L}} \cdot e^{-\mu_1 d_L^N(x, \mathcal{V}_{L,1}^N)},$$

for all $x \in \mathcal{V}_L^N$. Here, d_L^N is the N -particle graph distance as defined in Section 3.1 and

$$(2.14) \quad \mu_1 \equiv \mu_1(\Delta) := \ln \left(1 + \frac{(\Delta - 1)}{8} \right).$$

3. ESTIMATING EIGENFUNCTIONS

3.1. Fourier transform. For the entire section let $L, N \in \mathbb{N}$ with $N < L$. Since we are mainly interested in the N -particle subspace \mathcal{H}_L^N , we introduce the graph of N -particle configurations first.

Recall that the spin ring is described by the graph $\mathcal{G}_L := (\mathcal{V}_L, \mathcal{E}_L)$. The corresponding graph distance between two sites $j, k \in \mathcal{V}_L$ is given by

$$(3.1) \quad d_L(j, k) = L/2 - ||j - k| - L/2|.$$

Following [FS18], we construct the N -th symmetric product $\mathcal{G}_L^N := (\mathcal{V}_L^N, \mathcal{E}_L^N)$ of \mathcal{G}_L , where

$$(3.2) \quad \mathcal{V}_L^N := \{x \subseteq \mathcal{V}_L : |x| = N\} \quad \text{and} \quad \mathcal{E}_L^N := \{\{x, y\} \subseteq \mathcal{V}_L^N : x \Delta y \in \mathcal{E}_L\}.$$

Here, $x\Delta y$ denotes the symmetric difference between the two subsets $x, y \subseteq \mathcal{V}_L$. We also write $x \sim y$ for $\{x, y\} \in \mathcal{E}_L^N$. Finally, $d_L^N(\cdot, \cdot)$ denotes the graph distance on \mathcal{G}_L^N . As in [FS18], we identify $\mathbb{H}_L \cong \ell^2(\mathcal{P}(\mathcal{V}_L))$ and $\mathbb{H}_L^N \cong \ell^2(\mathcal{V}_L^N)$.

In order to exploit the ring's translational symmetry, we define a suitable Fourier-transform. To this end, for any $\gamma \in \mathbb{Z}$, we define the translations $T_L^\gamma : \mathcal{P}(\mathcal{V}_L) \rightarrow \mathcal{P}(\mathcal{V}_L)$ by

$$(3.3) \quad T_L^\gamma x = \{(j + \gamma) \bmod L : j \in x\} \quad \text{for all } x \subseteq \mathcal{V}_L.$$

For every $\gamma \in \mathbb{Z}$, the unitary translation operator $\tilde{T}_L^\gamma : \ell^2(\mathcal{P}(\mathcal{V}_L)) \rightarrow \ell^2(\mathcal{P}(\mathcal{V}_L))$ is given by

$$(3.4) \quad (\tilde{T}_L^\gamma \psi)(x) = \psi(T_L^\gamma x) \quad \text{for all } \psi \in \ell^2(\mathcal{P}(\mathcal{V}_L)), x \in \mathcal{P}(\mathcal{V}_L).$$

Due to translational symmetry of H_L , we then get $[\tilde{T}_L^\gamma, H_L] = 0$ for any $\gamma \in \mathbb{Z}$.

Now, let “ \approx ” denote the equivalence relation on $\mathcal{V}_L^N \times \mathcal{V}_L^N$ defined as

$$(3.5) \quad x \approx y :\Leftrightarrow \exists \gamma \in \{0, 1, \dots, L-1\} \text{ such that } T_L^\gamma x = y.$$

Moreover, let $\hat{\mathcal{V}}_L^N \subset \mathcal{V}_L^N$ be a fixed set of representatives of each equivalence class induced by “ \approx ”. For an element $\hat{x} \in \hat{\mathcal{V}}_L^N$, we denote the corresponding equivalence class by $[\hat{x}]$. We define $\hat{d}_L^N : \hat{\mathcal{V}}_L^N \times \hat{\mathcal{V}}_L^N \rightarrow \mathbb{N}_0$ by

$$(3.6) \quad \hat{d}_L^N(\hat{x}, \hat{y}) := \min_{\gamma \in \mathbb{Z}} d_L^N(\hat{x}, T_L^\gamma \hat{y}) \text{ for all } \hat{x}, \hat{y} \in \hat{\mathcal{V}}_L^N.$$

Lemma 3.1. \hat{d}_L^N is a metric on $\hat{\mathcal{V}}_L^N$.

Proof. Since d_L^N is a metric, if $\hat{d}_L^N(\hat{x}, \hat{y}) = 0$, this means that there exists a $\gamma \in \{0, 1, \dots, L-1\}$ such that $\hat{x} = T_L^\gamma \hat{y}$. By definition of $\hat{\mathcal{V}}_L^N$, this implies that $\hat{x} = \hat{y}$.

Now, for any $\hat{x}, \hat{y} \in \hat{\mathcal{V}}_L^N$ let us consider

$$(3.7) \quad \hat{d}_L^N(\hat{x}, \hat{y}) = \min_{\gamma \in \mathbb{Z}} d_L^N(\hat{x}, T_L^\gamma \hat{y}) = \min_{\gamma} d_L^N(T_L^{-\gamma} \hat{x}, \hat{y}) = \min_{\gamma} d_L^N(\hat{y}, T_L^{-\gamma} \hat{x}) = \hat{d}_L^N(\hat{y}, \hat{x}).$$

Finally, for any $\hat{x}, \hat{y}, \hat{z} \in \hat{\mathcal{V}}_L^N$ and any $\sigma \in \{0, 1, \dots, L-1\}$ consider

$$(3.8) \quad \begin{aligned} \hat{d}_L^N(\hat{x}, \hat{z}) &= \min_{\gamma} d_L^N(\hat{x}, T_L^\gamma \hat{z}) \leq \min_{\gamma} (d_L^N(\hat{x}, T_L^\gamma \hat{y}) + d_L^N(T_L^\sigma \hat{y}, T_L^\gamma \hat{z})) \\ &= d_L^N(\hat{x}, T_L^\sigma \hat{y}) + \min_{\gamma} d_L^N(\hat{y}, T_L^{\gamma-\sigma} \hat{z}) = d_L^N(\hat{x}, T_L^\sigma \hat{y}) + \hat{d}_L^N(\hat{y}, \hat{z}). \end{aligned}$$

Minimizing over $\sigma \in \mathbb{Z}$ now yields the desired triangle inequality $\hat{d}_L^N(\hat{x}, \hat{z}) \leq \hat{d}_L^N(\hat{x}, \hat{y}) + \hat{d}_L^N(\hat{y}, \hat{z})$ and thus the lemma. \square

We note that not all equivalence classes have the same cardinality. In fact, for any $\hat{x} \in \hat{\mathcal{V}}_L^N$ the number of elements in $[\hat{x}]$ is given by

$$(3.9) \quad n_{\hat{x}} := |[\hat{x}]| = \min\{\gamma \in \mathbb{N} : T_L^\gamma \hat{x} = \hat{x}\}.$$

Moreover, for any $\hat{x} \in \hat{\mathcal{V}}_L^N$ the number $n_{\hat{x}}$ divides L . Let us now define the unitary Fourier transform. To this end, let

$$(3.10) \quad \mathbb{S}_L^N := \left\{ \phi \in \ell^2(\mathcal{V}_L \times \hat{\mathcal{V}}_L^N) : \forall \hat{x} \in \hat{\mathcal{V}}_L^N \forall \gamma \notin \frac{L}{n_{\hat{x}}} \{0, \dots, n_{\hat{x}} - 1\} \text{ we have } \phi(\gamma, \hat{x}) = 0 \right\}.$$

The scalar product $\langle \cdot, \cdot \rangle_{\mathbb{S}_L^N}$ on this space is defined in the following way:

$$(3.11) \quad \langle \phi_1, \phi_2 \rangle_{\mathbb{S}_L^N} := \sum_{\gamma \in \mathcal{V}_L} \sum_{\hat{x} \in \hat{\mathcal{V}}_L^N} \frac{1}{L/n_{\hat{x}}} \overline{\phi_1(\gamma, \hat{x})} \phi_2(\gamma, \hat{x}),$$

for any $\phi_1, \phi_2 \in \mathbb{S}_L^N$. Moreover, for any $f \in \mathbb{S}_L^N$, we define $\|f\|_{\mathbb{S}_L^N} := \sqrt{\langle f, f \rangle_{\mathbb{S}_L^N}}$. The Fourier transform \mathfrak{F}_L^N is given by

$$(3.12) \quad \begin{aligned} \mathfrak{F}_L^N : \quad & \ell^2(\mathcal{V}_L^N) \rightarrow \mathbb{S}_L^N \\ (\mathfrak{F}_L^N \psi)(\gamma, \hat{x}) & := \frac{1}{\sqrt{L}} \sum_{z=0}^{L-1} e^{-\frac{2\pi i}{L} \gamma z} \psi(T_L^z \hat{x}) \end{aligned}$$

Lemma 3.2. *The Fourier transform is well-defined. Furthermore, it is unitary and its adjoint is given by*

$$(3.13) \quad \begin{aligned} (\mathfrak{F}_L^N)^* : \quad & \mathbb{S}_L^N \rightarrow \ell^2(\mathcal{V}_L^N) \\ ((\mathfrak{F}_L^N)^* \phi)(x) & := \frac{1}{\sqrt{L}} \sum_{\gamma=0}^{L-1} e^{\frac{2\pi i}{L} \gamma z} \phi(\gamma, \hat{x}), \end{aligned}$$

where $\hat{x} \in \hat{\mathcal{V}}_L^N$ and $z \in \{0, \dots, n_{\hat{x}} - 1\}$ are uniquely determined by $x = T_L^z \hat{x}$.

Proof. Firstly, let us prove that \mathfrak{F}_L^N is well-defined by showing that it indeed maps into \mathbb{S}_L^N . For $\psi \in \ell^2(\mathcal{V}_L^N)$, $\hat{x} \in \hat{\mathcal{V}}_L^N$ and $\gamma \notin (L/n_{\hat{x}})\mathbb{Z} \cap \{0, \dots, n_{\hat{x}} - 1\}$ consider

$$(3.14) \quad \begin{aligned} (\mathfrak{F}_L^N \psi)(\gamma, \hat{x}) & = \frac{1}{\sqrt{L}} \sum_{\zeta=0}^{n_{\hat{x}}-1} \sum_{k=0}^{L/n_{\hat{x}}-1} e^{-\frac{2\pi i}{L} (\zeta + kn_{\hat{x}}) \gamma} \psi(T_L^{\zeta + kn_{\hat{x}}} \hat{x}) \\ & = \frac{1}{\sqrt{L}} \sum_{\zeta=0}^{n_{\hat{x}}-1} e^{-\frac{2\pi i}{L} \zeta \gamma} \psi(T_L^{\zeta} \hat{x}) \left[\sum_{k=0}^{L/n_{\hat{x}}-1} e^{\frac{2\pi i}{L/n_{\hat{x}}} k \gamma} \right] = 0. \end{aligned}$$

In the first step of (3.14) we used that for every $z \in \mathcal{V}_L$ there exists unique $\zeta \in \{0, \dots, n_{\hat{x}} - 1\}$ and $k \in \{0, \dots, L/n_{\hat{x}} - 1\}$ such that $z = \zeta + kn_{\hat{x}}$. The last equality is due to the fact that the sum over all the $L/n_{\hat{x}}$ -th roots of unity is equal to zero.

Let us now show that the adjoint of \mathfrak{F}_L^N is indeed given by (3.13). To this end let $\psi \in \ell^2(\mathcal{V}_L^N)$ and $\phi \in \mathbb{S}_L^N$. Then

$$(3.15) \quad \begin{aligned} \langle \phi, \mathfrak{F}_L^N \psi \rangle_{\mathbb{S}_L^N} & = \frac{1}{\sqrt{L}} \sum_{\gamma=0}^{L-1} \sum_{\hat{x} \in \hat{\mathcal{V}}_L^N} \sum_{\zeta=0}^{n_{\hat{x}}-1} \sum_{k=0}^{L/n_{\hat{x}}-1} \frac{1}{L/n_{\hat{x}}} \overline{\phi(\gamma, \hat{x})} e^{-\frac{2\pi i}{L} \gamma (\zeta + kn_{\hat{x}})} \psi(T_L^{\zeta + kn_{\hat{x}}} \hat{x}) \\ & = \frac{1}{\sqrt{L}} \sum_{\hat{x} \in \hat{\mathcal{V}}_L^N} \sum_{\zeta=0}^{n_{\hat{x}}-1} \left[\sum_{\gamma=0}^{L-1} e^{\frac{2\pi i}{L} \gamma \zeta} \phi(\gamma, \hat{x}) \left[\frac{1}{L/n_{\hat{x}}} \sum_{k=0}^{L/n_{\hat{x}}-1} e^{\frac{2\pi i}{L/n_{\hat{x}}} \gamma k} \right] \right] \psi(T_L^{\zeta} \hat{x}) \end{aligned}$$

We note for any $\gamma \in (L/n_{\hat{x}})\mathbb{Z}$ that $\frac{1}{L/n_{\hat{x}}} \sum_{k=0}^{L/n_{\hat{x}}-1} e^{\frac{2\pi i}{L/n_{\hat{x}}}\gamma k} = 1$. Hence (3.15) is equal to

$$(3.16) \quad \sum_{x \in \mathcal{V}_L^N} \overline{((\mathfrak{F}_L^N)^* \phi)(x)} \psi(x) = \langle (\mathfrak{F}_L^N)^* \phi, \psi \rangle.$$

To show that indeed $(\mathfrak{F}_L^N)^* = (\mathfrak{F}_L^N)^{-1}$, take any $\psi \in \ell^2(\mathcal{V}_L^N)$ and $x \in \mathcal{V}_L^N$. There exist unique $\hat{x} \in \hat{\mathcal{V}}_L^N$ and $z \in \{0, \dots, n_{\hat{x}} - 1\}$ such that $x = T_L^z \hat{x}$. We obtain

$$(3.17) \quad ((\mathfrak{F}_L^N)^* \mathfrak{F}_L^N \psi)(T_L^z \hat{x}) = \frac{1}{L} \sum_{\gamma \in \mathcal{V}_L, \gamma \in (L/n_{\hat{x}})\mathbb{Z}} \sum_{\zeta=0}^{L-1} e^{\frac{2\pi i}{L}\gamma z} e^{-\frac{2\pi i}{L}\gamma \zeta} \psi(T_L^\zeta \hat{x}),$$

where we used that $\mathfrak{F}_L^N \psi \in \mathbb{S}_L^N$. By applying the coordinate shift $\sigma = \frac{\gamma}{L/n_{\hat{x}}}$ we see that (3.17) is equal to

$$(3.18) \quad \frac{1}{L} \sum_{\sigma=0}^{n_{\hat{x}}-1} \sum_{\xi=0}^{n_{\hat{x}}-1} \sum_{k=0}^{L/n_{\hat{x}}-1} e^{\frac{2\pi i}{n_{\hat{x}}}(z - (\xi + kn_{\hat{x}}))\sigma} \psi(T_L^\xi \hat{x}) = \frac{1}{n_{\hat{x}}} \sum_{\xi=0}^{n_{\hat{x}}-1} \left[\sum_{\sigma=0}^{n_{\hat{x}}-1} e^{\frac{2\pi i}{n_{\hat{x}}}(z - \xi)\sigma} \right] \psi(T_L^\xi \hat{x}) = \psi(T_L^z \hat{x}).$$

It can be shown analogously that $\mathfrak{F}_L^N (\mathfrak{F}_L^N)^* = 1$. \square

3.2. The Schrödinger operator formulation. Let again $L, N \in \mathbb{N}$ with $N < L$ be fixed. In [FS18] it was shown that the N -particle Hamiltonian is equivalent to a discrete Schrödinger operator acting on $\mathbb{H}_L^N \cong \ell^2(\mathcal{V}_L^N)$. More specifically,

$$(3.19) \quad H_L^N \cong -\frac{1}{2\Delta} A_L^N + W_L^N,$$

where A_L^N denotes the adjacency operator on \mathcal{G}_L^N

$$(3.20) \quad (A_L^N \psi)(x) := \sum_{y: x \sim y} \psi(y),$$

while W_L^N is a multiplication by the function $W : \mathcal{P}(\mathcal{V}_L) \rightarrow \mathbb{N}_0$ restricted to \mathbb{H}_L^N which counts the number of connected components of a configuration $x \in \mathcal{P}(\mathcal{V}_L)$

$$(3.21) \quad W(x) := \frac{1}{2} |\{ \{\alpha, \beta\} \in \mathcal{E}_L : \alpha \in x, \beta \notin x \}|.$$

Let us now consider the Fourier transform of the Hamiltonian $\hat{H}_L^N := \mathfrak{F}_L^N H_L^N (\mathfrak{F}_L^N)^*$, with \hat{A}_L^N and \hat{W}_L^N being defined analogously.

Lemma 3.3. *For any $\phi \in \mathbb{S}_L^N$, $\hat{x} \in \hat{\mathcal{V}}_L^N$ and $\gamma \in \mathcal{V}_L$ we have*

$$(3.22) \quad (\hat{H}_L^N \phi)(\gamma, \hat{x}) = -\frac{1}{2\Delta} \sum_{\hat{y} \in \hat{\mathcal{V}}_L^N} a_{L,\gamma}^N(\hat{x}, \hat{y}) \phi(\gamma, \hat{y}) + W(\hat{x}) \phi(\gamma, \hat{x}),$$

where the matrix elements of $a_{L,\gamma}^N$ are given by

$$(3.23) \quad a_{L,\gamma}^N(\hat{x}, \hat{y}) = \sum_{\substack{z \in \{0, \dots, n_{\hat{y}}-1\} \\ T_L^z \hat{y} \sim \hat{x}}} e^{\frac{2\pi i}{L}\gamma z}.$$

Proof. Firstly, for the potential W_L^N , observe that for any $\phi \in \mathbb{S}_L^N$ we get

$$(3.24) \quad (\mathfrak{F}_L^N W_L^N (\mathfrak{F}_L^N)^* \phi)(\gamma, \hat{x}) = W(\hat{x})\phi(\gamma, \hat{x}).$$

Let us now consider the adjacency operator A_L^N .

$$(3.25) \quad \begin{aligned} (\mathfrak{F}_L^N A_L^N (\mathfrak{F}_L^N)^* \phi)(\gamma, \hat{x}) &= \frac{1}{\sqrt{L}} \sum_{z=0}^{L-1} e^{-i\frac{2\pi}{L}\gamma z} (A_L^N (\mathfrak{F}_L^N)^* \phi)(T_L^z \hat{x}) \\ &= \frac{1}{\sqrt{L}} \sum_{z=0}^{L-1} \sum_{y: y \sim T_L^z \hat{x}} e^{-i\frac{2\pi}{L}\gamma z} ((\mathfrak{F}_L^N)^* \phi)(y) \end{aligned}$$

For any $y \in \mathcal{V}_L^N$ there exist unique $\hat{y} \in \hat{\mathcal{V}}_L^N$ and $\sigma \in \{0, \dots, n_{\hat{y}} - 1\}$ such that $y = \tilde{T}_L^\sigma \hat{y}$. Hence,

$$(3.26) \quad (\mathfrak{F}_L^N A_L^N (\mathfrak{F}_L^N)^* \phi)(\gamma, \hat{x}) = \frac{1}{L} \sum_{z=0}^{L-1} \sum_{\hat{y}} \sum_{\substack{\sigma \in \{0, \dots, n_{\hat{y}} - 1\} \\ T_L^{\sigma-z} \hat{y} \sim \hat{x}}} \sum_{\xi=0}^{L-1} e^{-i\frac{2\pi}{L}z(\gamma-\xi)} e^{i\frac{2\pi}{L}\xi(\sigma-z)} \phi(\xi, \hat{y}).$$

Since $\phi \in \mathbb{S}_L^N$, we have $\phi(\xi, \hat{y}) = 0$ for any $\hat{y} \in \hat{\mathcal{V}}_L^N$ and $\xi \notin (L/n_{\hat{y}})\{0, \dots, n_{\hat{y}} - 1\}$. We therefore consider only $\xi \in (L/n_{\hat{y}})\{0, \dots, n_{\hat{y}} - 1\}$. The second factor in (3.26) is subsequently given by

$$(3.27) \quad e^{i\frac{2\pi}{L}\xi(\sigma-z)} = e^{i\frac{2\pi}{n_{\hat{y}}}\frac{\xi}{L/n_{\hat{y}}}(\sigma-z) \bmod n_{\hat{y}}}.$$

By changing the summation index in (3.26) from σ to $\zeta := (\sigma - z) \bmod n_{\hat{y}}$ we conclude that (3.26) is equal to

$$(3.28) \quad \frac{1}{L} \sum_{\hat{y}} \sum_{\substack{\zeta \in \{0, \dots, n_{\hat{y}} - 1\} \\ T_L^\zeta \hat{y} \sim \hat{x}}} \sum_{\xi=0}^{L-1} \left[\sum_{z=0}^{L-1} e^{-i\frac{2\pi}{L}z(\gamma-\xi)} \right] e^{i\frac{2\pi}{L}\xi\zeta} \phi(\xi, \hat{y}) = \sum_{\hat{y}} \sum_{\substack{\zeta \in \{0, \dots, n_{\hat{y}} - 1\} \\ T_L^\zeta \hat{y} \sim \hat{x}}} e^{i\frac{2\pi}{L}\gamma\zeta} \phi(\gamma, \hat{y}).$$

This concludes the proof. \square

Remark 3.4. The operator \hat{A}_L^N is selfadjoint on \mathbb{S}_L^N , since it is unitarily equivalent to the selfadjoint operator A_L^N . This implies in particular that for all $\gamma \in \mathcal{V}_L$ and $\hat{x}, \hat{y} \in \hat{\mathcal{V}}_L^N$ we obtain

$$(3.29) \quad \frac{1}{L/n_{\hat{x}}} a_{L,\gamma}^N(\hat{x}, \hat{y}) = \frac{1}{L/n_{\hat{y}}} a_{L,\gamma}^N(\hat{y}, \hat{x}).$$

Now, we decompose \mathbb{S}_L^N into fiber spaces corresponding to the fiber index $\gamma \in \mathcal{V}_L$. We obtain

$$(3.30) \quad \mathbb{S}_L^N = \bigoplus_{\gamma=0}^{L-1} \mathbb{S}_{L,\gamma}^N,$$

where

$$(3.31) \quad \mathbb{S}_{L,\gamma}^N := \{\phi \in \mathbb{S}_L^N : \forall \hat{x} \in \hat{\mathcal{V}}_L^N, \forall \sigma \in \mathcal{V}_L, \sigma \neq \gamma, \text{ we have } \phi(\sigma, \hat{x}) = 0\}.$$

In Lemma 3.3 it is shown that for each $\gamma \in \mathcal{V}_L$ the subspace $\mathbb{S}_{L,\gamma}^N$ reduces \hat{H}_L^N . Consequently, we decompose

$$(3.32) \quad \hat{H}_L^N = \bigoplus_{\gamma=0}^{L-1} \hat{H}_{L,\gamma}^N,$$

where $\hat{H}_{L,\gamma}^N := \hat{H}_L^N \upharpoonright_{\mathbb{S}_{L,\gamma}^N}$. Analogously, we set $\hat{A}_{L,\gamma}^N := \hat{A}_L^N \upharpoonright_{\mathbb{S}_{L,\gamma}^N}$ and $\hat{W}_{L,\gamma}^N := \hat{W}_L^N \upharpoonright_{\mathbb{S}_{L,\gamma}^N}$ and thus obtain

$$(3.33) \quad \hat{H}_L^N = \bigoplus_{\gamma=0}^{L-1} \hat{H}_{L,\gamma}^N = \bigoplus_{\gamma=0}^{L-1} \left(-\frac{1}{2\Delta} \hat{A}_{L,\gamma}^N + \hat{W}_{L,\gamma}^N \right)$$

and

$$(3.34) \quad \sigma(H_L^N) = \sigma(\hat{H}_L^N) = \bigcup_{\gamma=0}^{L-1} \sigma(\hat{H}_{L,\gamma}^N).$$

3.3. Combes–Thomas estimate on fiber operators and proof of Theorem 2.3. Let again $L, N \in \mathbb{N}$ with $N < L$ be fixed. For the reader's convenience we will omit the indices N and L in the following proofs. However, every quantity may depend on N and L unless stated otherwise.

Lemma 3.5. *For all $\gamma \in \mathcal{V}_L$ the operator $\hat{A}_{L,\gamma}^N$ satisfies*

$$(3.35) \quad -2\hat{W}_{L,\gamma}^N \leq \hat{A}_{L,\gamma}^N \leq 2\hat{W}_{L,\gamma}^N.$$

Proof. It is sufficient to prove only the upper bound. The lower bound follows analogously by considering $-\hat{A}_\gamma$.

Let $\hat{x} \in \ell^2(\hat{\mathcal{V}})$. Equation (3.23) implies

$$(3.36) \quad \sum_{\hat{y} \in \hat{\mathcal{V}}} |a_\gamma(\hat{x}, \hat{y})| \leq \sum_{\hat{y} \in \hat{\mathcal{V}}} \sum_{\substack{z \in \{0, \dots, n_{\hat{y}}\}: \\ T^z \hat{y} \sim \hat{x}}} |e^{\frac{2\pi i}{L} \gamma z}| = \sum_{\substack{y \in \mathcal{V}: \\ y \sim \hat{x}}} 1.$$

According to (3.21) we get

$$(3.37) \quad \sum_{\hat{y} \in \hat{\mathcal{V}}} |a_\gamma(\hat{x}, \hat{y})| \leq 2W(\hat{x}).$$

Now, consider an arbitrary $\phi \in \mathbb{S}_\gamma$. Then

$$(3.38) \quad \begin{aligned} \langle \phi, \hat{A}_\gamma \phi \rangle_{\mathbb{S}} &= \sum_{\hat{x}, \hat{y} \in \hat{\mathcal{V}}} \overline{\phi(\gamma, \hat{x})} \frac{1}{L/n_{\hat{x}}} a_\gamma(\hat{x}, \hat{y}) \phi(\gamma, \hat{y}) \\ &\leq \left[\sum_{\hat{x}, \hat{y} \in \hat{\mathcal{V}}} |\phi(\gamma, \hat{x})|^2 \frac{1}{L/n_{\hat{x}}} |a_\gamma(\hat{x}, \hat{y})| \right]^{1/2} \left[\sum_{\hat{x}, \hat{y} \in \hat{\mathcal{V}}} |\phi(\gamma, \hat{y})|^2 \frac{1}{L/n_{\hat{x}}} |a_\gamma(\hat{x}, \hat{y})| \right]^{1/2}. \end{aligned}$$

By the identity (3.29), we obtain

$$(3.39) \quad \langle \phi, \hat{A}_\gamma \phi \rangle_{\mathbb{S}} \leq \sum_{\hat{x}, \hat{y} \in \hat{\mathcal{V}}} |\phi(\gamma, \hat{x})|^2 \frac{1}{L/n_{\hat{x}}} |a_\gamma(\hat{x}, \hat{y})|.$$

Hence by applying (3.37) we arrive at

$$(3.40) \quad \langle \phi, \hat{A}_\gamma \phi \rangle_{\mathbb{S}} \leq 2 \sum_{\hat{x}} |\phi(\gamma, \hat{x})|^2 \frac{1}{L/n_{\hat{x}}} W(\hat{x}) = 2 \langle \phi, \hat{W}_\gamma \phi \rangle_{\mathbb{S}}.$$

□

We are now able to prove a Combes-Thomas estimate on a fiber. The following is an adaptation of the proof of a similar result on the unbounded XXZ-chain [ARFS20, EKS18b, EKS18a].

Theorem 3.6. *For any $\gamma \in \{0, 1, \dots, L-1\}$ and any multiplication operator $\hat{Y}_{L,\gamma}^N : \mathbb{S}_{L,\gamma}^N \rightarrow \mathbb{S}_{L,\gamma}^N$, consider the Hamiltonian $\hat{O}_{L,\gamma}^N = -\frac{1}{2\Delta} \hat{A}_{L,\gamma}^N + \hat{W}_{L,\gamma}^N + \hat{Y}_{L,\gamma}^N$. Moreover, let $z \notin \sigma(\hat{O}_{L,\gamma}^N)$ be such that*

$$(3.41) \quad \|(\hat{W}_{L,\gamma}^N)^{1/2}(\hat{O}_{L,\gamma}^N - z)^{-1}(\hat{W}_{L,\gamma}^N)^{1/2}\| \leq \frac{1}{\kappa_L^N(z)} < \infty,$$

for some $\kappa_L^N(z) > 0$. Then for all $\mathcal{A}, \mathcal{B} \subset \hat{\mathcal{V}}_L^N$, we have

$$(3.42) \quad \|1_{\mathcal{A}}(\hat{O}_{L,\gamma}^N - z)^{-1}1_{\mathcal{B}}\| \leq \frac{2}{\kappa_L^N(z)} e^{-\eta_L^N(z)\hat{d}(\mathcal{A},\mathcal{B})},$$

where $\hat{d}(\mathcal{A}, \mathcal{B}) := \inf\{\hat{d}(\hat{x}, \hat{y}) : \hat{x} \in \mathcal{A}, \hat{y} \in \mathcal{B}\}$ for all $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\hat{\mathcal{V}}_L^N)$ and

$$(3.43) \quad \eta_L^N(z) = \ln \left(1 + \frac{\kappa_L^N(z)\Delta}{2} \right).$$

Proof. Firstly, observe that (3.35) implies that for any $\gamma \in \{0, \dots, L-1\}$

$$(3.44) \quad -2 \leq (\hat{W}_\gamma)^{-1/2} \hat{A}_\gamma (\hat{W}_\gamma)^{-1/2} \leq 2.$$

Now, for any $\mathcal{A} \subseteq \hat{\mathcal{V}}$, let $\rho_{\mathcal{A},\gamma} : \mathbb{S}_\gamma \rightarrow \mathbb{S}_\gamma$ be the operator of multiplication by $\hat{d}(\mathcal{A}, \cdot)$, i.e., $(\rho_{\mathcal{A},\gamma}\phi)(\gamma, \hat{x}) := \hat{d}(\mathcal{A}, \hat{x})\phi(\gamma, \hat{x})$ for any $\phi \in \mathbb{S}_\gamma$. For any $\eta > 0$ let us define

$$(3.45) \quad \hat{O}_{\eta,\gamma} := e^{-\eta\rho_{\mathcal{A},\gamma}} \hat{O}_\gamma e^{\eta\rho_{\mathcal{A},\gamma}}$$

and $\hat{B}_{\eta,\gamma} := \hat{O}_{\eta,\gamma} - \hat{O}_\gamma$. Observe that

$$(3.46) \quad \hat{B}_{\eta,\gamma} = -\frac{1}{2\Delta} (e^{-\eta\rho_{\mathcal{A},\gamma}} \hat{A}_\gamma e^{\eta\rho_{\mathcal{A},\gamma}} - \hat{A}_\gamma).$$

Now, for any $\phi \in \mathbb{S}_\gamma$, consider

$$(3.47) \quad \begin{aligned} & \|\hat{W}_\gamma^{-1/2} \hat{B}_{\eta,\gamma} \hat{W}_\gamma^{-1/2} \phi\|_{\mathbb{S}}^2 \\ &= \frac{1}{4\Delta^2} \sum_{\hat{x}} \frac{1}{L/n_{\hat{x}}} \left| \sum_{\hat{y}} \hat{W}^{-1/2}(\hat{x}) \hat{W}^{-1/2}(\hat{y}) (e^{\eta(\rho_{\mathcal{A},\gamma}(\hat{y}) - \rho_{\mathcal{A},\gamma}(\hat{x}))} - 1) a_\gamma(\hat{x}, \hat{y}) \phi(\gamma, \hat{y}) \right|^2 \end{aligned}$$

We note that for all $\gamma \in \{0, \dots, L-1\}$ and all $\hat{x}, \hat{y} \in \hat{\mathcal{V}}$ we have $|a_\gamma(\hat{x}, \hat{y})| \leq a_0(\hat{x}, \hat{y})$ which follows from (3.23). Furthermore we have $|e^{\eta(\hat{d}(\hat{x}, \mathcal{A}) - \hat{d}(\hat{y}, \mathcal{A}))} - 1| \leq (e^\eta - 1)$ for all $\hat{x}, \hat{y} \in \hat{\mathcal{V}}$ with $\hat{d}(\hat{x}, \hat{y}) = 1$. Hence (3.47) is bounded by

$$(3.48) \quad \begin{aligned} & \frac{1}{4\Delta^2} (e^\eta - 1)^2 \sum_{\hat{x}} \frac{1}{L/n_{\hat{x}}} \left[\sum_{\hat{y}} \hat{W}^{-1/2}(\hat{x}) \hat{W}^{-1/2}(\hat{y}) a_0(\hat{x}, \hat{y}) |\phi(\gamma, \hat{y})| \right]^2 \\ & \leq \frac{1}{4\Delta^2} (e^\eta - 1)^2 \|\hat{W}_0^{-1/2} \hat{A}_0 \hat{W}_0^{-1/2} \tilde{\phi}\|^2, \end{aligned}$$

where $\tilde{\phi} \in \mathbb{S}_0$ is defined by $\tilde{\phi}(\tilde{\gamma}, \hat{x}) := \delta_{\tilde{\gamma}, 0} |\phi(\gamma, \hat{x})|$ for all $\hat{x} \in \hat{\mathcal{V}}$ and $\tilde{\gamma} \in \{0, \dots, L-1\}$. The function $\tilde{\phi}$ is indeed an element of \mathbb{S}_0 , since for all $\hat{x} \in \hat{\mathcal{V}}$ we have $0 \in L/n_{\hat{x}}\{0, \dots, n_{\hat{x}} - 1\}$. Clearly, $\|\tilde{\phi}\| = \|\phi\|$. By using (3.44) we further estimate the left hand side of (3.47) and eventually get

$$(3.49) \quad \|\hat{W}_\gamma^{-1/2} \hat{B}_{n,\gamma} \hat{W}_\gamma^{-1/2}\| \leq \frac{1}{\Delta} (e^\eta - 1).$$

For $\eta \equiv \eta(z)$ as in (3.43) it now follows that

$$(3.50) \quad \begin{aligned} \|\hat{W}_\gamma^{-1/2} \hat{B}_{n,\gamma} (\hat{O}_\gamma - z)^{-1} \hat{W}_\gamma^{1/2}\| &= \|\hat{W}_\gamma^{-1/2} \hat{B}_{n,\gamma} \hat{W}_\gamma^{-1/2} \hat{W}_\gamma^{1/2} (\hat{O}_\gamma - z)^{-1} \hat{W}_\gamma^{1/2}\| \\ &\leq \frac{(e^\eta - 1)}{\Delta \kappa(z)} = \frac{1}{2}. \end{aligned}$$

Using the resolvent identity we get

$$(3.51) \quad \hat{W}_\gamma^{1/2} (\hat{O}_{n,\gamma} - z)^{-1} \hat{W}_\gamma^{1/2} (I + \hat{W}_\gamma^{-1/2} \hat{B}_{n,\gamma} (\hat{O}_\gamma - z)^{-1} \hat{W}_\gamma^{1/2}) = \hat{W}_\gamma^{1/2} (\hat{O}_\gamma - z)^{-1} \hat{W}_\gamma^{1/2}.$$

By further applying the elementary inequality $\|(I + C)^{-1}\| \leq (1 - \|C\|)^{-1}$ for any $C \in L(\mathbb{S}_\gamma)$, $\|C\| < 1$, we obtain from (3.41) and (3.50) that

$$(3.52) \quad \begin{aligned} & \|\hat{W}_\gamma^{1/2} (\hat{O}_{n,\gamma} - z)^{-1} \hat{W}_\gamma^{1/2}\| \\ & \leq \|\hat{W}_\gamma^{1/2} (\hat{O}_\gamma - z)^{-1} \hat{W}_\gamma^{1/2}\| \|(I + \hat{W}_\gamma^{-1/2} \hat{B}_{n,\gamma} (\hat{O}_\gamma - z)^{-1} \hat{W}_\gamma^{1/2})^{-1}\| \leq \frac{2}{\kappa(z)}. \end{aligned}$$

We conclude

$$(3.53) \quad \begin{aligned} & \|1_{\mathcal{A}} \hat{W}_\gamma^{1/2} (\hat{O}_\gamma - z)^{-1} \hat{W}_\gamma^{1/2} 1_{\mathcal{B}}\| = \|1_{\mathcal{A}} e^{\eta\rho_{\mathcal{A}}} \hat{W}_\gamma^{1/2} (\hat{O}_{n,\gamma} - z)^{-1} \hat{W}_\gamma^{1/2} e^{-\eta\rho_{\mathcal{A}}} 1_{\mathcal{B}}\| \\ & \leq \|\hat{W}_\gamma^{1/2} (\hat{O}_{n,\gamma} - z)^{-1} \hat{W}_\gamma^{1/2}\| \|e^{-\eta\rho_{\mathcal{A}}} 1_{\mathcal{B}}\| \leq \frac{2}{\kappa(z)} e^{-\eta\hat{d}(\mathcal{A}, \mathcal{B})}, \end{aligned}$$

which is the desired result. \square

We use the Combes-Thomas estimate to deduce pointwise upper bounds to eigenfunctions of the fiber operators. These estimates apply uniformly to all eigenstates corresponding to eigenvalues in a certain energy range. These energy ranges are associated with configurations of K or less clusters $\hat{\mathcal{V}}_{L,K}^N := \{\hat{x} \in \hat{\mathcal{V}}_L^N : W(\hat{x}) \leq K\}$ and are given by

$$(3.54) \quad I_{K,\delta} := \left[1 - \frac{1}{\Delta}, (K + 1 - \delta) \left(1 - \frac{1}{\Delta} \right) \right],$$

where $\delta \in (0, 1)$ and $K \in \mathbb{N}$.

For $K \in \mathbb{N}$ with $K \leq \|\hat{W}_L^N\|$ and $\gamma \in \mathcal{V}_L$, let $\hat{P}_{L,K,\gamma}^N : \mathbb{S}_{L,\gamma}^N \rightarrow \mathbb{S}_{L,\gamma}^N$ be the orthogonal projection given by

$$(3.55) \quad \hat{P}_{L,K,\gamma}^N := 1_{\leq K}(\hat{W}_{L,\gamma}^N).$$

Let further the whole projection $\hat{P}_{L,K}^N : \mathbb{S}_L^N \rightarrow \mathbb{S}_L^N$ be defined by

$$(3.56) \quad \hat{P}_{L,K}^N := \bigoplus_{\gamma \in \mathcal{V}_L} \hat{P}_{L,K,\gamma}^N.$$

Theorem 3.7. *Let $K \in \mathbb{N}$ with $K \leq \|\hat{W}_L^N\|$, $\delta \in (0, 1)$ and $\gamma \in \mathcal{V}_L$. For any $E \in \sigma(\hat{H}_{L,\gamma}^N) \cap I_{K,\delta}$ let $\phi_{L,\gamma}^N \equiv \phi_{L,\gamma}^N(\Delta, E) \in \mathbb{S}_{L,\gamma}^N$ be a corresponding eigenstate. Then, for any $\mathcal{A} \subseteq \hat{\mathcal{V}}_L^N$ we obtain*

$$(3.57) \quad \|1_{\mathcal{A}}\phi_{L,\gamma}^N\| \leq \frac{2(K+1)^2}{\delta} \cdot e^{-\mu_K \hat{d}_L^N(\mathcal{A}, \hat{\mathcal{V}}_{L,K}^N)} \|\hat{P}_{L,K,\gamma}^N \phi_{L,\gamma}^N\|,$$

where

$$(3.58) \quad \mu_K \equiv \mu_K(\delta, \Delta) := \ln \left(1 + \frac{\delta(\Delta-1)}{2(K+1)} \right).$$

Proof. Let us define the following multiplication operator $\hat{Y}_{K,\gamma} := (K+1)(1-1/\Delta)\hat{P}_{K,\gamma}$. Then

$$(3.59) \quad \begin{aligned} & \hat{W}_\gamma^{-1/2}(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E)\hat{W}_\gamma^{-1/2} \\ &= -\frac{1}{2\Delta}\hat{W}_\gamma^{-1/2}\hat{A}_\gamma\hat{W}_\gamma^{-1/2} + 1 + (K+1)\left(1 - \frac{1}{\Delta}\right)\hat{P}_{K,\gamma}\hat{W}_\gamma^{-1} - E\hat{W}_\gamma^{-1} \end{aligned}$$

By using the result of Lemma 3.5 as well as $E \in I_{K,\delta}$ we estimate

$$(3.60) \quad -\frac{1}{2\Delta}\hat{W}_\gamma^{-1/2}\hat{A}_\gamma\hat{W}_\gamma^{-1/2} + 1 \geq \left(1 - \frac{1}{\Delta}\right).$$

Moreover,

$$(3.61) \quad (K+1)\left(1 - \frac{1}{\Delta}\right)\hat{P}_{K,\gamma}\hat{W}_\gamma^{-1} - E\hat{P}_{K,\gamma}\hat{W}_\gamma^{-1} \geq \delta\left(1 - \frac{1}{\Delta}\right)\hat{P}_{K,\gamma}$$

and

$$(3.62) \quad -E(1 - \hat{P}_{K,\gamma})\hat{W}_\gamma^{-1} \geq -\frac{E}{K+1}(1 - \hat{P}_{K,\gamma}) \geq \left(1 - \frac{1}{\Delta}\right)\left(-1 + \frac{\delta}{K+1}\right)(1 - \hat{P}_{K,\gamma})$$

Hence (3.59) is estimated from below by

$$(3.63) \quad \hat{W}_\gamma^{-1/2}(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E)\hat{W}_\gamma^{-1/2} \geq \frac{\delta}{K+1}\left(1 - \frac{1}{\Delta}\right).$$

This implies that $E \notin \sigma(\hat{H}_\gamma + \hat{Y}_{K,\gamma})$ and in particular, we get

$$(3.64) \quad \|\hat{W}_\gamma^{1/2}(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E)^{-1}\hat{W}_\gamma^{1/2}\| \leq \frac{(K+1)\Delta}{\delta(\Delta-1)}.$$

By Theorem 3.6, this implies that

$$(3.65) \quad \|1_{\mathcal{A}}(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E)^{-1}1_{\mathcal{B}}\| \leq \frac{2\Delta(K+1)}{\delta(\Delta-1)} \cdot \left(1 + \frac{\delta(\Delta-1)}{2(K+1)}\right)^{-\hat{d}(\mathcal{A},\mathcal{B})}$$

for any $\mathcal{A}, \mathcal{B} \in \widehat{\mathcal{V}}_L^N$. Now, consider

$$(3.66) \quad \begin{aligned} & \|1_{\mathcal{A}}\phi_\gamma\|_{\mathbb{S}} \\ &= \|1_{\mathcal{A}}(\widehat{H}_\gamma + \widehat{Y}_{K,\gamma} - E)^{-1}(\widehat{H}_\gamma + \widehat{Y}_{K,\gamma} - E)\phi_\gamma\|_{\mathbb{S}}. \end{aligned}$$

We note that since ϕ_γ is an eigenfunction of \widehat{H}_γ we have $(\widehat{H}_\gamma - E)\phi_\gamma = 0$. Hence (3.66) is equal to

$$(3.67) \quad \begin{aligned} & (K+1)\left(1 - \frac{1}{\Delta}\right) \|1_{\mathcal{A}}(\widehat{H}_\gamma + \widehat{Y}_{K,\gamma} - E)^{-1}\widehat{P}_{K,\gamma}\phi_\gamma\|_{\mathbb{S}} \\ & \leq \frac{2(K+1)^2}{\delta} \cdot \left(1 + \frac{\delta(\Delta-1)}{2(K+1)}\right)^{-\widehat{d}(\mathcal{A}, \widehat{\mathcal{V}}_K)} \|\widehat{P}_{K,\gamma}\phi_\gamma\|_{\mathbb{S}}, \end{aligned}$$

which is the desired result. \square

Applying this result on fiber operators to the full N -particle Hamiltonian yields Theorem 2.3. In fact, our result can be applied to obtain estimates for eigenfunctions with eigenenergy in the K -cluster band $I_{K,\delta}$ for any K and not just $K = 1$.

Corollary 3.8. *Let $K \in \mathbb{N}$. For every $E \in I_{K,\delta} \cap \sigma(H_L^N)$ there exists an eigenstate $|\psi_L^N\rangle \equiv |\psi_L^N(\Delta, E)\rangle \in \mathbb{H}_L^N$ such that for all $x \in \mathcal{V}_L^N$ we obtain*

$$(3.68) \quad |\langle \delta_x^L, \psi_L^N \rangle| \leq \frac{2(K+1)^2}{\delta\sqrt{L}} \cdot e^{-\mu_K d_L^N(x, \mathcal{V}_{L,K}^N)},$$

where $\mu_K(\delta, \Delta)$ was defined in (3.58).

Proof. According to (3.34), for every $E \in I_{K,\delta} \cap \sigma(H)$ there exists a fiber index $\gamma \in \{0, \dots, L\}$ such that $E \in \sigma(\widehat{H}_\gamma)$. Let $\phi(\Delta, E) \in \mathbb{S}_\gamma$ be a normalized eigenvector of \widehat{H}_γ to E . Let $|\psi(\Delta, E)\rangle \cong (\mathfrak{F})^*\phi(\Delta, E)$ be the corresponding eigenstate of H to E .

Since $\phi \in \mathbb{S}_\gamma$ and by (3.13) we have

$$(3.69) \quad \psi(T^z \hat{x}) = \frac{1}{\sqrt{L}} e^{\frac{2\pi i}{L} z \gamma} \phi(\gamma, \hat{x}) \quad \text{for all } z \in \mathbb{Z} \text{ and } \hat{x} \in \widehat{\mathcal{V}}.$$

The result now follows from Theorem 3.7, since

$$(3.70) \quad |\langle \delta_x, \psi \rangle| = \frac{1}{\sqrt{L}} |\phi(\gamma, \hat{x})| = \frac{1}{\sqrt{L}} \|1_{\{\hat{x}\}}\phi\| \leq \frac{2(K+1)^2}{\delta\sqrt{L}} \cdot e^{-\mu_K \widehat{d}(\hat{x}, \widehat{\mathcal{V}}_K)}$$

and

$$(3.71) \quad \widehat{d}(\hat{x}, \widehat{\mathcal{V}}_K) = \min_\gamma d(T^\gamma \hat{x}, \widehat{\mathcal{V}}_K) = \min_\gamma d(x, T^\gamma \widehat{\mathcal{V}}_K) = d(x, \mathcal{V}_K),$$

where we used that $\bigcup_\gamma T^\gamma \widehat{\mathcal{V}}_K = \mathcal{V}_K$. \square

Proof of Theorem 2.3. Recall that $I_1 = [1 - \frac{1}{\Delta}, 2(1 - \frac{1}{\Delta})]$. According to Lemma A.1 we have $\sigma(H_L^N) \cap (1, 2(1 - 1/\Delta)) = \emptyset$, since $\Delta > 3$. Hence $I_1 \cap \sigma(H_L^N) = I_{1,1/2} \cap \sigma(H_L^N)$. The claim now follows immediately from Corollary 3.8 with $\delta = 1/2$, $K = 1$ and $\mu_1(\Delta) = \tilde{\mu}_1(\Delta, 1/2)$. \square

Remark 3.9. (i) In Lemma A.1, it was shown that for $\Delta > 2$ and all $\gamma \in \mathcal{V}_L$, each fiber operator $\hat{H}_{L,\gamma}^N$ has exactly one eigenvalue $E_\gamma \in [(1 - 1/\Delta), 2(1 - 1/\Delta))$. Let $\{|\varphi_{L,\gamma}^N(\Delta)\rangle\}_{\gamma \in \mathcal{V}_L} \subset \mathbb{H}_L^N$ be the orthonormal set of corresponding eigenstates, which is unique up to phase factors.

(ii) From Lemma A.2 it follows that $E_0 < E_\gamma$ for any $\gamma \neq 0$. This implies in particular that $|\varphi_{L,0}^N(\Delta)\rangle$ is the unique ground state of H_L^N .

3.4. The Ising-limit. Let again $N, L \in \mathbb{N}$ with $N < L$. The main idea of Theorem 2.1 is to view it as a perturbative result of the Ising limit “ $\Delta = \infty$ ”. From Corollary 3.8 it readily follows that for all $\gamma \in \mathcal{V}_L$ the density $\rho(\varphi_{L,\gamma}^N(\gamma, \Delta))$ converges weakly to

$$(3.72) \quad \rho_{L,\gamma}^N := \rho\left(\sum_{\zeta \in \mathcal{V}_L} \frac{1}{\sqrt{L}} e^{\frac{2\pi i}{L} \zeta \gamma} |\delta_{T_L^\zeta \hat{x}_0}^L\rangle\right) = \sum_{\xi, \zeta \in \mathcal{V}_L} \frac{1}{L} e^{\frac{2\pi i}{L} (\zeta - \xi) \gamma} |\delta_{T_L^\zeta \hat{x}_0}^L\rangle \langle \delta_{T_L^\xi \hat{x}_0}^L|,$$

where $\hat{x}_0 \in \hat{\mathcal{V}}_L^N \cap \mathcal{V}_{L,1}^N$ is the unique representative of all droplets in \mathcal{V}_L^N . As we will see in the following, the entanglement entropy of $\rho_{L,\gamma}^N$ satisfies the desired logarithmic correction to the area law.

In order to calculate the entanglement entropy of a given pure state $|\psi\rangle \in \mathbb{H}_L$ recall that it is necessary to determine its partial trace first.

Lemma 3.10. Let $\Lambda \subset \mathcal{V}_L$. For any state $|\psi\rangle \in \mathbb{H}_L^N$ and for all $n \in \{0, \dots, \min\{|\Lambda|, N\}\}$ there exists $\rho_\Lambda^n(\psi) \in L(\mathbb{H}_\Lambda^n)$ such that

$$(3.73) \quad \text{tr}_{\Lambda^c} \{\rho(\psi)\} = \bigoplus_{n=0}^{\min\{|\Lambda|, N\}} \rho_\Lambda^n(\psi).$$

Furthermore for any $n \in \{0, \dots, \min\{|\Lambda|, N\}\}$ and any $y, y' \in \mathbb{H}_L^n$ we have

$$(3.74) \quad \langle \delta_y^\Lambda, \rho_\Lambda^n(\psi) \delta_{y'}^\Lambda \rangle = \sum_{\substack{z \in \mathcal{P}(\Lambda^c), \\ |z|=N-n}} \langle \delta_{y \cup z}^L, \rho(\psi) \delta_{y' \cup z}^L \rangle.$$

Proof. This statement is shown by applying the definition of the partial trace, since $\mathcal{P}(\mathcal{V}_L) = \{y \cup z : y \in \mathcal{P}(\Lambda) \text{ and } z \in \mathcal{P}(\Lambda^c)\}$ and therefore by (2.9)

$$(3.75) \quad \text{tr}_{\Lambda^c} \{\rho(\psi)\} = \sum_{y, y' \in \mathcal{P}(\Lambda)} \left[\sum_{z \in \mathcal{P}(\Lambda^c)} \langle \delta_{y \cup z}^L, \rho(\psi) \delta_{y' \cup z}^L \rangle \right] |\delta_y^\Lambda\rangle \langle \delta_{y'}^\Lambda|.$$

□

Remark 3.11. In the following let $N < L/2$. Furthermore, for $\lambda_-, \lambda_+ \in \mathcal{V}_L$ with $\lambda_+ - \lambda_- \in (N, L/2)$, let $\Lambda \equiv \Lambda(\lambda_-, \lambda_+) = \{\lambda_-, \dots, \lambda_+\} \subset \mathcal{V}_L$.

(i) Let us consider the reduced density matrix of $\rho_{L,\gamma}^N$. By Lemma 3.10, for each $n \in \{0, \dots, N\}$ there exists $\rho_{L,\Lambda,\gamma}^n \in L(\mathbb{H}_\Lambda^n)$ such that

$$(3.76) \quad \rho_{L,\Lambda,\gamma} := \text{tr}_{\Lambda^c} \{\rho_{L,\gamma}^N\} = \bigoplus_{n=0}^N \rho_{L,\Lambda,\gamma}^n.$$

For $n \in \{1, \dots, N-1\}$ these operators are given by

$$(3.77) \quad \rho_{L,\Lambda,\gamma}^n := \frac{1}{L} (|\delta_{y_+^n}\rangle\langle\delta_{y_+^n}| + |\delta_{y_-^n}\rangle\langle\delta_{y_-^n}|),$$

where $y_{\pm}^n := \lambda_{\pm} \mp \{0, \dots, n-1\}$.

(ii) The entanglement entropy of $\rho_{L,\Lambda,\gamma}^n$ for any $n \in \{1, \dots, N-1\}$ is given by

$$(3.78) \quad \text{tr} \{s(\rho_{L,\Lambda,\gamma}^n)\} = \frac{2 \ln L}{L},$$

where $s : [0, 1] \rightarrow \mathbb{R}$, $t \mapsto -t \ln t$. Hence

$$(3.79) \quad \text{tr}_{\Lambda} \{s(\rho_{L,\Lambda,\gamma})\} \geq \sum_{n=1}^{N-1} \text{tr} \{s(\rho_{L,\Lambda,\gamma}^n)\} = 2 \frac{N-1}{L} \ln L.$$

4. ESTIMATING THE REDUCED STATE

4.1. Some technical preliminaries. For the entirety of this section, let $L, N \in \mathbb{N}$ with $N < L$ be fixed. Let $I(N) := \{1, \dots, N\}$. We first concern ourselves with the peculiar geometry of the graph \mathcal{V}_L^N and its graph norm.

For any two points $x, y \in \mathcal{V}_L^N$ let

$$(4.1) \quad \mathbf{u} := (u^{(0)}, \dots, u^{(k)}) \in \bigtimes_{l=0}^k \mathcal{V}_L^N = (\mathcal{V}_L^N)^{k+1}$$

be called a path from x to y of length $k \in \mathbb{N}_0$, if and only if $u^{(0)} = x$, $u^{(k)} = y$ and $d_L^N(u^{(l-1)}, u^{(l)}) = 1$ for all $l \in \{1, \dots, k\}$. If $k = d_L^N(x, y)$, then we call \mathbf{u} a shortest path from x to y .

Lemma 4.1. *Let $x, y \in \mathcal{V}_L^N$ and let \mathbf{u} be a shortest path from x to y of length $k = d_L^N(x, y)$. Let $k_0 \in \{0, \dots, k\}$,*

$$(4.2) \quad \mathbf{v} := (u^{(0)}, \dots, u^{(k_0)}) \quad \text{and} \quad \mathbf{w} := (u^{(k_0)}, \dots, u^{(k)}).$$

Then \mathbf{v} is a shortest path from x to $u^{(k_0)}$ and \mathbf{w} is a shortest path from $u^{(k_0)}$ to y . Moreover,

$$(4.3) \quad d_L^N(x, y) = d_L^N(x, u^{(k_0)}) + d_L^N(u^{(k_0)}, y).$$

Proof. The path \mathbf{v} is a path from x to $u^{(k_0)}$ of length k_0 and therefore $d_L^N(x, u^{(k_0)}) \leq k_0$. Analogously, \mathbf{w} is a path from $u^{(k_0)}$ to y of length $k - k_0$ with $d_L^N(u^{(k_0)}, y) \leq k - k_0$. Hence,

$$(4.4) \quad k = d_L^N(x, y) \leq d_L^N(x, u^{(k_0)}) + d_L^N(u^{(k_0)}, y) \leq k_0 + (k - k_0) = k.$$

This implies equality in (4.4) and consequently $d_L^N(x, u^{(k_0)}) = k_0$ and $d_L^N(u^{(k_0)}, y) = k - k_0$. \square

In what follows, it will be useful to consider each element $z \in \mathcal{V}_L^N$ as a set of N distinguishable, hard-core particles. We use the following convention to label each individual particle: For each $z \in \mathcal{V}_L^N$ there exists a unique $(z_1, \dots, z_N) \in (\mathcal{V}_L)^N$ with $z_1 < z_2 < \dots < z_N$ such that $z = \{z_j : j \in I(N)\}$. This enables us to track each individual particle along the path \mathbf{u} from x to y . To this end, we now construct a sequence $(\tilde{u}^{(l)})_{l \leq k} \subseteq (\mathcal{V}_L)^N$ with the property that

$u^{(l)} = \{\tilde{u}_j^{(l)} : j \in I(N)\}$ for all $0 \leq l \leq k$. Firstly, we set $\tilde{u}^{(0)} := (z_1, \dots, z_N)$. For all $1 \leq l \leq k$, we then define

$$(4.5) \quad \begin{cases} \tilde{u}_j^{(l)} := \tilde{u}_j^{(l-1)} & \text{for all } j \in I(N) \text{ with } \tilde{u}_j^{(l-1)} \in u^{(l)}, \\ \tilde{u}_j^{(l)} \in u^{(l)} \setminus u^{(l-1)} & \text{else.} \end{cases}$$

Note that $\tilde{u}^{(l)}$ is well-defined for all $1 \leq l \leq k$. The configuration $u^{(l)}$ is obtained by moving exactly one particle in $u^{(l-1)}$ to an unoccupied neighbouring site in \mathcal{V}_L . Hence there is always exactly one $j_0 \in I(N)$ such that $\tilde{u}_{j_0}^{(l+1)} \neq \tilde{u}_{j_0}^{(l)}$.

For the next lemma we require the following definition. For any $j \in I(N)$ we denote by

$$(4.6) \quad L_j^{\mathbf{u}} := \sum_{l=1}^k d_L(\tilde{u}_j^{(l-1)}, \tilde{u}_j^{(l)})$$

the distance that the j -th particle has traveled along the path \mathbf{u} .

Lemma 4.2. *For any $x, y \in \mathcal{V}_L^N$ the graph distance is given by*

$$(4.7) \quad d_L^N(x, y) = \min_{\sigma \in \mathfrak{S}_N^{cyc}} \sum_{j=1}^N d_L(x_j, y_{\sigma(j)}) = \min_{\sigma \in \mathfrak{S}_N} \sum_{j=1}^N d_L(x_j, y_{\sigma(j)}).$$

Here, $\mathfrak{S}_N^{(cyc)}$ denotes the set of (cyclic) permutations of the set $I(N)$.

Proof. Let \mathbf{u} be an arbitrary shortest path from x to y of length $k := d_L^N(x, y)$. For any $l \in \{0, \dots, k\}$ let $\tau_l \equiv \tau_l(\mathbf{u}) \in \mathfrak{S}_N$ be the uniquely defined permutation such that

$$(4.8) \quad 0 \leq \tilde{u}_{\tau_l(1)}^{(l)} < \tilde{u}_{\tau_l(2)}^{(l)} < \dots < \tilde{u}_{\tau_l(N)}^{(l)} \leq L - 1.$$

We now claim that $\tau_l \in \mathfrak{S}_N^{cyc}$ for all $0 \leq l \leq k$, which we prove by induction. For the base case $l = 0$, the statement is true since $\tau_0 = \text{id} \in \mathfrak{S}_N^{cyc}$. Now assume that for $l < k$, there exists a $\tau_l \in \mathfrak{S}_N^{cyc}$ such that (4.8) is satisfied. To show that the statement is then also true for $l + 1$, we distinguish three cases:

- First case: $\tilde{u}_{\tau_l(1)}^{(l)} = 0$ and $\tilde{u}_{\tau_l(1)}^{(l+1)} = L - 1$. This implies $\tilde{u}_{\tau_l(N)}^{(l+1)} < L - 1$. According to the induction hypothesis we have $0 = \tilde{u}_{\tau_l(1)}^{(l)} < \tilde{u}_{\tau_l(2)}^{(l)} < \dots < \tilde{u}_{\tau_l(N)}^{(l)} < L - 1$, therefore we conclude

$$(4.9) \quad 0 < \tilde{u}_{\tau_l(2)}^{(l+1)} < \tilde{u}_{\tau_l(3)}^{(l+1)} < \dots < \tilde{u}_{\tau_l(N)}^{(l+1)} < \tilde{u}_{\tau_l(1)}^{(l+1)} = L - 1.$$

The permutation $\tau_{l+1} = \tau_l \circ \sigma$ satisfies (4.8) at the position $l + 1$, where $\sigma \in \mathfrak{S}_N^{cyc}$ is the uniquely defined cyclic permutation with $\sigma(1) = 2$. Clearly, this implies that $\tau_{l+1} \in \mathfrak{S}_N^{cyc}$, since the composition of two cyclic permutations is cyclic.

- Second case: $\tilde{u}_{\tau_l(N)}^{(l)} = L - 1$ and $\tilde{u}_{\tau_l(N)}^{(l+1)} = 0$. By a completely analogous argument as for the first case, we get $\tau_{l+1} = \tau_l \circ \sigma^{-1} \in \mathfrak{S}_N^{cyc}$.
- The third case covers any other situation. Let $j_0 \leq N$ be the unique index for which $\{\tilde{u}_{\tau_l(j_0)}^{(l)}, \tilde{u}_{\tau_l(j_0)}^{(l+1)}\} \in \mathcal{E}_L$. Since the previous two cases have been excluded, observe that the

only two possibilities are $\tilde{u}_{\tau(j_0)}^{(l+1)} = \tilde{u}_{\tau(j_0)}^{(l)} \pm 1 \neq \tilde{u}_{\tau(j_0 \pm 1)}^{(l)}$. In either case, it is important to note that this implies

$$(4.10) \quad \tilde{u}_{\tau(j_0-1)}^{(l)} = \tilde{u}_{\tau(j_0-1)}^{(l+1)} < \tilde{u}_{\tau(j_0)}^{(l+1)} < \tilde{u}_{\tau(j_0+1)}^{(j+1)} = \tilde{u}_{\tau(j_0+1)}^{(l)}.$$

Hence, $\tau_{l+1} = \tau_l \in \mathfrak{S}_N^{cyc}$.

Since each step on the path moves exactly one particle to a neighboring position we have

$$(4.11) \quad k = \sum_{l=1}^k \sum_{j=1}^N d_L(\tilde{u}_j^{(l)}, \tilde{u}_j^{(l-1)}) = \sum_{j=1}^N L_j^{\mathbf{u}}.$$

Moreover, for any $j \in I(N)$ we have

$$(4.12) \quad d_L(x_j, y_{\tau_k^{-1}(j)}) = d_L(\tilde{u}_j^{(0)}, \tilde{u}_j^{(k)}) \leq \sum_{l=1}^k d_L(\tilde{u}_j^{(l-1)}, \tilde{u}_j^{(l)}) = L_j^{\mathbf{u}}.$$

Since $\tau_k^{-1} \in \mathfrak{S}_N^{cyc}$, this implies – together with (4.11):

$$(4.13) \quad \min_{\tau \in \mathfrak{S}_N^{cyc}} \sum_{j=1}^N d_L(x_j, y_{\tau(j)}) \leq \sum_{j=1}^N L_j^{\mathbf{u}} = k.$$

On the other hand, it was shown in [FS18, Appendix A] that $d_L^N(x, y) = \min_{\sigma \in \mathfrak{S}_N} \sum_{j=1}^N d_L(x_j, y_{\sigma(j)})$. Since $\mathfrak{S}_N^{cyc} \subseteq \mathfrak{S}_N$, we immediately obtain

$$(4.14) \quad k = d_L^N(x, y) = \min_{\sigma \in \mathfrak{S}_N} \sum_{j=1}^N d_L(x_j, y_{\sigma(j)}) \leq \min_{\tau \in \mathfrak{S}_N^{cyc}} \sum_{j=1}^N d_L(x_j, y_{\tau(j)}),$$

which concludes the proof. \square

Corollary 4.3. *Let $x, y \in \mathcal{V}_L^N$, $k := d_L^N(x, y)$, and let \mathbf{u} be a shortest path between x and y . Then*

$$(4.15) \quad L_j^{\mathbf{u}} = d_L(\tilde{u}_j^{(0)}, \tilde{u}_j^{(k)}) \leq L/2$$

and $\tilde{u}_j^{(k)} \in \{(x_j \pm L_j^{\mathbf{u}}) \bmod L\}$ for all $j \in I(N)$.

Proof. Equation (4.7) immediately implies (4.12) and (4.13). Due to the definition of d_L we have

$$(4.16) \quad L_j^{\mathbf{u}} = d_L(\tilde{u}_j^{(0)}, \tilde{u}_j^{(k)}) = d_L(x_j, y_{\tau_k^{-1}(j)}) \leq L/2.$$

This already yields $\tilde{u}_j^{(k)} \in \{(x_j \pm L_j^{\mathbf{u}}) \bmod L\}$ for all $j \in I(N)$. \square

Corollary 4.3 implies that along any shortest path \mathbf{u} from x to y of length k , each individual particle moves, if at all, either clockwise or counter-clockwise. We therefore define

$$(4.17) \quad I_{\pm}^{\mathbf{u}} := \{j \leq N : L_j^{\mathbf{u}} \neq 0 \text{ and } \exists l \in \{0, \dots, k\} \text{ with } \tilde{u}_j^{(l)} = (\tilde{u}_j^{(0)} \pm 1) \bmod L\} \quad \text{and}$$

$$(4.18) \quad I_0^{\mathbf{u}} := \{j \leq N : L_j^{\mathbf{u}} = 0\}.$$

Note that $I(N) = I_+^{\mathbf{u}} \cup I_-^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ for any shortest path \mathbf{u} , where the union is disjoint. The definition of $L_j^{\mathbf{u}}$ as well as (4.15) imply that

$$(4.19) \quad \{u_j^{(l)} : 0 \leq l \leq k\} = \{(x_j \pm \xi) \bmod L : 0 \leq \xi \leq L_j^{\mathbf{u}}\} \quad \text{for all } j \in I_{\pm}^{\mathbf{u}} \cup I_0^{\mathbf{u}}.$$

Let $\kappa_j^{(l)} \equiv \kappa_j^{(l)}(\mathbf{u}) \in \{0, \dots, L_j^{\mathbf{u}}\}$ for any $j \in I(N)$ and any $l \in \{0, \dots, k\}$ such that

$$(4.20) \quad \tilde{u}_j^{(l)} = (\tilde{u}_j^{(0)} \pm \kappa_j^{(l)}) \bmod L \quad \text{for all } j \in I_{\pm}^{\mathbf{u}} \cup I_0^{\mathbf{u}}.$$

It follows from the previous observations that the quantity $\kappa_j^{(l)}$ is well-defined.

The next lemma further establishes that we can indeed consider \mathbf{u} as a path of hard core particles.

Lemma 4.4 (hard-core particle property). *Let $e_0 := \{0, L-1\}$, $c \in \mathcal{V}_{L,1}^N$ with $e_0 \not\subseteq c$ and $x \in \mathcal{V}_L^N$. Moreover, let \mathbf{u} be a shortest path from c to x of length $k := d_L^N(c, x)$.*

Then,

(i) *If $i, j \in I_-^{\mathbf{u}}$ with $i < j$ then $\kappa_i^{(l)} \geq \kappa_j^{(l)}$ for all $l \in \{0, \dots, k\}$.*

(ii) *If $i, j \in I_+^{\mathbf{u}}$ with $i < j$ then $\kappa_i^{(l)} \leq \kappa_j^{(l)}$ for all $l \in \{0, \dots, k\}$.*

(iii) *We have the following inequalities:*

$$(4.21) \quad \sup I_-^{\mathbf{u}} \leq \inf(I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}) \quad \text{and} \quad \sup(I_-^{\mathbf{u}} \cup I_0^{\mathbf{u}}) \leq \inf I_+^{\mathbf{u}},$$

where we use the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

(iv) *If $1 \in I_-^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ and $N \in I_+^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ then*

$$(4.22) \quad L_1^{\mathbf{u}} + L_N^{\mathbf{u}} \leq L - N.$$

Proof. Since $c \in \mathcal{V}_{L,1}^N$ with $e_0 \not\subseteq c$ we have $c_j = c_1 + j - 1$ for all $j \in I(N)$. Let $J := \{0, \dots, k\}$.

(i) It suffices to show that if $j, j+1 \in I_-^{\mathbf{u}}$, then $\kappa_j^{(l)} \geq \kappa_{j+1}^{(l)}$ for all $l \in J$. Let us define $f : J \rightarrow \mathbb{Z}$, $l \mapsto \kappa_j^{(l)} - \kappa_{j+1}^{(l)}$. Suppose there exists a $k_0 \in J$ such that $f(k_0) < 0$. Due to the definition of the path we have $f(0) = 0$ and $|f(l) - f(l-1)| \in \{0, 1\}$ for all $l \in J \setminus \{0\}$. Now, suppose there exists a $k_0 \in J$ such that $f(k_0) < 0$. This would imply that there exist a $k_1 \in J$, $k_1 \leq k_0$, with $f(k_1) = -1$. Hence

$$(4.23) \quad u_j^{(k_1)} = (c_j - \kappa_j^{(k_1)}) \bmod L = (c_j - f(k_1) - \kappa_{j+1}^{(k_1)}) \bmod L = u_{j+1}^{(k_1)},$$

since $c_j - f(k_1) = c_{j+1}$, which is a contradiction.

(ii) Analogous to (i).

(iii) We only prove the first inequality in (4.21). The right-hand side follows analogously. If $I_-^{\mathbf{u}} = \emptyset$, then the result is trivial. So, from now on, we assume that $I_-^{\mathbf{u}} \neq \emptyset$. Suppose $\inf(I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}) < \sup I_-^{\mathbf{u}}$. This implies that there exists a $j \in I_-^{\mathbf{u}}$ such that $j-1 \in I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}$. Consider the function $g : J \rightarrow \mathbb{Z}$, $l \mapsto \kappa_j^{(l)} + \kappa_{j-1}^{(l)}$. We again have $g(0) = 0$, $g(k) \geq L_j^{\mathbf{u}} \geq 1$ and $|g(l) - g(l-1)| \in \{0, 1\}$ for all $l \in J \setminus \{0\}$. Hence, there exists $k_0 \in J$ with $g(k_0) = 1$. This implies

$$(4.24) \quad \tilde{u}_{j-1}^{(k_0)} = (c_{j-1} + \kappa_{j-1}^{(k_0)}) \bmod L = (c_{j-1} + g(k_0) - \kappa_j^{(k_0)}) \bmod L = \tilde{u}_j^{(k_0)},$$

which is a contradiction.

- (iv) Let us consider the function $h : J \rightarrow \mathbb{N}_0$, $l \mapsto \kappa_1^{(l)} + \kappa_N^{(l)}$. Suppose $L_1^{\mathbf{u}} + L_N^{\mathbf{u}} > L - N$. Since $h(k) = L_1^{\mathbf{u}} + L_N^{\mathbf{u}}$ and $h(l) - h(l-1) \in \{0, 1\}$ for all $l \in J \setminus \{0\}$, we conclude that there exists a $k_0 \in J$ with $h(k_0) = L - N + 1$. Hence, by using $c_N = c_1 + N - 1 = c_1 - h(k_0) + L$, we obtain

$$(4.25) \quad u_1^{(k_0)} = (c_1 - \kappa_1^{(k_0)}) \bmod L = (c_1 - h(k_0) + \kappa_N^{(k_0)}) \bmod L = u_N^{(k_0)},$$

which is a contradiction. □

To calculate the distance of two configurations in the graph norm on the ring, we rely on the results obtained for the line. The only difference between the graph of the ring \mathcal{G}_L and the graph of the line is, that the edge $e_0 = \{0, L - 1\}$ is not an element of the edge set of the line. We therefore are interested in paths, for which no particle crosses this particular edge.

Given a path \mathbf{u} of length k and an edge $e \in \mathcal{E}_L$, we say that \mathbf{u} does not cross e , if $u^{(l-1)} \Delta u^{(l)} \neq e$ for all $l \in \{1, \dots, k\}$.

Lemma 4.5 (cutting lemma). *Let $N < L/2$. Let $c \in \mathcal{V}_{L,1}^N$ and $x \in \mathcal{V}_L^N$. Then, there exists a shortest path \mathbf{u} from c to x and an edge $e \in \mathcal{E}_L$ with $\min_{m \in c} d_L(m, e) \geq \lfloor L/2 \rfloor - N$ such that \mathbf{u} does not cross e .*

Proof. Due to the translational symmetry of the system we may assume w.l.o.g. that $c = \{0, \dots, N - 1\}$. Let \mathbf{u} be a shortest path of length $k := d_L^N(c, x)$ connecting c and x . To prove the lemma we have to show that there exists an $e \in \mathcal{E}_L$ such that for all $l \in \{1, \dots, k\}$ we have $u^{(l-1)} \Delta u^{(l)} \in \mathcal{E}_L \setminus \{e\}$. However, it suffices to show that for all $j \in I(N)$ we have $e \not\subseteq \{u_j^{(l)} : 0 \leq l \leq k\}$. We distinguish between three cases.

- First case: $I_-^{\mathbf{u}} = \emptyset$, which means that no particle moves clockwise. Let $e := \{N - 1 + \lfloor L/2 \rfloor, N + \lfloor L/2 \rfloor\}$. For all $l \in \{0, \dots, k\}$ and for all $j \in I(N)$ we have

$$(4.26) \quad \{u_j^{(l)} : 0 \leq l \leq k\} = j - 1 + \{0, \dots, L_j^{\mathbf{u}}\} \subseteq \{0, \dots, N - 1 + \lfloor L/2 \rfloor\},$$

where we used Corollary 4.3 for the second inclusion. Since e is not a subset of the right-hand side of (4.26), this proves the claim for this case.

- Second case: $I_+^{\mathbf{u}} = \emptyset$ or all particle move either clockwise or not at all. Analogously to the first case, we see that the edge $e := \{\lfloor L/2 \rfloor - 1, \lfloor L/2 \rfloor\}$ satisfies the claim.
- Third case: both $I_-^{\mathbf{u}}$ and $I_+^{\mathbf{u}}$ are non-empty. Let us first note that due to Lemma 4.4 (iii) $\max I_-^{\mathbf{u}} \leq \min(I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}})$ and $\max(I_0^{\mathbf{u}} \cup I_-^{\mathbf{u}}) \leq \min I_+^{\mathbf{u}}$. This implies that $N \in I_+^{\mathbf{u}}$ and $1 \in I_-^{\mathbf{u}}$. According to Lemma 4.4 (iv) we have $L_1^{\mathbf{u}} + L_N^{\mathbf{u}} \leq L - N$ and hence

$$(4.27) \quad \tilde{u}_1^{(k)} = L - L_1^{\mathbf{u}} > N + L_N^{\mathbf{u}} - 1 = \tilde{u}_N^{(k)}.$$

Let $e \in \mathcal{E}_L$ with $e \subset \{\tilde{u}_N^{(k)}, \dots, \tilde{u}_1^{(k)}\}$. For any $j \in I_+^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ and any $l \in \{0, \dots, k\}$ we have

$$(4.28) \quad \tilde{u}_j^{(l)} \in j - 1 + \{0, \dots, \kappa_j^{(l)}\} \subseteq \{0, \dots, \tilde{u}_N^{(k)}\} =: \mathcal{A}_+,$$

where we used Lemma 4.4 (ii) for the second inclusion. Analogously, for any $j \in I_-^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ and any $l \in \{0, \dots, k\}$ we have

$$(4.29) \quad u_j^{(l)} \in \{0, \dots, N\} \cup \{L - L_1^{\mathbf{u}}, \dots, L - 1\} =: \mathcal{A}_-$$

Since $e \notin \mathcal{A}_\pm$, this shows that \mathbf{u} does not cross e .

Moreover, since $\tilde{u}_N^{(k)} \leq N - 1 + \lfloor L/2 \rfloor$ and $\tilde{u}_1^{(k)} \geq \lfloor L/2 \rfloor$ there exists at least one edge $e \in \mathcal{E}_L$ with $e \subseteq \{\lfloor L/2 \rfloor, \dots, N - 1 + \lfloor L/2 \rfloor\} \subseteq \{\tilde{u}_1^{(k)}, \dots, \tilde{u}_N^{(k)}\}$ such that e has a distance of at least $\lfloor L/2 \rfloor - N$ to any $m \in c$. \square

Thus, when trying to find a shortest path between an arbitrary configuration to a given droplet, the previous lemma tells us that we can always cut the ring open alongside an edge e and rather consider the graph of the thus obtained line. This is useful, since it enables us to draw from the results in [ARFS20]. Let us summarize these results in the following proposition:

Proposition 4.6. (i) Let $\mathcal{V} := \mathbb{Z}$ and $\mathcal{E} := \{\{j, j + 1\} : j \in \mathbb{Z}\}$. Moreover, for any $N \in \mathbb{N}$, let the graph of N -particle configurations be given by $\mathcal{G}^N := (\mathcal{V}^N, \mathcal{E}^N)$ where $\mathcal{V}^N := \{x \subseteq \mathcal{V} : |x| = N\}$ and $\mathcal{E}^N := \{\{x, y\} \subseteq \mathcal{V}^N : x \Delta y \in \mathcal{E}\}$. Then, for any $x = \{x_1 < x_2 < \dots < x_N\}, y = \{y_1 < y_2 < \dots < y_N\} \in \mathcal{V}^N$, the graph distance d^N is explicitly given by:

$$(4.30) \quad d^N(x, y) = \sum_{j=1}^N |x_j - y_j|.$$

(ii) For any $m \in \mathcal{V}$, and $N \in \mathbb{N}$ let us define the droplet centered around m by

$$(4.31) \quad c_m^N := m + \left\{ -\left\lfloor \frac{N-1}{2} \right\rfloor, \dots, \left\lceil \frac{N-1}{2} \right\rceil \right\},$$

and let $\mathcal{V}_1^N := \{c_m^N : m \in \mathcal{V}\}$ denote the set of all droplets. We are interested in the droplets closest to a given $x \subseteq \mathcal{V}$, $|x| > 0$. Let us define

$$(4.32) \quad \mathcal{W}(x) := \{m \in \mathcal{V} : d^N(x, c_m^{|x|}) = d^N(x, \mathcal{V}_1^{|x|})\}.$$

According to [ARFS20, Lemma A.1], we have

$$(4.33) \quad \mathcal{W}(x) = \begin{cases} \{x_\kappa\} & \text{if } N \text{ is odd,} \\ \{x_\kappa, \dots, x_{\kappa+1} - 1\} & \text{if } N \text{ is even,} \end{cases}$$

for all $x \in \mathcal{V}^N$, where $\kappa := \lfloor (N + 1)/2 \rfloor$.

Let us now introduce a notation for a droplet in \mathcal{V}_L^N centered around a site $m \in \mathcal{V}_L$

$$(4.34) \quad c_{L,m}^N := \{j \bmod L : j \in c_m^N\}.$$

Analogously to Remark 4.6 (ii), for any $x \subseteq \mathcal{V}_L$, $|x| > 0$, we define the set of centers of droplets that are closest to this configuration

$$(4.35) \quad \mathcal{W}_L(x) := \{m \in \mathcal{V}_L : d_L^{|x|}(x, c_{L,m}^{|x|}) = d_L^{|x|}(x, \mathcal{V}_{L,1}^{|x|})\}.$$

Lemma 4.7. Let $N < L/2$ and $x \in \mathcal{V}_L^N$. Then,

$$(4.36) \quad \mathcal{W}_L(x) \cap x \neq \emptyset.$$

Furthermore, let $m \in \mathcal{W}_L(x) \cap x$. If there exists a shortest path \mathbf{u} from $c_{L,m}^N$ to x that does not cross $e_0 := \{0, L - 1\}$ and $e_0 \notin c_{L,m}^N$, then $m = x_\kappa$, where $\kappa := \lfloor (N + 1)/2 \rfloor$.

Proof. Let $\nu \in \mathcal{W}_L(x)$ and define $c := c_{L,\nu}^N$. According to Lemma 4.5, there exists a shortest path \mathbf{u} from c to x and an edge $e \in \mathcal{E}_L$ with $\min_{j \in c} d_L(j, e) > \lfloor L/2 \rfloor - L/2 \geq 0$ such that \mathbf{u} does not cross e . This implies $e \not\subseteq c$. Let us assume w.l.o.g. that $e = e_0$. In any other case we can choose a suitable translation by $\gamma \in \mathbb{Z}$ such that $T_L^\gamma e = e_0$ and consider $T_L^\gamma x$, $T_L^\gamma c$ and the path $\mathbf{u}_\gamma := (T_L^\gamma u^{(0)}, \dots, T_L^\gamma u^{(k)})$ instead.

Since \mathbf{u} does not cross e_0 , it can also be viewed as a path on the infinite line \mathcal{G}^N . Therefore, according to Proposition 4.6 we have $d_L^N(x, c) = d^N(x, c)$. Let us consider the droplet $c' := c_{x_\kappa}^N = c_{L,x_\kappa}^N$. Then, Proposition 4.6 (i) states that $\{x_\kappa\} = \mathcal{W}(x) \cap x$. Hence,

$$(4.37) \quad d_L^N(x, \mathcal{V}_{L,1}^N) = d^N(x, c) \geq d^N(x, c') = \sum_{j=1}^N |c'_j - x_j| \geq \sum_{j=1}^N d_L(c'_j, x_j) \geq d_L^N(c', x),$$

where we applied Lemma 4.2 to achieve the final estimate. Since $c' \in \mathcal{V}_{L,1}^N$ we have equality in (4.37) and therefore $x_\kappa \in \mathcal{W}_L(x) \cap x$. Moreover, if $\nu \in x$ it follows immediately that $\nu = x_\kappa$, since $d^N(x, c) = d^N(x, c')$ and the set $\mathcal{W}(x) \cap x$ contains only one element. \square

By making further assumptions on the configuration x , we are able to determine $\mathcal{W}_L(x) \cap x$ precisely. For now, let us only consider configurations that are contained in a sufficiently small sector of the ring.

Here, a sector of size $\theta \in (0, 1/2)$ around a site $m \in \mathcal{V}_L^N$ is given by

$$(4.38) \quad \mathcal{S}_{L,m}(\theta) := \{k \in \mathcal{V}_L : d_L(k, m) < \theta L\}.$$

Lemma 4.8. *Let $M := \lfloor (L-1)/2 \rfloor$, $\beta \in (0, 1/4)$, $N < \beta L$ and $x \in \mathcal{V}^N(\mathcal{S}_{L,M}(1/4 - \beta/2))$. Then,*

$$(4.39) \quad \{x_\kappa\} = \mathcal{W}_L(x) \cap x,$$

where $\kappa := \lfloor (N+1)/2 \rfloor$.

Furthermore, no shortest path \mathbf{u} from c_{L,x_κ}^N to x crosses $e_0 := \{0, L-1\}$. Moreover, we have $\{1, \dots, \kappa\} \subseteq I_0^{\mathbf{u}} \cup I_-^{\mathbf{u}}$ and $\{\kappa, \dots, N\} \subseteq I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}$.

Proof. Let $\nu \in \mathcal{S}_{L,M}(1/4 - \beta/2)$. We claim that no shortest path \mathbf{u} from $c_{L,\nu}^N$ to x crosses the edge e_0 . Let $c := c_{L,\nu}^N$ and $k := d_L^N(x, c_{L,\nu}^N)$. As in the proof of Lemma 4.5, it is sufficient to show that for any $j \in I(N)$ we have $e_0 \not\subseteq \{\tilde{u}_j^{(l)} : 0 \leq l \leq k\}$.

Suppose there exists a $j \in I(N)$ such that $e_0 \subseteq \mathcal{Z}_j := \{\tilde{u}_j^{(l)} : 0 \leq l \leq k\}$, which readily implies $j \notin I_0^{\mathbf{u}}$. But according to (4.19), we know that for any $j \in I_\pm^{\mathbf{u}}$, we have

$$(4.40) \quad |\mathcal{Z}_j| = |\{c_j, \dots, (c_j \pm L_j^{\mathbf{u}}) \bmod L\}| = L_j^{\mathbf{u}} + 1 \leq \lfloor L/2 \rfloor + 1,$$

where we used Corollary 4.3 to estimate $L_j^{\mathbf{u}}$. W.l.o.g., let us assume $j \in I_-^{\mathbf{u}}$. Since we assumed $e_0 \subseteq \mathcal{Z}_j$, we deduce from (4.19) that

$$(4.41) \quad \{c_j, (c_j - L_j^{\mathbf{u}}) \bmod L\} \cup (\mathcal{S}_{L,M}(1/4))^c \subseteq \mathcal{Z}_j.$$

Notice that both $c_j \in \mathcal{S}_{L,M}(1/4)$ and $c_j - L_j^{\mathbf{u}} \in x \subseteq \mathcal{S}_{L,M}(1/4)$. Hence,

$$(4.42) \quad |\mathcal{Z}_j| \geq 2 + |(\mathcal{S}_{L,M}(1/4))^c| \geq 2 + \lfloor L/2 \rfloor,$$

which is a contradiction.

Since no shortest path \mathbf{u} from $c_{L,\nu}^N$ to x crosses e_0 , observe that \mathbf{u} can also be viewed as a path on the graph induced by N particles on the infinite line \mathcal{G}^N . Hence,

$$(4.43) \quad d_L^N(x, c_{L,x_j}^N) = d^N(x, c_{L,x_j}^N)$$

and $\mathcal{W}_L(x) \cap x = \mathcal{W}(x) \cap x = \{x_\kappa\}$ according to Proposition 4.6 (ii).

Let us now consider a shortest path \mathbf{v} from $c' := c_{L,x_\kappa}^N$ to x . Lemma 4.4 implies that $\tilde{v}_j^{(k)} = x_j$ and $L_j^\mathbf{v} = d_L(x_j, c'_j)$ for all $j \in I(N)$. Hence $L_\kappa^\mathbf{v} = 0$ and therefore $\kappa \in I_0^\mathbf{v}$. The rest of the statement follows from Lemma 4.4 (iii). \square

Lemma 4.9. *Let $N < L/2$ and $\mu \geq \ln 2$. Then,*

$$(4.44) \quad \sum_{x \in \mathcal{V}_L^N} e^{-\mu d_L^N(x, \mathcal{V}_{L,1}^N)} \leq L(1 + 2^9 e^{-\mu}).$$

Proof. For any $m \in \mathcal{V}_L$ let

$$(4.45) \quad \mathcal{B}_{L,m}^N := \{x \in \mathcal{V}_L^N : m \in \mathcal{W}_L(x) \cap x\}.$$

By Lemma 4.7 we have

$$(4.46) \quad \bigcup_{m \in \mathcal{V}_L} \mathcal{B}_{L,m}^N = \mathcal{V}_L^N.$$

Let $x \in \mathcal{B}_{L,m}^N$. According to Lemma 4.5 there exists an edge e with $\max_{j \in e} d_L(j, e) > 0$ and a shortest path \mathbf{u} from $c := c_{L,m}^N$ to x that does not cross e . Pick $\gamma \in \mathbb{Z}$ such that $T_L^\gamma e = e_0 := \{0, L-1\}$. Let $x' := T_L^\gamma x$, $c' := T_L^\gamma c$ and $\mathbf{v} := \{T_L^\gamma u^{(0)}, \dots, T_L^\gamma u^{(k)}\}$, where $k := d_L^N(x, c)$. Let us define $\chi_- \equiv \chi_-(x) \in \mathbb{N}_0^{\kappa-1}$, $\chi_+ \equiv \chi_+(x) \in \mathbb{N}_0^{N-\kappa}$ by

$$(4.47) \quad \chi_{-,j} := d_L(x'_{\kappa-j}, c'_{\kappa-j}) \quad \text{for } j \leq \kappa - 1,$$

$$(4.48) \quad \chi_{+,j} := d_L(x'_{\kappa+j}, c'_{\kappa+j}) \quad \text{for } j \leq N - \kappa.$$

We want to show that $\chi_+ \in \mathcal{X}^{N-\kappa}$ and $\chi_- \in \mathcal{X}^{\kappa-1}$. Since \mathbf{v} does not cross e_0 we have $v_j^{(k)} = x'_j$ for all $j \in I(N)$. By Lemma 4.7 we have $x'_\kappa = c'_\kappa$. Hence $\kappa \in I_0^\mathbf{v}$, since $L_\kappa^\mathbf{v} = d_L(x'_\kappa, c'_\kappa) = 0$. As a consequence of Lemma 4.4 (iii) we conclude $\{1, \dots, \kappa\} \subseteq I_0^\mathbf{v} \cup I_-^\mathbf{v}$ and $\{\kappa, \dots, N\} \subseteq I_0^\mathbf{v} \cup I_+^\mathbf{v}$.

By Lemma 4.4 (i), we get

$$(4.49) \quad \chi_{-,j} = L_{\kappa-j}^\mathbf{v} \leq L_{\kappa-j-1}^\mathbf{v} = \chi_{-,j+1} \quad \text{for all } 1 \leq j < \kappa - 1,$$

and therefore $\chi_- \in \mathcal{X}^{\kappa-1}$. Analogously $\chi_+ \in \mathcal{X}^{N-\kappa}$. Furthermore

$$(4.50) \quad d_L^N(x, c) = d_L^N(x', c') = \sum_{j=1}^N L_j^\mathbf{v} = |\chi_-|_1 + |\chi_+|_1.$$

Note that each pair $(\chi_-, \chi_+) \in \mathcal{X}^{\kappa-1} \times \mathcal{X}^{N-\kappa}$ corresponds to one $x \in \mathcal{B}_{L,m}^N$ only, since

$$(4.51) \quad x = \{(m + j + \chi_{+,j}) \bmod L : j \leq N - \kappa\} \cup \{m\} \cup \{(m - j - \chi_{-,j}) \bmod L : j \leq \kappa - 1\}.$$

$$(4.52) \quad \sum_{x \in \mathcal{B}_{L,m}^N} e^{-\mu d_L^N(x, c)} \leq \sum_{\chi_- \in \mathcal{X}^{\kappa-1}} \sum_{\chi_+ \in \mathcal{X}^{N-\kappa}} e^{-\mu(|\chi_-|_1 + |\chi_+|_1)} \leq (1 + 30e^{-\mu})^2 \leq 1 + 2^9 e^{-\mu}$$

where we applied Lemma A.3 and $\mu \geq \ln 2$. Together with (4.46), this concludes the proof. \square

4.2. Moving particles to the boundary. Let us first introduce some notation. In the following, let $\epsilon \in (0, 1/16)$ and $\theta \in (\epsilon, 1/16)$ be fixed. Let $L \in \mathbb{N}$ and $N \equiv N(\epsilon, L) := \lfloor \epsilon L \rfloor$. Let

$$(4.53) \quad \Lambda_L \equiv \Lambda_L(\theta) := \mathcal{S}_{L,M}(\theta) = \{\lambda_-, \dots, \lambda_+\} \subseteq \mathcal{V}_L,$$

where $M := \lfloor (L-1)/2 \rfloor$, $\lambda_- := \min \Lambda_L$ and $\lambda_+ := \max \Lambda_L$.

We are ultimately interested in taking the partial trace over the Fock-space associated with Λ_L^c . This entails summing up all contributions of configurations in $\mathcal{P}(\Lambda_L^c)$, as we have seen in Lemma 3.10. Thus, let us introduce some additional notation to classify configurations in $\mathcal{P}(\Gamma^c)$, for any connected subset $\Gamma = \{\gamma_-, \dots, \gamma_+\} \subseteq \mathcal{S}_{L,M}(1/4)$ with $\gamma_- < \gamma_+$.

For $x \subseteq \mathcal{V}_L$ let $x^{in} \equiv x^{in}(\Gamma) := x \cap \Gamma$ and $x^{out} \equiv x^{out}(\Gamma) := x \setminus \Gamma$. If $|x^{in}|, |x^{out}| > 0$, let $\ell, r \in \mathbb{N}_0$ be arbitrary such that $\ell + r = |x^{out}|$. The idea is to further split x^{out} into a configuration of ℓ particles which are thought of as being close to γ_- and a configuration of r particles close to γ_+ . Note that for every such ℓ and r , there exists a unique permutation $\sigma_{\ell,r} \equiv \sigma_{\ell,r}(x, \Gamma) \in \mathfrak{S}_N^{cyc}$ with the property that

$$(4.54) \quad x^{in} = \{x_{\sigma_{\ell,r}(j)} : \ell < j \leq N - r\}.$$

We then define

$$(4.55) \quad x_-^{\ell, out} \equiv x_-^{\ell, out}(x, \Gamma) := \{x_{\sigma_{\ell,r}(j)} : j \leq \ell\} \quad \text{and} \quad x_+^{r, out} \equiv x_+^{r, out}(x, \Gamma) := \{x_{\sigma_{\ell,r}(j)} : j > N - r\}.$$

We have $x^{out} = x_-^{\ell, out} \cup x_+^{r, out}$.

For any $j \in I(N)$ let

$$(4.56) \quad a_{\pm, j}(\Gamma) := \gamma_{\pm} \pm j.$$

Then there exist unique $\chi_-^\ell \equiv \chi_-^\ell(x, \Gamma) \in \mathcal{X}^\ell$ and $\chi_+^r \equiv \chi_+^r(x, \Gamma) \in \mathcal{X}^r$ with $\chi_{-, \ell}^\ell, \chi_{+, r}^r < L$ such that

$$(4.57) \quad x_{\sigma_{\ell,r}(\zeta)} = (a_{-, \ell - \zeta + 1} - \chi_{-, \ell - \zeta + 1}^\ell) \bmod L \quad \text{for all } 1 \leq \zeta \leq \ell,$$

$$(4.58) \quad x_{\sigma_{\ell,r}(N+1-\xi)} = (a_{+, r - \xi + 1} + \chi_{+, r - \xi + 1}^r) \bmod L \quad \text{for all } 1 \leq \xi \leq r.$$

Finally let us denote the special configuration in $\mathcal{V}_L^{\ell+r}(\Gamma^c)$ that consists of two clusters of size ℓ and r at the boundaries of Γ^c by

$$(4.59) \quad b_{\ell,r} \equiv b_{\ell,r}(\Gamma) := \{a_{-, j} : j \leq \ell\} \cup \{a_{+, j} : j \leq r\}.$$

Lemma 4.10. *Let $\Gamma = \{\gamma_-, \dots, \gamma_+\} \subseteq \mathcal{S}_{L,M}(\theta + 2\epsilon)$. Moreover, let $n \in \mathbb{N}$ with $n < N$ and $x \in \mathcal{V}_L^N$ with $|x^{in}(\Gamma)| = n$. Let $r, \ell \in \mathbb{N}_0$ such that $\ell + r = N - n$ and $c \in \mathcal{V}_{L,1}^N$ with $\{c_j : \ell < j \leq N - r\} \subseteq \Gamma$. Assume that \mathbf{u} is a shortest path from c to x of length $k := d_L^N(c, x)$ such that*

$$(4.60) \quad \tilde{u}_j^{(k)} = x_{\sigma_{\ell,r}(j)} \quad \text{for all } j \in I(N)$$

and in addition that

$$(4.61) \quad \{1, \dots, \ell\} \subseteq I_0^{\mathbf{u}} \cup I_-^{\mathbf{u}} \quad \text{and} \quad c_\zeta \geq a_{-, \ell - \zeta + 1} \quad \text{for } \zeta \leq \ell$$

$$(4.62) \quad \{N - r + 1, \dots, N\} \subseteq I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}} \quad \text{and} \quad c_{N+1-\xi} \leq a_{+, r - \xi + 1} \quad \text{for } \xi \leq r.$$

Then there exists a shortest path \mathbf{v} from c to x with

$$(4.63) \quad v^{(k_0)} = x^{in} \cup b_{\ell,r}(\Gamma) \subseteq \mathcal{S}_{L,M}(1/4 - \epsilon/2)$$

for some $k_0 \in \{0, \dots, k\}$. Furthermore,

$$(4.64) \quad k - k_0 = \sum_{\zeta=1}^{\ell} \chi_{-, \zeta}^{\ell}(x, \Gamma) + \sum_{\xi=1}^r \chi_{+, \xi}^r(x, \Gamma).$$

Proof. For all $j \in I(N)$ let

$$(4.65) \quad \mathcal{Z}_j := \{\tilde{u}_j^{(l)} : 0 \leq l \leq k\}.$$

We claim that for all $j \in \{\ell + 1, \dots, N - r\} \cap (I_0^{\mathbf{u}} \cup I_{\pm}^{\mathbf{u}})$ one has

$$(4.66) \quad \mathcal{Z}_j = \{c_j, \dots, (c_j \pm L_j^{\mathbf{u}}) \bmod L\} \subseteq \Gamma.$$

Suppose this is not true. This would imply that there exists a $j \in I(N)$ with $\Gamma^c \cup \{\tilde{u}_j^{(0)}, \tilde{u}_j^{(k)}\} \subseteq \mathcal{Z}_j$. By Assumption (4.60) we have $\tilde{u}_j^{(k)} = x_{\sigma_{\ell, r}(j)} \in \Gamma$ and $\tilde{u}_j^{(0)} = c_j \in \Gamma$ for all $\ell < j \leq N - r$. Hence,

$$(4.67) \quad |\mathcal{Z}_j| = |\Gamma^c| + 2 > L/2 + 1 \geq L_j^{\mathbf{u}} + 1 = |\mathcal{Z}_j|,$$

where we used $|\Gamma^c| \geq |(\mathcal{S}_{L, M}(1/4))^c| > L/2 - 1$. This is a contradiction.

We claim that for all $\zeta \leq \ell$ and for all $\xi \leq r$, it holds that

$$(4.68) \quad a_{-, \ell - \zeta + 1} \equiv a_{-, \ell - \zeta + 1}(\Gamma) \in \mathcal{Z}_{\zeta} = \{c_{\zeta}, \dots, (c_{\zeta} - L_{\zeta}^{\mathbf{u}}) \bmod L\},$$

$$(4.69) \quad a_{-, r - \xi + 1} \equiv a_{-, r - \xi + 1}(\Gamma) \in \mathcal{Z}_{N+1-\xi} = \{c_{N+1-\xi}, \dots, (c_{N+1-\xi} + L_{N+1-\xi}^{\mathbf{u}}) \bmod L\}.$$

We present a proof for (4.68), since (4.69) follows analogously. For $\zeta = \ell$ we have $\tilde{u}_{\ell}^{(k)} \in x^{\text{out}}(\Gamma) \subseteq \Gamma^c$ according to Assumption (4.60). Together with Assumption (4.61) this implies $\gamma_- - 1 = a_{-, 1} \in \mathcal{Z}_{\ell}$. The claim now follows from an inductive argument as well as from Lemma 4.4 (i).

Let us define

$$(4.70) \quad K_{-, \ell} := \begin{cases} d_L(c_{\ell}, a_{-, 1}) & \text{for } \ell > 0, \\ 0 & \text{for } \ell = 0, \end{cases} \quad \text{and} \quad K_{+, r} := \begin{cases} d_L(c_{N+1-r}, a_{+, 1}) & \text{for } r > 0, \\ 0 & \text{for } r = 0. \end{cases}$$

Then, (4.68) and (4.69) imply that for all $\zeta \leq \ell$ and for all $\xi \leq r$

$$(4.71) \quad K_{-, \ell} = d_L(c_{\zeta}, a_{-, \ell - \zeta + 1}) \leq L_{\zeta}^{\mathbf{u}} \quad \text{and} \quad K_{+, r} = d_L(c_{N+1-\xi}, a_{-, r - \xi + 1}) \leq L_{N+1-\xi}^{\mathbf{u}}.$$

Let us now give an iterative construction of a path $\mathbf{v} = (v^{(0)}, \dots, v^{(k)})$ starting from c . To this end, set $\tilde{v}^{(0)} := (c_1, \dots, c_N)$. For $\zeta \in \{1, \dots, \ell\}$ and $l \in (\zeta - 1)K_{-, \ell} + \{1, \dots, K_{-, \ell}\}$, let

$$(4.72) \quad \tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \dots, \tilde{v}_{\zeta}^{(l-1)} - 1, \dots, \tilde{v}_N^{(l-1)}).$$

Let $k_1 := \ell K_{-, \ell}$. For $\xi \in \{1, \dots, r\}$ and $l \in k_1 + (\xi - 1)K_{+, r} + \{1, \dots, K_{+, r}\}$ let

$$(4.73) \quad \tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \dots, \tilde{v}_{N+1-\xi}^{(l-1)} + 1, \dots, \tilde{v}_N^{(l-1)}).$$

The path \mathbf{v} has the property that it moves all particles of the configuration $\{c_j : j \leq \ell \text{ or } j \geq N - r + 1\}$ into the configuration $b_{\ell, r}(\Gamma)$ outside the boundary of Γ .

In the next step, we move the particles that are still remaining inside of Γ into the configuration $x^{in} = x^{in}(\Gamma)$. Let $k_2 := k_1 + rK_{+,r}$. Let $\ell' := |\{j \in I_-^{\mathbf{u}} : j > \ell\}|$, $r' := |\{j \in I_+^{\mathbf{u}} : j \leq N - r\}|$. For $\zeta \in \{1, \dots, \ell'\}$ and $l \in k_2 + \sum_{j=1}^{\zeta-1} L_{\ell+j}^{\mathbf{u}} + \{1, \dots, L_{\zeta}^{\mathbf{u}}\}$ we set

$$(4.74) \quad \tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \dots, \tilde{v}_{\ell+\zeta}^{(l-1)} - 1, \dots, \tilde{v}_N^{(l-1)}).$$

Let $k_3 := k_2 + \sum_{j=1}^{\ell'} L_{\ell+j}^{\mathbf{u}}$. For $\xi \in \{1, \dots, r'\}$ and $l \in k_3 + \sum_{j=1}^{\xi-1} L_{N-r-j}^{\mathbf{y}} + \{1, \dots, L_{N-r-\xi}^{\mathbf{y}}\}$ we set

$$(4.75) \quad \tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \dots, \tilde{v}_{N-r-\xi}^{(l-1)} + 1, \dots, \tilde{v}_N^{(l-1)}).$$

The fact that this construction is well-defined follows from the statement in (4.66), since no particle with index $j \in \{\ell + 1, \dots, N - r\}$ leaves Γ and therefore does not intersect the configuration $b_{\ell,r}$. In the last step, we move the configuration $b_{\ell,r}$ into $x^{out}(\Gamma)$. Let $k_0 := k_3 + \sum_{j=1}^{r'} L_{\ell+j}^{\mathbf{u}}$. By construction we have $v^{(k_0)} = b_{\ell,r} \cup x^{in}(\Gamma)$. Notice that by definition of $\chi_-^{\ell} \equiv \chi_-^{\ell}(x, \Gamma)$ and $\chi_+^r \equiv \chi_+^r(x, \Gamma)$, we obtain for all $\zeta \leq \ell$ and $\xi \leq r$ that

$$(4.76) \quad \chi_{-, \ell-\zeta+1}^{\ell} = d_L(\tilde{u}_{\zeta}^{(k)}, a_{-, \ell-\zeta+1}) = L_{\zeta}^{\mathbf{u}} - K_{-, \ell},$$

$$(4.77) \quad \chi_{+, r-\xi+1}^r = d_L(\tilde{u}_{N+1-\xi}^{(k)}, a_{+, r-\xi+1}) = L_{N+1-\xi}^{\mathbf{u}} - K_{+, r}.$$

For all $\zeta \in \{1, \dots, \ell\}$ and $l \in k_0 + \sum_{j=1}^{\zeta-1} \chi_{-, \ell-j+1}^{\ell} + \{1, \dots, \chi_{-, \ell-\zeta+1}^{\ell}\}$ we set

$$(4.78) \quad \tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \dots, (\tilde{v}_{\zeta}^{(l-1)} - 1) \bmod L, \dots, \tilde{v}_N^{(l-1)}).$$

Let $k_4 := k_0 + \sum_{\zeta=1}^{\ell} \chi_{-, \zeta}^{\ell}$. For $\xi \in \{1, \dots, r\}$ and $l \in k_4 + \sum_{j=1}^{\xi-1} \chi_{+, r-j+1}^r + \{1, \dots, \chi_{+, r-\xi+1}^r\}$ we set

$$(4.79) \quad \tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \dots, (\tilde{v}_{N+1-\xi}^{(l-1)} + 1) \bmod L, \dots, \tilde{v}_N^{(l-1)}).$$

By construction, \mathbf{v} is a shortest path from c to x , since it has a length of $k = \sum_{j=1}^N L_j^{\mathbf{u}}$ and $\tilde{v}^{(k)} = x$. Moreover, by construction as well as (4.76) and (4.77), we get

$$(4.80) \quad k - k_0 = \sum_{\zeta=1}^{\ell} \chi_{-, \zeta}^{\ell} + \sum_{\xi=1}^r \chi_{+, \xi}^r$$

Finally, we note that $v^{(k_0)} \subseteq \mathcal{S}_{L,M}(1/4 - \epsilon/2)$. This follows from the fact that for all $m \in b_{\ell,r}$, one has $d_L(m, M) < (\theta + 2\epsilon)L + (\ell + r) \leq (1/4 - \epsilon/2)L$, where we used $\ell + r \leq N/2 < \epsilon/2$ and $\epsilon < \theta < 1/16$. \square

Lemma 4.11. *Let $\Lambda'_L := \mathcal{S}_{L,M}(\theta + 2\epsilon)$. Fix $n \in \mathbb{N}$ such that $N/2 < n < N$ and $x \in \mathcal{V}_L^N$ with $x^{in}(\Lambda'_L) \in \mathcal{V}^n(\Lambda'_L)$. Let $c \in \mathcal{V}_{L,1}^N$, with $c \subseteq \Lambda'_L$ and set $k := d_L^N(x, c)$. Then there exists $r, \ell \in \mathbb{N}_0$ with $r + \ell = N - n$ and a shortest path \mathbf{v} from c to x such that $\tilde{v}_j^{(k)} = x_{\sigma_{\ell,r}(j)}$ for all $j \in I(N)$. Moreover, there exists a $k_0 \in \{0, \dots, k\}$ such that*

$$(4.81) \quad v^{(k_0)} = x^{in}(\Lambda'_L) \cup b_{\ell,r}(\Lambda'_L) \subseteq \mathcal{S}_{L,M}(1/4 - \epsilon/2).$$

Proof. Let $\lambda'_- := \min \Lambda'_L$ and $\lambda'_+ := \max \Lambda'_L$. Then $\Lambda'_L = \{\lambda'_-, \dots, \lambda'_+\}$.

Let \mathbf{u} be a shortest path from c to x . According to Lemma 4.5 there exists an edge $e \in \mathcal{E}_L$ with $\max_{j \in I(N)} d_L(c_j, e) \geq (1/2 - \epsilon)L$, which is not crossed by the path \mathbf{u} . Notice that this implies $e \subseteq (\Lambda'_L)^c$. We define the index sets

$$(4.82) \quad J_{\pm}^{\mathbf{u}, in} := \{j \in I_{\pm}^{\mathbf{u}} : \tilde{u}_j^{(k)} \in x^{in}(\Lambda'_L)\} \quad \text{and} \quad J_{\pm}^{\mathbf{u}, out} := \{j \in I_{\pm}^{\mathbf{u}} : \tilde{u}_j^{(k)} \in x^{out}(\Lambda'_L)\}.$$

We claim that

$$(4.83) \quad \max J_{+}^{\mathbf{u}, in} \leq \min J_{+}^{\mathbf{u}, out} \quad \text{and} \quad \max J_{-}^{\mathbf{u}, out} \leq \min J_{-}^{\mathbf{u}, in}.$$

These statements are a consequence of Lemma 4.4. Here, we only prove the first inequality, the other one follows analogously. Suppose that $\max J_{+}^{\mathbf{u}, in} > \min J_{+}^{\mathbf{u}, out}$. This implies that there exists a $j \in J_{+}^{\mathbf{u}, out}$ such that $j+1 \in J_{+}^{\mathbf{u}, in}$. According to Lemma 4.4 (i) we have $L_j^{\mathbf{u}} \leq L_{j+1}^{\mathbf{u}}$ and therefore also

$$(4.84) \quad \lambda'_+ \geq \tilde{u}_{j+1}^{(k)} = c_{j+1} + L_{j+1}^{\mathbf{u}} > c_j + L_j^{\mathbf{u}} = \tilde{u}_j^{(k)} \geq \lambda'_-,$$

which is a contradiction to $j \in J_{+}^{\mathbf{u}, out}$.

Let $\ell := |J_{-}^{\mathbf{u}, out}|$ and $r := |J_{+}^{\mathbf{u}, out}|$. Together with (4.83) this implies $\{j : j \leq \ell\} = J_{-}^{\mathbf{u}, out} \subseteq I_{-}^{\mathbf{u}}$ and $\{j : j > N - r\} = J_{+}^{\mathbf{u}, out} \subseteq I_{+}^{\mathbf{u}}$. Furthermore, Lemma 4.4 yields $\tilde{u}_j^{(k)} = x_{\sigma_{\ell, r}(j)}$ for all $j \in I(N)$. Moreover, for all $\zeta \in \{1, \dots, \ell\}$ holds $c_{\zeta} \geq \lambda_- > a_{-, \ell - \zeta + 1}(\Lambda'_L)$ and for all $\xi \in \{1, \dots, r\}$ holds $c_{N+1-\xi} \leq \lambda_+ < a_{+, r - \xi + 1}(\Lambda'_L)$, since $c \subseteq \Lambda'_L$.

Lemma 4.10 now yields the proposition. \square

Let us now define the set of negligible configurations that are in a large enough distance to the set of cluster configurations. This set is given by

$$(4.85) \quad \mathcal{C}_L^N := \{x \in \mathcal{V}_L^N : d_L^N(x, \mathcal{V}_{L,1}^N) \geq L^{3/2}\}.$$

Lemma 4.12. *There exists a $L_0 \equiv L_0(\epsilon) > 0$ such that for all $L \geq L_0$, $n \in \mathbb{N}$ with $N/2 < n < N$ and $x \in \mathcal{V}_L^N \setminus \mathcal{C}_L^N$ with $x \cap \Lambda_L \in \mathcal{V}^n(\Lambda_L)$ one has*

$$(4.86) \quad \mathcal{W}_L(x) \subseteq \mathcal{S}_{L,M}(\theta + \epsilon).$$

Moreover, for all $m \in \mathcal{W}_L(x)$:

$$(4.87) \quad c_{L,m}^N \subseteq \mathcal{S}_{L,M}(\theta + 2\epsilon).$$

Proof. Let us first show (4.86). Let $\nu \in (\mathcal{S}_{L,M}(\theta + \epsilon))^c$. Then for all $\xi \in c_{L,\nu}^N$ one has

$$(4.88) \quad d_L(\xi, \Lambda_L) \geq d_L(\nu, \Lambda_L) - \lceil (N+1)/2 \rceil \geq \epsilon L - 2\epsilon L/3$$

for all $L \geq L_1$ with $L_1 \equiv L_1(\epsilon) := 9/\epsilon$. Since n particles of x are located inside of Λ_L we have

$$(4.89) \quad d_L^N(x, c_{L,\nu}^N) \geq n\epsilon L/3 \geq \epsilon^2 L^2/6,$$

where we applied Lemma 4.2. Let $L_0 \equiv L_0(\epsilon) > L_1$, such that $\epsilon^2 L_0^{1/2}/6 > 1$. For any $L \geq L_0$ this implies $d_L^N(x, c_{L,\nu}^N) \geq L^{3/2}$. Since $x \notin \mathcal{C}_L^N$ by assumption, we conclude that $\nu \notin \mathcal{W}_L(x)$.

For all $L \geq L_0$ and $m \in \mathcal{W}_L(x) \subseteq \mathcal{S}_{L,M}(\theta + \epsilon)$, observe that for all $\xi \in c_{L,m}^N$, we have

$$(4.90) \quad d_L(\xi, M) \leq d_L(\xi, m) + d_L(m, M) < \lceil (N+1)/2 \rceil + (\theta + \epsilon)L \leq (\theta + 2\epsilon)L,$$

from which we conclude $c_{L,m}^N \subseteq \mathcal{S}_{L,M}(\theta + 2\epsilon)$. This shows the lemma. \square

Lemma 4.13. *Let $L \geq L_0$, with L_0 as in Lemma 4.12. Let $n \in \mathbb{N}$ with $N/2 < n < N$, $y \in \mathcal{V}^n(\Lambda_L)$, $x \in \mathcal{V}_L^N \setminus \mathcal{C}_L^N$ with $x^{in}(\Lambda_L) = y$. Then there exist $\ell, r \in \mathbb{N}_0$ with $r + \ell = N - n$ such that $y_{\kappa-\ell} \in \mathcal{W}_L(x)$ for $\kappa := \lfloor (N+1)/2 \rfloor$ and*

$$(4.91) \quad c_{L, y_{\kappa-\ell}}^N \subseteq \mathcal{S}_{L, M}(1/4 - \epsilon/2).$$

Furthermore, there exists a shortest path \mathbf{v} from $c_{L, y_{\kappa-\ell}}^N$ to x and a $k_0 \in \{1, \dots, k\}$ with $k := d_L^N(x, \mathcal{V}_{L, 1}^N)$ such that

$$(4.92) \quad v^{(k_0)} = y \cup b_{\ell, r}(\Lambda_L) \subseteq \mathcal{S}_{L, M}(1/4 - \epsilon/2),$$

and

$$(4.93) \quad k - k_0 = \sum_{\zeta=1}^{\ell} \chi_{-, \zeta}^{\ell}(x, \Lambda_L) + \sum_{\xi=1}^r \chi_{+, \xi}^r(x, \Lambda_L).$$

Proof. By Lemma 4.7 there exists $m \in \mathcal{W}_L(x) \cap x$. From Lemma 4.12, we know

$$(4.94) \quad c := c_{L, m}^N \subseteq \Lambda'_L := \mathcal{S}_{L, M}(\theta + 2\epsilon) \subseteq \mathcal{S}_{L, M}(1/4 - \epsilon/2).$$

According to Lemma 4.11 there exists a shortest path \mathbf{u} from c to x and $\ell', r' \in \mathbb{N}_0$ with $\ell' + r' = N - |x^{in}(\Lambda'_L)|$ and $k_1 \in \{0, \dots, k\}$ such that

$$(4.95) \quad z := u^{(k_1)} = b_{\ell', r'}(\Lambda'_L) \cup x^{in}(\Lambda'_L) \subseteq \mathcal{S}_{L, M}(1/4 - \epsilon/2),$$

and $\tilde{u}_j^{(k)} = x_{\sigma_{\ell', r'}(j)}$ for all $j \in I(N)$.

Let us define

$$(4.96) \quad \ell := \ell' + |\{\nu \in x^{in}(\Lambda'_L) : \nu < \lambda_-\}| = |\{j : z_j < \lambda_-\}| \quad \text{and}$$

$$(4.97) \quad r := r' + |\{\nu \in x^{in}(\Lambda'_L) : \nu > \lambda_+\}| = |\{j : z_j > \lambda_+\}|.$$

These quantities satisfy $\ell + r = N - |\{j : z_j \in \Lambda_L\}| = N - n$. Note that by this definition $\sigma_{\ell', r'}(x, \Lambda'_L) = \sigma_{\ell, r}(x, \Lambda_L)$. Hence, $\tilde{u}_j^{(k)} = x_{\sigma_{\ell, r}(j)}$ for all $j \in I(N)$ and $y_j = z_{j+\ell}$ for all $j \in \{1, \dots, n\}$.

Next, we show that $m = z_{\kappa}$. First, we claim that

$$(4.98) \quad m \in \mathcal{W}_L(z).$$

Take any $\nu \in \mathcal{W}_L(z)$ and any shortest path \mathbf{v} from $c' := c_{L, \nu}^N$ to z . Lemma 4.1 indicated that

$$(4.99) \quad k_2 := d_L^N(z, c') = d_L^N(z, \mathcal{V}_{L, 1}^N) \leq d_L^N(z, c) = k_1.$$

The path $\{v^{(0)}, \dots, v^{(k_2)}, u^{(k_1+1)}, \dots, u^{(k)}\}$ is therefore a path from c' to x of length $k + (k_2 - k_1)$ and therefore – using (4.99) – we get

$$(4.100) \quad k = d_L^N(x, \mathcal{V}_{L, 1}^N) \leq k + (k_2 - k_1) \leq k.$$

Hence $k_1 = k_2$. Equality in (4.99) implies $m \in \mathcal{W}_L(z)$. According to Lemma 4.12 we have $\mathcal{W}_L(z) \subseteq \mathcal{S}_{L, M}(\theta + \epsilon) \subseteq \Lambda'_L$. Hence

$$(4.101) \quad m \in x \cap \mathcal{W}_L(z) = x \cap \mathcal{W}_L(z) \cap \Lambda'_L = z \cap \mathcal{W}_L(z) = \{z_{\kappa}\},$$

where we used $z \subseteq \mathcal{S}_{L, M}^N(1/4 - \epsilon/2)$ together with Lemma 4.8. Therefore $m = z_{\kappa} = y_{\kappa-\ell}$.

Lemma 4.1 states that $\mathbf{w} := (u^{(0)}, \dots, u^{(k_1)})$ is a shortest path from c to z . According to Lemma 4.8 we have $\kappa \in I_0^{\mathbf{w}}$, since $m = z_\kappa$. This also implies $\kappa \in I_0^{\mathbf{u}}$, since by construction of the path \mathbf{u} , for all $l \in \{k_1, \dots, k\}$ one has $\tilde{u}_\kappa^{(l)} = \tilde{u}_\kappa^{(k_1)}$. In this case, it follows from Lemma 4.4 (iii) that $\{j : j \leq \ell\} \subseteq \{0, \dots, \kappa\} \subseteq I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}$ and for all $\zeta \leq \ell$, we have

$$(4.102) \quad a_{-, \ell - \zeta + 1} \leq z_{\ell+1} - (\ell - \zeta + 1) \leq c_{\ell+1} - (\ell - \zeta + 1) = c_\zeta.$$

Analogously, we have $\{j : j > N - r\} \subseteq \{\kappa, \dots, N\} \subseteq I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}$ and for all $\xi \leq r$ one therefore gets

$$(4.103) \quad a_{+, r - \xi + 1} \geq c_{N+1-\xi}.$$

According to Lemma 4.10 there therefore exists a path \mathbf{v} with all the properties stated in the proposition. \square

4.3. Estimates based on geometric series. In this section, we compute and estimate various geometric sum, which will be necessary for estimating the partial trace later.

Lemma 4.14. *There exists $L_0 \equiv L_0(\epsilon) > 0$ such that for all $L \geq L_0$, $n \in \mathbb{N}$ with $N/2 < n < N$, $y \in \mathcal{V}^n(\Lambda_L)$ and $\mu \geq \ln 2$ one has*

$$(4.104) \quad \sum_{\substack{z \in \mathcal{V}^{N-n}(\Lambda_L^c), \\ z \cup y \notin \mathcal{V}_{L,1}^N}} e^{-\mu d_L^N(y \cup z, \mathcal{V}_{L,1}^N)} \leq 333 e^{-\mu} e^{-\mu h_L^n(y)},$$

with $h_L^n : \mathcal{V}^n(\Lambda_L) \rightarrow (0, \infty)$,

$$(4.105) \quad h_L^n(y) := \begin{cases} \min \{d_L^{n+1}(y \cup \{a_{\pm,1}(\Lambda_L)\}, \mathcal{V}_{L,1}^{n+1}), L^{5/4}\} - 1 & \text{for } y \notin \{y_+^n, y_-^n\}, \\ 0 & \text{for } y \in \{y_+^n, y_-^n\}, \end{cases}$$

where $y_\pm^n := \lambda_\pm \mp \{0, \dots, n-1\}$.

Proof. Let

$$(4.106) \quad \mathcal{A}'(y) := \{z \in \mathcal{V}^{N-n}(\Lambda_L^c) : y \cup z \notin \mathcal{C}_L^N\} \subseteq \mathcal{V}^{N-n}(\Lambda_L^c),$$

$$(4.107) \quad \mathcal{A}(y) := \{z \in \mathcal{A}'(y) : y \cup z \notin \mathcal{V}_{L,1}^N\} \subseteq \mathcal{A}'(y).$$

There exists L_1 such that $L^{1/2} - \ln L / \ln 2 \geq L^{1/4}$ for all $L \geq L_1$. Hence for all $L \geq L_1$ we get

$$(4.108) \quad \sum_{z \in (\mathcal{A}(y))^c} e^{-\mu d_L^N(y \cup z, \mathcal{V}_{L,1}^N)} \leq |(\mathcal{A}(y))^c| e^{-\mu L^{3/2}} \leq e^{-\mu L^{5/4}} \leq e^{-\mu} e^{-\mu h_L^n(y)},$$

where we used $|(\mathcal{A}(y))^c| \leq |\Lambda_L^c|^{N-n} \leq L^L$ as well as $\mu \geq \ln 2$. We partition $\mathcal{A}'(y)$ into smaller subsets. For any $\ell, r \in \mathbb{N}_0$ with $\ell + r = N - n$ let $c^\ell := c_{L, y_{\kappa-\ell}}^N$ with $\kappa := \lfloor (N+1)/2 \rfloor$. Let us further define

$$(4.109) \quad \begin{aligned} \mathcal{A}_{\ell,r}^{(\prime)}(y) &:= \{z \in \mathcal{A}'(y) : d_L^N(y \cup z, \mathcal{V}_{L,1}^N) = d_L^N(c^\ell, y \cup b_{\ell,r}) + d_L^N(y \cup b_{\ell,r}, y \cup z), \\ d_L^N(y \cup b_{\ell,r}, y \cup z) &= \sum_{\zeta=1}^{\ell} \chi_{-, \zeta}^\ell(x, \Lambda_L) + \sum_{\xi=1}^r \chi_{+, \xi}^r(x, \Lambda_L)\}, \end{aligned}$$

where $\chi_-^\ell(x, \Lambda_L) \in \mathcal{X}^\ell$ and $\chi_+^r(x, \Lambda_L) \in \mathcal{X}^r$ where defined in (4.57) and (4.58).

Lemma 4.1 and Lemma 4.13 imply immediately, that there exists a $L_2 \equiv L_2(\epsilon) > L_1$ such that for all $L \geq L_2$, we get the equality

$$(4.110) \quad \mathcal{A}(y) = \bigcup_{\substack{\ell, r \in \mathbb{N}_0, \\ \ell + r = N - n}} \mathcal{A}_{\ell, r}(y) = \mathcal{A}_{N-n, 0}(y) \cup \mathcal{A}_{0, N-n}(y) \cup \bigcup_{\substack{\ell, r \in \mathbb{N}, \\ \ell + r = N - n}} \mathcal{A}'_{\ell, r}(y).$$

Let us first consider the case $\ell, r \in \mathbb{N}$. Definition (4.109) implies, together with Lemma A.3 for $\mu \geq \ln 2$, that

$$(4.111) \quad \sum_{z \in \mathcal{A}'_{\ell, r}(y)} e^{-\mu d_L^N(y \cup z, \mathcal{V}_{L,1}^N)} \leq e^{-\mu d_L^N(c^\ell, y \cup b_{\ell, r})} \left(\sum_{\chi^\ell \in \mathcal{X}^\ell} e^{-\mu |\chi^\ell|_1} \right) \left(\sum_{\chi^r \in \mathcal{X}^r} e^{-\mu |\chi^r|_1} \right) \leq e^{-\mu d_L^N(c^\ell, y \cup b_{\ell, r})} (1 + 30e^{-\mu})^2.$$

Now, we estimate the first factor on the right hand side of (4.111) uniformly in ℓ, r . Both c^ℓ and $y \cup b_{\ell, r}$ are subsets of $\mathcal{S}_{L, M}(1/4 - \epsilon/2)$. By Lemma 4.8 we have $y_{\kappa-\ell} \in \mathcal{W}_L(y \cup b_{\ell, r})$ with

$$(4.112) \quad d_L^N(y \cup b_{\ell, r}, c^\ell) = d^N(y \cup b_{\ell, r}, c^\ell) \geq \sum_{j=1}^n |y_j - c_{j+\ell}^\ell| + |a_{-,1} - c_\ell^\ell| + |a_{+,1} - c_{N-r+1}^\ell|,$$

where we applied (4.43) and Remark 4.6. For all $\ell \in \{1, \dots, N - n - 1\}$ we have

$$(4.113) \quad |a_{-,1} - c_\ell^\ell| + |a_{+,1} - c_{N-r+1}^\ell| \geq |a_{+,1} - a_{-,1}| - |c_{N-r+1}^\ell - c_\ell^\ell| \geq 2\theta L - (n+1) \geq \epsilon L,$$

where we used that $a_{+,1} - a_{-,1} \geq d_L(a_{+,1}, M) - d_L(a_{-,1}, M) \geq 2\theta L$, as well as $n+1 \leq N < \epsilon L$ and $\theta > \epsilon$. Hence for all $y \in \mathcal{V}^n(\Lambda_L)$ we have either $|a_{-,1} - c_\ell^\ell| \geq \epsilon L/4 + 1$ or $|a_{+,1} - c_{r+\ell+1}^\ell| \geq \epsilon L/4 + 1$ for all $L \geq L_0 = L_0(\epsilon) = \max\{L_2, 4/\epsilon\}$. This implies, together with (4.112) that

$$(4.114) \quad d_L^N(y \cup b_{\ell, r}, c^\ell) - 1 \geq h_L^n(y) + \epsilon L/4.$$

Hence, by combining (4.111) and (4.114) we find

$$(4.115) \quad \sum_{\substack{\ell, r \in \mathbb{N}, \\ \ell + r = N - n}} \sum_{z \in \mathcal{A}'_{\ell, r}(y)} e^{-\mu d_L^N(y \cup z, \mathcal{V}_{L,1}^N)} \leq (N - n) e^{-\mu \epsilon L/4} (1 + 30e^{-\mu})^2 e^{-\mu} e^{-\mu h_L^n(y)} \leq 272 e^{-\mu} e^{-\mu h_L^n(y)},$$

where we used that $(N - n) e^{-\mu \epsilon L/4} \leq (\epsilon L/2) 2^{-\epsilon L/4} \leq \frac{2}{e \ln 2}$ for all $\mu \geq \ln 2$.

Let us now consider the case $\ell = 0$ or $r = 0$. There are only two configurations $y \in \mathcal{V}^n(\Lambda_L)$ such that there exists a $z \in \mathcal{V}^{N-n}(\Lambda_L^c)$ with $y \cup z \in \mathcal{V}_{L,1}^N$, namely y_+^n and y_-^n . The configurations $z_-^n := b_{N-n, 0}$ and $z_+^n := b_{0, N-n}$ satisfy $y_\pm^n \cup z_\pm^n \in \mathcal{V}_{L,1}^n$. There are no other configurations in $\mathcal{V}^{N-n}(\Lambda_L^c)$ with this property. We further restrict ourselves to the case $\ell = N - n$ and $r = 0$. The other case can be treated analogously. Now, our approach depends on whether $y = y_-^n$ or not. We have

$$(4.116) \quad \mathcal{A}_{N-n, 0}(y) \subseteq \begin{cases} \mathcal{A}'_{N-n, 0}(y) & \text{for } y \neq y_-^n, \\ \mathcal{A}'_{N-n, 0}(y) \setminus \{z_-^n\} & \text{for } y = y_-^n. \end{cases}$$

Analogous to (4.111) we obtain

$$(4.117) \quad \sum_{z \in \mathcal{A}_{N-n, 0}(y)} e^{-\mu d_L^N(y \cup z, \mathcal{V}_{L,1}^N)} \leq e^{-\mu d_L^N(y \cup b_{N-n, 0}, c^{N-n})} \begin{cases} \sum_{\chi \in \mathcal{X}^{N-n}} e^{-\mu |\chi|_1} & \text{for } y \neq y_-^n, \\ \sum_{\chi \in \mathcal{X}^{N-n} \setminus \{0\}} e^{-\mu |\chi|_1} & \text{for } y = y_-^n, \end{cases}$$

and for all $y \in \mathcal{V}_L^n(\Lambda_L)$

$$(4.118) \quad d_L^N(y \cup b_{N-n,0}, c^{N-n}) \geq d_L^{n+1}(y \cup \{a_{-,1}\}, \mathcal{V}_{L,1}^{n+1}) \geq \begin{cases} h_L^n(y) + 1 & \text{for } y \neq y_-^n, \\ h_L^n(y) & \text{for } y = y_-^n. \end{cases}$$

Lemma A.3 then implies

$$(4.119) \quad \sum_{z \in \mathcal{A}_{N-n,0}(y)} e^{-\mu d_L^N(y \cup z, \mathcal{V}_{L,1}^N)} \leq \begin{cases} (1 + 30e^{-\mu})e^{-\mu}e^{-\mu h_L^n(y)} & \text{for } y \neq y_-^n, \\ 30e^{-\mu}e^{-\mu h_L^n(y)} & \text{for } y = y_-^n. \end{cases}$$

Hence, by (4.110), (4.115) and (4.119), as well as the definition of h_L

$$(4.120) \quad \sum_{z \in \mathcal{A}(y)} e^{-\mu d_L^N(y \cup z, \mathcal{V}_{L,1}^N)} \leq 332e^{-\mu}e^{-\mu h_L^n(y)}$$

where we used $\mu \geq \ln 2$. Together with (4.108), this concludes the proposition. \square

Lemma 4.15. *There exists $L_0 \equiv L_0(\epsilon) > 0$ such that for all $L \geq L_0$, $n \in \mathbb{N}$ with $N/2 < n < N$ and $\mu \geq \ln 2$ holds*

$$(4.121) \quad \sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y_{\pm}^n\}} e^{-\mu h_L^n(y)} \leq 2^{11} e^{-\mu}.$$

Proof. For any $y \in \mathcal{V}^n(\Lambda_L)$ holds $y' := y \cup \{a_{+,1}\} \subseteq \mathcal{S}_{L,M}(1/4 - \epsilon/2)$. Hence, according to Lemma 4.8, we have $y_{\kappa} \in \mathcal{W}_L(y')$ with $\kappa := \lfloor (n+2)/2 \rfloor$. For any $j \in \{0, \dots, |\Lambda_L| - n\}$ let

$$(4.122) \quad \mathcal{B}_j^n := \{y \in \mathcal{V}^n(\Lambda_L) : y_{\kappa} = \lambda_+ - (n - \kappa + j)\}.$$

Hence

$$(4.123) \quad \mathcal{V}^n(\Lambda_L) = \bigcup_{j=0}^{|\Lambda_L|-n} \mathcal{B}_j^n.$$

Let us now consider a $y \in \mathcal{B}_j^n$ for a $j \in \{0, \dots, |\Lambda_L| - n\}$. Let $c := c_{L, y_{\kappa}}^{n+1}$. We define $\chi'_- \equiv \chi'_-(y) \in \mathcal{X}^{\kappa-1}$ and $\chi'_+ \equiv \chi'_+(y) \in \mathcal{X}^{n-\kappa}$ such that

$$(4.124) \quad \chi'_{-,j}(y) := |(y_{\kappa} - j) - c_{\kappa-j}| \quad \text{for } 1 \leq j \leq \kappa - 1,$$

$$(4.125) \quad \chi'_{+,j}(y) := |(y_{\kappa} + j) - c_{\kappa+j}| \quad \text{for } 1 \leq j \leq n - \kappa.$$

Then,

$$(4.126) \quad d_L^{n+1}(y \cup \{a_{+,1}\}, \mathcal{V}_{L,1}^{n+1}) = \sum_{j=1}^{\kappa-1} \chi'_{-,j} + \sum_{j=1}^{n-\kappa} \chi'_{+,j} + j.$$

Hence, according to Lemma A.3, we obtain for all $j \in \{0, \dots, |\Lambda_L| - n\}$ and all $\mu \geq \ln 2$,

$$(4.127) \quad \sum_{y \in \mathcal{B}_j^n} e^{-\mu d_L^{n+1}(y \cup \{a_{+,1}\}, \mathcal{V}_{L,1}^{n+1})} \leq e^{-\mu j} (1 + 30e^{-\mu})^2.$$

Therefore, by (4.123) we have

$$(4.128) \quad \sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y_{\pm}^n\}} e^{-\mu d_L^{n+1}(y \cup \{a_{+,1}\}, \mathcal{V}_{L,1}^{n+1})} \leq \left(1 + \frac{e^{-\mu}}{1 - e^{-\mu}}\right) (1 + 30e^{-\mu})^2 - 1 \leq 1022e^{-\mu}$$

where we used, that $\mu \geq \ln 2$. By an analogous method we obtain the same bound for the sum over $\exp(d_L^{n+1}(y \cup \{a_{-,1}\}, \mathcal{V}_{L,1}^{n+1}))$.

Let $L_0 > 0$ such that $L_0^{1/4} - \ln L_0 > 2$. By using $n \leq N < \mu L$, we get for all $L \geq L_0$ that

$$(4.129) \quad \sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y_{\pm}^n\}} e^{-\mu(L^{1+\alpha/2}-1)} \leq L^n e^{-\mu(L^{1+\alpha/2}-1)} \leq e^{-\mu(L(L^{\alpha/2}-\ln L)-1)} \leq e^{-\mu}$$

By the definition of h_L^n in (4.105) we obtain

$$(4.130) \quad \sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y_{\pm}^n\}} e^{-\mu h_L^n(y)} \leq \sum_{\eta \in \{\pm\}} \sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y_{\eta}^n\}} e^{\mu} e^{-\mu d_L^{n+1}(y \cup \{a_{\eta,1}\}, \mathcal{V}_{L,1}^{n+1})} + e^{-\mu} \leq 2^{11} e^{-\mu},$$

where we used (4.128). □

5. PERTURBATION OF THE ISING LIMIT

For the whole section let $\epsilon \in (0, 1/16)$ and $\theta \in (\epsilon, 1/16)$. For $L \in \mathbb{N}$, let $N \equiv N(L) := \lfloor \epsilon L \rfloor$. As before, for the reader's convenience, we will omit the indices N and L in the following proofs. Let $\Lambda_L := \mathcal{S}_{L,M}(\theta)$, where $M := \lfloor (L-1)/2 \rfloor$ and the sector $\mathcal{S}_{L,M}(\theta)$ was defined in (4.38).

5.1. The mass of the droplet configurations. Firstly, given any low-energy eigenstate let us examine the contribution of the droplet configurations.

Lemma 5.1. *Let $\Delta > 3$ such that $\mu_1(\Delta) \geq \ln 2$, where μ_1 was defined in Corollary 3.8. Moreover, let $\gamma \in \mathcal{V}_L$. Then*

$$(5.1) \quad \frac{1}{L} \left(1 - 2^{17} e^{-2\mu_1}\right) \leq |\langle \delta_x^L, \varphi_{L,\gamma}^N \rangle|^2 \leq \frac{1}{L}$$

for all $x \in \mathcal{V}_{L,1}^N$, where $|\varphi_{L,\gamma}^N(\Delta)\rangle$ was defined in Remark 3.9.

Proof. Analogously to (3.69), the definition of $|\varphi_{\gamma}\rangle$ implies that

$$(5.2) \quad |\langle \delta_x, \varphi_{\gamma} \rangle| = |\langle \delta_{x_0}, \varphi_{\gamma} \rangle|$$

for all droplets $x \in [x_0]$, where $x_0 \in \hat{\mathcal{V}} \cap \mathcal{V}_1$ is the unique representative in $\hat{\mathcal{V}}$ of a droplet. Hence, by the results of Corollary 3.8, we have

$$(5.3) \quad 1 = L |\langle \delta_{x_0}, \varphi_{\gamma} \rangle|^2 + \sum_{x \in \mathcal{V} \setminus \mathcal{V}_1} |\langle \delta_x, \varphi_{\gamma} \rangle|^2 \leq L |\langle \delta_{x_0}, \varphi_{\gamma} \rangle|^2 + \sum_{x \in \mathcal{V} \setminus \mathcal{V}_1} \frac{2^6}{L \delta^2} e^{-2\mu_1 d(x, \mathcal{V}_1)}.$$

The first equality already yields the upper bound in (5.1). For the lower bound, we still need to estimate the second term of the right hand side in (5.3).

Lemma 4.9 allows us to estimate the last term on the right hand side of (5.3) in the following way

$$(5.4) \quad 1 \leq L |\langle \delta_{x_0}, \varphi_{\gamma} \rangle|^2 + \frac{2^{15}}{\delta^2} e^{-2\mu_1}.$$

This concludes the proof. □

Lemma 5.2. *Let $\Delta > 3$ such that $\mu_1 \geq \ln 2$. Moreover, let $\gamma \in \mathcal{V}_L$. Then for all $x, x' \in \mathcal{V}_L^N$*

$$(5.5) \quad |\langle \delta_x^L, (\rho(\varphi_{L,\gamma}^N) - \rho_{L,\gamma}^N) \delta_{x'}^L \rangle| \leq \frac{2^{17}}{L} \begin{cases} e^{-2\mu_1} & \text{if } x, x' \in [\hat{x}_0], \\ e^{-\mu_1(d_L^N(x, \mathcal{V}_{L,1}^N) + d_L^N(x', \mathcal{V}_{L,1}^N))} & \text{else.} \end{cases}$$

Proof. Let again $x_0 \in \hat{\mathcal{V}} \cap \mathcal{V}_1$. We only need to discuss the case $x, x' \in [x_0]$. All other cases follow immediately from Corollary 3.8. Let $x = T^\zeta \hat{x}_0$ and $x' = T^\xi x_0$ for some $\xi, \zeta \in \{0, \dots, L-1\}$. Remark 3.9 implies that

$$(5.6) \quad \langle \delta_x, \rho(\varphi_\gamma) \delta_{x'} \rangle = e^{\frac{2\pi i}{L}(\zeta - \xi)\gamma} \langle \delta_{x_0}, \rho(\varphi_\gamma) \delta_{x_0} \rangle = e^{\frac{2\pi i}{L}(\zeta - \xi)\gamma} |\langle \delta_{x_0}, \varphi_\gamma \rangle|^2,$$

while Definition (3.72) implies

$$(5.7) \quad \langle \delta_x, \rho_\gamma \delta_{x'} \rangle = e^{\frac{2\pi i}{L}(\zeta - \xi)\gamma} \langle \delta_{x_0}, \rho_\gamma \delta_{x_0} \rangle = \frac{1}{L} e^{\frac{2\pi i}{L}(\zeta - \xi)\gamma}.$$

By applying Lemma 5.1 we obtain

$$(5.8) \quad |\langle \delta_x^L, (\rho(\varphi_\gamma) - \rho_\gamma) \delta_{x'} \rangle| \leq \frac{2^{17}}{L} e^{-2\mu_1}.$$

□

Lemma 5.3. *Let $\Delta > 3$ such that $\mu_1 \geq \ln 2$ and let $\gamma \in \mathcal{V}_L$. Then, there exists an $L_0 \equiv L_0(\epsilon)$ such that for all $L \geq L_0$, all $n \in \mathbb{N}$, $N/2 < n < N$, and all $y, y' \in \mathcal{V}^n(\Lambda_L)$, we have*

$$(5.9) \quad |\langle \delta_y^{\Lambda_L}, (\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N) - \rho_{L,\Lambda_L,\gamma}^n) \delta_{y'}^{\Lambda_L} \rangle| \leq \frac{2^{34}}{L} e^{-\mu_1} e^{-\mu_1(h_L^n(y) + h_L^n(y'))}$$

where h_L^n was defined in Lemma 4.14.

Proof. Let $z_-^n := b_{N-n,0}$ and $z_+^n := b_{0,N-n}$. Then for all $y \in \mathcal{V}^n(\Lambda)$ and all $z \in \mathcal{V}^{N-n}(\Lambda^c) \setminus \{z_\pm^n\}$ we have that $y \cup z \notin \mathcal{V}_1$ is not a droplet configuration. By Lemma 3.10 we obtain for all $y, y' \in \mathcal{V}^n(\Lambda_L)$ that

$$(5.10) \quad \begin{aligned} |\langle \delta_y^\Lambda, (\rho_\Lambda^n(\varphi_\gamma) - \rho_{\Lambda,\gamma}^n) \delta_{y'}^\Lambda \rangle| &\leq \sum_{\eta \in \{\pm\}} |\langle \delta_{y \cup z_\eta^n}, (\rho(\varphi_\gamma) - \rho_\gamma) \delta_{y' \cup z_\eta^n} \rangle| \\ &+ \sum_{z \in \mathcal{V}_\Lambda^{N-n} \setminus \{z_\pm^n\}} |\langle \delta_{y \cup z}, \rho(\varphi_\gamma) \delta_{y' \cup z} \rangle|. \end{aligned}$$

By Theorem 2.3 and Lemma 4.14, using Cauchy-Schwarz, we further estimate

$$(5.11) \quad \sum_{z \in \mathcal{V}_\Lambda^{N-n} \setminus \{z_\pm^n\}} |\langle \delta_{y \cup z}, \rho(\varphi_\gamma) \delta_{y' \cup z} \rangle| \leq \frac{2^8}{L} 333 e^{-2\mu_1} e^{-\mu_1(h^n(y) + h^n(y'))},$$

for all $L \geq L_0$, where $L_0 \equiv L_0(\epsilon)$ was given in Lemma 4.14. Moreover, according to Lemma 5.2 we derive the estimate

$$(5.12) \quad |\langle \delta_{y \cup z_\eta^n}, (\rho(\varphi_\gamma) - \rho_\gamma) \delta_{y' \cup z_\eta^n} \rangle| \leq \frac{2^{17}}{L} \begin{cases} e^{-2\mu_1} & \text{if } y = y' = y_\eta^n, \\ e^{-\mu_1(d(y \cup z_\eta^n, \mathcal{V}_1) + d(y' \cup z_\eta^n, \mathcal{V}_1))} & \text{else} \end{cases}$$

for all $\eta \in \{\pm\}$. Lemma 4.14 implies for all $\eta \in \{\pm\}$ and all $y \in \mathcal{V}^n(\Lambda) \setminus \{y_\eta^n\}$ that

$$(5.13) \quad e^{-\mu_1 d(y \cup z_\eta^n, \mathcal{V}_1)} \leq 333 e^{-\mu_1} e^{-\mu_1 h^n(y)},$$

since in this case $y \cup z_\eta^n \notin \mathcal{V}_1^n$. On the other hand, if $y = y_\eta^n$ we have

$$(5.14) \quad e^{-\mu_1 d(y_\eta^n \cup z_\eta^n, \mathcal{V}_1)} = 1 = e^{-\mu_1 h^n(y_\eta^n)}.$$

Finally, if $y, y' \in \{y_\pm^n\}$ with $y' \neq y$ we have

$$(5.15) \quad e^{-\mu_1 (d(y \cup z_\eta^n, \mathcal{V}_1) + d(y' \cup z_\eta^n, \mathcal{V}_1))} \leq e^{-\mu_1}.$$

By combining (5.11), (5.12), (5.13), (5.14) and (5.15), we conclude the proof. \square

5.2. Eigenvalue estimates. For this section let $n \in \mathbb{N}$ with $N/2 < n < N$ be fixed. The objective here is to prove the convergence of $\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N(\Delta))$ to $\rho_{\Lambda_L,\gamma}^n$ in the Schatten-quasinorm $\|\cdot\|_{1/p}$ for a any $p \in (1, \infty)$. Firstly, we establish estimates for the eigenvalues of the afore mentioned operator. For an operator $A \in L(\mathbb{H})$ acting on a finite dimensional Hilbert space \mathbb{H} let $\{\lambda_j(A)\}_{j \leq \dim \mathbb{H}}$ denote the singular values of A in descending order. If A is a self adjoint operator, these are also the absolute values of the eigenvalues of A .

Let $\Xi_L^n : \{1, \dots, |\mathcal{V}^n(\Lambda_L)|\} \rightarrow \mathcal{V}^n(\Lambda_L)$ be a bijective map such that $h_L^n \circ \Xi_L^n$ is monotonously decreasing. In a way, the function Ξ_L^n orders the configurations $y \in \mathcal{V}^n(\Lambda_L)$ with respect to $h_L^n(y)$.

Lemma 5.4. *Let $\Delta > 3$ such that $\mu_1 \geq \ln 2$ and let $\gamma \in \mathcal{V}_L$. Then, there exists a $L_0 > 0$ such that for all $L \geq L_0$ and all $j \in \mathbb{N}$ with $j \leq \dim \mathbb{H}_{\Lambda_L}^n$ holds*

$$(5.16) \quad \lambda_j(\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N) - \rho_{L,\Lambda_L,\gamma}^n) \leq \frac{2^{45} e^{-\mu_1}}{L} e^{-\mu_1 h_L^n \circ \Xi_L^n(j/2)}.$$

Proof. Let $A^n := (\rho_{\Lambda}^n(\varphi_\gamma) - \rho_{\Lambda,\gamma}^n)$ and $m := \dim \mathbb{H}_{\Lambda}^n = |\mathcal{V}^n(\Lambda)|$. Now, we split up A^n into a sum of operators of lower rank. For all $j \in \{1, \dots, m\}$ we define

$$(5.17) \quad A_j^n := \langle \delta_{\Xi^n(j)}^\Lambda, A^n \delta_{\Xi^n(j)}^\Lambda \rangle |\delta_{\Xi^n(j)}^\Lambda\rangle \langle \delta_{\Xi^n(j)}^\Lambda| + \sum_{l>j} (\langle \delta_{\Xi^n(l)}^\Lambda, A^n \delta_{\Xi^n(j)}^\Lambda \rangle |\delta_{\Xi^n(l)}^\Lambda\rangle \langle \delta_{\Xi^n(j)}^\Lambda| + h.c.)$$

These are operators of rank less than or equal to two. Moreover,

$$(5.18) \quad A^n = \sum_{j=1}^m A_j^n.$$

First, we estimate the largest eigenvalue of $R_j^n := \sum_{k>j} A_k^n$ for all $j \in \{0, \dots, m\}$. We obtain

$$(5.19) \quad \lambda_1(R_j^n) \leq \sup_{\psi \neq 0} \frac{\|R_j^n \psi\|_\infty}{\|\psi\|_\infty} = \max_{k>j} \sum_{l>j} |\langle \delta_{\Xi^n(k)}^\Lambda, A^n \delta_{\Xi^n(l)}^\Lambda \rangle|,$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbb{H}_{\Lambda}^n \cong \mathbb{C}^m$. According to Lemma 4.15 and Lemma 5.3 there exists a $L_0 \equiv L_0(\alpha, \epsilon)$ such that

$$(5.20) \quad \lambda_1(R_j^n) \leq \frac{2^{34} e^{-\mu_1}}{L} e^{-\mu_1 h^n \circ \Xi^n(j+1)} \sum_{l>j} e^{-\mu_1 h^n \circ \Xi^n(l)} \leq \frac{2^{45} e^{-\mu_1}}{L} e^{-\mu_1 h^n \circ \Xi^n(j+1)}$$

for all $L \geq L_0$, where we used the monotonicity of $h^n \circ \Xi^n$ and $\mu_1 \geq \ln 2$. Since $R_0^n = A^n$ this also implies

$$(5.21) \quad \lambda_1(A^n) \leq \frac{2^{45} e^{-\mu_1}}{L}.$$

Let $S_j^n := A^n - R_j^n$ for all $j \in \{0, \dots, m\}$. Hence $\text{rank}(S_j^n) \leq \sum_{k=1}^j \text{rank}(A_k^n) \leq 2j$, and therefore also

$$(5.22) \quad \lambda_{2j+1}(S_j^n) = 0.$$

The operator A^n is self adjoint and $A^n = S_j^n + R_j^n$ for all $j \in \{0, \dots, m\}$. By a well-known inequality for singular values [Woj91], we deduce that for all $j \in \mathbb{N}$ with $2j + 1 \leq m$ or $2j + 2 \leq m$, it holds

$$(5.23) \quad \lambda_{2j+2}(A^n) \leq \lambda_{2j+1}(A^n) \leq \lambda_{2j+1}(S_j^n) + \lambda_1(R_j^n) \leq \frac{2^{45}e^{-\mu_1}}{L}e^{-\mu_1 h^n \circ \Xi^n(j+1)},$$

where we used (5.22). Lastly, we note that for all $j \leq 2$, we have

$$(5.24) \quad \lambda_j(A^n) \leq \lambda_1(A^n) \leq \frac{2^{45}e^{-\mu_1}}{L}e^{-\mu_1 h^n \circ \Xi^n(1)},$$

where we used that $h^n(\Xi^n(1)) = h^n(y_{\pm}^n) = 0$. \square

Lemma 5.5. *Let $p \in (1, \infty)$ and $\Delta > 3$ such that $\mu_1/p \geq \ln 2$ and let $\gamma \in \mathcal{V}_L$. There exists a $L_0 \equiv L_0(\epsilon) > 0$ such that for all $L \geq L_0$ we obtain*

$$(5.25) \quad \|\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N) - \rho_{L,\Lambda_L,\gamma}^n\|_{1/p} \leq \frac{2^{56}}{L^{1/p}}e^{-\mu_1/p}.$$

Proof. Let us again define $A^n := (\rho_{\Lambda}^n(\varphi_{\gamma}) - \rho_{\Lambda,\gamma}^n)$ and $m := \dim \mathbb{H}_{\Lambda}^n$. Then

$$(5.26) \quad \|A^n\|_{1/p}^{1/p} = \sum_{j=1}^m \lambda_j^{1/p}(A^n).$$

We remark that $\{y_{\pm}^n\} \subseteq \{y : h^n(y) = 0\}$. Therefore, by Lemma 4.15 and Lemma 5.4, there exists a $L_0 \equiv L_0(\epsilon) > 0$ such that for all $L \geq L_0$ holds

$$(5.27) \quad \|A^n\|_{1/p}^{1/p} \leq \frac{2^{45/p}e^{-\mu_1/p}}{L^{1/p}}(2 + 2^{11}e^{-\mu_1/p}) \leq \frac{2^{56}}{L^{1/p}}e^{-\mu_1/p},$$

where we used that $\mu_1/p \geq \ln 2$. \square

5.3. Proof of Theorem 2.1. We are now prepared to prove the logarithmically corrected area law as stated in Theorem 2.1. As we already showed in Section 3.4, the entanglement entropy of the density $\rho_{L,\gamma}^N$ in the Ising-limit “ $\Delta = \infty$ ” satisfies this scaling behavior.

We use the formalism of spectral shift functions to control the difference in the entanglement entropy. For a finite dimensional vector space \mathbb{H} , the spectral shift function of a selfadjoint operator $A \in L(\mathbb{H})$ and a selfadjoint perturbation $B \in L(\mathbb{H})$ is given by $\xi(\cdot; A, A+B) : \mathbb{R} \rightarrow \mathbb{R}$,

$$(5.28) \quad \xi(E; A, A+B) := \text{tr}\{1_{\leq E}(A+B) - 1_{\leq E}(A)\}.$$

According to [CHN01], for any $p \in [1, \infty)$ the L^p -norm of the spectral shift functions satisfies

$$(5.29) \quad \|\xi(\cdot; A, A+B)\|_p \leq \|B\|_{1/p}^{1/p}.$$

Lemma 5.6. *Let \mathbb{H} be a finite dimensional Hilbert space, $A, B \in L(\mathbb{H})$ be self adjoint operators such that $\sigma(A), \sigma(A+B) \subseteq [0, 1]$. Let $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. Then*

$$(5.30) \quad |\text{tr}\{s(A+B) - s(A)\}| \leq \|B\|_{1/p}^{1/p}(1 + \|\ln(\cdot)1_{(0,1)}(\cdot)\|_q).$$

Proof. Kreĭn's theorem for the spectral shift function [Sch12] states that

$$(5.31) \quad \mathrm{tr}\{f(A+B) - f(A)\} = \int_{\mathbb{R}} f'(t)\xi(t; A, A+B) dt$$

for any compactly supported and smooth function $f \in C_c^\infty(\mathbb{R})$. Since s is not differentiable in 0, we cannot apply this result directly. We therefore define a family of suitable auxiliary functions $(s_\eta)_{\eta \in \mathbb{N}} \in C_0^\infty(\mathbb{R})$ such that $\lim_{\eta \rightarrow \infty} s_\eta(t) = s(t)$ for all $t \in [0, 1]$. Let $\chi \in C^\infty(\mathbb{R})$ be a function, such that $\chi(\mathbb{R}) = [0, 1]$, $\chi(t) = 0$ for $t \leq 1/2$ and $\chi(t) = 1$ for $t \geq 1$. For $\eta \in \mathbb{N}$ and $\tau \in \mathbb{R}$ let

$$(5.32) \quad s_\eta(\tau) := \chi(2 - \tau) \int_0^\tau s'(t)\chi(\eta t) dt.$$

Since both $s_\eta(0) = s(0) = 0$ and $\lim_{\eta \rightarrow \infty} \|(s'_\eta - s')1_{(0,1)}\|_p = 0$ for all $p \in [1, \infty)$ we conclude that $\lim_{\eta \rightarrow \infty} s_\eta(t) = s(t)$ for all $t \in [0, 1]$. Both, A and $A+B$ have a finite number of eigenvalues. Hence

$$(5.33) \quad \lim_{\eta \rightarrow \infty} |\mathrm{tr}\{s_\eta(A+B) - s_\eta(A)\} - \mathrm{tr}\{s(A+B) - s(A)\}| = 0.$$

For any $p, q \in (1, \infty)$, $1/p + 1/q = 1$, and $\eta \in \mathbb{N}$ we obtain by applying (5.31) to s_η that

$$(5.34) \quad |\mathrm{tr}\{s_\eta(A+B) - s_\eta(A)\}| \leq \|\xi(\cdot; A, A+B)\|_p \|s'_\eta 1_{(0,1)}\|_q,$$

where we used that $\xi(t; A, A+B) = 0$ for $t > 1$. The first term of the right hand side can be estimated by (5.29). We estimate the second term by

$$(5.35) \quad \|s'_\eta 1_{(0,1)}\|_q = \|s'(\cdot)\chi(\eta \cdot)1_{(0,1)}(\cdot)\|_q \leq \|s'1_{(0,1)}\|_q \leq 1 + \|\ln(\cdot)1_{(0,1)}(\cdot)\|_q.$$

Together with (5.33) this yields (5.30). \square

Remark 5.7. Observe that for all $q \in (1, \infty)$ we obtain the elementary estimate

$$(5.36) \quad \|\ln(\cdot)1_{(0,1)}(\cdot)\|_q = \Gamma(q+1)^{1/q} \leq 2q,$$

where Γ denotes the Gamma function.

Proof of Theorem 2.1. By (3.34), for every $E \in \sigma(H_L^N) \cap I_1$, there exists at least one $\gamma \in \mathcal{V}_L$ such that $E = \inf \sigma(\hat{H}_{L,\gamma}^N)$. Let $|\varphi_{L,\gamma}^N\rangle$ be the corresponding eigenvector.

Let $n \in \mathbb{N}$ with $N/2 < n < N$ and $p, q > 1$ such that $1 = 1/p + 1/q$. Let $\Delta > 3$, such that $\mu_1(\Delta)/p \geq \ln 2$. According to Lemma 5.5, Lemma 5.6 and Remark 5.7, there exists a $L'_0 \equiv L'_0(\epsilon) > e^2$ such that

$$(5.37) \quad |\mathrm{tr}\{s(\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N)) - s(\rho_{L,\Lambda_L,\gamma}^n)\}| \leq \frac{2^{56}}{L^{1/p}} e^{-\mu_1/p} (1 + 2q)$$

for all $L \geq L'_0$. We now choose $p, q > 1$ to depend on L as follows. Let $q \equiv q(L) := \ln(L)$ and $p \equiv p(L) := (1 - 1/\ln(L))^{-1}$. This implies $L^{1/p} = e^{-1}L$. For all $L \geq e^2$, we have $1/p \geq 1/2$ and if $\Delta > 25$, then this implies $\mu_1(\Delta)/2 \geq \ln 2$. For $L \geq L'_0$ we bound (5.37) by

$$(5.38) \quad |\mathrm{tr}\{s(\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N)) - s(\rho_{L,\Lambda_L,\gamma}^n)\}| \leq \frac{2^{57}}{L} e^{-\mu_1/2} (1 + 2\ln(L)).$$

Corollary 3.8 implies that $\mu_1(\Delta) \rightarrow \infty$ for $\Delta \rightarrow \infty$. Therefore, there exists a $\Delta_0 \geq 25$ such that

$$(5.39) \quad 2^{57} e^{-\mu_1/2} \leq 1/2$$

for all $\Delta \geq \Delta_0$. Hence, by applying (3.78) as well as (5.39), we obtain

$$(5.40) \quad \text{tr}\{s(\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N))\} \geq \text{tr}\{s(\rho_{L,\Lambda_L,\gamma}^n)\} - |\text{tr}\{s(\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N)) - s(\rho_{L,\Lambda_L,\gamma}^n)\}| \geq \frac{\ln L - 1}{L}.$$

This implies for the entanglement entropy that

$$(5.41) \quad S(\varphi_{L,\gamma}^N, \Lambda_L) \geq \sum_{\substack{n \in \mathbb{N}, \\ N/2 < n < N}} \text{tr}\{s(\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N))\} \geq (N/2 - 1) \frac{\ln L - 1}{L}.$$

We notice, that $\lim_{L \rightarrow \infty} \frac{N/2-1}{L} = \epsilon/2$. Hence, for all $\Delta \geq \Delta_0$ we have

$$(5.42) \quad \liminf_{L \rightarrow \infty} \frac{S(\varphi_{L,\gamma}^N, \Lambda_L)}{\ln L} \geq \frac{\epsilon}{2}.$$

□

APPENDIX A.

A.1. Uniqueness of fiber operator ground states.

Lemma A.1. *Let $\Delta > 2$, let $L, N \in \mathbb{N}$ with $1 < N < L - 1$. Then for any $\gamma \in \mathcal{V}_L$, the operator $\hat{H}_{L,\gamma}^N$ has exactly one eigenvalue in $[1 - \frac{1}{\Delta}, 1]$ and no eigenvalue in $(1, 2 - \frac{2}{\Delta})$.*

Proof. By Lemma 3.5, we get

$$(A.1) \quad \hat{H}_{L,\gamma}^N = -\frac{1}{2\Delta} \hat{A}_{L,\gamma}^N + \hat{W}_{L,\gamma}^N \geq \left(1 - \frac{1}{\Delta}\right) \hat{W}_{L,\gamma}^N.$$

There exists exactly one element $\hat{x}_0 \in \hat{\mathcal{V}}_L^N \cap \mathcal{V}_{L,1}^N$. It satisfies $W(\hat{x}_0) = 1$. For any other $\hat{x} \in \hat{\mathcal{V}}_L^N \setminus \{\hat{x}_0\}$ we have $W(\hat{x}) \geq 2$. Let $\phi_{L,\gamma,0}^N \in \mathbb{S}_{L,\gamma}^N$ be defined by $\phi_{L,\gamma,0}^N(\sigma, \hat{x}) := \delta_{\gamma,\sigma} \delta_{\hat{x}_0, \hat{x}}$.

Hence, the operator

$$(A.2) \quad \hat{H}_{L,\gamma}^N + \left(1 - \frac{1}{\Delta}\right) |\phi_{L,\gamma,0}^N\rangle\langle\phi_{L,\gamma,0}^N| \geq \left(1 - \frac{1}{\Delta}\right) \left(\hat{W}_{L,\gamma}^N + |\phi_{L,\gamma,0}^N\rangle\langle\phi_{L,\gamma,0}^N|\right) \geq \left(2 - \frac{2}{\Delta}\right)$$

is a rank-one perturbation of $\hat{H}_{L,\gamma}^N$. Therefore the unperturbed operator $\hat{H}_{L,\gamma}^N$ has at most one eigenvalue below $(2 - \frac{2}{\Delta})$. On the other hand, since $\langle\phi_{L,\gamma,0}^N, \hat{H}_{L,\gamma}^N \phi_{L,\gamma,0}^N\rangle = 1$, there exists at least one eigenvalue which is less than or equal to 1. Since for $\Delta > 2$, we get $1 < 2 - \frac{2}{\Delta}$, this concludes the proof. □

For $\gamma = 0$, it follows from the explicit structure of the fiber operator $\hat{H}_{L,0}^N$ that it has a unique ground state $\hat{\varphi}_{L,0}^N$ which can be chosen to be strictly positive. The same is true for the original operator H_L^N . This will allow us to conclude that $\varphi_{L,0}^N := (\mathfrak{F}_L^N)^* \hat{\varphi}_{L,0}^N$ is the ground state of H_L^N . The main tool for our result will be an idea presented in [YY66], where the existence of a strictly positive ground state for the XXZ model on the ring was established, however

let us also point out that this piece of the proof also follows from the Allegretto-Piepenbrink theorem shown in [HK11].

Lemma A.2. *Let $N, L \in \mathbb{N}$, $0 < N < L$. Moreover, let $E_0 \equiv E_0(L, N, \Delta) = \inf \sigma(\hat{H}_{L,0})$. Then, E_0 is non-degenerate and the corresponding eigenvector $\hat{\varphi}_{L,0}^N \in \mathbb{S}_{L,0}^N$ can be chosen such that $\|\hat{\varphi}_{L,0}^N\| = 1$ and $\hat{\varphi}_{L,0}^N(0, \hat{x}) > 0$ for all $\hat{x} \in \hat{\mathcal{V}}_L^N$. In addition, $\varphi_{L,0}^N := (\mathfrak{F}_L^N)^* \hat{\varphi}_{L,0}^N$ is the unique ground state of H_L^N .*

Proof. Firstly, note that if we choose the constant $C > 2N \geq \|W_L^N\| = \|\hat{W}_{L,0}^N\|$, we then get that the matrix representations of both operators $A_1 := (C\mathbb{1}_{\mathbb{H}_L^N} - H_L^N)$ and $A_2 := (C\mathbb{1}_{\mathbb{S}_{L,0}^N} - \hat{H}_{L,0}^N)$ have only non-negative entries. Moreover, note that since A_1 and A_2 are irreducible, we can choose $D \geq \dim(\mathbb{H}_L^N)$ large enough, such that the matrix entries of A_1^D and A_2^D will all be strictly positive. Hence, by the Perron-Frobenius Theorem, the largest eigenvalue of each of these operators A_1^D and A_2^D is positive, non-degenerate and the corresponding eigenfunctions can be chosen to be strictly positive. Let $\varphi_{L,0}^N$ and $\hat{\varphi}_{L,0}^N$ denote the eigenfunctions for A_1^D and A_2^D respectively, that satisfy these properties. Clearly, $\varphi_{L,0}^N$ and $\hat{\varphi}_{L,0}^N$ will then be the eigenfunctions of H_L^N and $\hat{H}_{L,0}^N$ corresponding to the respective minima E_0 and \hat{E}_0 of the spectra. Now, since H_L^N and $\hat{H}_{L,0}^N$ are unitarily equivalent via the Fourier transform \mathfrak{F}_L^N , the function $(\mathfrak{F}_L^N)^* \hat{\varphi}_{L,0}^N$ is also an eigenfunction of H_L^N and thus $\hat{E}_0 \in \sigma(H_L^N)$. However, from the explicit form of $(\mathfrak{F}_L^N)^*$ as given in (3.13), one sees that since $\hat{\varphi}_{L,0}^N \in \mathbb{S}_{L,0}^N$ we get that $(\mathfrak{F}_L^N)^* \hat{\varphi}_{L,0}^N$ is a strictly positive eigenfunction of H_L^N . Thus, we conclude that $(\mathfrak{F}_L^N)^* \hat{\varphi}_{L,0}^N = \varphi_{L,0}^N$ and consequently $E_0 = \hat{E}_0$. \square

A.2. An auxiliary result. The following lemma is an adaptation of a similar result in [ARFS20, Thm. 6.1].

Lemma A.3. *Let $N \in \mathbb{N}$ and*

$$(A.3) \quad \mathcal{X}^N := \{\chi = (\chi_1, \dots, \chi_N) \subseteq \mathbb{N}_0^N : \chi_1 \leq \dots \leq \chi_N\}.$$

Then, for all $\mu \geq \ln 2$ we have

$$(A.4) \quad \sum_{\chi \in \mathcal{X}^N \setminus \{0\}} e^{-\mu|\chi|_1} \leq 30e^{-\mu},$$

where $|\cdot|_1$ -denotes the ℓ^1 -norm of \mathbb{Z}^N .

Proof. Let $\Psi : \mathbb{N}_0^N \rightarrow \mathcal{X}^N$ with

$$(A.5) \quad x \mapsto \Psi(x) := (\Psi_1(x), \dots, \Psi_N(x)),$$

where for each $j \in \{1, 2, \dots, N\}$, we have defined $\psi_j(x) := \sum_{i=1}^j x_i$. Note that Ψ is a bijection.

For any $x \in \mathbb{N}_0^N$, we therefore get

$$(A.6) \quad \sum_{j=1}^N \Psi_j(x) = \sum_{k=1}^N (N - k + 1)x_k$$

and Ψ is a bijection, we have

$$(A.7) \quad \sum_{\chi \in \mathcal{X}^N} e^{-\mu|\chi|_1} = \sum_{x \in \mathbb{N}_0^N} e^{-\mu|\Psi(x)|_1} = \sum_{x \in \mathbb{N}_0^N} \prod_{k=1}^N e^{-\mu(N-k+1)x_k},$$

which yields

$$(A.8) \quad \sum_{\chi \in \mathcal{X}^N} e^{-\mu|\chi|_1} = \prod_{k=1}^N \sum_{y=0}^{\infty} e^{-\mu y(N-k+1)} = \prod_{k=1}^N \frac{1}{1 - e^{-\mu(N-k+1)}}.$$

This gives us the following estimate, which is uniform in N :

$$(A.9) \quad \sum_{\chi \in \mathcal{X}^N} e^{-\mu|\chi|_1} \leq \prod_{m=1}^{\infty} \frac{1}{1 - e^{-\mu m}} \leq \exp\left(\frac{2e^{-\mu}}{1 - e^{-\mu}}\right),$$

where we have used that $\ln(1 - \lambda)^{-1} \leq 2\lambda$, whenever $\lambda \in (0, 1/2)$. Hence,

$$(A.10) \quad \sum_{\chi \in \mathcal{X}^N \setminus \{0\}} e^{-\mu|\chi|_1} \leq \exp\left(\frac{2e^{-\mu}}{1 - e^{-\mu}}\right) - 1 \leq 4e^2 e^{-\mu},$$

since $e^x - 1 \leq xe^x$ for all $x \geq 0$ and $e^{-\mu} \leq 2^{-1}$ for $\mu \geq \ln 2$. \square

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