# HÖLDER CONTINUITY OF ABSOLUTELY CONTINUOUS SPECTRAL MEASURE FOR THE EXTENDED HARPER'S MODEL

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Abstract. We establish sharp results on the modulus of continuity of the distribution of the spectral measure for the extended Harper's model in the absolutely continuous spectrum regime.

## 1. Introduction

In this paper, we consider the following extended Harper's model:

$$(H_{\lambda,\alpha,\theta}u)_n = c(\theta+n\alpha)u_{n+1} + \tilde{c}(\theta+(n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta+n\alpha)u_n$$

where $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the phase, $\alpha \in \mathbb{T}$ is the frequency, $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ is the coupling constant and

$$c(\theta) := \lambda_1 e^{-2\pi i(\theta+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta+\frac{\alpha}{2})},$$
$$\tilde{c}(\theta) := \lambda_1 e^{2\pi i(\theta+\frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{-2\pi i(\theta+\frac{\alpha}{2})}.$$

Proposed by Thouless [39] in 1983, the extended Harper's model (EHM) attracted significant attention in physics literature, and has been studied rigorously by Bellissard [9], Helffer [22], Shubin [35] and others. Physically, it describes the influence of a transversal magnetic field of flux $\alpha$ on a single tight-binding electron in a 2-dimensional crystal layer. The 2-dimensional electron is permitted to hop to both nearest (expressed through $\lambda_2$) and next-nearest neighboring (NNN) interaction between lattice sites (expressed through $\lambda_1$ and $\lambda_3$). Since this operator is symmetric in $\lambda$, one may always assume $0 \leq \lambda_2$, $0 \leq \lambda_1 + \lambda_3$ and at least one of $\lambda_1, \lambda_2, \lambda_3$ to be positive.

The extended Harper's model is a generalization of the following central model in quasi-periodic Schrödinger operators called almost Mathieu operator (AMO) (in the physics literature also known as Harper's model):

$$(H_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + 2\lambda\cos 2\pi(\theta+n\alpha)u_n.$$

To see this, one only needs to take $\lambda_1 = \lambda_3 = 0$.

Recently, there are several progresses in the studying of spectral type of EHM, see [7, 19, 20, 25, 42], the structure of the spectrum, see [18, 34, 41], and the regularity of Lyapunov exponents (LE)[1], see [24, 28, 38]. All these results show that in general, although various spectral properties of EHM are similar to those of AMO, this model is technically more complicated to

[1]See Section 2.3 for the definition.

deal with. The complexity comes from the so-called singularities (i.e. the zeroes of $c(\theta)$)[2] of quasi-periodic Jacobi cocycles.

In the present paper, we mainly focus on the regularity of the universal spectral measure $\mu_\theta = \mu^{e_0}_{\lambda,\alpha,\theta} + \mu^{e_1}_{\lambda,\alpha,\theta}$ (see Section 2.2 for details) for extended Harper's model. Note that if $\mu_\theta = \mu_\theta^{pp}$, then the distribution of $\mu_\theta$ is not even continuous. Thus a more suitable question is the regularity of the distribution of $\mu_\theta$ when $\mu_\theta = \mu_\theta^{ac}$. This question was answered by Avila and Jitomirskaya [6] for one-frequency Schrödinger operator under the assumption that the frequency is Diophantine [3].

Note that the result in [6] only works for one-frequency quasi-periodic operators, since a crucial technique in [6] is *almost reducibility* developed by Avila and Jitomirskaya in [5] based on quantitative Aubry duality and it seems non-trivial to generalize the method in [5] to multi-frequency case. The results in [6] was recently generalized by Zhao in [44] to the multi-frequency case based on the classical KAM theory developed in [12–15, 23, 30], since results in [12–15, 30] do work in any dimension. In this paper, we generalize the results in [6, 44] to *singular Jacobi operators* which indicates the KAM theory also works for Jacobi operators.

More precisely, if one formulates Avila and Jitomirskaya's result [6] for the almost Mathieu operator, one can get the result that if the frequency is Diophantine, then the spectral measure of subcritical AMO ($|\lambda| < 1$) is 1/2-Hölder continuous. We will show that the same result holds for subcritical extended Harper's model. Since the extended Harper's model is more complicated (its coupling constant is a three dimensional vector), let us first explain the meaning of subcritical. The definition of subcritical comes from Avila's global theory [2]. In [28], for EHM, Jitomirskaya and Marx explicitly computed the Lyapunov exponent (Consult Section 2.4 for details.) for all $\lambda$ and all irrational $\alpha$. They showed the subcritical regime of EHM is exact as follow:

$$\begin{aligned}\Lambda =& \{0 \le \lambda_1 + \lambda_3 < \lambda_2, \lambda_2 > 1\} \\ & \cup (\{\max\{1, \lambda_2\} < \lambda_1 + \lambda_3, \lambda_2 > 0\} \cap \{\lambda_1 \neq \lambda_3\})\end{aligned}$$

Our main Theorem is

**Theorem 1.1.** *Assume $\alpha \in DC_1$ and $\lambda \in \Lambda$, there exists $C = C(\lambda, \alpha)$ such that for any $E \in \mathbb{R}$, there holds*

$$\mu_\theta(E - \epsilon, E + \epsilon) \le C\epsilon^{\frac{1}{2}},$$

*for any $\epsilon > 0$ and any $\theta \in \mathbb{T}$.*

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[2] See Section 2 for more explanations.

[3] $\alpha \in \mathbb{T}^d$ is called *Diophantine*, denoted by $\alpha \in \mathrm{DC}_d(\kappa, \tau)$, if there exist $\kappa > 0$ and $\tau > d - 1$ such that

$$\mathrm{DC}_d(\kappa,\tau) := \left\{\alpha \in \mathbb{T}^d : \inf_{j\in\mathbb{Z}} |\langle n, \alpha\rangle - j| > \frac{\kappa}{|n|^\tau}, \quad \forall\, n \in \mathbb{Z}^d \backslash \{0\}\right\}. \tag{1.1}$$

Let $\mathrm{DC}_d := \bigcup_{\kappa>0,\, \tau>d-1} \mathrm{DC}_d(\kappa, \tau)$.

**Remark 1.1.** *The $\frac{1}{2}$-modulus of continuity is sharp even for the integrated density of states (IDS). One may see [33] for an argument for quasi-periodic Schrödinger operators.*

**Remark 1.2.** *In general, one can't expect any Hölder continuity of the spectral measure in the singular spectrum region. See [8, 11] for counterexamples. Since if $\lambda \notin \Lambda$, $\mu_\theta$ has singular part, our result is also sharp in the aspect.*

We now give a brief review of the histories on the regularity of LE (IDS) for quasi-periodic Schrödinger/Jacobi operators. For analytic quasi-periodic potentials, in the positive Lyapunov exponent regime, Goldstein and Schlag [16] proved that for real analytic potentials with strong Diophantine frequency, LE is Hölder continuous (one-frequency) or weak Hölder continuous (multi-frequency). For the almost Mathieu operator, Bourgain [10] proved that for Diophantine $\alpha$ and large enough $\lambda$, LE is $\frac{1}{2}-\epsilon$-Hölder continuous for any $\epsilon > 0$. Later, Goldstein and Schlag [17] generalized Bourgain's result [10], and proved that if the potential is in a small $L^\infty$ neighborhood of a trigonometric polynomial of degree $k$, then the IDS is $\frac{1}{2k}-\epsilon$-Hölder continuous for all $\epsilon > 0$. Moreover, they further proved ([17]) that IDS is absolutely continuous for $a.e.$ $\alpha$. We refer the reader to [21, 43] for some recent results on the regularity of Lyapunov exponents in the positive Lyapunov regime.

In the zero Lyapunov exponent regime, based on Eliasson's perturbative KAM scheme [14], Amor [1] got $\frac{1}{2}$-Hölder continuity of IDS for quasi-periodic cocycles in $SL(2,\mathbb{R})$ with Diophantine frequency. Besides, Avila and Jitomirskaya [5] used almost localization and Aubry duality to obtain the same result with one frequency in the non-perturbative regime. A recent breakthrough belongs to Avila [2]: for one-frequency Schrödinger operators with general analytic potentials and irrational frequency, Avila [2] has established the fantastic global theory saying that Lyapunov exponent is a $C^\omega$-stratified function of the energy.

For analytic quasi-periodic Jacobi operators, in the positive Lyapunov exponent regime, Tao [37], Tao and Voda [38] proved that the Lyapunov exponent is the Hölder continuous if the frequency is Diophantine. Recently, Jian and Shi [24] proved 1/2-Hölder continuity of Lyapunov exponent for EHM with Liouvillean frequencies.

For the lower regularity case, Klein [29] proved that for Schrödinger operators with potentials in some Gevrey class, the Lyapunov exponent is weak Hölder continuous in the positive Lyapunov regime. Recently, Wang and Zhang [40] obtained the weak Hölder continuity of Lyapunov exponent as a function of energies, for a class of $C^2$ quasi-periodic potentials and for any Diophantine frequency. More recently, Cai, Chavaudret, You and Zhou [12] proved sharp Hölder continuity of Lyapunov exponent for quasi-periodic Schrödinger operator with small finitely differential potentials and Diophantine frequencies.

Up to now, the regularity result of the distribution of individual spectral measure for quasi-periodic Schrödinger operators is few. We mention that recently, Avila and Jitomirskaya [6] proved sharp Hölder continuity of $\mu_\theta$ in the non-perturbative regime for Diophantine frequencies. Later, Liu and Yuan [31] generalized this result to Liouvillean frequencies. Zhao [44] generalized this result to the multi-frequency case and Sun-Wang [36] generalized this result to the smooth case. Munger and Ong [32] showed Hölder continuity of spectral measure for extended CMV matrix (another class of Jacobi operators).

Finally, we give the structure of this paper. Some basic concepts and propositions are given in Section 2. In Section 3, we obtain some quantitative estimates by the KAM scheme developed in [12, 30]. Based on these quantitative estimates on conjugation transformation and constant matrix, the proof of the main theorem with some arguments in [6, 44] is given in Section 4.

## 2. Preliminaries

For a given function $f$ defined on a strip $\{|\Im z| < h\}$, we define $|f|_h := \sup_{|\Im z|<h} |f(z)|$. Analogously, for $f$ defined on $\mathbb{T}$, we set $|f|_0 := \sup_{x\in\mathbb{T}} |f(x)|$. For any $\theta \in \mathbb{R}$, we also set $\|\theta\|_{\mathbb{T}} := \inf_{j\in\mathbb{Z}} |\theta - j|$.

### 2.1. Continued Fraction Expansion.

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Suppose $\{\frac{p_n}{q_n}\}_n$ to be the best rational approximations of $\alpha$, we have $\|k\alpha\|_{\mathbb{T}} \geq \|q_{n-1}\alpha\|_{\mathbb{T}}$, $\forall\ 1 \leq k < q_n$, and

$$\frac{1}{2q_{n+1}} \leq \|q_n\alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}}.$$

Denote $\beta(\alpha) := \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}$. Equivalently, we have

$$\beta(\alpha) = \limsup_{n\to\infty} \frac{1}{|n|} \ln \frac{1}{\|n\alpha\|_{\mathbb{T}}}.$$

We can deduce if $\alpha \in \mathrm{DC}$, then $\beta(\alpha) = 0$.

### 2.2. Extended harper model.

For any $\psi \in \ell^2(\mathbb{Z})$, let $\mu^{\psi}_{\lambda,\alpha,\theta}$ be the spectral measure of $H_{\lambda,\alpha,\theta}$ corresponding to $\psi$:

$$\langle (H_{\lambda,\alpha,\theta} - E)^{-1}\psi, \psi\rangle = \int_{\mathbb{R}} \frac{1}{E - E'} d\mu^{\psi}_{\lambda,\alpha,\theta}(E'), \quad \forall E \in \mathbb{C}\backslash\Sigma_{\lambda,\alpha}.$$

We denote $\mu_\theta := \mu^{e_0}_{\lambda,\alpha,\theta} + \mu^{e_1}_{\lambda,\alpha,\theta}$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. This measure can serve as a universal spectral measure for $H_{\lambda,\alpha,\theta}$ because of the cyclic property.

We introduce another operator $\tilde{H}_{\lambda,\alpha,\theta}$ which is unitary equivalent to $H_{\lambda,\alpha,\theta}$ but its entries are real as follows

$$(\tilde{H}_{\lambda,\alpha,\theta} u)_n = |c|(\theta + n\alpha)u_{n+1} + |c|(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n$$

with $|c|(\theta) := \sqrt{c(\theta)\tilde{c}(\theta)}$ and $\tilde{H}_{\lambda,\alpha,\theta} = UH_{\lambda,\alpha,\theta}U^{-1}$, where $U$ is an unitary operator defined as follows:

$$\langle e_i, Ue_j\rangle = \delta_{ij}\prod_{n=0}^{j-1} e^{2\pi i \arg c(\theta+n\alpha)}.$$

Thus $H_{\lambda,\alpha,\theta}$ and $\tilde{H}_{\lambda,\alpha,\theta}$ share the same spectral property. So if we want to study the property of spectrum (from topology or measure theory) we could study $\tilde{H}_{\lambda,\alpha,\theta}$ instead of $H_{\lambda,\alpha,\theta}$.

The *integrated density of states* (IDS) is the function $N_{\lambda,\alpha} : \mathbb{R} \to [0, 1]$ defined by

$$N_{\lambda,\alpha}(E) = \int_{\mathbb{T}} \mu^{e_0}_{\lambda,\alpha,\theta}(-\infty, E]d\theta. \tag{2.1}$$

It is a continuous non-decreasing surjective function. Let $\tilde{N}_{\lambda,\alpha}$ be the IDS of $\{\tilde{H}_{\lambda,\alpha,\theta}\}_{\theta\in\mathbb{T}}$. We have $N_{\lambda,\alpha}(E) = \tilde{N}_{\lambda,\alpha}(E)$ (see [20]).

2.3. **Quasi-periodic cocycles.** Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $A \in C^0(\mathbb{T}, M_2(\mathbb{C}))$ be measurable with $\log\|A(x)\| \in L^1(\mathbb{T})$. The quasi-periodic *cocycle* $(\alpha, A)$ is the dynamical system on $\mathbb{T}\times\mathbb{C}^2$ defined by $(\alpha, A)(x, v) = (x + \alpha, A(x)v)$. The iterates of $(\alpha, A)$ are of the form $(\alpha, A)^n = (n\alpha, A_n)$, where

$$A_n(x) := \begin{cases} A(x + (n-1)\alpha)\cdots A(x + \alpha)A(x), & n \geq 0 \\ A^{-1}(x + n\alpha)A^{-1}(x + (n+1)\alpha)\cdots A^{-1}(x-\alpha), & n < 0 \end{cases}.$$

The *Lyapunov exponent* is defined by $L(\alpha, A) := \frac{1}{n}\int_{\mathbb{T}} \ln\|A_n(x)\|dx$.

Assume now that $A \in C^0(\mathbb{T}, \mathrm{SL}(2,\mathbb{R}))$ is homotopic to the identity. Then there exist $\phi : \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} \to \mathbb{R}$ and $v : \mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z} \to \mathbb{R}^+$ such that

$$A(x)\begin{pmatrix}\cos 2\pi y\\ \sin 2\pi y\end{pmatrix} = v(x, y)\begin{pmatrix}\cos 2\pi(y + \phi(x, y))\\ \sin 2\pi(y + \phi(x, y))\end{pmatrix}.$$

The function $\phi$ is called a lift of $A$. Let $\mu$ be any probability on $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$ which is invariant under the continuous map $T : (x, y) \to (x+\alpha, y+\phi(x, y))$, projecting over Lebesgue measure on the first coordinate. Then the number

$$\rho(\alpha, A) = \int \phi d\mu \bmod \mathbb{Z}$$

is independent of the choices of $\phi$ and $\mu$, and is called the *fibered rotation number* of $(\alpha, A)$.

We say that $(\alpha, A)$ is *uniformly hyperbolic* if for every $x \in \mathbb{T}$, there exists a continuous splitting $\mathbb{C}^2 = E^s(x) \oplus E^u(x)$ such that for every $n \geq 0$,

$$|A_n(x)v| \leq Ce^{-cn}|v|, \quad v \in E^s(x),$$
$$|A_n(x)^{-1}v| \leq Ce^{-cn}|v|, \quad v \in E^u(x + n\alpha),$$

for some constants $C, c > 0$. This splitting is invariant by the dynamics, i.e.,

$$A(x)E^*(x) = E^*(x + \alpha), \quad * = \text{“}s\text{” or “}u\text{”}, \quad \forall x \in \mathbb{T}.$$

We consider the cocycle $(\alpha, A_{\lambda,E})$ associated with extended Harper's model $\{H_{\lambda,\alpha,\theta}\}_{\theta\in\mathbb{T}}$. Any formal solution of $H_{\lambda,\alpha,\theta}u = Eu$ satisfies

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_{\lambda,E}(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad \forall n \in \mathbb{Z},$$

where

$$A_{\lambda,E}(\theta) = \frac{1}{c(\theta)} \begin{pmatrix} E - 2\cos 2\pi\theta & -\tilde{c}(\theta-\alpha) \\ c(\theta) & 0 \end{pmatrix}.$$

We introduce a normalized cocycle $(\alpha, \tilde{A}_{\lambda,E})$ associated with the operator $\tilde{H}_{\lambda,\alpha,\theta}$, where

$$\begin{aligned} \tilde{A}_{\lambda,E}(\theta) &= \frac{1}{\sqrt{|c|(\theta)|c|(\theta-\alpha)}} \begin{pmatrix} E - 2\cos 2\pi\theta & -|c|(\theta-\alpha) \\ |c|(\theta) & 0 \end{pmatrix} \\ &= Q_\lambda^{-1}(\theta+\alpha) A_{\lambda,E}(\theta) Q_\lambda(\theta), \end{aligned}$$

where $Q_\lambda$ is analytic on $|\Im\theta| \le h(\lambda)$ for some $h$ that is independent of $E$ when $\lambda$ belongs to the subcritical region (see [20]).

2.4. **Avila's global theory and its application to extended Harper's model.** Let us make a short review of Avila's global theory of one frequency $SL(2,\mathbb{R})$-cocycles [2]. Suppose that $A \in C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$ admits a holomorphic extension to $\{|\Im z| < h\}$. Then for $|\epsilon| < h$, we define $A_\epsilon \in C^\omega(\mathbb{T}, SL(2,\mathbb{C}))$ by $A_\epsilon(\cdot) = A(\cdot + i\epsilon)$. The cocycles which are not uniformly hyperbolic are classified into three classes: subcritical, critical, and supercritical. In particular, $(\alpha, A)$ is said to be subcritical if there exists $h > 0$ such that $L(\alpha, A_\epsilon) = 0$ for $|\epsilon| < h$.

A cornerstone in Avila's global theory is the "Almost Reducibility Conjecture" (ARC), which says that $(\alpha, A)$ is almost reducible if it is subcritical. Recall that the cocycle $(\alpha, A)$ is said to be almost reducible if there exist $h_* > 0$, and a sequence $B_n \in C^\omega_{h_*}(\mathbb{T}, PSL(2,R))$ such that $B_n^{-1}(\theta+\alpha)A(\theta)B(\theta)$ converges to constant uniformly in $|\Im\theta| < h_*$. The complete solution of ARC was recently given by Avila [3, 4], in the case $\beta(\alpha) = 0$, it is the following:

**Theorem 2.1.** *For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\beta(\alpha) = 0$, and $A \in C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$, $(\alpha, A)$ is almost reducible if it is subcritical.*

For the extended Harper's model, Jitomirskaya and Marx [28] identified subcritical, critical, and supercritical energies for all values of $\lambda$ and all irrational $\alpha$ via an explicit computation of the complexified Lyapunov exponent. In Avila, Jitomirskaya and Marx's paper [7], they summarized the results in the following theorem,

**Theorem 2.2** ([7, 28])**.** *For $\alpha$ irrational, all energies in the spectrum of extended Harper's model are*

(1) *supercritical for all $\lambda \in I^\circ \cup \{\lambda_1 + \lambda_3 = 0, 0 < \lambda_2 < 1\}$,*
(2) *subcritical for all $\lambda \in II^\circ \cup \{\lambda_1 + \lambda_3 = 0, \lambda_2 > 1\}$,*

*(3) subcritical for all* $\lambda \in III^\circ$ *if* $\lambda_1 \neq \lambda_3$,
*(4) critical for all* $\lambda \in III^\circ$ *if* $\lambda_1 = \lambda_3$,
*(5) critical for all* $\lambda \in L_I \cup L_{II} \cup L_{III}$,

*where*

**region** $I$**:** $0 \leq \lambda_1 + \lambda_3 \leq 1,\ 0 < \lambda_2 \leq 1$,
**region** $II$**:** $0 \leq \lambda_1 + \lambda_3 \leq \lambda_2,\ 1 \leq \lambda_2$,
**region** $III$**:** $\max\{1, \lambda_2\} \leq \lambda_1 + \lambda_3,\ \lambda_2 > 0$,

*and*

*(1)* $L_I := \{\ \lambda_1 + \lambda_3 = 1,\ 0 < \lambda_2 \leq 1\}$,
*(2)* $L_{II} := \{0 \leq \lambda_1 + \lambda_3 \leq \lambda_2,\ \lambda_2 = 1\}$,
*(3)* $L_{III} := \{\ 1 \leq \lambda_1 + \lambda_3 = \lambda_2\}$.

They also gave the precise computation of the singularities of extended Harper's model in the following proposition,

**Proposition 2.1** ([7])**.** *Letting* $z = \theta + i\epsilon$, $\theta \in \mathbb{T}$, $c(z)$ *has at most two zeros. Necessary conditions for real roots are* $\lambda_1 = \lambda_3$ *or* $\lambda_1 + \lambda_3 = \lambda_2$. *Moreover, (a) for* $\lambda_1 = \lambda_3$, $c(z)$ *has real roots if and only if* $2\lambda_3 \geq \lambda_2$, *determined by*

$$2\lambda_3 \cos(2\pi(\theta + \frac{\alpha}{2})) = -\lambda_2, \tag{2.2}$$

*and giving rise to a double root at* $\theta = \frac{1}{2} - \frac{\alpha}{2}$ *if* $\lambda_2 = 2\lambda_3$.
*(b) for* $\lambda_1 \neq \lambda_3$, $c(z)$ *has only one simple real root at* $\theta = \pm\frac{1}{2} - \frac{\alpha}{2}$ *if* $\lambda_1 + \lambda_3 = \lambda_2$.

**Remark 2.1.** *Thus, if* $\lambda$ *belongs to* $\Lambda$, $|c|(\cdot)$ *is analytic and non-zero on a small strip where* $c(\cdot)\tilde{c}(\cdot) \in \mathbb{C}\backslash(-\infty, 0]$.

2.5. $m$ **function for singular Jacobi operators.**

The Borel transform of the spectral measure $\mu_\theta$ takes the form

$$\begin{aligned} M(z) := F_\mu(z) &= \int \frac{d\mu(E)}{E - z} \\ &= \langle e_0, (H_{\lambda,\alpha,\theta} - zI)^{-1} e_0 \rangle + \langle e_1, (H_{\lambda,\alpha,\theta} - zI)^{-1} e_1 \rangle. \end{aligned} \tag{2.3}$$

It is standard that $M(z)$ has a close relation to the well-known Weyl-Titchmarsh $m$-function. Given $z \in \mathbb{C}^+$, then there are non-zero solutions $u_z^\pm$ of $H_{\lambda,\alpha,\theta} u_z^\pm = z u_z^\pm$ which are $\ell^2$ at $\pm\infty$. The Weyl-Titchmarsh $m$-functions are defined by

$$m_z^\pm = \mp \frac{u_z^\pm(1)}{u_z^\pm(0)},$$

we refer readers to consult [26, 27] for more details of the definitions.

As discussed in [27],

$$M(z) = \frac{m_z^+ m_z^- - 1}{m_z^+ + m_z^-}.$$

For $k \in \mathbb{Z}^+$ and $E \in \mathbb{R}$, let

$$P_k(E) = \sum_{j=1}^{k} A_{2j-1}^*(E) A_{2j-1}(E),$$

where $A_n(E) = A_{\lambda,E}(\theta + (n-1)\alpha) \cdots A_{\lambda,E}(\theta+\alpha) A_{\lambda,E}(\theta)$.

The following propositions which were proved in [6] for Schrödinger operators are very important for our applications. They also work for Jacobi operators. For completeness, we give the proof in the Appendix.

**Proposition 2.2** ([6])**.** *For any $E \in \mathbb{R}$ and $\epsilon_k = \sqrt{\frac{1}{4 \det P_k(E)}}$, we have*

$$\mu(E-\epsilon_k, E+\epsilon_k) \le 2\epsilon_k \Im M(E + i\epsilon_k) \le 4(5+\sqrt{24})\epsilon_k^2 \|P_k(E)\|.$$

**Proposition 2.3** ([6])**.** *For any $E \in \mathbb{R}$ and $\epsilon_k = \sqrt{\frac{1}{4 \det P_k(E)}}$, let $(u_n^\beta)_{n\ge 0}$ satisfy*

$$A_{\lambda,E}(\theta + n\alpha) \begin{pmatrix} u_n^\beta \\ u_n^\beta \end{pmatrix} = \begin{pmatrix} u_{n+1}^\beta \\ u_{n+1}^\beta \end{pmatrix},$$

$$u_0^\beta \cos\beta + u_1^\beta \sin\beta = 0, \quad |u_0^\beta|^2 + |u_1^\beta|^2 = 1.$$

*Then we have*

$$\det P_k(E) = \inf_\beta \|u^\beta\|_{2k}^2 \|u^{\beta+\pi/2}\|_{2k}^2,$$

*where for any integer $L$*

$$\|u\|_L = (\sum_{n=1}^{L} |u_n|^2)^{\frac{1}{2}}.$$

The following proposition is proved in [6, 44], we write it here for completeness.

**Proposition 2.4** ([6, 44])**.** *Assume that*

$$T = \begin{pmatrix} e^{2\pi i\theta} & c \\ 0 & e^{-2\pi i\theta} \end{pmatrix},$$

*let $X_k = \sum\limits_{j=1}^{k} (T^{2j-1})^*(T^{2j-1})$, then*

$$X_k = \begin{pmatrix} k & x_{k,1} \\ \bar{x}_{k,1} & x_{k,2} \end{pmatrix}$$

*where*

$$x_{k,1} = ce^{-2\pi i\theta} \sum_{j=1}^{k} \frac{e^{-4\pi i\theta(2j-1)} - 1}{e^{-4\pi i\theta} - 1},$$

$$x_{k,2} = k + |c|^2 \sum_{j=1}^{k} (\frac{\sin 2\pi(2j-1)\theta}{\sin 2\pi\theta})^2.$$

## 3. Quantitative almost reducibility

### 3.1. **Global to local reduction.** In this subsection, we introduce some reducibility results. Firstly by Avila's well-known "Almost Reducibility Conjecture" (Theorem 2.1) we have the following corollary,

**Corollary 3.1.** *Let $\alpha \in DC_1$ and $\lambda \in \Lambda$. There exists $r = r(\lambda, \alpha) > 0$ such that for any $\eta > 0$, $E \in \Sigma_{\lambda,\alpha}$, there exist $\Phi_E \in C^\omega_r(\mathbb{T}, \mathrm{PSL}(2,\mathbb{R}))$ and $\Gamma = \Gamma(\lambda, \alpha, \eta)$ such that*

$$\Phi_E(\theta+\alpha)^{-1}\tilde{A}_{\lambda,E}(\theta)\Phi_E(\theta) = R_{\phi(E)}e^{f_E(\theta)}. \tag{3.1}$$

*Moreover, $\|\Phi_E\|_r \le \Gamma$ and $\|f_E\|_r \le \eta$.*

**Remark 3.1.** *Note that if $F_E \in C^\omega_r(\mathbb{T}, \mathrm{gl}(2,\mathbb{R}))$ is small, we always can find some $f_E \in C^\omega_r(\mathbb{T}, \mathrm{sl}(2,\mathbb{R}))$ such that $R_{\phi(E)} + F_E(\theta) = R_{\phi(E)}e^{f_E(\theta)}$.*

*Proof.* The crucial fact in this corollary is that we can choose $r$ to be independent of $E$ and $\eta$, and choose $\eta$ to be independent of $E$. The ideas of the proof are essentially contained in Proposition 5.1 and Proposition 5.2 of [30], we include the proof here for completeness.

For any $E \in \Sigma_{\lambda,\alpha}$ and $\lambda \in \Lambda$, the cocycle $(\alpha, \tilde{A}_{\lambda,E})$ is subcritical, hence it is almost reducible by Theorem 2.1, i.e. there exists $h_0 = h_0(\lambda, \alpha, E) > 0$, such that for any $\eta > 0$, there are $\Phi_E \in C^\omega_{h_0}(\mathbb{T}, \mathrm{PSL}(2,\mathbb{R}))$, $F_E \in C^\omega_{h_0}(\mathbb{T}, \mathrm{gl}(2,\mathbb{R}))$ and $\phi(E) \in \mathbb{T}$ such that

$$\Phi_E(\cdot+\alpha)^{-1}\tilde{A}_{\lambda,E}(\cdot)\Phi_E(\cdot) = R_{\phi(E)} + F_E(\cdot),$$

with $\|F_E\|_{h_0} < \eta/2$ and $\|\Phi_E\|_{h_0} < \tilde{\Gamma}$ for some $\tilde{\Gamma} = \Gamma(\lambda, \alpha, \eta, E) > 0$. As a consequence, for any $E' \in \mathbb{R}$, one has

$$\left\|\Phi_E(\theta+\alpha)^{-1}\tilde{A}_{\lambda,E'}(\theta)\Phi_E(\theta) - R_{\phi(E)}\right\|_{h_0} < \frac{\eta}{2} + \left|\frac{1}{\sqrt{|c|(\theta)|c|(\theta-\alpha)}}\right| |E-E'| \|\Phi_E\|^2_{h_0}.$$

It follows that with the same $\Phi_E$, we have

$$\|\Phi_E(\theta+\alpha)^{-1}\tilde{A}_{\lambda,E'}(\theta)\Phi_E(\theta) - R_{\phi(E)}\|_{h_0} < \eta$$

for any energy $E'$ in a neighborhood $\mathcal{U}(E)$ of $E$. Since $\Sigma_{\lambda,\alpha}$ is compact, by compactness argument, we can select $h_0(\lambda, \alpha, E)$, $\Gamma(\lambda, \alpha, \eta, E) > 0$ to be independent of the energy $E$. □

### 3.2. **Local quantitative almost reducibility.** In this section, we concentrate on the following analytic quasi-periodic $SL(2,\mathbb{R})$ cocycle:

$$\begin{aligned}(\alpha, A_0e^{f_0(\theta)}) : \mathbb{T}^d \times \mathbb{R}^2 &\to \qquad \mathbb{T}^d \times \mathbb{R}^2 \\ (\theta, v) &\mapsto (\theta+\alpha, A_0e^{f_0(\theta)} \cdot v),\end{aligned}$$

where $f_0 \in C^\omega_{r_0}(\mathbb{T}^d, \mathrm{sl}(2,\mathbb{R}))$, $r_0 > 0$, $d \in \mathbb{N}^+$, and $\alpha \in DC_d$. Notice that $A_0$ has eigenvalues $\{e^{i\xi}, e^{-i\xi}\}$ with $\xi \in \mathbb{C}$.

The following quantitative almost reducibility proposition is proved in [44].

**Proposition 3.1** ([44]). *For any $0 < r < r_0$, $\kappa > 0$, $\tau > d-1$. Suppose that $\alpha \in DC_d(\kappa, \tau)$. Then there exist $B_n \in C_r^\omega(\mathbb{T}^d, PSL(2,\mathbb{R}))$ and $A_n \in SL(2,\mathbb{R})$ satisfying*

$$B_n^{-1}(\theta+\alpha)A_0 e^{f_0(\theta)}B_n(\theta) = A_n e^{f_n(\theta)},$$

*provided that $\|f_0\|_{r_0} < \epsilon_*$ for some $\epsilon_* > 0$ depending on $A_0, \kappa, \tau, r, r_0, d$, with the following estimates*

$$\|f_n\|_r \le \epsilon_n, \tag{3.2}$$

$$\|B_n\|_0 \le \epsilon_{n-1}^{-\frac{1}{800}}. \tag{3.3}$$

*Moreover, there exists unitary $U_n \in SL(2,\mathbb{C})$ such that*

$$U_n A_n U_n^{-1} = \begin{pmatrix} e^{i\xi_n} & c_n \\ 0 & e^{-i\xi_n} \end{pmatrix},$$

*and*

$$|c_n|\|B_n\|_0^8 \le 4\|A_0\|, \tag{3.4}$$

*with $\xi_n, c_n \in \mathbb{C}$.*

**Remark 3.2.** *$\varepsilon_*$ doesn't depend on $A_0$ as $A_0$ is a rotation matrix.*

### 3.3. Global quantitative almost reducibility.

**Proposition 3.2.** *Let $\alpha \in DC_1$ and $\lambda \in \Lambda$. Then there exist $B_n \in C_r^\omega(\mathbb{T}^d, PSL(2,\mathbb{R}))$ and $A_n \in SL(2,\mathbb{R})$ satisfying*

$$B_n^{-1}(\theta+\alpha)A_{\lambda,E}(\theta)B_n(\theta) = A_n e^{f_n(\theta)},$$

*with the following estimates*

$$\|f_n\|_r \le \epsilon_n, \tag{3.5}$$

$$\|B_n\|_0 \le C(\lambda,\alpha)\epsilon_{n-1}^{-\frac{1}{800}}. \tag{3.6}$$

*Moreover, there exists unitary $U_n \in SL(2,\mathbb{C})$ such that*

$$U_n A_n U_n^{-1} = \begin{pmatrix} e^{i\xi_n} & c_n \\ 0 & e^{-i\xi_n} \end{pmatrix},$$

*and*

$$|c_n|\|B_n\|_0^8 \le C(\lambda,\alpha), \tag{3.7}$$

*with $\xi_n, c_n \in \mathbb{C}$.*

*Proof.* Since $r_0(\lambda,\alpha)$ in Corollary 3.1 is fixed, and it is independent of $\eta$, then one can always take $\eta$ small enough such that

$$\eta \le \epsilon_*(\alpha, r, r_0),$$

where $\epsilon_*$ is the constant defined in Proposition 3.1 with $d = 1$. Note that by Remark 3.2, the constant $\epsilon_*$ given by Proposition 3.1 can be taken uniformly with respect to $R_\phi \in \mathrm{SO}(2,\mathbb{R})$. By Corollary 3.1, there exists $\Phi_E \in C^\omega_{r_0}(\mathbb{T}, PSL(2,\mathbb{R}))$ such that

$$\begin{aligned}\Phi_E^{-1}(\theta+\alpha)\tilde{A}_{\lambda,E}(\theta)\Phi_E(\theta) &= \Phi_E^{-1}(\theta+\alpha)Q_\lambda^{-1}(\theta+\alpha)A_{\lambda,E}(\theta)Q_\lambda(\theta)\Phi_E(\theta)\\ &= R_{\phi(E)}e^{f_E(\theta)}.\end{aligned}$$

with

$$\|\Phi_E\|_{r_0} \leq \Gamma, \tag{3.8}$$

and $\|f_E\|_{r_0} \leq \eta$. By our selection, $\|f_E\|_{r_0} \leq \eta \leq \epsilon_*(\alpha, r_0, r)$, then we can apply Proposition 3.1, and obtain $\bar{B}_n \in C^\omega_r(\mathbb{T}, PSL(2,\mathbb{R}))$ and $A_n \in SL(2,\mathbb{R})$, such that

$$\bar{B}_n(\theta+\alpha)^{-1}R_{\phi(E)}e^{f_E(\theta)}\bar{B}_n(\theta) = A_n e^{f_n(\theta)}.$$

with the following estimates

$$\|f_n\|_r \leq \epsilon_n, \tag{3.9}$$

$$\|\bar{B}_n\|_0 \leq \epsilon_{n-1}^{-\frac{1}{800}}. \tag{3.10}$$

Moreover, there exists unitary $U_n \in SL(2,\mathbb{C})$ such that

$$U_n A_n U_n^{-1} = \begin{pmatrix} e^{i\xi_n} & c_n \\ 0 & e^{-i\xi_n} \end{pmatrix},$$

and

$$|c_n|\|\bar{B}_n\|_0^8 \leq 4, \tag{3.11}$$

with $\xi_n, c_n \in \mathbb{C}$.

Let $B_n(\theta) = Q_\lambda(\theta)\Phi_E(\theta)\bar{B}_n(\theta)$, then

$$B_n(\theta+\alpha)^{-1}A_{\lambda,E}(\theta)B_n(\theta) = A_n e^{f_n(\theta)}.$$

By (3.8) and (3.10), we have

$$\|B_n\|_0 = \|Q_\lambda\Phi_E\bar{B}_n\|_0 \leq \|Q_\lambda\|\Gamma\epsilon_{n-1}^{-\frac{1}{800}} \leq C(\lambda,\alpha,r_0,r)\epsilon_{n-1}^{-\frac{1}{800}}$$

By (3.8), (3.10) and (3.11), we have

$$|c_n|\|B_n\|_0^8 \leq |c_n|\|Q_\lambda\Phi_E\bar{B}_n\|_0^8 \leq 4\|Q_\lambda\|_0^8\Gamma^8 \leq C(\lambda,\alpha,r_0,r).$$

Since $r_0, r$ only depend on $\lambda, \alpha$, thus $C$ only depends on $\lambda, \alpha$. We finish the whole proof. □

## 4. Proof of Theorem 1.1

We denote by

$$P_k(E)=\sum_{j=1}^{k}((A_{\lambda,E}(\theta))_{2j-1})^*(A_{\lambda,E}(\theta))_{2j-1}.$$

To prove Theorem 1.1, we only need to prove the following Lemma.

**Lemma 4.1.** *Assume $\alpha\in DC_1$ and $\lambda\in\Lambda$, there exists $C=C(\lambda,\alpha)>0$ for any $E\in\Sigma_{\lambda,\alpha}$, we have $\|P_k(E)\|\leq C\|P_k^{-1}(E)\|^{-3}$.*

**Proof of Theorem 1.1:** Since $P_k(E)$ is a self-adjoint matrix, we have $\|P_k(E)\|=\det P_k(E)\|P_k^{-1}(E)\|$. If we let $\det P_k(E)=\frac{1}{4\epsilon_k^2}$, by Lemma 4.1, we have

$$\|P_k(E)\|=\frac{1}{4\epsilon_k^2}\|P_k^{-1}(E)\|\leq\frac{C}{\epsilon_k^2}\|P_k(E)\|^{-\frac{1}{3}},$$

thus

$$\|P_k(E)\|\leq C\epsilon_k^{-\frac{3}{2}}.$$

By Proposition 2.2, we have

$$\frac{\Im M(E+i\epsilon_k)}{\epsilon_k}\leq C\epsilon_k^{-\frac{3}{2}}.$$

On the one hand, since $\|A_{\lambda,E}\|_0\leq C(\lambda,\alpha)$ for any $\lambda\in\Lambda$, thus any solution $u$ we have $\|u\|_{2(k+1)}\leq C(\lambda,\alpha)\|u\|_{2k}$, by Proposition 2.3 we have $\det P_{k+1}(E)\leq C\det P_k(E)$, thus $\epsilon_{k+1}\geq c\epsilon_k$. On the other hand, we can check easily in (5.1) that $\frac{\Im M(E+i\epsilon)}{\epsilon}$ is monotonic with respect to $\epsilon$, thus for any $\epsilon>0$, there exists $k$ such that $\epsilon_{k+1}<\epsilon<\epsilon_k$, combining this with the fact $\epsilon_{k+1}\geq c\epsilon_k$, we have

$$\frac{\Im M(E+i\epsilon)}{\epsilon}\leq\frac{\Im M(E+i\epsilon_{k+1})}{\epsilon_{k+1}}\leq C\epsilon_{k+1}^{-\frac{3}{2}}\leq C\epsilon_k^{-\frac{3}{2}}\leq C\epsilon^{-\frac{3}{2}}.$$

By (5.2), we have

$$\mu_\theta(E-\epsilon,E+\epsilon)\leq 2\epsilon\Im M(E+\epsilon)\leq C\epsilon^{\frac{1}{2}},\tag{4.1}$$

for $E\in\Sigma_{\lambda,\alpha}$ and $\theta\in\mathbb{T}^d$.

For any $E\in\mathbb{R}$, we have the following two cases

**Case 1:** $(E-\epsilon,E+\epsilon)\cap\Sigma_{\alpha,\lambda V}=\emptyset$, we have

$$\mu_\theta(E-\epsilon,E+\epsilon)=0\leq C\epsilon^{\frac{1}{2}}.$$

**Case 2:** $(E-\epsilon,E+\epsilon)\cap\Sigma_{\alpha,\lambda V}\neq\emptyset$, there exists $E'\in(E-\epsilon,E+\epsilon)\cap\Sigma_{\alpha,\lambda V}$, then

$$(E-\epsilon,E+\epsilon)\subset(E'-2\epsilon,E'+2\epsilon).$$

Thus

$$\mu_\theta(E-\epsilon,E+\epsilon)\leq\mu_\theta(E'-2\epsilon,E'+2\epsilon)\leq C(2\epsilon)^{\frac{1}{2}}\leq C\epsilon^{\frac{1}{2}}.$$

□

**Proof of Lemma 4.1:** Note that the proof of Lemma 4.1 is similar with the proof in [44]. However, there are two differences. The first one is for EHM there is a normalization transformation. The second one is in the almost reducible theorem of EHM, there is a global to local reduction. Thankfully, both of these two transformations only depend on $\lambda, \alpha$ (do not depend on $E$ !). The details are the followings:

Since $\alpha \in DC_1$ and $\lambda \in \Lambda$, by Proposition 3.2, there exist $r > 0$, $B_n \in C_r^\omega(\mathbb{T}^d, PSL(2,\mathbb{R}))$ and $A_n \in SL(2,\mathbb{R})$ satisfying

$$B_n^{-1}(\theta+\alpha)A_{\lambda,E}(\theta)B_n(\theta) = A_n e^{f_n(\theta)},$$

with the following estimates

$$\|f_n\|_r \le \epsilon_n, \tag{4.2}$$

$$\|B_n\|_0 \le C(\lambda,\alpha)\epsilon_{n-1}^{-\frac{1}{800}}. \tag{4.3}$$

Moreover, there exists unitary $U_n \in SL(2,\mathbb{C})$ such that

$$U_n A_n U_n^{-1} = \begin{pmatrix} e^{i\xi_n} & c_n \\ 0 & e^{-i\xi_n} \end{pmatrix},$$

and

$$|c_n|\|B_n\|_0^8 \le C(\lambda,\alpha), \tag{4.4}$$

with $\xi_n, c_n \in \mathbb{C}$.

For $E \in \Sigma_{\lambda,\alpha}$, we always have that $|\Im\xi_n| \le \epsilon_n^{\frac{1}{4}}$ since $(\alpha, A_n e^{f_n})$ is not uniformly hyperbolic. Let $\tilde{B}_n(\theta) = B_n(\theta)U_n^{-1} \in C_r^\omega(\mathbb{T}, PSL(2,\mathbb{C}))$ we have

$$\tilde{B}_n^{-1}(\theta+\alpha)A_{\lambda,E}(\theta)\tilde{B}_n(\theta) = \tilde{A}_n e^{\tilde{f}_n(\theta)}, \tag{4.5}$$

where $\tilde{A}_n = \begin{pmatrix} e^{2\pi i\gamma_n} & \tilde{c}_n \\ 0 & e^{-2\pi i\gamma_n} \end{pmatrix}$ with $\gamma_n = \frac{\Re\xi_n}{2\pi}$ and

$$\|\tilde{f}_n\|_r \le \epsilon_n^{\frac{1}{4}}, \tag{4.6}$$

$$\|\tilde{B}_n\|_0 \le \|B_n\|_0\|U_n^{-1}\| = \|B_n\|_0 \le C(\lambda,\alpha)\epsilon_{n-1}^{-\frac{1}{800}}, \tag{4.7}$$

$$|\tilde{c}_n|\|\tilde{B}_n\|_0^8 \le C(\lambda,\alpha). \tag{4.8}$$

For any $k \in (\epsilon_0^{-\frac{1}{40}}, \infty)$, we denote by

$$X_k = \sum_{j=1}^{k} (\tilde{A}_n^{2j-1})^* \tilde{A}_n^{2j-1}, \quad k \in I_n := (\epsilon_{n-1}^{-\frac{1}{40}}, \epsilon_n^{-\frac{1}{30}}),$$

$$\tilde{X}_k(\theta) = \sum_{j=1}^{k} ((\tilde{A}_n e^{\tilde{f}_n(\theta)})_{2j-1})^* (\tilde{A}_n e^{\tilde{f}_n(\theta)})_{2j-1}, \quad k \in I_n := (\epsilon_{n-1}^{-\frac{1}{40}}, \epsilon_n^{-\frac{1}{30}}).$$

We divide the remaining proof into three steps.

**STEP 1: Estimation of $X_k$.** The following Proposition is essentially proved by Avila and Jitomirskaya (Lemma 4.3 in [6]). See also Zhao (Step 1 on page 13 in [44]) for the present form.

**Proposition 4.1** ([6, 44]). *For any $k \in I_n$, we always have*

$$\|(X_k)^{-1}\|^{-1} \geq ck, \tag{4.9}$$

$$\|X_k\| \leq Ck(1+k^2|\tilde{c}_n|^2). \tag{4.10}$$

**STEP 2: Estimation of $\tilde{X}_k(\theta)$.**

We need the following Lemma in [6],

**Lemma 4.2** ([6]). *Assume $T = \begin{pmatrix} e^{2\pi i\theta} & c \\ 0 & e^{-2\pi i\theta} \end{pmatrix} \in SL(2,\mathbb{C})$ and $\tilde{T} \in C^0(\mathbb{T}^d, SL(2,\mathbb{C}))$, let $\tilde{T}_k(x) = \sum_{j=1}^{k} \tilde{T}^*_{2j-1}(x)\tilde{T}_{2j-1}(x)$ and $T_k = \sum_{j=1}^{k}(T^{2j-1})^*T^{2j-1}$, if $\|\tilde{T}-T\|_0 \leq \frac{1}{100}k^{-2}(1+2ck)^{-2}$, we have*

$$\|\tilde{T}_k - T_k\|_0 \leq 1.$$

Note that by (4.6) we have $\|\tilde{A}_n e^{\tilde{f}_n} - \tilde{A}_n\|_0 \leq 2\epsilon_n^{\frac{1}{4}}$. Since $k \in I_n$, we have $\epsilon_n^{\frac{1}{4}} \leq \frac{1}{100}k^{-2}(1+2\tilde{c}_n k)^{-2}$, by Lemma 4.2, we have

$$\|\tilde{X}_k - X_k\|_0 \leq 1,$$

thus

$$\|(\tilde{X}_k)^{-1} - (X_k)^{-1}\|_0 \leq \|(\tilde{X}_k)^{-1}\|_0 \|\tilde{X}_k - X_k\|_0 \|(X_k)^{-1}\|_0 \leq 1.$$

By (4.9) and (4.10), for any $k \in I_n$, we have

$$\|(\tilde{X}_k)^{-1}\|^{-1} \geq ck.$$

$$\|\tilde{X}_k\| \leq Ck(1+k^2|\tilde{c}_n|^2).$$

**STEP 3: Estimation of $P_k(E)$.**

For $k \in I_n$, by equation (4.5), we have

$$\|P_k(E)\|_0 \leq \|\tilde{B}_n\|_0^4 \|\tilde{X}_k\|_0 \leq C\|\tilde{B}_n\|_0^4 k(1+k^2|\tilde{c}_n|^2),$$

$$\|P_k^{-1}(E)\|_0^{-1} \geq \|\tilde{B}_n\|_0^{-4}\|(\tilde{X}_k)^{-1}\|_0^{-1} \geq c\|\tilde{B}_n\|_0^{-4}k,$$

thus

$$\frac{\|P_k(E)\|_0}{\|P_k^{-1}(E)\|_0^{-3}} \leq C\|\tilde{B}_n\|_0^{16}|\tilde{c}_n|^2 + C\|\tilde{B}_n\|_0^{16}k^{-2}.$$

On the one hand, $k^{-2} \leq \epsilon_{n-1}^{\frac{1}{20}}$, by (4.7), we have $C\|\tilde{B}_n\|_0^{16}k^{-2} \leq C(\lambda,\alpha)$. On the other hand, by (4.8), we have $C\|\tilde{B}_n\|_0^{16}|\tilde{c}_n|^2 \leq C(\lambda,\alpha)$, thus for any $k \in (\epsilon_0^{-\frac{1}{40}}, \infty)$

$$\frac{\|P_k(E)\|_0}{\|P_k^{-1}(E)\|_0^{-3}} \leq C(\lambda,\alpha). \tag{4.11}$$

For $k \in (0, \epsilon_0^{-\frac{1}{40}})$, it is obvious that there exists $C = C(\lambda, \alpha)$ such that

$$\frac{\|P_k(E)\|_0}{\|P_k^{-1}(E)\|_0^{-3}} \leq C. \tag{4.12}$$

(4.11) and (4.12) implies that for all $k \in \mathbb{N}$, we have

$$\frac{\|P_k(E)\|_0}{\|P_k^{-1}(E)\|_0^{-3}} \leq C(\lambda, \alpha).$$

This finish the proof of Lemma 4.1. □

## 5. Appendix

In this section, we give the proof of several propositions in the preliminary.
**Proof of Proposition 2.2:** By (2.3), for any $\epsilon > 0$, we have

$$\Im M(E + i\epsilon) = \int \frac{\epsilon}{(E' - E)^2 + \epsilon^2} d\mu(E'), \tag{5.1}$$

thus

$$\begin{aligned} \Im M(E + i\epsilon) &\geq \int_{E-\epsilon}^{E+\epsilon} \frac{\epsilon}{(E' - E)^2 + \epsilon^2} d\mu(E') \\ &\geq \frac{1}{2\epsilon} \int_{E-\epsilon}^{E+\epsilon} d\mu(E') \\ &= \frac{1}{2\epsilon} \mu(E - \epsilon, E + \epsilon). \end{aligned} \tag{5.2}$$

We denote $\psi(z) = \sup_\beta |R_{-\beta/2\pi} z|$, by the argument in Section 4.1 in [6], one has

$$|M(E + i\epsilon)| \leq \psi(m^+_{E+i\epsilon}). \tag{5.3}$$

By Lemma 4.2 [4]in [6], one has

$$\psi(m^+_{E+i\epsilon_k}) \leq 2(5 + \sqrt{24})\epsilon_k \|P_k(E)\|. \tag{5.4}$$

(5.2), (5.3) and (5.4) imply that

$$\begin{aligned} \mu(E - \epsilon_k, E + \epsilon_k) &\leq 2\epsilon_k \Im M(E + i\epsilon_k) \leq 2\epsilon_k \psi(m^+_{E+i\epsilon_k}) \\ &\leq 4(5 + \sqrt{24})\epsilon_k^2 \|P_k(E)\|. \end{aligned}$$

□

**Proof of Proposition 2.3:** By the definition of $P_k(E)$, we have

$$\|u^\beta\|_{2k}^2 = \langle P_k(E) \begin{pmatrix} u_1^\beta \\ u_0^\beta \end{pmatrix}, \begin{pmatrix} u_1^\beta \\ u_0^\beta \end{pmatrix} \rangle.$$

[4]Although Avila and Jitomirskaya proved Lemma 4.2 for Schrödinger operator, their results work for Jacobi operators without any change of the proof.

Since $P_k(E)$ is self-adjoint, it immediately follows that

$$\det P_k(E) = \inf_{\beta} \|u^{\beta}\|_{2k}^2 \|u^{\beta+\pi/2}\|_{2k}^2.$$

□

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