Critical phenomena, arithmetic phase transitions, and universality: some recent results on the almost Mathieu operator.

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Abstract. This is an expanded version of the notes of lectures given at the conference "Current Developments in Mathematics 2019" held at Harvard University on November 22–23, 2019. We present an overview of some recent developments.

Contents

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1 Introduction

Consider a two-dimensional discrete Laplacian: an operator on $\ell^2(\mathbb{Z}^2)$ of the form

$$(H\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + \psi_{m,n-1} + \psi_{m,n+1} \tag{1}$$

Fourier transform makes it unitarily equivalent to multiplication by $2(\cos 2\pi x + \cos 2\pi y)$ on $L^2(\mathbb{T}^2)$ where $T^2 := \mathbb{R}^2/\mathbb{Z}^2$. Thus its spectrum[2] is the segment $[-4, 4]$.

In physics this is a tight binding model of a single electron confined to a 2D crystal layer. What happens if we put this crystal in a uniform magnetic field with flux orthogonal to the lattice plane? Of course, we have a freedom of gauge choice, but all the resulting operators are unitarily equivalent, so we may as well choose one, the so called Landau gauge, leading to the discrete magnetic Laplacian[3] operator

$$(H(\alpha)\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + e^{-i\alpha m}\psi_{m,n-1} + e^{i\alpha m}\psi_{m,n+1} \tag{2}$$

Even though incorporating those phase factors may seem innocent enough, basic quantum mechanics teaches us that magnetic fields may have a profound effect on allowed energies. In the continuum model, subjecting the electron plane to a perpendicular magnetic field of flux α changes the standard Laplacian into a direct integral of shifted harmonic oscillators, and thus the $[0, \infty)$ spectrum of the Laplacian turns into a discrete set of infinitely degenerate Landau levels, at $c|\alpha|(n + 1/2), n \in \mathbb{N}$. It turns out that in the discrete setting, the situation is even more dramatic and also much more rich and interesting. For any irrational α, the spectrum of $H(\alpha)$ is a Cantor

[2] See Section 2 for a quick reminder on the basics of spectral theory and ergodic operators

[3] The name "discrete magnetic Laplacian" first appeared in [81]

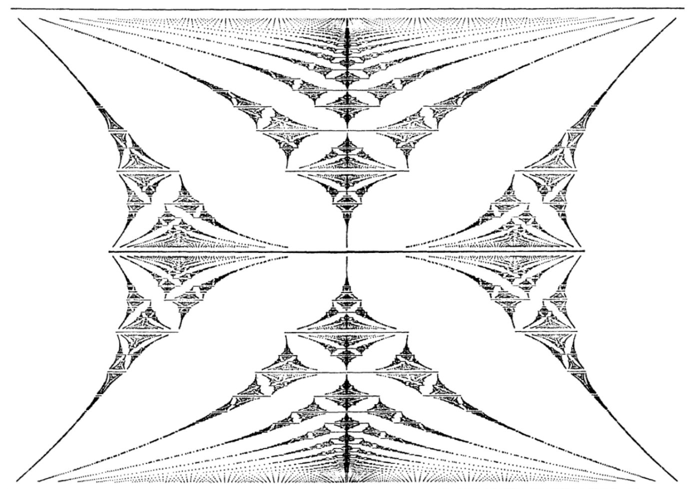

Figure 1: This picture is a plot of spectra of $H(\alpha)$ for 50 rational values of α [48]. The fluxes $\alpha = p/q$ are listed on the vertical line, and the corresponding horisontal sections are spectra of $H(\alpha)$.

set of measure zero, and the spectra for rational α, plotted together, form a beautiful self-similar structure, shown on Fig.1, called Hofstadter's butterfly.

Operator $H(\alpha)$, to the best of our knowledge, was introduced by Peierls in [79], and later studied by his student Harper. The first predictions of Cantor spectrum with arithmetic, continued-fraction based hierarchy of both the spectrum and eigenfunctions was made by Mark Azbel [21], remarkably, before any numerics was even possible. Yet, the model got a particular prominence only after Hofstadter's numerical discovery [48].

It was noticed already by Peierls in [79] that, similarly to the described above Landau gauge solutions for free electrons in a uniform magnetic field, the Landau gauge in the discrete setting, as in (2), also makes the Hamiltonian separable and turns it into the direct integral in θ of operators $H_{\alpha,\theta} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, of the form

$$(H_{\alpha,\theta}\phi)(n) = \phi(n-1) + \phi(n+1) + 2\cos 2\pi(\alpha n + \theta)\phi(n). \tag{3}$$

In this sense, $H_{\alpha,\theta}$ can be viewed as the tight-binding analogue of the harmonic oscillator. Here α is a magnetic flux per unit cell, and θ is a phase parameter characterizing plane waves in the direction perpendicular to the vector-potential, so has no meaning to the physics of the original 2D operators. Usually, one introduces also another parameter λ, characterizing the anisotropy of the lattice: it is the the ratio between the length of a unit cell in the direction of the vector potential and its length in the transversal direction, leading to the 2D operator

$$(H(\alpha,\lambda)\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + \lambda e^{-i\alpha m}\psi_{m,n-1} + \lambda e^{i\alpha m}\psi_{m,n+1} \tag{4}$$

and the family

$$(H_{\alpha,\theta,\lambda}\phi)(n) = \phi(n-1) + \phi(n+1) + 2\lambda\cos 2\pi(\alpha n + \theta)\phi(n). \tag{5}$$

In physics literature, this family has appeared under the names Harper's, Azbel-Hofstadter, and Aubry-Andre model (with the first two names also used for the discrete magnetic Laplacian $H(\alpha,\lambda)$) and often restricted to the isotropic case $\lambda = 1$. In mathematics, the name almost Mathieu operator is used universally, so we also use it for these lectures. This name was originally introduced by Barry Simon [82] in analogy with the Mathieu equation $-f'' + 2\lambda\cos x f(x) = Ef(x)$. For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $H_{\alpha,\theta,\lambda}$ is an ergodic (and minimal) family, so (see Sections 2.3, refergodic) the spectra $\sigma(H_{\alpha,\theta,\lambda})$ do not depend on θ and coincide with the spectrum $\sigma_{\alpha,\lambda}$ of the 2D operator $H(\alpha,\lambda)$. In general (which is only relevant for rational α), we have $\sigma_{\alpha,\lambda} = \cup_\theta \sigma(H_{\alpha,\theta,\lambda})$, and this is what the Hofstadter butterfly represents.

These lectures are devoted to some recent (roughly last 3 years) advances on this model. They are by no means comprehensive, neither historically, as we only mention past papers directly relevant to the presented results, nor even in terms of very recent advances, as it is a fast developing field with many exciting developments even in the last few years.

In physics, this model is the theoretical underpinning of the Quantum Hall Effect (QHE), as proposed by D.J. Thouless in 1983, and is therefore directly related to two Nobel prizes: von Klitzig (1998, for his experimental discovery of the Integer QHE) and Thouless (2016, for the theory behind the QHE and related topological insulators). Thouless theory is illustrated by Fig. 2, where Chern numbers corresponding to each gap are produced using the equations in [91], and color-coded, with warmer colors corresponding to positive numbers, and colder colors to negative ones.

The model also has very strong relationship to the theory of graphene (Geim and Novoselov, Nobel prize 2010), a robust 2D magnetic material

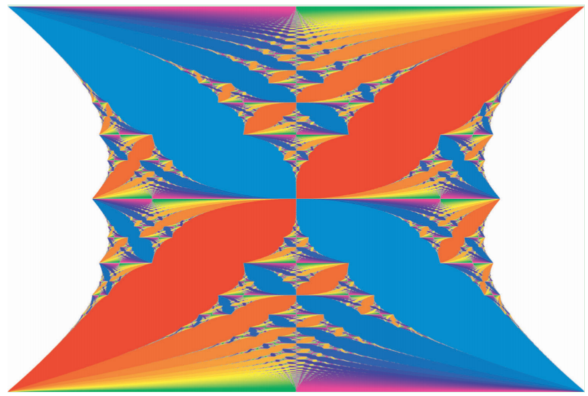

Figure 2: This picture is produced by Avron-Osadchy-Seiler [19].

whose spectra also form similar butterflies, and quasicrystals (Schechtman, Nobel prize 2011), as it is a standard model of a 1D quasicrystal. To make it a total of five Nobel prizes, one can also argue a weak relationship to the Anderson localization Nobel prize (Anderson, 1977), for Anderson localization is one property of this family for certain parameters, and, more importantly, it features the metal-insulator transition (something only seen, but prominently yet mysteriously so, in 3D or higher, for the random model). Then, one can also add a 2014 Fields medal and the 2020 Heineman prize to the list!

One of the most interesting features of the almost Mathieu family is sharp phase transitions in its several parameters, for various properties. The system, in particular, has distinct behaviors for $\lambda < 1$ and $\lambda > 1$. These two regimes have traditionally been approached perturbatively, by different KAM-type schemes, and then non-perturbative methods have been developed [51, 53], allowing to obtain the a.e. results up to the phase transition value $\lambda = 1$. Since then, even sharper localization [7, 59, 60] and reducibility [95] techniques have been developed, allowing to treat various delicate questions on both $\lambda > 1$ and $\lambda < 1$ sides. None of these methods work for the actual transition point $\lambda = 1$, and the operator at the critical value remains least understood. Yet it corresponds to the isotropic model, so is the most important operator in the one-parameter family from the physics viewpoint. From the dynamical systems point of view, the critical case is also special: the transfer-matrix cocycle for energies on the spectrum is critical in the sense of Avila's global theory (see Sections 2.5, 2.7), and thus non-amenable to either supercritical (localization) or subcritical (reducibility) methods. The global theory tells us that critical cocycles are rare in many ways, so it is almost tempting to ignore them in a large mathematical picture. Yet, as models coming from physics tend to be entirely critical on their spectra in this sense, one can actually argue that it is their study that is the most important.

After the preliminaries, we start with two very recent results on the critical case: the singular-continuous nature of the spectrum and Hausdorff dimension of the spectrum as a set, both subject to long-standing conjectures. Our solution of both conjectures is based on exploring certain hidden singularity of the model. The developed technique allowed also to obtain sharp estimate on the Hausdorff dimension of the spectrum for another interesting model, quantum graph graphene, where singularity is also present. The study of the Hausdorff dimension of course only makes sense once we know the spectrum has measure zero. This was proved by Last [72] for a.e. irrational α, but remarkably resisted treatment for the remaining zero measure set, that included the golden mean, the most popular irrational number in the physics community. Barry Simon listed the problem to obtain the result for the remaining parameters in his list of mathematical problems for the XXI century [85]. It was solved by Avila-Krikorian [10] who were able to treat Diophantine α using deep dynamical methods (for $\lambda \neq 1$ the solution was given in [56]). Our proof of the Hausdorff dimension estimate [57] (joint with Igor Krasovsky) allows also to give a very simple proof of this theorem, simultaneously for all irrational α.

Another very interesting feature of the almost Mathieu family is that, while α is a parameter coming from physics, the system behaves differently depending on whether α is rational or irrational. While this aspect was well understood already in the 60s, and the metal-insulator transition at $\lambda = 1$ was discovered by the physicists, Aubry and Andre [1], the physicists missed further dependence on the arithmetics within the class of irrational numbers. In mathematics, it was soon understood by Avron and Simon [17], based on Gordon [40], that within the super-critical regime the arithmetics of α plays a role, and later, in [66], that so does the arithmetics of θ. In [50] we conjectured that there is the second sharp transition governed by the arithmetics of the continued fraction expansion of α and the exponential rate of phase-resonances. The recent proof of this conjecture, joint with Wencai Liu, for both the frequency and phase cases, is discussed in Sec. 6.

A very captivating question and a longstanding theoretical challenge is to explain the self-similar hierarchical structure visually obvious in the Hofstadter's butterfly, as well as the hierarchical structure of eigenfunctions, as related to the continued fraction expansion of the magnetic flux. Such structure was first predicted in the work of Azbel in 1964 [21], some 12 years before Hofstadter [48] and before numerical experimentation was possible.

The simplest mathematical feature of the spectrum for irrational α one observes in the Hofstadter's picture, is that it is a Cantor set. Mark Kac offered ten martinis in 1982 for the proof of Azbel's 1964 Cantor set conjecture. It was dubbed the Ten Martini problem by Barry Simon, who advertised it in his lists of 15 mathematical physics problems [84] and later, mathematical physics problems for the XXI century [85]. Most substantial partial solutions were made by Bellissard, Simon, Sinai, Helffer, Sjöstrand, Choi, Eliott, Yui, and Last, between 1983-1994. J. Puig [80] solved it for Diophantine α by noticing that localization at $\theta = 0$ [53] leads to gaps at corresponding (dense) energies. Final solution was given in [7]. Cantor spectrum is also generic for general one-frequency operators with analytic potential: in the subcritical regime [8], and, by very different methods, in the supercritical regime [39] (and it is conjectured [9] also in the critical regime, which is actually non-generic in itself [5]). Moreover, even all gaps predicted by the gap labeling are open in the non-critical almost Mathieu case [8, 15]. Ten Martini and its dry version were very important challenges in themselves, even though these results, while strongly indicate, do not describe or explain the hierarchical structure, and the problem of its description/explanation remains open, even in physics. As for the understanding the hierarchical behavior of the eigenfunctions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions [93], it has also remained an important open challenge even at the physics level. Certain results indicating the hierarchical structure in the corresponding semi-classical/perturbative regimes were previously obtained in the works of Sinai, Helffer-Sjostrand, and Buslaev-Fedotov (see [30, 46, 87], and also [98] for a different model).

In Secs. 7,8 we present the solution of the latter problem in the exponential regime. We describe the universal self-similar exponential structure of eigenfunctions throughout the entire localization region. In particular, we determine explicit universal functions $f(k)$ and $g(k)$, depending only on the Lyapunov exponent and the position of k in the hierarchy defined by the denominators q_n of the continued fraction approximants of the flux α, that completely define the exponential behavior of, correspondingly, eigenfunctions and norms of the transfer matrices of the almost Mathieu operators, for all eigenvalues corresponding to a.e. phase, see Theorem 8.1. Our result holds for *all* frequency and coupling pairs in the localization regime. Since the behavior is fully determined by the frequency and does not depend on the phase, it is the same, eventually, around any starting point, so is also seen unfolding at different scales when magnified around local eigenfunction maxima, thus describing the exponential universality in the hierarchical structure, see, for example, Theorems 7.2, 7.4.

Moreover, our proof of the phase part of the arithmetical spectral transition conjecture uncovers a universal structure of the eigenfunctions throughout the corresponding pure point spectrum regime, Theorem 8.1, which, in presence of exponentially strong resonances, demonstrates a new phenomenon that we call a *reflective hierarchy*, when the eigenfunctions feature self-similarity upon proper reflections (Theorem 8.2). This phenomenon was not even previously described in the (vast) physics literature. This joint work with Wencai Liu will also be presented in Sections 7,8.

In the next section we list the basic definitions/necessary facts. Sections 3-5 are devoted to the critical almost Mathieu operator, and Sections 6-8 to sharp arithmetic spectral transitions and universal structure of eigenfunctions in the (supercritical) regime of localization.

2 The basics

2.1 The spectrum

The spectrum of a bounded linear operator H on a Hilbert space $\mathcal{H}$, denoted $\sigma(H)$, is the set of energies E for which $H - E$ does not have a bounded inverse. If $\mathcal{H}$ is finite-dimensional, it clearly coincides with the set of the eigenvalues. For an infinite-dimensional space, however, there are more ways not to be invertible than to have a kernel.

Example: Let (X, μ) be a measure space. Given bounded $f : X \to \mathbb{R}$, define the multiplication operator H_f by

$$H_f : L^2(X, \mu) \to L^2(X, \mu),\ H_f(g) = fg.$$

Then the formal inverse of $H_f - E$ is, of course, $H_{\frac{1}{f-E}}$, and it is easy to show that $\sigma(H_f)$ is the μ-essential range of f, that is $\{E : \mu(E - \epsilon, E + \epsilon) > 0, \text{any }\ \epsilon > 0.\}$

Note that the specrum is a unitary invariant, and it turns out that the example above is in this sense all there is:

Spectral theorem: Every self-adjoint $A : \mathcal{H} \to \mathcal{H}$ is unitarily equivalent to H_f for some f, X, μ.

It should be noted that no uniqueness of either of f, X or μ is claimed (or holds) here; in fact the more standard statement is with f fixed as x.

Example 1: If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a self-adjoint matrix with distinct eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n$, one can take $X = \mathbb{R}$, μ any measure that lives on $\cup_{i=1}^n \lambda_i$ and gives non-zero weight to each λ_i, and $f = x$. Then $L^2(X, \mu)$ is just $\mathbb{R}^n$ and the spectral theorem boils down to the diagonalization theorem for self-adjont matrices. In case of higher dimensional eigenspaces, one can take X equal to the union of k copies of $\mathbb{R}$, with k equal to the largest multiplicity of an eigenvalue, and modify the μ accordingly, keeping $f = x$.

Example 2: By Fourier transforming $\ell^2(\mathbb{Z}^2)$ into $L^2(\mathbb{T}^2)$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the discrete 2D Laplacian

$$(H\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + \psi_{m,n-1} + \psi_{m,n+1}$$

is unitarily equivalent to $H_{2\cos x + 2\cos y}$ on $L^2(\mathbb{T}^2)$, so $\sigma(H) = [-4, 4]$.

2.2 Spectral measure of a self-adjoint operator

Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. The time evolution of a wave function is described in the Schrödinger picture of quantum mechanics by

$$i\frac{\partial \psi}{\partial t} = H\psi.$$

The solution with initial condition $\psi(0) = \psi_0$ is then given by

$$\psi(t) = e^{-itH}\psi_0.$$

Another version of the spectral theorem says that for any $\psi_0 \in \mathcal{H}$, there is a unique finite measure μ_{ψ_0} (called the spectral measure of $\psi_0 \in \mathcal{H}$) such that

$$(e^{-itH}\psi_0, \psi_0) = \int_{\mathbb{R}} e^{-it\lambda} d\mu_{\psi_0}(\lambda). \tag{6}$$

2.3 Spectral decompositions

Every finite measure on $\mathbb{R}$ is uniquely decomposed into three mutually singular parts

$$\mu = \mu_{pp} + \mu_{sc} + \mu_{ac},$$

where *pp* stands for pure point, the atomic part of the measure, *ac* stands for absoltely continuous with respect to Lebesgue measure, and *sc* stands for singular continuous, that is all the rest: the part that is singular (with respect to Lebesgue), yet continuous (has no atoms).
Define

$$\mathcal{H}_\gamma = \{\phi \in \mathcal{H} : \mu_\phi \text{ is } \gamma\}$$

where $\gamma \in \{pp, sc, ac\}$. Then we have $\mathcal{H} = \mathcal{H}_{pp} \bigoplus \mathcal{H}_{sc} \bigoplus \mathcal{H}_{ac}$.

H preserves each $\mathcal{H}_\gamma$, so we can define: $\sigma_\gamma(H) = \sigma(H|_{\mathcal{H}_\gamma})$, $\gamma \in \{pp, sc, ac\}$. The set $\sigma_{pp}(H)$ admits a direct characterization as the closure of the set of all eigenvalues

$$\sigma_{pp}(H) = \overline{\sigma_p(H)},$$

where

$$\sigma_p(H) = \{\lambda : \text{ there exists a nonzero vector } \psi \in \mathcal{H} \text{ such that } H\psi = \lambda\psi\}.$$

2.4 Ergodic operators

We are going to study Schrödinger operators with potentials related to dynamical systems. Let $H = \Delta + V$ be defined by

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n) \tag{7}$$

on a Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$. Here $V : \mathbb{Z} \to \mathbb{R}$ is the potential. Let (Ω, P) be a probability space. A measure-preserving bijection $T : \Omega \to \Omega$ is called ergodic, if any T-invariant measurable set $A \subset \Omega$ has either $P(A) = 1$ or $P(A) = 0$. By a dynamically defined potential we understand a family $V_\omega(n) = v(T^n\omega), \omega \in \Omega$, where $v : \Omega \to \mathbb{R}$ is a measurable function. The corresponding family of operators $H_\omega = \Delta + V_\omega$ is called an ergodic family. More precisely,

$$(H_\omega u)(n) = u(n+1) + u(n-1) + v(T^n\omega)u(n). \tag{8}$$

Theorem 2.1 (Pastur [78]; Kunz-Souillard [71])**.** *There exists a full measure set Ω_0 and $\sum$, $\sum_{pp}$, $\sum_{sc}$, $\sum_{ac} \subset \mathbb{R}$ such that for all $\omega \in \Omega_0$, we have $\sigma(H_\omega) = \sum$, and $\sigma_\gamma(H_\omega) = \sum_\gamma$, $\gamma = pp, sc, ac$.*

Theorem 2.2. *[Avron-Simon [18],Last-Simon [75]]If T is minimal, then $\sigma(H_\omega) = \sum$, and $\sigma_{ac}(H_\omega) = \sum_{ac}$ for all $\omega \in \Omega$.*

Theorem 2.2 does not hold for $\sigma_\gamma(H_\omega)$ with $\gamma \in \{sc, pp\}$ [66], but it is an interesting and difficult open problem whether it holds for $\sigma_{sing}(H_\omega)$.

2.5 Cocycles and Lyapunov exponents

By an $SL(2,\mathbb{R})$ cocycle, we mean a pair (T, A), where $T : \Omega \to \Omega$ is ergodic, A is a measurable 2×2 matrix valued function on Ω and $\det A = 1$.

We can regard it as a dynamical system on $\Omega \times \mathbb{R}^2$ with

$$(T, A) : (x, f) \longmapsto (Tx, A(x)f), \ (x, f) \in \Omega \times \mathbb{R}^2.$$

For $k > 0$, we define the k-step transfer matrix as

$$A_k(x) = \prod_{l=k}^{1} A(T^{l-1}x). \tag{9}$$

For $k < 0$, define

$$A_k(x) = A_{-k}^{-1}(T^k x). \tag{10}$$

Denote $A_0 = I$, where I is the 2×2 identity matrix. Then $f_k(x) = \ln ||A_k(x)||$ is a subadditive ergodic process. The (non-negative) Lyapunov exponent for the cocycle (α, A) is given by

$$L(T, A) = \inf_n \frac{\int_\Omega \ln \|A_n(x)\| dx}{n} = \lim_n \frac{\int_\Omega \ln \|A_n(x)\| dx}{n} \overset{\text{a.e. } x}{=\!=\!=} \lim_{n\to\infty} \frac{\ln \|A_n(x)\| dx}{n}. \tag{11}$$

with both the second and the third equality in (11) guaranteed by Kingman's subadditive ergodic theorem. Cocycles with positive Lyapunov exponent are called hyperbolic. Here one should distinguish uniform hyperbolicity where there exists a continuous splitting of $\mathbb{R}^2$ into expanding and contracting directions, and nonuniform hyperbolicity, where $L > 0$ but such splitting does not exist. Nevertheless, we have

Theorem 2.3 (Oseledets)**.** *Suppose $L(T, A) > 0$. Then, for almost every $x \in \Omega$, there exist solutions $v^+, v^- \in \mathbb{C}^2$ such that $||A_k(x)v^\pm||$ decays exponentially at $\pm\infty$, respectively, at the rate $-L(T, A)$. Moreover, for every vector w which is linearly independent with v^+ (resp., v^-), $||A_k(x)w||$ grows exponentially at $+\infty$ (resp., $-\infty$) at the rate $L(T, A)$.*

Suppose u is an eigensolution of $H_x u = Eu$, where H_x is given by (8). Then

$$\begin{bmatrix} u(n+m) \\ u(n+m-1) \end{bmatrix} = A_n(T^m x) \begin{bmatrix} u(m) \\ u(m-1) \end{bmatrix}, \tag{12}$$

where $A_n(x)$ is the n-step transfer matrix of $(T, A_E(x))$ and

$$A_E(x) = \begin{bmatrix} TE - v(x) & -1 \\ 1 & 0 \end{bmatrix}.$$

Such $(T, A_E(x))$ are called Schrödinger cocycles. Denote by $L(E)$ the Lyapunov exponent of a Schrödinger cocycle (we omit the dependence on T and v). It turns out that (at least for uniquely ergodic dynamics) the resolvent set of H_x is precisely the set of E such that the Schrödinger cocycle $(T, A_E(x))$ is uniformly hyperbolic. The set $\sigma \cap \{L(E) > 0\}$ is therefore the set of non-uniform hyperbolicity for the one-parameter family of cocycles $(T, A_E(x))_{E\in\mathbb{R}}$, and is our main interest. Then Oseledets theorem can be reformulated as

Theorem 2.4. *Suppose that $L(E) > 0$. Then, for every $x \in \Omega_E$ (where Ω_E has full measure), there exist solutions ϕ^+, ϕ^- of $H_x\phi = E\phi$ such that $\phi^\pm$ decays exponentially at $\pm\infty$, respectively, at the rate $-L(E)$. Moreover, every solution which is linearly independent of ϕ^+ (resp., ϕ^-) grows exponentially at $+\infty$ (resp., $-\infty$) at the rate $L(E)$.*

It turns out that the set where the Lyapunov exponent vanishes fully determines the absolutely continuous spectrum.

Theorem 2.5 (Ishii-Pastur-Kotani)**.** $\sigma_{ac}(H_x) = \overline{\{E \in \mathbb{R} : L(E) = 0\}}^{ess}$ *for almost every $x \in \Omega$.*

The inclusion "$\subseteq$" was proved by Ishii and Pastur [49, 78]. The other, a lot more difficult, inclusion was proved by Kotani [70, 83].

2.6 Continuity of the Lyapunov exponent

Lyapunov exponent $L(\alpha, A) := L(R_\alpha, A)$ is generally not a very nice function of its parameters. It can be a discontinuous function of α at $\alpha \in \mathbb{Q}$ (almost Mathieu cocycle is one example), is generally discontinuous in A in C^0 and can be discontinuous in A even in C^∞ [92]. It is a remarkable fact, enabling much of the related theory, that it is continuous in the analytic category

Theorem 2.6. *[29, 55]* $L(\beta + .,.) : \mathbb{T} \times \mathcal{C}^\omega(\mathbb{T}, SL(2,\mathbb{R})) \to \mathbb{R}$ *is jointly continuous at irrational* β *.*

For the almost Mathieu operator, it leads to

Theorem 2.7. *[29] For every* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\lambda \in \mathbb{R}$ *and* $E \in \sigma(H_{\lambda,\alpha,\theta})$*, one has* $L_{\lambda,\alpha}(E) = \max\{\ln|\lambda|, 0\}$.

2.7 Implications of Avila's global theory

Continuity of the Lyapunov exponent in the analytic category [29, 55] makes it possible to make conclusions from the study of its behavior for complexified cocycles, and Avila [5] discovered a remarkable related structure. Analytic cocycles $A(x)$ can be classified depending on the behavior of the Lyapunov exponent L^ϵ of the complexified cocycle $A(x + i\epsilon)$. Namely, we distinguish three cases:

Subcritical: $L^\epsilon = 0, \epsilon < \delta,\ \delta > 0.$

Supercritical: $L^0 > 0$

Critical: Otherwise, that is $L^0 = 0, L^\epsilon > 0, \epsilon > 0.$

Avila observed that, for a given cocycle, L^ϵ is a convex function of ϵ, and proved that it has quantized derivative in ϵ. This has enabled the global theory [5], where Avila shows, in particular, that prevalent potentials are acritical, that is have no critical transfer-matrix cocycles for energies in their spectrum. The almost reducibility conjecture [5, 8] states that subcritical cocycles are almost reducible, that is have constant cocycles in the closure of their analytic conjugacy class. It was solved by Avila for the Liouville case in [3] and the solution for the Diophantine case has been announced [4]. Both almost reducible and supercritical cocycles are well studied and their basic spectral theory is understood.

For the almost Mathieu cocycle, quantization of acceleration allows to exactly compute $L^\epsilon(E)$ for E in the spectrum, leading to

Subcritical, i.e. $\lambda < 1$**:** In this case, $L^\epsilon(E) = 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$ and $\epsilon \leq \frac{-\ln\lambda}{2\pi}$. $H_{\lambda,\alpha,\theta}$ has purely ac spectrum [8, 16].

Critical, i.e. $\lambda = 1$**:** In this case, for $E \in \sigma(H_{\lambda,\alpha,\theta})$ the cocycle is critical

Supercritical, i.e. $\lambda > 1$**:** $L(E) = \ln\lambda > 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$.

We now quickly review the basics of continued fraction approximations.

2.8 Continued fraction expansion

Define, as usual, for $0 \leq \alpha < 1$,

$$a_0 = 0, \alpha_0 = \alpha,$$

and, inductively for $k > 0$,

$$a_k = [\alpha_{k-1}^{-1}], \alpha_k = \alpha_{k-1}^{-1} - a_k.$$

We define

$$\begin{aligned} p_0 &= 0, \quad q_0 = 1, \\ p_1 &= 1, \quad q_1 = a_1, \end{aligned}$$

and inductively,

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2}, \\ q_k &= a_k q_{k-1} + q_{k-2}. \end{aligned}$$

Recall that $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of denominators of best rational approximants to irrational number α, since it satisifies

$$\text{for any } 1 \leq k < q_{n+1}, \|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq ||q_n\alpha||_{\mathbb{R}/\mathbb{Z}}. \tag{13}$$

Moreover, we also have the following estimate,

$$\frac{1}{2q_{n+1}} \leq \Delta_n \triangleq \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{q_{n+1}}. \tag{14}$$

- α is called Diophantine if there exists $\kappa, \nu > 0$ such that $||k\alpha|| \geq \frac{\nu}{|k|^\kappa}$ for any $k \neq 0$, where $||x|| = \min_{k \in \mathbb{Z}} |x - k|$.

- α is called Liouville if

$$\beta(\alpha) = \limsup_{k \to \infty} \frac{-\ln ||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n} > 0 \tag{15}$$

- α is called weakly Diophantine if $\beta(\alpha) = 0$.

Clearly, Diophantine implies weakly Diophantine. By Borel-Cantelli lemma, Diophantine α form a set of full Lebesgue measure.

3 Do critical almost Mathieu operators ever have eigenvalues?

The critical almost Mathieu operator $H_{\alpha,\theta}$ given by

$$(H_{\alpha,\theta}\phi)(n) = \phi(n-1) + \phi(n+1) + 2\cos 2\pi(\alpha n + \theta)\phi(n), \tag{16}$$

has been long (albeit not from the very beginning [82][4]) conjectured to have purely singular continuous spectrum for every $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ and every θ. Since the spectrum (which is θ-independent for $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ [18]) has Lebesgue measure zero [10], the problem boils down to the proof of absence of eigenvalues, see e.g. problem 7 in [52]. This simple question has a surprisingly rich (and dramatic) history.

Aside from the results on topologically generic absence of point spectrum [17, 66] that hold in a far greater generality, all the proofs were, in one way or another, based on the Aubry duality [1], a Fourier-type transform for which the family $\{H_{\alpha,\theta}\}_\theta$ is a fixed point. One manifestation of the Aubry duality is: if $u \in \ell^2(\mathbb{Z})$ solves the eigenvalue equation $H_{\alpha,\theta}u = Eu$, then $v_n^x := e^{2\pi i n\theta}\hat{u}(x + n\alpha)$ solves

$$H_{\alpha,x}v^x = Ev^x \tag{17}$$

for a.e. x, where $\hat{u}(x) = \sum e^{2\pi i n x}u_n$ is the Fourier transform of u. This led Delyon [35] to prove that there are no ℓ^1 solutions of $H_{\alpha,\theta}u = Eu$, for

[4] It is the paper where the name *almost Mathieu* was introduced.

otherwise (17) would hold also for $x = \theta$, leading to a contradiction. Thus any potential eigenfunctions must be decaying slowly. Chojnacki [32] used duality-based C^*-algebraic methods to prove the existence of some continuous component, but without ruling out the point spectrum. [41] gave a duality-based argument for no point spectrum for a.e. θ, but it had a gap, as it was based on the validity of Deift-Simon's [33] theorem on a.e. mutual singularity of singular spectral measures, which is only proved in [33] in the hyperbolic case, and is still open in the regime of zero Lyapunov exponents. Avila and Krikorian (see [6]) used convergence of renormalization [11] and non-perturbative reducibility [29] to show that for every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, eigenvalues may only occur for countably many θ. Then Avila [6] found a simple proof of the latter fact, also characterizing this potentially exceptional set of phases explicitly: these are phases θ that are α-rational, i.e. $2\theta + k\alpha \in \mathbb{Z}$, for some k. The argument of [6] was incorporated in [9], where it was developed to prove a.e. absence of point spectrum for the extended Harper's model (EHM) in the entire critical region (the EHM result was later further improved by Han [42]). The proof in [6, 9] has as a starting point the dynamical formulation of the Aubry duality: if v_n^x solves the eigenvalue equation $H_{\alpha,x}v = Ev$, then so does its complex conjugate $\bar{v}_n^x$, and this can be used to construct an L^2-reducibility of the transfer-matrix cocycles to the rotation by θ, given independence of v and $\bar{v}$. Unfortunately those vectors are always linearly dependent if θ is α-rational. Thus the argument hopelessly breaks down for $2\theta + k\alpha \in \mathbb{Z}$.

Moreover, it was noted in [9] that in the bulk of the critical region, for α-rational phases θ, the extended Harper's operator actually does have eigenvalues. Also, supercritical almost Mathieu with Diophantine α, has eigenvalues (with exponentially decaying eigenfunctions) for α-rational phases as well [55]. All this increased the uncertainty about whether eigenvalues may exist for the α-rational phases also for the critical almost Mathieu.

We will present the fully self-contained proof of

Theorem 3.1. *[54] $H_{\alpha,\theta}$ does not have eigenvalues for any α, θ (and thus has purely singular-continuous spectrum for all $\alpha \notin \mathbb{Q}$).*

In our proof we replace the Aubry duality by a new transform, inspired by the chiral gauge transform of [57].

4 Proof of Theorem 3.1.

Given $u \in \ell^2(\mathbb{Z})$, set

$$u(x) = \sum_{n=-\infty}^{\infty} u_n e^{\pi i n(\theta + n\alpha - 2x)} \tag{18}$$

and

$$u_n^x = u(x + n\alpha) e^{\pi i n(x + \frac{n\alpha - 3\theta}{2})} \tag{19}$$

where u_n^x is defined for a.e. x.

Let $\tilde{H}_\alpha^x : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}, x \in \mathbb{R}/2\mathbb{Z}$, be given by

$$(\tilde{H}_\alpha^x v)_n = 2\cos\pi(x + n\alpha) v_{n-1} + 2\cos\pi(x + (n+1)\alpha) v_{n+1} \tag{20}$$

Lemma 4.1. *If $u \in \ell^2(\mathbb{Z})$ solves $H_{\alpha,\theta} u = Eu$, then $u^x \in \mathbb{R}^{\mathbb{Z}}$ is a formal solution of the difference equation*

$$\tilde{H}_\alpha^{x + \frac{\theta - \alpha}{2}} u^x = E u^x \tag{21}$$

for a.e. x.

Proof. If $(Tu)_n := u_{n+1} + u_{n-1}$, and $(Su)_n := \cos 2\pi(\theta + n\alpha) u_n$, we obtain $(Tu)(x) = u(x - \alpha) e^{\pi i(\theta + \alpha - 2x)} + u(x + \alpha) e^{\pi i(-\theta + \alpha + 2x)}$ and $(Su)(x) = u(x - \alpha) e^{2\pi i \theta} + u(x + \alpha) e^{-2\pi i \theta}$, leading, by a straightforward computation, to $((T + S)u)^x = \tilde{H}_\alpha^{x + \frac{\theta - \alpha}{2}} u^x$. □

We note that the family $\{\tilde{H}_\alpha^x\}_{x \in \mathbb{R}/2\mathbb{Z}}$ is self-dual with respect to the Aubry-type duality. Namely, the following holds. For $x \in \mathbb{R}/2\mathbb{Z}, v \in \ell^2(\mathbb{Z})$ for a.e. β, we can define $w^\beta \in \mathbb{R}^{\mathbb{Z}}$ by

$$w_n^\beta = \hat{v}(\frac{\beta + n\alpha}{2}) e^{\pi i n(x + \frac{\alpha}{2})}. \tag{22}$$

Lemma 4.2. *If $v \in \ell^2(\mathbb{Z})$ solves $\tilde{H}_\alpha^x v = Ev$, then, for a.e. β, $w^\beta \in \mathbb{R}^{\mathbb{Z}}$ is a formal solution of the difference equation*

$$\tilde{H}_\alpha^{\beta - \frac{\alpha}{2}} w^\beta = E w^\beta. \tag{23}$$

Proof. A similar direct computation. □

Let now $u \in \ell^2(\mathbb{Z})$ with $\|u\|_2 = 1$ be a solution of $H_{\alpha,\theta} u = Eu$. By Lemma 4.1, (21) holds, which implies that we also have, for a.e. x,

$$\tilde{H}_{\alpha}^{x+\frac{\theta-\alpha}{2}}\bar{u}^x = E\bar{u}^x \tag{24}$$

thus the Wronskian of u^x and $\bar{u}^x$ is constant in n. That is

$$\cos\pi(x+n\alpha)\mathrm{Im}\,(u(x+n\alpha)\bar{u}(x+(n-1)\alpha)e^{\pi i(x+n\alpha+ia(\alpha,\theta))}) = c(x) \tag{25}$$

for some $c(x)$, all n and a.e. x. Here and below $a(\alpha,\theta)$ stands for (an explicit) real-valued function that does not depend on n, x. Its exact form is not important. $a(\alpha,\theta)$ may stand for different such functions in different expressions.

By ergodicity, this implies that, for a.e. x and some constant c,

$$\cos\pi x(u(x)\bar{u}(x-\alpha)e^{\pi ix+ia(\alpha,\theta)} - u(x-\alpha)\bar{u}(x)e^{-\pi ix-ia(\alpha,\theta)}) = c. \tag{26}$$

It follows by Cauchy-Schwarz that $u(x)\bar{u}(x-\alpha)e^{\pi ix+ia(\alpha,\theta)} \in L^1$, which implies that $c=0$. We note that a similar argument was used by R. Han in [42]. Thus we have

$$u(x)\bar{u}(x-\alpha)e^{\pi ix+ia(\alpha,\theta)} - u(x-\alpha)\bar{u}(x)e^{-\pi ix-ia(\alpha,\theta)} = 0 \tag{27}$$

for a.e. x.

Lemma 4.3. *For a.e. x, we have $u(x) \neq 0$.*

Proof. Indeed, otherwise, by the ergodic theorem, there would exist (in fact, a full measure of, but it is not important) x such that u_n^x solves (21) and $u_n^x = 0$ for infinitely many n (in fact, only four such n suffice for the argument). Let $n_i < n_{i+1} - 1, i \in \mathbb{Z}$, be the labeling of zeros of such u_n^x. Clearly, if $v \in \mathbb{R}^{\mathbb{Z}}$ is a solution of (20) with $v_n = v_m = 0$, we have that $v_{[n,m]} \in \ell^2(\mathbb{Z})$ defined by $(v_{[n,m]})_k = \begin{cases} v_k, k \in [n+1, m-1] \\ 0, \text{otherwise} \end{cases}$ is also a solution of (21). Set $v^{x,i} := u^x_{[n_i,n_{i+1}]}$.

Clearly, for any $I \subset \mathbb{Z}$ the collection $\{v^{x,i}\}_{i\in I}$ is linearly independent in $\ell^2(\mathbb{Z})$. This implies that the corresponding Aubry dual collection $\{w^{x,i,\beta}\}_{i\in I}$ constructed by (22) from $\{v^{x,i}\}_{i\in I}$, is linearly independent in $\mathbb{R}^{\mathbb{Z}}$. Thus, by Lemmas 4.1,4.2 we obtain, for a.e. β, infinitely many linearly independent

$w^{x,i,\beta} \in \mathbb{R}^{\mathbb{Z}}$, that all solve (23). This is in contradiction with the fact that the space of solutions of (23) is two-dimensional for a.e. β. $\square$

Therefore we can define for a.e. x, a unimodular measurable function on $\mathbb{R}/2\mathbb{Z}$

$$\phi(x) := \frac{u(x)}{\bar{u}(x)} e^{\pi i x + i a(\alpha,\theta)} \tag{28}$$

By (5.2), (28) we have that, for a.e. x,

$$\phi(x) = \phi(x-\alpha) e^{-2\pi i x + i a(\alpha,\theta)}, \tag{29}$$

and expanding $\phi(x)$ into the Fourier series, $\phi(x) = \sum_{k=-\infty}^{\infty} a_k e^{\pi i k x}$, we obtain $|a_{k+2}| = |a_k|$, a contradiction. $\square$

5 Thouless' Hausdorff dimension conjecture

The spectrum of $H_{\alpha,\theta}$ for irrational α is a θ-independent[5] fractal, beautifully depicted via the Hofstadter butterfly [48]. There have been many numerical and heuristic studies of its fractal dimension in physics literature (e.g., [38, 68, 88, 94]). A conjecture attributed to Thouless (e.g., [94]), and appearing already in the early 1980's, is that the dimension is equal to 1/2. It has been rethought after rigorous and numerical studies demonstrated that the Hausdorff dimension can be less than 1/2 (and even be zero) for some α [12, 74, 94], while packing/box counting dimension can be higher (even equal to one) for some (in fact, of the same!) α [67]. However, all these are Lebesgue measure zero sets of α, and the conjecture may still hold, in some sense. There is also a conjecture attributed to J. Bellissard (e.g., [45, 74]) that the dimension of the spectrum is a property that only depends on the tail in the continued fraction expansion of α and thus should be the same for a.e. α (by the properties of the Gauss map). We discuss the history of rigorous results on the dimension in more detail below.

In the past few years, there was an increased interest in the dimension of the spectrum of the critical almost Mathieu operator, leading to a number of other rigorous results mentioned above. Those include zero Hausdorff dimension for a subset of Liouville α by Last and Shamis [74], also extended to all weakly Liouville[6] α by Avila, Last, Shamis, Zhou [12]; the full packing (and therefore box counting) dimension for weakly Liouville α [67], and

[5] Also for any $\lambda \neq 0$.

[6] We say α is weakly Liouville if $\beta(\alpha) := -\limsup \frac{\ln \|n\alpha\|}{n} > 0$, where $\|\theta\| = \mathrm{dist}(\theta, \mathbb{Z})$.

existence of a dense positive Hausdorff dimension set of Diophantine α with positive Hausdorff dimension of the spectrum by Helffer, Liu, Qu, and Zhou [45]. All those results, as well as heuristics by Wilkinson-Austin [94] and, of course, numerics, hold for measure zero sets of α. Recently, B. Simon listed the problem to determine the Hausdorff dimension of the spectrum of the critical almost Mathieu on his new list of hard unsolved problems [86].

The equality in the original conjecture can be viewed as two inequalities. In a joint work with Igor Krasovsky [57] we prove one of those for *all* irrational α. This is also the first result on the fractal dimension that holds for more than a measure zero set of α. Denote the spectrum of an operator K by $\sigma(K)$, the Lebesgue measure of a set A by $|A|$, and its Hausdorff dimension by $\dim_{\mathrm{H}}(A)$. We have

Theorem 5.1. *[57] For any irrational α and real θ,* $\dim_{\mathrm{H}}(\sigma(H_{\alpha,\theta})) \leq 1/2$.

Of course, it only makes sense to discuss upper bounds on the Hausdorff dimension of a set on the real line once its Lebesgue measure is shown to be zero. The Aubry-Andre conjecture stated that the measure of the spectrum of $H_{\alpha,\theta,\lambda}$ is equal to $4|1-|\lambda||$, so to 0 if $\lambda = 1$, for any irrational α. This conjecture was popularized by B. Simon, first in his list of 15 problems in mathematical physics [84] and then, after it was proved by Last for a.e. α [72, 73], again as Problem 5 in [85], which was to prove this conjecture for the remaining measure zero set of α, namely, for α of bounded type.[7] The arguments of [72, 73] did not work for this set, and even though the semi-classical analysis of Hellfer-Sjöstrand [46] applied to some of this set for $H_{\alpha,\theta}$, it did not apply to other such α, including, most notably, the golden mean — the subject of most numerical investigations. For the non-critical case, the proof for all α of bounded type was given in [57], but the critical "bounded-type" case remained difficult to crack. This remaining problem for zero measure of the spectrum of $H_{\alpha,\theta}$ was finally solved by Avila-Krikorian [10], who employed a deep dynamical argument. We note that the argument of [10] worked not for all α, but for a full measure subset of Diophantine α. Here we give a very simple argument that recovers this theorem and thus gives an elementary solution to Problem 5 of [85]. Moreover, our argument works simultaneously for all irrational α.

Theorem 5.2. *For any irrational α and real θ,* $|\sigma(H_{\alpha,\theta})| = 0$.

[7] That is α with all coefficients in the continued fraction expansion bounded by some M.

The proofs are based on two key ingredients. We introduce what we call the chiral gauge transform and show that the direct sum in θ of operators $H_{2\alpha,\theta}$ is isospectral with the direct sum in θ of $\widehat{H}_{\alpha,\theta}$ given by

$$(\widehat{H}_{\alpha,\theta}\phi)(n) = 2\sin 2\pi(\alpha(n-1)+\theta)\phi(n-1) + 2\sin 2\pi(\alpha n+\theta)\phi(n+1). \quad (30)$$

This representation of the almost Mathieu operator corresponds to choosing the chiral gauge for the perpendicular magnetic field applied to the electron on the square lattice,

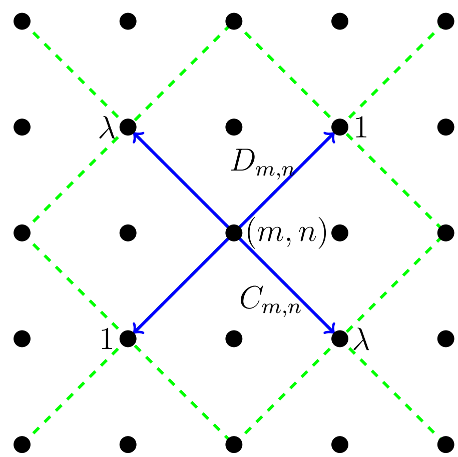

Figure 3

Any choice of gauge such that

$$C_{m,n} + D_{m+1,n-1} - C_{m+1,n-1} - D_{m,n} = 2\pi \cdot 2\alpha, \quad (31)$$

leads to an operator on $\ell^2(\mathbb{Z}^2)$

$$(H_{C,D}\psi)_{m,n} = e^{iC_{m,n}}\psi_{m+1,n-1} + e^{iD_{m,n}}\psi_{m+1,n+1} + e^{-iC_{m-1,n+1}}\psi_{m-1,n+1} + e^{-iD_{m-1,n-1}}\psi_{m-1,n-1} \quad (32)$$

which represents the Hamiltonian of an electron in a uniform perpendicular magnetic field with flux $2\pi\alpha$. Here $4\pi\alpha$ is the total flux through each doubled cell.

The chiral gauge that corresponds to (30) is given by

$$\begin{cases} C_{m,n} \equiv 0 \\ D_{m,n} = 4\pi m\alpha \end{cases}$$

It was previously discussed non-rigorously in [77, 93]. The advantage of (30) is that it is a *singular* Jacobi matrix, that is one with off-diagonal elements not bounded away from zero, so that the matrix quasi-separates into blocks.

This alone is already sufficient to conclude Theorem 5.2 because $\hat{H}$ is represented by a matrix with off-diagonal terms nearly vanishing along a subsequence. Singular Jacobi matrices are trace-class perturbations of direct sums of finite blocks, thus never have absolutely continuous spectrum. Therefore, by Kotani theory (that does extend to the singular case), and the fact that the Lyapunov exponent is zero on the spectrum, as easily follows from the formula for the invariance of the IDS under the gauge transform and a Thouless-type formula for the Lyapunov exponent, the measure of the spectrum must be zero.

The second key ingredient is a general result on almost Lipshitz continuity of spectra for *singular* quasiperiodic Jacobi matrices. The modulus of continuity statements have, in fact, been central in previous literature. We consider a general class of quasiperiodic C^1 Jacobi matrices, that is operators on $\ell^2(\mathbb{Z})$ given by

$$(H_{v,b,\alpha,\theta}\phi)(n) = b(\theta+(n-1)\alpha)\phi(n-1)+b(\theta+n\alpha)\phi(n+1)+v(\theta+n\alpha)\phi(n), \tag{33}$$

with $b(x), v(x) \in C^1(\mathbb{R})$, and periodic with period 1.

Let $M_{v,b,\alpha}$ be the direct sum of $H_{v,b,\alpha,\theta}$ over $\theta \in [0,1)$,

$$M_{v,b,\alpha} = \oplus_{\theta\in[0,1)} H_{v,b,\alpha,\theta}. \tag{34}$$

Continuity in α of $\sigma(M_{v,b,\alpha})$ in the Hausdorff metric was proved in [18]. Continuity of the measure of the spectrum is a more delicate issue, since, in particular, $|\sigma(M_\alpha)|$ can be (and is, for the almost Mathieu operator) discontinuous at rational α. Establishing continuity at irrational α requires quantitative estimates on the Hausdorff continuity of the spectrum. In the Schrödinger case, that is for $b = 1$, Avron, van Mouche, and Simon [20] obtained a very general result on Hölder-$\frac{1}{2}$ continuity (for arbitrary $v \in C^1$), improving Hölder-$\frac{1}{3}$ continuity obtained earlier by Choi, Elliott, and Yui [31]. It was argued in [20] that Hölder continuity of *any* order larger than 1/2 would imply the desired continuity property of the measure of the spectrum for *all* α. Lipshitz continuity of gaps was proved by Bellissard [23] for a large class of quasiperiodic operators, however without a uniform Lipshitz constant, thus not allowing to conclude continuity of the measure of the spectrum. In [56] (see also [62]) we showed a uniform almost Lipshitz continuity for Schrödinger operators with analytic potentials and Diophantine frequencies in the regime of positive Lyapunov exponents, which, in particular, allowed us to complete the proof of the Aubry-Andre conjecture for the non-critical case.

Namely, a Jacobi matrix (33) is called *singular* if for some θ_0, $b(\theta_0) = 0$. We assume that the number of zeros of b on its period is finite. In this case, uniform almost Lipshitz continuity (with a logarithmic correction) holds [57] which allows to conclude continuity of the measure of the spectrum for general singular Jacobi matrices:

Theorem 5.3. *For singular $H_{v,b,\alpha,\theta}$ as above, for any irrational α there exists a subsequence of canonical approximants $\frac{p_{n_j}}{q_{n_j}}$ such that*

$$|\sigma(M_{v,b,\alpha})| = \lim_{j\to\infty} \left|\sigma\left(M_{v,b,\frac{p_{n_j}}{q_{n_j}}}\right)\right|. \tag{35}$$

In the case of Schrödinger operators (i.e., for $b = 1$), the statement (35) was previously established in various degrees of generality in the regime of positive Lyapunov exponents [56, 65] and, in all regimes for analytic [63] or sufficiently smooth [97] v. Typically, proofs that work for $b = 1$ extend also to the case of non-vanishing b, that is *non-singular* Jacobi matrices, and there is no reason to believe the results of [63, 97] should be an exception. On the other hand, extending various Schrödinger results to the singular Jacobi case is technically non-trivial and adds a significant degree of complexity (e.g. [9, 44, 64]). Our proof however is based on showing that a singularity can be *exploited*, rather than circumvented, to establish enhanced continuity of spectra and therefore Theorem 5.3. Of course, Theorem 5.2 also follows immediately from the chiral gauge representation, the bound (36) below, and Theorem 5.3, providing a third proof of Problem 5 of [85].

Moreover, enhanced continuity combined with the chiral gauge representation allows to immediately prove Theorem 5.1 by an argument of [72]. Indeed, the original intuition behind Thouless' conjecture on the Hausdorff dimension 1/2 is based on another fascinating Thouless' conjecture [89, 90]: that for the critical almost Mathieu operator $H_{\alpha,\theta}$, in the limit $p_n/q_n \to \alpha$, we have $q_n|\sigma(M_{p_n/q_n})| \to c$ where $c = 32C_c/\pi$, C_c being the Catalan constant. Thouless argued that if $\sigma(M_\alpha)$ is "economically covered" by $\sigma(M_{p_n/q_n})$ and if all bands are of about the same size then the spectrum, being covered by q_n intervals of size $\frac{c}{q_n^2}$, has the box counting dimension 1/2. Clearly, the exact value of $c > 0$ is not important for this argument. An upper bound of the form

$$q_n|\sigma(M_{p_n/q_n})| < C, \qquad n = 1, 2, \ldots, \tag{36}$$

was proved by Last [72][8], which, combined with Hölder-$\frac{1}{2}$ continuity, led him in [72] to the bound $\leq \frac{1}{2}$ for the Hausdorff dimension for irrational α satisfying $\lim_{n\to\infty} |\alpha - p_n/q_n|q_n^4 = 0$. Such α form a zero measure set. The almost Lipschitz continuity and (36) allow us to obtain the result (Theorem 5.1) for *all* irrational α.

Since our proof of Theorem 5.1 only requires an estimate such as (36) and the existence of isospectral family of singular Jacobi matrices, it applies equally well to all other situations where the above two facts are present. For example, Becker et al [24] recently introduced a model of graphene as a quantum graph on the regular hexagonal lattice and studied it in the presence of a magnetic field with a constant flux Φ, with the spectrum denoted σ^Φ. Upon identification with the interval $[0, 1]$, the differential operator acting on each edge is then the maximal Schrödinger operator $\frac{d^2}{dx^2} + V(x)$ with domain H^2, where V is a Kato-Rellich potential symmetric with respect to 1/2. We then have

Theorem 5.4. *For any symmetric Kato-Rellich potential* $V \in L^2$*, the Hausdorff dimension* $\dim_H(\sigma^\Phi) \leq 1/2$*, for all irrational* Φ*.*

This result was proved in [24] for a topologically generic but measure zero set of α.

The basic idea behind the proof that singularity leads to enhanced continuity is that creating approximate eigenfunctions by cutting at near-zeros of the off-diagonal terms leads to smaller errors in the kinetic term. However, without apriori estimates on the behavior of solutions (and it is in fact natural for solutions to be large around the singularity) this in itself is insufficient to achieve an improvement over the Hölder exponent 1/2, so the argument ends up being not entirely straightforward.

6 Small denominators and arithmetic spectral transitions

In general, localization for quasiperiodic operators is a classical case of a small denominator problem, and has been traditionally approached in a per-

[8]with $C = 8e$.

turbative way: through KAM-type schemes for large couplings [36, 37, 87] (which, being KAM-type schemes, all required Diophantine conditions on frequencies). The opposite regime of very Liouville frequencies allowed proofs of delocalization by perturbation of periodic operators. Unlike the random case, where, in dimension one, localization holds for all couplings, a distinctive feature of quasiperiodic operators is the presence of metal-insulator transitions as couplings increase. Even when non-perturbative methods, for the almost Mathieu and then for general analytic potentials, were developed in the 90s [27, 51, 53], allowing to obtain localization for a.e. frequency throughout the regime of positive Lyapunov exponents, they still required Diophantine conditions, and exponentially approximated frequencies that are neither far from nor close enough to rationals remained a challenge, as for them there was nothing left to perturb about or to remove. It has gradually become clear that small denominators are not simply a nuisance, but lead to actual change in the spectral behavior.

The transitions in coupling between absolutely continuous and singular spectrum are governed by vanishing/non-vanishing of the Lyapunov exponent. It turns out that in the regime of positive Lyapunov exponents (also called supercritical in the analytic case, with the name inspired by the almost Mathieu operator) small denominators lead also to more delicate transitions: between localization (point spectrum with exponentially decaying eigenfunctions) and singular continuous spectrum. They are governed by the resonances: eigenvalues of box restrictions that are too close to each other in relation to the distance between the boxes, leading to small denominators in various expansions. All known proofs of localization, are based, in one way or another, on avoiding resonances and removing resonance-producing parameters, while all known proofs of singular continuous spectrum and even some of the absolutely continuous one are based on showing their abundance.

For quasiperiodic operators, one category of resonances are the ones determined entirely by the frequency. Indeed, for smooth potentials, large coefficients in the continued fraction expansion of the frequency lead to almost repetitions and thus resonances, regardless of the values of other parameters. Such resonances were first understood and exploited to show singular continuous spectrum for Liouville frequencies in [17], based on [40] . The strength of frequency resonances is measured by the arithmetic parameter

$$\beta(\alpha) = \limsup_{k\to\infty} -\frac{\ln ||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} \tag{37}$$

where $||x||_{\mathbb{R}/\mathbb{Z}} = \inf_{\ell \in \mathbb{Z}} |x - \ell|$. Another class of resonances, appearing for all *even* potentials, was discovered in [66], where it was shown for the first time that the arithmetic properties of the phase also play a role and may lead to singular continuous spectrum even for the Diophantine frequencies. Indeed, for even potentials, phases with almost symmetries lead to resonances, regardless of the values of other parameters. The strength of phase resonances is measured by the arithmetic parameter

$$\delta(\alpha, \theta) = \limsup_{k \to \infty} -\frac{\ln ||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} \tag{38}$$

In both these cases, the strength of the resonances is in competition with the exponential growth controlled by the Lyapunov exponent. It was conjectured in 1994 [50] that for the almost Mathieu family- the prototypical quasiperiodic operator - the two above types of resonances are the only ones that appear, and the competition between the Lyapunov growth and resonance strength resolves, in both cases, in a sharp way.

Recall that α is called weakly Diophantine if $\beta(\alpha) = 0$, and θ is called α-Diophantine if $\delta(\alpha, \theta) = 0$. By a simple Borel-Cantelli argument, both weakly Diophantine and α-Diophantine numbers form sets of full Lebesque measure (for any α). Separating frequency and phase resonances, the frequency conjecture was that for α-Diophantine phases, there is a transition from singular continuous to pure point spectrum precisely at $\beta(\alpha) = L$, where L is the Lyapunov exponent. The phase conjecture was that for weakly Diophantine frequencies, there is a transition from singular continuous to pure point spectrum precisely at $\delta(\alpha, \theta) = L$.

Operator H is said to have Anderson localization if it has pure point spectrum with exponentially decaying eigenfunctions. We have

Theorem 6.1. *[Phase,[60]] For weakly Diophantine α,*

1. *$H_{\lambda,\alpha,\theta}$ has Anderson localization if $|\lambda| > e^{\delta(\alpha,\theta)}$,*
2. *$H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $1 < |\lambda| < e^{\delta(\alpha,\theta)}$.*
3. *$H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum if $|\lambda| < 1$.*

and

Theorem 6.2. *[Frequency, [59]]*
For α-Diophantine θ,

1. *$H_{\lambda,\alpha,\theta}$ has Anderson localization if $|\lambda| > e^{\beta(\alpha)}$,*

2. *$H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $1 < |\lambda| < e^{\beta(\alpha)}$.*

3. *$H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum if $|\lambda| < 1$.*

Remark

1. Part 2 of Theorem 6.1 holds for all irrational α, and part 2 of Theorem 6.2 ([14], see also a footnote in [7]) holds for all θ. Both also hold for general Lipshitz v replacing the $\cos$.

2. Part 3 of both theorems is known for all α, θ [6] and is included here for completeness.

3. Parts 1 and 2 of both Theorems put together verify the conjecture in [50], as stated there. The frequency half was first proved in [14] in a measure-theoretic sense (for a.e. θ).

For $\beta = \delta = 0$ (which is a.e. α, θ) the result follows from [53]. Proofs of the localization part of both theorems are based on the method developed in [53]. However, since the arithmetic transitions happen within the excluded measure zero set where the resonances are exponentially strong, new ideas were needed to handle those. A progress towards the localization side of the above conjecture in the frequency case was made in [7] (localization for $|\lambda| > e^{\frac{16}{9}\beta}$, as a step in solving the Ten Martini problem). The method developed in [7] that allowed to approach exponentially small denominators on the localization side was brought to its technical limits in [76], where the result for $|\lambda| > e^{\frac{3}{2}\beta}$ was obtained.

There have been no previous results on the transition in phase for $0 < \delta < \infty$. Singular continuous spectrum was first established for $1 < |\lambda| < e^{c\delta(\alpha,\theta)}$ (correspondingly, $1 < |\lambda| < e^{c\beta(\alpha)}$ for sufficiently small c [18, 66]. One can see that even with tight upper semicontinuity bounds the argument of [66] does not work for $c > 1/4$, New ideas to remove the factor of 4 and approach the actual threshold were required to prove Theorems 6.1, 6.2 in, correspondingly, [59, 60] . The singular continuous spectrum up to the threshold for frequency was established in [7, 14].

7 Exact asymptotics and universal hierarchical structure for frequency resonances

In this section we describe the universal self-similar exponential structure of eigenfunctions throughout the entire localization regime. We present explicit universal functions $f(k)$ and $g(k)$, depending only on the Lyapunov exponent and the position of k in the hierarchy defined by the denominators q_n of the continued fraction approximants of the flux α, that completely define the exponential behavior of, correspondingly, eigenfunctions and norms of the transfer matrices of the almost Mathieu operators, for all eigenvalues corresponding to a.e. phase, see Theorem 8.1. This result holds for *all* frequency and coupling pairs in the localization regime. Since the behavior is fully determined by the frequency and does not depend on the phase, it is the same, eventually, around any starting point, so is also seen unfolding at different scales when magnified around local eigenfunction maxima, thus describing the exponential universality in the hierarchical structure, see, for example, Theorems 7.2,7.4.

Since we are interested in exponential growth/decay, the behavior of f and g becomes most interesting in case of frequencies with exponential rate of approximation by the rationals.

These functions allow to describe *precise* asymptotics of *arbitrary* solutions of $H_{\lambda,\alpha,\theta}\varphi = E\varphi$ where E is an eigenvalue. The precise asymptotics of the norms of the transfer-matrices, provides the first example of this sort for non-uniformly hyperbolic dynamics. Since those norms sometimes differ significantly from the reciprocals of the eigenfunctions, this leads to further interesting and unusual consequencies, for example exponential tangencies between contracted and expanded directions at the resonant sites.

From this point of view, this analysis also provides the first study of the dynamics of Lyapunov-Perron non-regular points, in a natural setting. An artificial example of irregular dynamics can be found in [22], p.23, however it is not even a cocycle over an ergodic transformation, and we are not aware of other such, even artificial, ergodic examples where the dynamics has been studied. Loosely, for a cocycle A over a transformation f acting on a space X (Lyapunov-Perron) non-regular points $x \in X$ are the ones at which Oseledets multiplicative ergodic theorem does not hold coherently in both directions. They therefore form a measure zero set with respect to any invariant measure on X. Yet, it is precisely the non-regular points that are of interest in the study of Schrödinger cocycles in the non-uniformly hyperbolic (positive Lyapunov exponent) regime, since spectral measures, for every fixed phase, are always supported on energies where there exists a solution polynomially bounded in both directions, so the (hyperbolic) cocycle defined at such energies is always non-regular at precisely the relevant phases. Thus the non-regular points capture the entire action from the point of view of spectral theory, so become the most important ones to study. One can also discuss stronger non-regularity notions: absence of forward regularity and, even stronger, non-exactness of the Lyapunov exponent [22]. While it is not difficult to see that energies in the support of singular continuous spectral measure in the non-uniformly hyperbolic regime always provide examples of non-exactness, our analysis gave the first non-trivial example of non-exactness with non-zero upper limit (Corollary 7.12). Finally, as we understand, it also provided the first natural example of an even stronger manifestation of the lack of regularity, the exponential tangencies (Corollary 7.13). Tangencies between contracted and expanded directions are a characteristic feature of nonuniform hyperbolicity (and, in particular, always happen at the maxima of the eigenfunctions). They complicate proofs of positivity of the Lyapunov exponents and are viewed as a difficulty to avoid through e.g. the parameter exclusion [25, 96]. However, when the tangencies are only subexponentially deep they do not in themselves lead to non-exactness. Corollary 7.13 presents the first natural example of *exponentially* strong tangencies (with the rate determined by the arithmetics of α and the positions precisely along the sequence of resonances.)

For the almost Mathieu operator the k-step transfer matrix defined by (9),(10), becomes

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha) \cdots A(\theta) \quad (39)$$

and

$$A_{-k}(\theta) = A_k^{-1}(\theta - k\alpha) \quad (40)$$

for $k \geq 1$, where $A(\theta) = \begin{pmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}$. As is clear from the definition, A_k also depends on θ and E but since those parameters will be usually fixed, we omit this from the notation.

Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ we define functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$ in the following way.

Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α. For any $\frac{q_n}{2} \le k < \frac{q_{n+1}}{2}$, define $f(k), g(k)$ as follows:

Case 1 $q_{n+1}^{\frac{8}{9}} \ge \frac{q_n}{2}$ or $k \ge q_n$.

If $\ell q_n \le k < (\ell+1) q_n$ with $\ell \ge 1$, set

$$f(k) = e^{-|k-\ell q_n| \ln|\lambda|} \bar{r}_\ell^n + e^{-|k-(\ell+1)q_n| \ln|\lambda|} \bar{r}_{\ell+1}^n, \tag{41}$$

and

$$g(k) = e^{-|k-\ell q_n| \ln|\lambda|} \frac{q_{n+1}}{\bar{r}_\ell^n} + e^{-|k-(\ell+1)q_n| \ln|\lambda|} \frac{q_{n+1}}{\bar{r}_{\ell+1}^n}, \tag{42}$$

where for $\ell \ge 1$,

$$\bar{r}_\ell^n = e^{-(\ln|\lambda| - \frac{\ln q_{n+1}}{q_n} + \frac{\ln \ell}{q_n}) \ell q_n}.$$

Set also $\bar{r}_0^n = 1$ for convenience.

If $\frac{q_n}{2} \le k < q_n$, set

$$f(k) = e^{-k \ln|\lambda|} + e^{-|k-q_n| \ln|\lambda|} \bar{r}_1^n, \tag{43}$$

and

$$g(k) = e^{k \ln|\lambda|}. \tag{44}$$

Case 2 $q_{n+1}^{\frac{8}{9}} < \frac{q_n}{2}$ and $\frac{q_n}{2} \le k \le \min\{q_n, \frac{q_{n+1}}{2}\}$.

Set

$$f(k) = e^{-k \ln|\lambda|}, \tag{45}$$

and

$$g(k) = e^{k \ln|\lambda|}. \tag{46}$$

Notice that f, g only depend on α and λ but not on θ or E. $f(k)$ decays and $g(k)$ grows exponentially, globally, at varying rates that depend on the position of k in the hierarchy defined by the continued fraction expansion of α, see Fig.4 and Fig.5.

We say that ϕ is a generalized eigenfunction of H with generalized eigenvalue E, if

$$H\phi = E\phi, \text{ and } |\phi(k)| \le \hat{C}(1 + |k|). \tag{47}$$

It turns out that in the entire regime $|\lambda| > e^{\beta}$, the exponential asymptotics of the generalized eigenfunctions and norms of transfer matrices at the generalized eigenvalues are completely determined by $f(k), g(k)$.

Theorem 7.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$. Suppose θ is Diophantine with respect to α, E is a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ and ϕ is the generalized eigenfunction. Let $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$. Then for any $\varepsilon > 0$, there exists K (depending on $\lambda, \alpha, \hat{C}, \varepsilon$) such that for any $|k| \geq K$, $U(k)$ and A_k satisfy*

$$f(|k|)e^{-\varepsilon|k|} \leq ||U(k)|| \leq f(|k|)e^{\varepsilon|k|}, \tag{48}$$

and

$$g(|k|)e^{-\varepsilon|k|} \leq ||A_k|| \leq g(|k|)e^{\varepsilon|k|}. \tag{49}$$

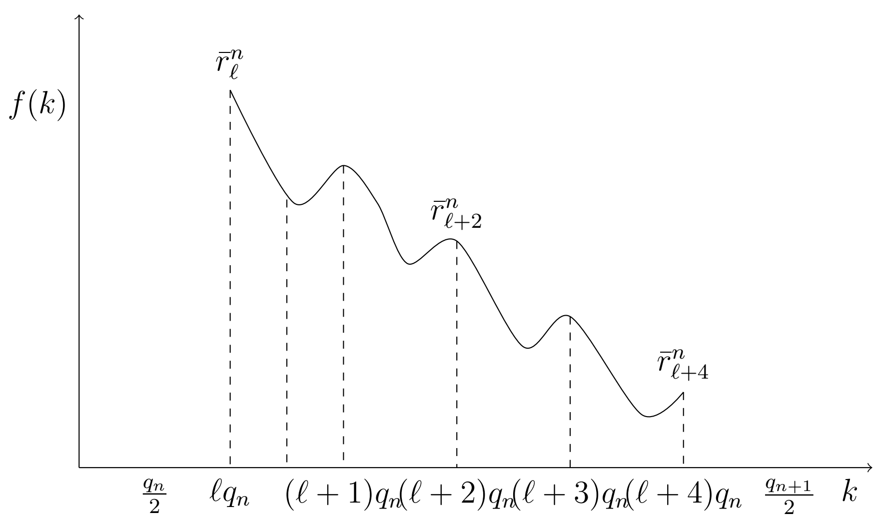

Figure 4

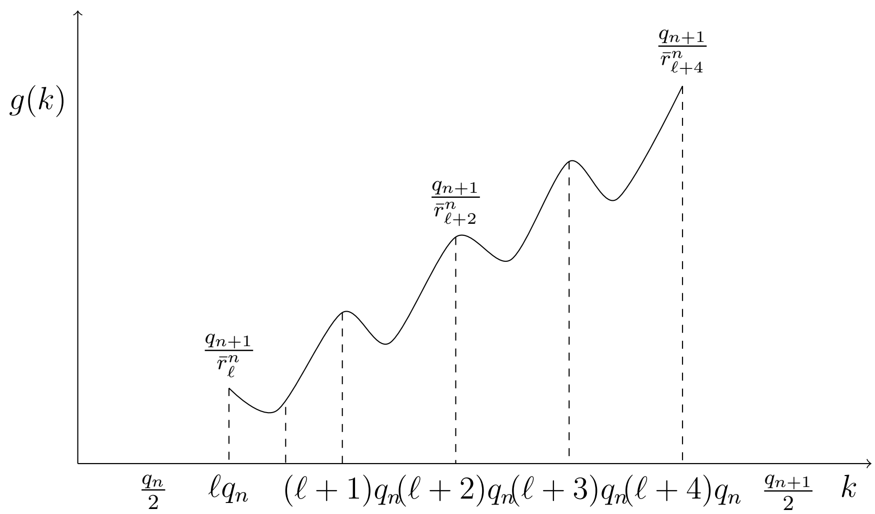

Figure 5

Certainly, there is nothing special about $k = 0$, so the behavior described in Theorem 8.1 happens around arbitrary point $k = k_0$. This implies the self-similar nature of the eigenfunctions): $U(k)$ behave as described at scale q_n but when looked at in windows of size $q_k, q_k \leq q_{n-1}$ will demonstrate the same universal behavior around appropriate local maxima/minima.

To make the above precise, let ϕ be an eigenfunction, and $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$. Let $I^j_{\varsigma_1,\varsigma_2} = [-\varsigma_1 q_j, \varsigma_2 q_j]$, for some $0 < \varsigma_1, \varsigma_2 \leq 1$. We will say k_0 is a local j-maximum of ϕ if $||U(k_0)|| \geq ||U(k)||$ for $k - k_0 \in I^j_{\varsigma_1,\varsigma_2}$. Occasionally, we will also use terminology (j,ς)-maximum for a local j-maximum on an interval $I^j_{\varsigma,\varsigma}$.

Fix $\kappa < \infty$, $\nu > 1$. We will say a local j-maximum k_0 is *nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{{q_{j-1}}^{\nu}},$$

for all $|k| \leq 2q_{j-1}$ and

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{50}$$

for all $2q_{j-1} < |k| \leq 2q_j$.

We will say a local j-maximum is *strongly nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{51}$$

for all $0 < |k| \leq 2q_j$.

An immediate corollary of Theorem 8.1 is the universality of behavior at all (strongly) nonresonant local maxima.

Theorem 7.2. *Given $\varepsilon > 0$, there exists $j(\varepsilon) < \infty$ such that if k_0 is a local j-maximum for $j > j(\epsilon)$, then the following two statements hold:*

If k_0 is nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \tag{52}$$

for all $2s \in I^j_{\varsigma_1,\varsigma_2}$, $|s| > \frac{q_{j-1}}{2}$.

If k_0 is strongly nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \tag{53}$$

for all $2s \in I^j_{\varsigma_1,\varsigma_2}$.

Remark 7.3.

1. For the neighborhood of a local j-maximum described in the Theorem 7.2 only the behavior of $f(s)$ for $q_{j-1}/2 < |s| \leq q_j/2$ is relevant. Thus f implicitly depends on j but through the scale-independent mechanism described in (41),(43) and (45).

2. Actually, one can formulate (52) in Theorem 7.2 with non-resonant condition (50) only required for $2q_{j-1} < |k| \leq q_j$ rather than for $2q_{j-1} < |k| \leq 2q_j$.

In case $\beta(\alpha) > 0$, Theorem 8.1 also guarantees an abundance (and a hierarchical structure) of local maxima of each eigenfunction. Let k_0 be a global maximum[9] .

[9] If there are several, what follows is true for each.

Universal hierarchical structure of an eigenfunction

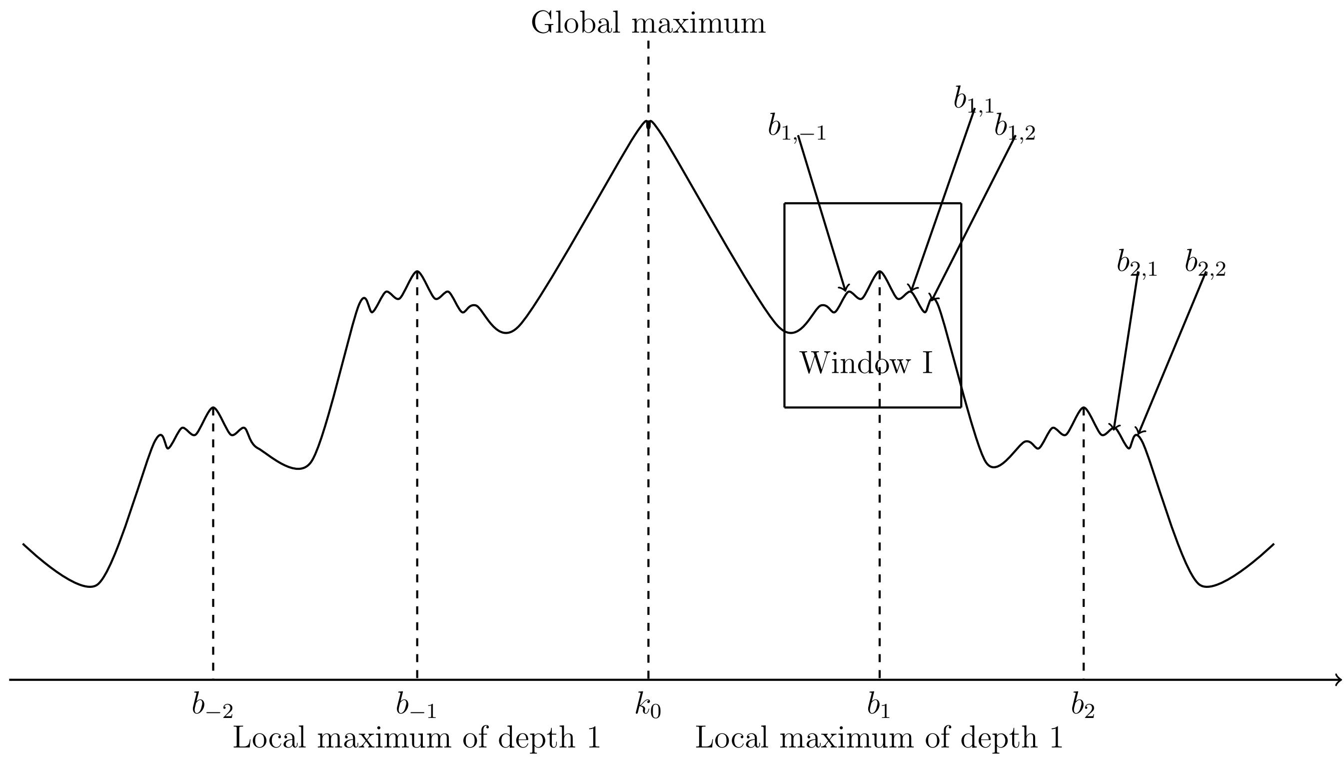

Figure 6

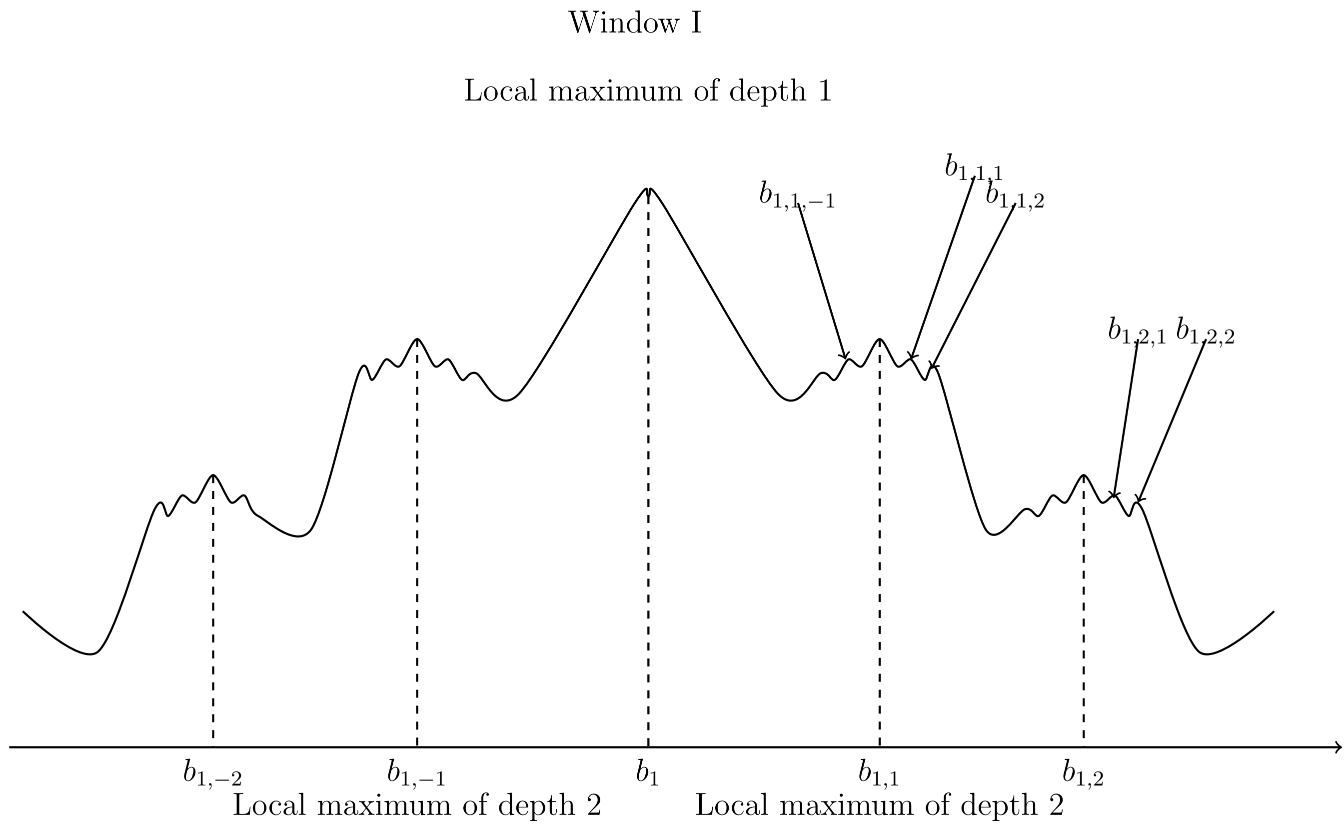

Figure 7

We first describe the hierarchical structure of local maxima informally. We will say that a scale n_{j_0} is exponential if $\ln q_{n_{j_0}+1} > cq_{n_{j_0}}$. Then there is a *constant* scale $\hat{n}_0$ thus a constant $C := q_{\hat{n}_0+1}$, such that for any exponential scale n_j and any eigenfunction there are local n_j-maxima within distance C of $k_0 + sq_{n_{j_0}}$ for each $0 < |s| < e^{cq_{n_{j_0}}}$. Moreover, these are all the local n_{j_0}-maxima in $[k_0 - e^{cq_{n_{j_0}}}, k_0 + e^{cq_{n_{j_0}}}]$. The exponential behavior of the eigenfunction in the local neighborhood (of size of order $q_{n_{j_0}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Note that only exponential behavior at the corresponding scale is determined by f and fluctuations of much smaller size are invisible. Now, let $n_{j_1} < n_{j_0}$ be another exponential scale. Denoting "depth 1" local maximum located near $k_0 + a_{n_{j_0}} q_{n_{j_0}}$ by $b_{a_{n_{j_0}}}$ we then have a similar picture around $b_{a_{n_{j_0}}}$: there are local n_{j_1}-maxima in the vicinity of $b_{a_{n_{j_0}}} + sq_{n_{j_1}}$ for each

$0 < |s| < e^{cq_{n_{j_1}}}$. Again, this describes all the local $q_{n_{j_1}}$-maxima within an exponentially large interval. And again, the exponential (for the n_{j_1} scale) behavior in the local neighborhood (of size of order $q_{n_{j_1}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Denoting those "depth 2" local maxima located near $b_{a_{n_{j_0}}} + a_{n_{j_1}} q_{n_{j_1}}$, by $b_{a_{n_{j_0}}, a_{n_{j_1}}}$ we then get the same picture taking the magnifying glass another level deeper, and so on. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ with each "depth $s+1$" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ being in the corresponding vicinity of the "depth s" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}}$, and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, yet the depth of the hierarchy that can be so achieved is at least $j/2 - C$, see Corollary 7.7. Fig. 6 schematically illustrates the structure of local maxima of depth one and two, and Fig. 7 illustrates that the neighborhood of a local maximum appropriately magnified looks like a picture of the global maximum.

We now describe the hierarchical structure precisely. Suppose

$$||2(\theta + k_0\alpha) + k\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^\nu}, \tag{54}$$

for any $k \in \mathbb{Z}\backslash\{0\}$. Fix $0 < \varsigma, \epsilon$, with $\varsigma + 2\epsilon < 1$. Let $n_j \to \infty$ be such that $\ln q_{n_j+1} \geq (\varsigma + 2\epsilon) \ln |\lambda| q_{n_j}$. Let $\mathfrak{c}_j = (\ln q_{n_j+1} - \ln |a_{n_j}|)/\ln |\lambda| q_{n_j} - \epsilon$. We have $\mathfrak{c}_j > \epsilon$ for $0 < a_{n_j} < e^{\varsigma \ln |\lambda| q_{n_j}}$. Then we have

Theorem 7.4. *There exists $\hat{n}_0(\alpha, \lambda, \kappa, \nu, \epsilon) < \infty$ such that for any $j_0 > j_1 > \cdots > j_k$, $n_{j_k} \geq \hat{n}_0 + k$, and $0 < a_{n_{j_i}} < e^{\varsigma \ln |\lambda| q_{n_{j_i}}}, i = 0, 1, \ldots, k$, for all $0 \leq s \leq k$ there exists a local n_{j_s}-maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ on the interval $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} + I^{n_{j_s}}_{\mathfrak{c}_{j_s}, 1}$ for all $0 \leq s \leq k$ such that the following holds:*

I $|b_{a_{n_{j_0}}} - (k_0 + a_{n_{j_0}} q_{n_{j_0}})| \leq q_{\hat{n}_0+1}$,

II *For any* $1 \leq s \leq k$, $|b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} - (b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}} + a_{n_{j_s}} q_{n_{j_s}})| \leq q_{\hat{n}_0+s+1}$.

III *if* $2(x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}) \in I^{n_{j_k}}_{\mathfrak{c}_{j_k}, 1}$ *and* $|x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}| \geq q_{\hat{n}_0+k}$, *then for each* $s = 0, 1, \ldots, k$,

$$f(x_s)e^{-\varepsilon|x_s|} \leq \frac{||U(x)||}{||U(b_{a_{n_{j_0}},a_{n_{j_1}},\dots,a_{n_{j_s}}})||} \leq f(x_s)e^{\varepsilon|x_s|}, \tag{55}$$

where $x_s = |x - b_{a_{n_{j_0}},a_{n_{j_1}},\dots,a_{n_{j_s}}}|$ *is large enough.*

Moreover, every local n_{j_s}*-maximum on the interval* $b_{a_{n_j},a_{n_{j_1}},\dots,a_{n_{j_{s-1}}}} + [-e^{\epsilon \ln \lambda q_{n_{j_s}}}, e^{\epsilon \ln \lambda q_{n_{j_s}}}]$ *is of the form* $b_{a_{n_{j_0}},a_{n_{j_1}},\dots,a_{n_{j_s}}}$ *for some* $a_{n_{j_s}}$.

Remark 7.5. By I of Theorem 7.4, the local maximum can be determined up to a constant $K_0 = q_{\hat{n}_0+1}$. Actually, if k_0 is only a local n_j+1-maximum, we can still make sure that I, II and III of Theorem 7.4 hold. This is the local version of Theorem 7.4

Remark 7.6. $q_{\hat{n}_0+1}$ is the scale at which *phase* resonances of $\theta + k_0\alpha$ still can appear. Notably, it determines the precision of pinpointing local n_{j_0}-maxima in a (exponentially large in $q_{n_{j_0}}$) neighborhood of k_0, for any j_0. When we go down the hierarchy, the precision decreases, but note that except for the very last scale it stays at least iterated logarithmically [10] small in the corresponding scale $q_{n_{j_s}}$

Thus for $x \in b_{a_{n_{j_0}},a_{n_{j_1}},\dots,a_{n_{j_s}}} + [-\frac{c_{j_s}}{2}q_{n_{j_s}}, \frac{1}{2}q_{n_{j_s}}]$, the behavior of $\phi(x)$ is described by the same universal f in each $q_{n_{j_s}}$-window around the corresponding local maximum $b_{a_{n_{j_0}},a_{n_{j_1}},\dots,a_{n_{j_s}}}, s = 0, 1, \dots, k$. We call such a structure *hierarchical*, and we will say that a local j-maximum is k-hierarchical if the complete hierarchy goes down at least k levels. We then have an immediate corollary

Corollary 7.7. *There exists* $C = C(\alpha, \lambda, \kappa, \nu, \epsilon)$ *such that every local* n_j*-maximum in* $[k_0 - e^{\varsigma \ln |\lambda| q_{n_j}}, k_0 + e^{\varsigma \ln |\lambda| q_{n_j}}]$ *is at least* $(j/2 - C)$*-hierarchical.*

Remark 7.8. The estimate on the depth of the hierarchy in the corollary assumes the worst case scenario when all scales after $\hat{n}_0$ are Liouville. Otherwise the hierarchical structure will go even much deeper. Note that a local n_j-maximum that is not an n_{j+1}-maximum cannot be k-hierarchical for $k > j$.

Another interesting corollary of Theorem 8.1 is

[10] for most scales even much less

Theorem 7.9. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$ and θ is Diophantine with respect to α. Then $H_{\lambda,\alpha,\theta}$ has Anderson localization, with eigenfunctions decaying at the rate* $\ln|\lambda| - \beta$.

This solves the arithmetic version of the second transition conjecture in that it establishes localization throughout the entire regime of (α, λ) where localization may hold for any θ (see the discussion in Section 6), for an arithmetically defined full measure set of θ.

Also, it could be added that, for all θ, $H_{\lambda,\alpha,\theta}$ has no localization (i.e., no exponentially decaying eigenfunctions) if $|\lambda| = e^{\beta}$

Let $\psi(k)$ denote any solution to $H_{\lambda,\alpha,\theta}\psi = E\psi$ that is linearly independent with respect to $\phi(k)$. Let $\tilde{U}(k) = \begin{pmatrix} \psi(k) \\ \psi(k-1) \end{pmatrix}$. An immediate counterpart of (49) is the following

Corollary 7.10. *Under the conditions of Theorem 8.1 for large k vectors $\tilde{U}(k)$ satisfy*

$$g(|k|)e^{-\varepsilon|k|} \leq ||\tilde{U}(k)|| \leq g(|k|)e^{\varepsilon|k|}. \tag{56}$$

Thus every solution is exponentially expanding at the rate $g(k)$ except for one that is exponentially decaying at the rate $f(k)$.

It is well known that for E in the spectrum the dynamics of the transfer-matrix cocycle A_k is nonuniformly hyperbolic. Moreover, E being a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ already implies that the behavior of A_k is non-regular. Theorem 8.1 provides precise information on how the non-regular behavior unfolds in this case. We are not aware of other non-artificially constructed examples of non-uniformly hyperbolic systems where non-regular behavior can be described with similar precision.

The information provided by Theorem 8.1 leads to many interesting corollaries. Here we only want to list a few immediate sharp consequences.

Corollary 7.11. *Under the condition of Theorem 8.1, we have*

i)

$$\limsup_{k\to\infty} \frac{\ln||A_k||}{k} = \limsup_{k\to\infty} \frac{\ln||\tilde{U}(k)||}{k} = \ln|\lambda|,$$

ii)

$$\liminf_{k\to\infty} \frac{\ln||A_k||}{k} = \liminf_{k\to\infty} \frac{\ln||\tilde{U}(k)||}{k} = \ln|\lambda| - \beta.$$

iii) Outside an explicit sequence of lower density zero, [11]

$$\lim_{k\to\infty}\frac{\ln||A_k||}{k}=\lim_{k\to\infty}\frac{\ln||\tilde{U}(k)||}{k}=\ln|\lambda|.$$

Therefore the Lyapunov behavior for the norm fails to hold only along a sequence of density zero. It is interesting that the situation is different for the eigenfunctions. While, just like the overall growth of $\|A_k\|$ is $\ln|\lambda|-\beta$, the overall rate of decay of the eigenfunctions is also $\ln|\lambda|-\beta$, they however decay at the Lyapunov rate only outside a sequence of positive upper density. That is

Corollary 7.12. *Under the condition of Theorem 8.1, we have*

i)

$$\limsup_{k\to\infty}\frac{-\ln||U(k)||}{k}=\ln|\lambda|,$$

ii)

$$\liminf_{k\to\infty}\frac{-\ln||U(k)||}{k}=\ln|\lambda|-\beta.$$

iii) There is an explicit sequence of upper density $1-\frac{1}{2}\frac{\beta}{\ln|\lambda|}$, [12], *along which*

$$\lim_{k\to\infty}\frac{-\ln||U(k)||}{k}=\ln|\lambda|.$$

iv) There is an explicit sequence of upper density $\frac{1}{2}\frac{\beta}{\ln|\lambda|}$,[13]*along which*

$$\limsup_{k\to\infty}\frac{-\ln||U(k)||}{k}<\ln|\lambda|.$$

The fact that g is not always the reciprocal of f leads also to another interesting phenomenon.

Let $0\le\delta_k\le\frac{\pi}{2}$ be the angle between vectors $U(k)$ and $\tilde{U}(k)$.

[11]The sequence with convergence to the Lyapunov exponent contains $q_n, n=1,\cdots$.

[12]The sequence contains $\lfloor\frac{q_n}{2}\rfloor, n=1,\cdots$.

[13]This sequence can have lower density ranging from 0 to $\frac{1}{2}\frac{\beta}{\ln|\lambda|}$ depending on finer continued fraction properties of α.

Corollary 7.13. *We have*

$$\limsup_{k\to\infty} \frac{\ln \delta_k}{k} = 0, \tag{57}$$

and

$$\liminf_{k\to\infty} \frac{\ln \delta_k}{k} = -\beta. \tag{58}$$

Thus neighborhoods of resonances q_n are the places of exponential tangencies between contracted and expanded directions, with the rate approaching $-\beta$ along a subsequence.[14] This means, in particular, that A_k with $k \sim q_n$ is exponentially close to a matrix with the trace $e^{(\ln|\lambda|-\beta)k}$. Exponential tangencies also happen around points of the form jq_n but at lower strength.

8 Asymptotics of eigenfunctions and universal hierarchical structure for phase resonances

Our proof of localization is, again, based on determining the *exact asymptotics* of the generalized eigenfunctions in the regime $|\lambda| > e^{\delta(\alpha,\theta)}$. However, the asymptotics (and the methods required) are very different in the case of phase resonances.

For any ℓ, let x_0 (we can choose any one if x_0 is not unique) be such that

$$|\sin \pi(2\theta + x_0\alpha)| = \min_{|x|\leq 2|\ell|} |\sin \pi(2\theta + x\alpha)|.$$

Let $\eta = 0$ if $2\theta + x_0\alpha \in \mathbb{Z}$, otherwise let $\eta \in (0,\infty)$ be given by the following equation,

$$|\sin \pi(2\theta + x_0\alpha)| = e^{-\eta|\ell|}. \tag{59}$$

Define $f : \mathbb{Z} \to \mathbb{R}^+$ as follows.

Case 1: $x_0 \cdot \ell \leq 0$. Set $f(\ell) = e^{-|\ell|\ln|\lambda|}$.

Case 2. $x_0 \cdot \ell > 0$. Set $f(\ell) = e^{-(|x_0|+|\ell-x_0|)\ln|\lambda|}e^{\eta|\ell|} + e^{-|\ell|\ln|\lambda|}$.

We say that ϕ is a generalized eigenfunction of H with generalized eigenvalue E, if

$$H\phi = E\phi, \text{ and } |\phi(k)| \leq \hat{C}(1+|k|). \tag{60}$$

[14]In fact, the rate is close to $-\frac{\ln q_{n+1}}{q_n}$ for any large n.

For a fixed generalized eigenvalue E and corresponding generalized eigenfunction ϕ of $H_{\lambda,\alpha,\theta}$, let $U(\ell) = \begin{pmatrix} \phi(\ell) \\ \phi(\ell-1) \end{pmatrix}$. We have

Theorem 8.1. *Assume* $\ln|\lambda| > \delta(\alpha,\theta)$. *Then for any* $\varepsilon > 0$, *there exists* K *such that for any* $|\ell| \geq K$, $U(\ell)$ *satisfies*

$$f(\ell)e^{-\varepsilon|\ell|} \leq ||U(\ell)|| \leq f(\ell)e^{\varepsilon|\ell|}. \tag{61}$$

In particular, the eigenfunctions decay at the rate $\ln|\lambda| - \delta(\alpha,\theta)$.

Remark

- For $\delta = 0$ we have that for any $\varepsilon > 0$,

  $$e^{-(\ln|\lambda|+\varepsilon)|\ell|} \leq f(\ell) \leq e^{-(\ln|\lambda|-\varepsilon)|\ell|}.$$

  This implies that the eigenfunctions decay precisely at the rate of Lyapunov exponent $\ln|\lambda|$.

- For $\delta > 0$, by the definition of δ and f, we have for any $\varepsilon > 0$,

  $$f(\ell) \leq e^{-(\ln|\lambda|-\delta-\varepsilon)|\ell|}. \tag{62}$$

- By the definition of δ again, there exists a subsequence $\{\ell_i\}$ such that

  $$|\sin\pi(2\theta + \ell_i\alpha)| \leq e^{-(\delta-\varepsilon)|\ell_i|}.$$

  By the DC on α, one has that

  $$|\sin\pi(2\theta + \ell_i\alpha)| = \min_{|x|\leq 2|\ell_i|} |\sin\pi(2\theta + x\alpha)|.$$

  Then

  $$f(\ell_i) \geq e^{-(\ln|\lambda|-\delta+\varepsilon)|\ell_i|}. \tag{63}$$

  This implies the eigenfunctions decay precisely at the rate $\ln|\lambda| - \delta(\alpha,\theta)$.

- If x_0 is not unique, by the DC on α, η is necessarily arbitrarily small. Then

  $$e^{-(\ln|\lambda|+\varepsilon)|\ell|} \leq ||U(\ell)|| \leq e^{-(\ln|\lambda|-\varepsilon)|\ell|}.$$

The behavior described in Theorem 8.1 happens around arbitrary point. This, coupled with effective control of parameters at the local maxima, allows to uncover the self-similar nature of the eigenfunctions. Hierarchical behavior of solutions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions [93] has remained an important open challenge even at the physics level. In the previous section we described universal hierarchical structure of the eigenfunctions for all frequencies α and phases with $\delta(\alpha,\theta)=0$. In studying the eigenfunctions of $H_{\lambda,\alpha,\theta}$ for $\delta(\alpha,\theta)>0$ Wencai Liu and I [60] obtained a different kind of universality throughout the pure point spectrum regime, which features a self-similar hierarchical structure upon proper *reflections*.

Assume phase θ satisfies $0<\delta(\alpha,\theta)<\ln\lambda$. Fix $0<\varsigma<\delta(\alpha,\theta)$.

Let k_0 be a global maximum of eigenfunction ϕ.[15] Let K_i be the positions of exponential resonances of the phase $\theta'=\theta+k_0\alpha$ defined by

$$||2\theta+(2k_0+K_i)\alpha||_{\mathbb{R}/\mathbb{Z}}\leq e^{-\varsigma|K_i|}, \tag{64}$$

This means that $|v(\theta'+\ell\alpha)-v(\theta'+(K_i-\ell)\alpha)|\leq Ce^{-\varsigma|K_i|}$, uniformly in ℓ, or, in other words, the potential $v_n=v(\theta+n\alpha)$ is $e^{-\varsigma|K_i|}$-almost symmetric with respect to $(k_0+K_i)/2$.

Since α is Diophantine, we have

$$|K_i|\geq ce^{c|K_{i-1}|}, \tag{65}$$

where c depends on ς and α through the Diophantine constants κ,τ. On the other hand, K_i is necessarily an infinite sequence.

Let ϕ be an eigenfunction, and $U(k)=\begin{pmatrix}\phi(k)\\ \phi(k-1)\end{pmatrix}$. We say k is a local K-maximum if $||U(k)||\geq||U(k+s)||$ for all $s-k\in[-K,K]$.

We first describe the hierarchical structure of local maxima informally. There exists a *constant* $\hat{K}$ such that there is a local cK_j-maximum b_j within distance $\hat{K}$ of each resonance K_j. The exponential behavior of the eigenfunction in the local cK_j-neighborhood of each such local maximum, normalized by the value at the local maximum is given by the *reflection* of f. Moreover, this describes the entire collection of local maxima of depth 1, that is K such that K is a cK-maximum. Then we have a similar picture in the

[15] Can take any one if there are several.

vicinity of b_j : there are local cK_i-maxima $b_{j,i}, i < j$, within distance $\hat{K}^2$ of each $K_j - K_i$. The exponential (on the K_i scale) behavior of the eigenfunction in the local cK_i-neighborhood of each such local maximum, normalized by the value at the local maximum is given by f. Then we get the next level maxima $b_{j,i,s}, s < i$ in the $\hat{K}^3$-neighborhood of $K_j - K_i + K_s$ and reflected behavior around each, and so on, with reflections alternating with steps. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{j_0,j_1,\dots,j_s}$, with each "depth $s+1$" local maximum $b_{j_0,j_1,\dots,j_s}$ being in the corresponding vicinity of the "depth s" local maximum $b_{j_0,j_1,\dots,j_{s-1}} \approx k_0 + \sum_{i=0}^{s-1}(-1)^i K_{j_i}$ and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, with $b_{j_0,j_1,\dots,j_{s-1}}$ determined with $\hat{K}^s$ precision, thus it presents an accurate picture as long as $K_{j_s} \gg \hat{K}^s$.

We now describe the hierarchical structure precisely.

Theorem 8.2. *[60] Assume sequence K_i satisfies (64) for some $\varsigma > 0$. Then there exists $\hat{K}(\alpha, \lambda, \theta, \varsigma) < \infty$[16] such that for any $j_0 > j_1 > \cdots > j_k \geq 0$ with $K_{j_k} \geq \hat{K}^{k+1}$, for each $0 \leq s \leq k$ there exists a local $\frac{\varsigma}{2\ln\lambda}K_{j_s}$-maximum[17] $b_{j_0,j_1,\dots,j_s}$ such that the following holds:*

I $|b_{j_0,j_1,\dots,j_s} - k_0 - \sum_{i=0}^{s}(-1)^i K_{j_i}| \leq \hat{K}^{s+1}$.

II *For any $\varepsilon > 0$, if $C\hat{K}^{k+1} \leq |x - b_{j_0,j_1,\dots,j_k}| \leq \frac{\varsigma}{4\ln\lambda}|K_{j_k}|$, where C is a large constant depending on $\alpha, \lambda, \theta, \varsigma$ and ε, then for each $s = 0, 1, \dots, k$,*

$$f((-1)^{s+1}x_s)e^{-\varepsilon|x_s|} \leq \frac{||U(x)||}{||U(b_{j_0,j_1,\dots,j_s})||} \leq f((-1)^{s+1}x_s)e^{\varepsilon|x_s|}, \qquad (66)$$

where $x_s = x - b_{j_0,j_1,\dots,j_s}$.

Thus the behavior of $\phi(x)$ is described by the same universal f in each $\frac{\varsigma}{2\ln\lambda}K_{j_s}$ window around the corresponding local maximum $b_{j_0,j_1,\dots,j_s}$ after alternating reflections. The positions of the local maxima in the hierarchy are determined up to errors that at all but possibly the last step are superlogarithmically small in K_{j_s}. We call such a structure *reflective hierarchy.*

[16] $\hat{K}$ depends on θ through $2\theta + k\alpha$, see (38).

[17] Actually, it can be a local $(\frac{\varsigma}{\ln\lambda} - \varepsilon)K_{j_s}$-maximum for any $\varepsilon > 0$.

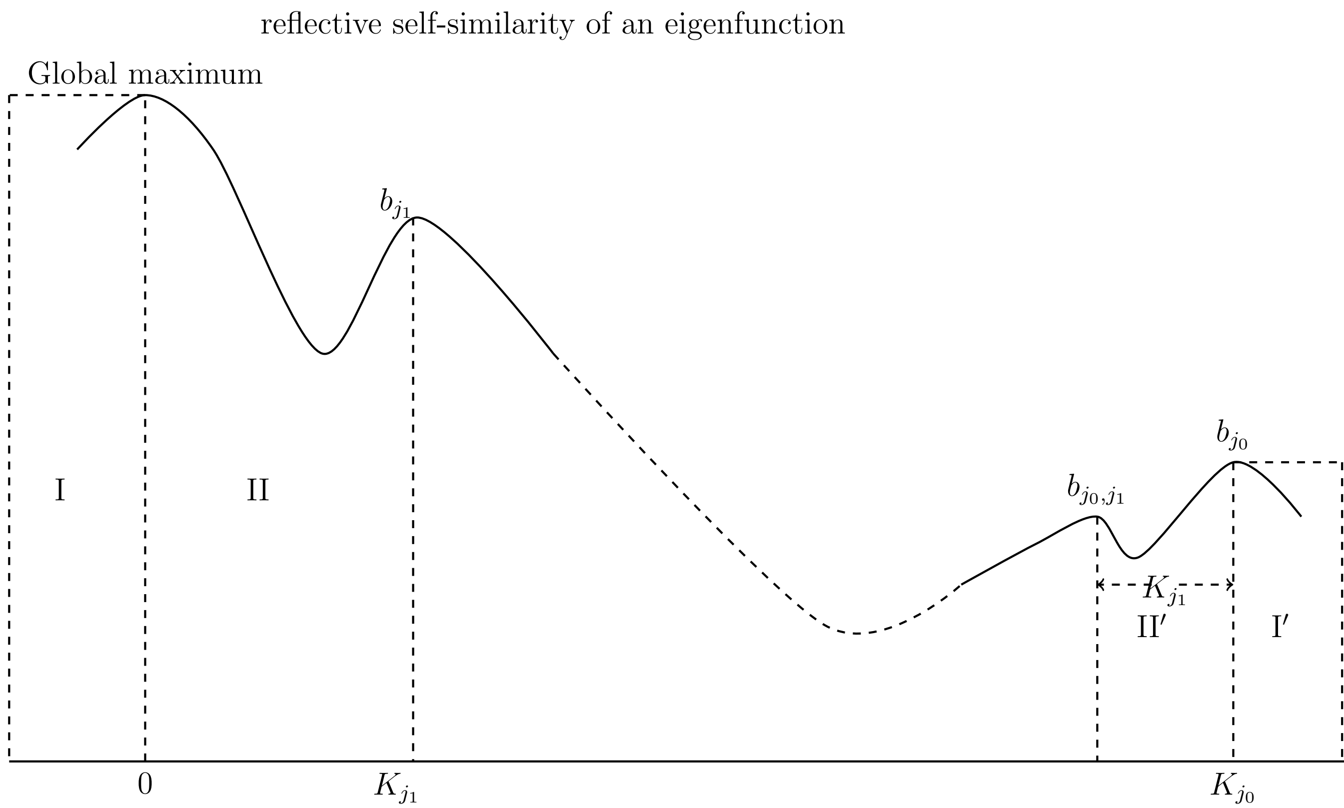

Figure 8: This depicts reflective self-similarity of an eigenfunction with global maximum at 0. The self-similarity: I′ is obtained from I by scaling the x-axis proportional to the ratio of the heights of the maxima in I and I′. II′ is obtained from II by scaling the x-axis proportional to the ratio of the heights of the maxima in II and II′. The behavior in the regions I′, II′ mirrors the behavior in regions I, II upon reflection and corresponding dilation.

Finally, as in the frequency resonance case, we discuss the asymptotics of the transfer matrices. Let, as before, $A_0 = I$ and for $k \geq 1$,

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha) \cdots A(\theta)$$

and

$$A_{-k}(\theta) = A_k^{-1}(\theta - k\alpha),$$

where $A(\theta) = \begin{pmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}$. Thus A_k is the (k-step) transfer matrix. It also depends on α and E but since those parameters will be fixed, we omit them from the notation.

We define a new function $g : \mathbb{Z} \to \mathbb{R}^+$ as follows.

Case 1: If $x_0 \cdot \ell \leq 0$ or $|x_0| > |\ell|$, set

$$g(\ell) = e^{|\ell| \ln |\lambda|}.$$

Case 2: If $x_0 \cdot \ell \geq 0$ and $|x_0| \leq |\ell| \leq 2|x_0|$, set

$$g(\ell) = e^{(\ln \lambda - \eta)|\ell|} + e^{|2x_0 - \ell| \ln |\lambda|}.$$

Case 3: If $x_0 \cdot \ell \geq 0$ and $|\ell| > 2|x_0|$, set

$$g(\ell) = e^{(\ln \lambda - \eta)|\ell|}.$$

We have

Theorem 8.3. *Under the conditions of Theorem 8.1, we have*

$$g(\ell)e^{-\varepsilon|\ell|} \leq ||A_\ell|| \leq g(\ell)e^{\varepsilon|\ell|}. \tag{67}$$

Let $\psi(\ell)$ denote any solution to $H_{\lambda,\alpha,\theta}\psi = E\psi$ that is linearly independent with $\phi(\ell)$. Let $\tilde{U}(\ell) = \begin{pmatrix} \psi(\ell) \\ \psi(\ell-1) \end{pmatrix}$. An immediate counterpart of (67) is the following

Corollary 8.4. *Under the conditions of Theorem 8.1, vectors $\tilde{U}(\ell)$ satisfy*

$$g(\ell)e^{-\varepsilon|\ell|} \leq ||\tilde{U}(\ell)|| \leq g(\ell)e^{\varepsilon|\ell|}. \tag{68}$$

Our analysis also gives

Corollary 8.5. *Under the conditions of Theorem 8.1, we have,*

i)

$$\limsup_{k\to\infty} \frac{\ln ||A_k||}{k} = \limsup_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda|,$$

ii)

$$\liminf_{k\to\infty} \frac{\ln ||A_k||}{k} = \liminf_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda| - \delta.$$

iii) outside a sequence of lower density $1/2$*,*

$$\lim_{k\to\infty} \frac{-\ln ||U(k)||}{|k|} = \ln|\lambda|, \tag{69}$$

iv) outside a sequence of lower density 0*,*

$$\lim_{k\to\infty} \frac{\ln ||A_k||}{|k|} = \ln|\lambda|. \tag{70}$$

Note that (70) also holds throughout the pure point regime of [59]. As in the previous section, the fact that g is not always the reciprocal of f leads to exponential tangencies between contracted and expanded directions with the rate approaching $-\delta$ along a subsequence. Tangencies are an attribute of nonuniform hyperbolicity and are usually viewed as a difficulty to avoid through e.g. the parameter exclusion. Our analysis allows to study them in detail and uncovers the hierarchical structure of exponential tangencies positioned precisely at phase resonances. The methods developed to prove these theorems have made it possible to determine also the *exact* exponent of the exponential decay rate in expectation for the two-point function [58], the first result of this kind for any model.

9 Further extensions

While the almost Mathieu family is precisely the one of main interest in physics literature, it also presents the simplest case of analytic quasiperiodic operator, so a natural question is which features discovered for the almost Mathieu would hold for this more general class. Not all do, in particular, the ones that exploit the self-dual nature of the family $H_{\lambda,\alpha,\theta}$ often cannot be expected to hold in general. In case of Theorems 6.2 and 7.1, we conjecture that they should in fact hold for general analytic (or even more general) potentials, for a.e. phase and with $\ln|\lambda|$ replaced by the Lyapunov exponent $L(E)$, but with otherwise the same or very similar statements. The hierarchical structure theorems 7.2 and 7.4 are also expected to hold universally for most (albeit not all, as in the present paper) appropriate local maxima. Some of our qualitative corollaries may hold in even higher generality. Establishing this fully would require certain new ideas since so far even an arithmetic version of localization for the Diophantine case has not been established for the general analytic family, the current state-of-the-art result by Bourgain-Goldstein [27] being measure theoretic in α. However, some ideas of our method can already be transferred to general trigonometric polynomials [62]. Moreover, our method was used recently in [43] to show that the same f and g govern the asymptotics of eigenfunctions and universality around the local maxima throughout the a.e. localization regime in another popular object, the Maryland model, as well as in several other scenarios (work in progress).

So we expect the same arithmetic frequency transition for general analytic potentials, but as far as the arithmetic phase transitions, we expect the same results to hold for general *even* analytic potentials for a.e. frequency, see more detail below. We note that the singular continuous part up to the conjectured transition is already established, even in a far greater generality, in [14, 60].

The universality of the hierarchical structure described in Sections 7, 8 is twofold: not only it is the same universal function that governs the behavior around each exponential frequency or phase resonance (upon reflection and renormalization), it is the same structure for all the parameters involved: any (Diophantine) frequency α, (any α-Diophantine phase θ) with $\beta(\alpha) < L$, ($\delta(\alpha, \theta) < L$), and any eigenvalue E. The universal reflective-hierarchical structure requires evenness of the function defining the potential, and moreover, in general, resonances of other types may also be present. However, we conjectured in [60] that for general even analytic potentials for a.e. frequency only finitely many other exponentially strong resonances will appear, thus the structure described here will hold for the corresponding class, with the $\ln \lambda$ replaced by the Lyapunov exponent $L(E)$ throughout.

The key elements of the technique developed for the treatment of arithmetic resonances are robust and have made it possible to approach other scenarios, and in particular, study delicate properties of the singular continuous regime, obtaining upper bounds on fractal dimensions of the spectral measure and quantum dynamics for the almost Mathieu operator [61], as well as potentials defined by general trigonometric analytic functions [62].

Finally, we briefly comment that for Schrödinger operators with analytic periodic potentials, almost Lipshitz continuity of gaps holds for Diophantine α for all non-critical (in the sense of Avila's global theory [5]) energies [63]. For critical energies, we do not have anything better than Hölder-$\frac{1}{2}$ regularity that holds universally. For the prototypical critical potential, the critical almost Mathieu, almost Lipshitz continuity of spectra also holds, because of the hidden singularity. This leads to two potentially related questions for analytic quasiperiodic Schrödinger operators:

1. Does some form of uniform almost Lipshitz continuity always hold?
2. Is there always a singularity hidden behind the criticality?

A positive answer to the second question would lead to a statement that critical operators never have eigenvalues and that Hausdorff dimension of the critical part of the spectrum is always bounded by 1/2.

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