Critical almost Mathieu operator: hidden singularity, gap continuity, and the Hausdorff dimension of the spectrum

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In memory of David Thouless, 1934–2019

We prove almost Lipshitz continuity of spectra of singular quasiperiodic Jacobi matrices and obtain a representation of the critical almost Mathieu family that has a singularity. This allows us to prove that the Hausdorff dimension of its spectrum is not larger than 1/2 for all irrational frequencies, solving a long-standing problem. Other corollaries include two very elementary proofs of zero measure of the spectrum (Problem 5 in [41]) and a similar Hausdorff dimension result for the quantum graph graphene.

1 Introduction

The critical almost Mathieu operator on $\ell^2(\mathbb{Z})$, that is

$$(H_{\alpha,\theta,\lambda}\phi)(n) = \phi(n-1) + \phi(n+1) + 2\lambda \cos 2\pi(\alpha n + \theta)\phi(n), \qquad n = \dots, -1, 0, 1, \dots \quad (1)$$

with $\lambda = 1$, represents a transition from "extended states" for the subcritical case, $0 \le \lambda < 1$, to "localization" for the supercritical case, $\lambda > 1$. The almost Mathieu family models an electron on the two-dimensional square lattice in a uniform perpendicular magnetic field with flux α [39]; the critical case corresponds to the isotropic lattice, while $\lambda \neq 1$ corresponds to anisotropy. The almost Mathieu operator $H_{\alpha,\theta} := H_{\alpha,\theta,1}$ at the critical value of the parameter λ is the one most important in physics, where it is also known as the Harper or Azbel-Hofstadter model (see e.g., [5] and references therein), and also the one least understood mathematically and even heuristically.

The spectrum of $H_{\alpha,\theta}$ for irrational α is a θ-independent[3] fractal, beautifully depicted via the Hofstadter butterfly [22]. There have been many numerical and heuristic studies of its fractal dimension in physics literature (e.g., [29, 14, 43, 48]). A conjecture, sometimes attributed to Thouless (e.g., [48]), and appearing already in the early 1980's, is that

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the dimension is equal to 1/2. It has been rethought after rigorous and numerical studies demonstrated that the Hausdorff dimension can be less than 1/2 (and even be zero) for some α [35, 4, 48], while packing/box counting dimension can be higher (even equal to one) for some (in fact, of the same!) α [28]. However, all these are Lebesgue measure zero sets of α, and the conjecture may still hold, in some sense. There is also a conjecture attributed to J. Bellissard (e.g., [35, 20]) that the dimension of the spectrum is a property that only depends on the tail in the continued fraction expansion of α and thus should be the same for a.e. α (by the properties of the Gauss map). We discuss the history of rigorous results on the dimension in more detail below.

The equality in the original conjecture can be viewed as two inequalities, and here we prove one of those for *all* irrational α. This is also the first result on the fractal dimension that holds for more than a measure zero set of α. Denote the spectrum of an operator K by $\sigma(K)$, the Lebesgue measure of a set A by $|A|$, and its Hausdorff dimension by $\dim_{\mathrm{H}}(A)$. We have

Theorem 1.1 *For any irrational α and real θ,* $\dim_{\mathrm{H}}(\sigma(H_{\alpha,\theta})) \leq 1/2$.

Of course, it only makes sense to discuss upper bounds on the Hausdorff dimension of a set on the real line once its Lebesgue measure is shown to be zero. The Aubry-Andre conjecture stated that the measure of the spectrum of $H_{\alpha,\theta,\lambda}$ is equal to $4|1-|\lambda||$, so to 0 if $\lambda = 1$, for any irrational α. This conjecture was popularized by B. Simon, first in his list of 15 problems in mathematical physics [40] and then, after it was proved by Last for a.e. α [33, 34], again as Problem 5 in [41], which was to prove this conjecture for the remaining measure zero set of α, namely, for α of bounded type.[4] The arguments of [33, 34] did not work for this set, and even though the semi-classical analysis of Hellfer-Sjöstrand [21] applied to some of this set for $H_{\alpha,\theta}$, it did not apply to other such α, including, most notably, the golden mean — the subject of most numerical investigations. For the non-critical case, the proof for all α of bounded type was given in [23], but the critical "bounded-type" case remained difficult to crack. This remaining problem for zero measure of the spectrum of $H_{\alpha,\theta}$ was finally solved by Avila-Krikorian [3], who employed a deep dynamical argument. We note that the argument of [3] worked not for all α, but for a full measure subset of Diophantine α. In the present paper, we provide a very simple argument that recovers this theorem and thus gives an elementary solution to Problem 5 of [41]. Moreover, our argument works simultaneously for all irrational α.

Theorem 1.2 *For any irrational α and real θ,* $|\sigma(H_{\alpha,\theta})| = 0$.

Our proofs are based on two key ingredients. In Section 3, we introduce the chiral gauge transform and show that the direct sum in θ of operators $H_{2\alpha,\theta}$ is isospectral with the direct sum in θ of $\widehat{H}_{\alpha,\theta}$ given by

$$(\widehat{H}_{\alpha,\theta}\phi)(n) = 2\sin 2\pi(\alpha(n-1)+\theta)\phi(n-1) + 2\sin 2\pi(\alpha n+\theta)\phi(n+1). \tag{2}$$

[4] That is α with all coefficients in the continued fraction expansion bounded by some M.

This representation of the almost Mathieu operator corresponds to choosing the chiral gauge for the perpendicular magnetic field applied to the electron on the square lattice. It was previously discussed non-rigorously in [38, 30, 47].[5] The advantage of (2) is that it is a *singular* Jacobi matrix, that is the one with off-diagonal elements not bounded away from zero, so that the matrix quasi-separates into blocks.

This is already sufficient to conclude Theorem 1.2 which we do in Section 4. Yet another proof of Theorem 1.2 follows from Theorem 1.3 below.

The second key ingredient is a general result on almost Lipshitz continuity of spectra for *singular* quasiperiodic Jacobi matrices, Theorem 5.2 in Section 5. The modulus of continuity statements have, in fact, been central in previous literature. We consider a general class of quasiperiodic C^1 Jacobi matrices, that is operators on $\ell^2(\mathbb{Z})$ given by

$$(H_{v,b,\alpha,\theta}\phi)(n) = b(\theta + (n-1)\alpha)\phi(n-1) + b(\theta + n\alpha)\phi(n+1) + v(\theta + n\alpha)\phi(n), \quad (3)$$

with $b(x), v(x) \in C^1(\mathbb{R})$, and periodic with period 1.

Let $M_{v,b,\alpha}$ be the direct sum of $H_{v,b,\alpha,\theta}$ over $\theta \in [0,1)$,

$$M_{v,b,\alpha} = \oplus_{\theta\in[0,1)} H_{v,b,\alpha,\theta}. \quad (4)$$

Continuity in α of $\sigma(M_{v,b,\alpha})$ in the Hausdorff metric was proved in [7, 13]. Continuity of the measure of the spectrum is a more delicate issue, since, in particular, $|\sigma(M_\alpha)|$ can be (and is, for the almost Mathieu operator) discontinuous at rational α. Establishing continuity at irrational α requires quantitative estimates on the Hausdorff continuity of the spectrum. In the Schrödinger case, that is for $b = 1$, Avron, van Mouche, and Simon [6] obtained a very general result on Hölder-$\frac{1}{2}$ continuity (for arbitrary $v \in C^1$), improving Hölder-$\frac{1}{3}$ continuity obtained earlier by Choi, Elliott, and Yui [12]. It was argued in [6] that Hölder continuity of *any* order larger than 1/2 would imply the desired continuity property of the measure of the spectrum for *all* α. Lipshitz continuity of gaps was proved by Bellissard [9] for a large class of quasiperiodic operators, however without a uniform Lipshitz constant, thus not allowing to conclude continuity of the measure of the spectrum. In [23] (see also [24]) we showed a uniform almost Lipshitz continuity for Schrödinger operators with analytic potentials and Diophantine frequencies in the regime of positive Lyapunov exponents, which, in particular, allowed us to complete the proof of the Aubry-Andre conjecture for the non-critical case.

A Jacobi matrix (3) is called *singular* if for some θ_0, $b(\theta_0) = 0$. We assume that the number of zeros of b on its period is finite. Theorem 5.2 establishes a uniform almost Lipshitz continuity in this case and allows to conclude continuity of the measure of the spectrum for general singular Jacobi matrices:

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[5] In the rational case $\alpha = p/q$, a similar representation for the *discriminant* of $H_{p/q,\theta}$ is easy to justify (see, e.g., the appendix of [31]) and is already useful [32].

Theorem 1.3 *For singular $H_{v,b,\alpha,\theta}$ as above, for any irrational α there exists a subsequence of canonical approximants $\frac{p_{n_j}}{q_{n_j}}$ such that*

$$|\sigma(M_{v,b,\alpha})| = \lim_{j\to\infty} \left|\sigma\left(M_{v,b,\frac{p_{n_j}}{q_{n_j}}}\right)\right|. \qquad (5)$$

Remark 1.4 We note that this form of continuity is usually sufficient for practical purposes, since mere existence of some sequence of periodic approximants along which the convergence happens is enough, as the measure of the spectrum can often be estimated for an arbitrary rational.

In the case of Schrödinger operators (i.e., for $b = 1$), the statement (5) was previously established in various degrees of generality in the regime of positive Lyapunov exponents [23, 27, 18] and, in all regimes (using [3]), for analytic [25] or sufficiently smooth [49] v. Typically, proofs that work for $b = 1$ extend also to the case of non-vanishing b, that is *non-singular* Jacobi matrices, and there is no reason to believe the results of [25, 49] should be an exception. On the other hand, extending various Schrödinger results to the singular Jacobi case is technically non-trivial and adds a significant degree of complexity (e.g. [26, 2, 16]). Here however, we show that a singularity can be *exploited*, rather than circumvented,[6] to establish enhanced continuity of spectra (Theorem 5.2) and therefore Theorem 1.3. Of course, Theorem 1.2 also follows immediately from the chiral gauge representation, the bound (6) below, and Theorem 1.3, providing a third proof of Problem 5 of [41].

Moreover, Theorem 5.2 combined with the chiral gauge representation allows to immediately prove Theorem 1.1 by an argument of [34]. Indeed, the original intuition behind Thouless' conjecture on the Hausdorff dimension 1/2 is based on another fascinating Thouless' conjecture [45, 46]: that for the critical almost Mathieu operator $H_{\alpha,\theta}$, in the limit $p_n/q_n \to \alpha$, we have $q_n|\sigma(M_{p_n/q_n})| \to c$ where $c = 32C_c/\pi$, C_c being the Catalan constant. Thouless argued that if $\sigma(M_\alpha)$ is "economically covered" by $\sigma(M_{p_n/q_n})$ and if all bands are of about the same size,[7] then the spectrum, being covered by q_n intervals of size $\frac{c}{q_n^2}$, has the box counting dimension 1/2. Clearly, the exact value of $c > 0$ is not important for this argument. An upper bound of the form

$$q_n|\sigma(M_{p_n/q_n})| < C, \qquad n = 1, 2, \ldots, \qquad (6)$$

was proved by Last [34][8], which, combined with Hölder-$\frac{1}{2}$ continuity, led him in [34] to the bound $\leq \frac{1}{2}$ for the Hausdorff dimension for irrational α satisfying $\lim_{n\to\infty} |\alpha - p_n/q_n|q_n^4 = 0$.

[6] Singularity has also been treated recently as a friend rather than foe in [19, 8] in some other contexts.

[7] In reality, the bands can decay exponentially with distance from the center at each step in the continued fraction hierarchy [21]. The central bands can be power-law small in $1/q$ [32]. However, "economically covered" is a physicist's way of stating a nice modulus of continuity, and Thouless' intuition does work for the upper bound.

[8] with $C = 8e$.

Such α form a zero measure set. The almost Lipschitz continuity of Theorem 5.2 and (6) allow us to obtain the result (Theorem 1.1) for *all* irrational α.

In the past few years, there was an increased interest in the dimension of the spectrum of the critical almost Mathieu operator, leading to a number of other rigorous results mentioned above. Those include zero Hausdorff dimension for a subset of Liouville α by Last and Shamis [35], also extended to all weakly Liouville[9] α by Avila, Last, Shamis, Zhou [4]; the full packing (and therefore box counting) dimension for weakly Liouville α [28], and existence of a dense positive Hausdorff dimension set of Diophantine α with positive Hausdorff dimension of the spectrum by Helffer, Liu, Qu, and Zhou [20]. All those results, as well as heuristics by Wilkinson-Austin [48] and, of course, numerics, hold for measure zero sets of α. Recently, B. Simon listed the problem to determine the Hausdorff dimension of the spectrum of the critical almost Mathieu on his new list of hard unsolved problems [42].

Since our proof of Theorem 1.1 only requires an estimate such as (6) and the existence of isospectral family of singular Jacobi matrices, it applies equally well to all other situations where the above two facts are present. For example, Becker et al [8] recently introduced a model of graphene as a quantum graph on the regular hexagonal lattice and studied it in the presence of a magnetic field with a constant flux Φ, with the spectrum denoted σ^{Φ}. Upon identification with the interval $[0,1]$, the differential operator acting on each edge is then the maximal Schrödinger operator $\frac{d^2}{dx^2} + V(x)$ with domain H^2, where V is a Kato-Rellich potential symmetric with respect to 1/2. We then have

Theorem 1.5 *For any symmetric Kato-Rellich potential* $V \in L^2$*, the Hausdorff dimension* $\dim_H(\sigma^{\Phi}) \leq 1/2$ *for all irrational* Φ*.*

This result was proved in [8] for a topologically generic but measure zero set of α.

The basic idea of the proof of Theorem 5.2 is that a singularity could lead to enhanced continuity because creating approximate eigenfunctions by cutting at near-zeros of the off-diagonal terms leads to smaller errors in the kinetic energy. However, without apriori estimates on the behavior of solutions (and it is in fact natural for solutions to be large around the singularity) this in itself is insufficient to achieve an improvement over the Hölder exponent 1/2. Our main technical achievement here is in finding a proper continuity statement and an argument that allows to exploit the singularity efficiently.

Finally, we briefly comment that for Schrödinger operators with analytic periodic potentials almost Lipshitz continuity of gaps holds for Diophantine α for all non-critical (in the sense of Avila's global theory [1]) energies [25]. For critical energies, we do not have anything better than Hölder-$\frac{1}{2}$ that holds universally. Here we prove that for the prototypical critical potential, the critical almost Mathieu, almost Lipshitz continuity of spectra also holds, because of the hidden singularity. This leads to two potentially related questions for analytic quasiperiodic Schrödinger operators:

[9] We say α is weakly Liouville if $\beta(\alpha) := -\limsup \frac{\ln \|n\alpha\|}{n} > 0$, where $\|\theta\| = \mathrm{dist}\,(\theta, \mathbb{Z})$.

1. Does some form of uniform almost Lipshitz continuity (a statement such as Theorem 5.2) always hold?

2. Is there always a singularity hidden behind the criticality?

2 Preliminaries

2.1 IDS and the Lyapunov exponent

For a family of operators H_θ on $\ell^2(\mathbb{Z})$ such that $T^{-1}H_\theta T = H_{\theta-\alpha}$, let ν_θ be the spectral measure of H_θ corresponding to $\delta_0 = (\dots, 0, \delta_0(0) = 1, 0, \dots)$. Namely for any Borel set A we have

$$\nu_\theta(A) = (\chi_A(H_\theta)\delta_0, \delta_0).$$

The *density of states measure* ρ is defined by

$$\rho(A) = \int_{\mathbb{T}} \nu_\theta(A)\ \mathrm{d}\theta.$$

We have [7] $\sigma(\oplus_\theta H_\theta) = \operatorname{supp}\rho$. The cumulative distribution function $N(E) := \rho((-\infty, E))$ of $\rho(A)$ is called the *integrated density of states (IDS)* of H_θ.

For a Jacobi matrix (3) we label the zeros of $b(\theta)$ on the period, whose number we assume to be finite, by $\theta_1, \theta_2, \dots, \theta_m$. (For $b(\theta) = 2\sin 2\pi\theta$ from (2), we have two zeros $\theta_1 = 0$, $\theta_2 = 1/2$.) Let $\Theta = \cup_{j=1}^m \cup_{k\in\mathbb{Z}} \{\theta_j + k\alpha\}$.

For $\theta \notin \Theta$, the eigenvalue equation $H_{v,b,\alpha,\theta}\phi = E\phi$ has the following dynamical reformulation:

$$\begin{pmatrix} \phi(n+1) \\ \phi(n) \end{pmatrix} = A^E(\theta + n\alpha) \begin{pmatrix} \phi(n) \\ \phi(n-1) \end{pmatrix},$$

where

$$\mathrm{GL}(2, \mathbb{C}) \ni A^E(\theta) = \frac{1}{b(\theta)} \begin{pmatrix} E - v(\theta) & -b(\theta-\alpha) \\ b(\theta) & 0 \end{pmatrix}$$

is called the *transfer matrix*. Let

$$A_n^E(\theta) = A^E(\theta + (n-1)\alpha) \cdots A^E(\theta+\alpha)A^E(\theta)$$

be the *n-step transfer matrix.*

The *Lyapunov exponent* of $H_{v,b,\alpha,\theta}$ at energy E is defined as

$$L(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \ln \|A_n^E(\theta)\|\ \mathrm{d}\theta. \tag{7}$$

The Thouless formula (e.g., [44]) links $N(E)$ and $L(E)$:

$$L(E) = -\int_{\mathbb{T}} \ln|b(\theta)|d\theta + \int_{\mathbb{R}} \ln|E - E_1|dN(E_1). \tag{8}$$

Note that both for $b(\theta) = 1$ and $b(\theta) = 2\sin 2\pi\theta$, we have

$$\int_{\mathbb{T}} \ln|b(\theta)|d\theta = 0. \tag{9}$$

2.2 Continued fraction expansion

Let $\alpha \in (0,1)$ be irrational. Then α has the following continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

with a_n positive integers for $n \geq 1$.

The reduced rational numbers

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_n}}}}, \qquad n = 1, 2, \ldots, \tag{10}$$

are called the *canonical approximants* of α. The following property is well-known:

$$\frac{1}{q_n(q_n + q_{n+1})} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{11}$$

3 Chiral gauge

Consider the following operator on $\ell^2(\mathbb{Z})$:

$$\widetilde{H}_{\alpha,\theta} = \widehat{H}_{\alpha,1/4+\alpha/2+\theta}, \tag{12}$$

in terms of the operator defined in (2).

Define the following unitary operators on $\ell^2(\mathbb{Z} \times \mathbb{T})$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$(T\phi)(n,\theta) = \phi(n+1,\theta), \qquad (S\phi)(n,\theta) = e^{2\pi i(\theta+n\alpha)}\phi(n,\theta), \tag{13}$$

$$(U_x\phi)(n,\theta) = e^{2\pi i n x(\theta+n\alpha/2)}\phi(n,\theta), \tag{14}$$

$$(R\phi)(n,\theta) = \sum_{k\in\mathbb{Z}} e^{-2\pi i k(\theta+n\alpha)} \int_{\mathbb{T}} e^{-2\pi i n\beta}\phi(k,\beta)d\beta. \tag{15}$$

Note that $H_{\alpha,\theta}$ and $\widetilde{H}_{\alpha,\theta}$ have the following representation in terms of S, T considered on the subspace of $\ell^2(\mathbb{Z}\times\mathbb{T})$ for a fixed θ (clearly these subspaces are invariant w.r.t. S, T, but not w.r.t. R):

$$H_{\alpha,\theta} = T + T^{-1} + S + S^{-1}, \tag{16}$$

$$\widetilde{H}_{\alpha,\theta} = e^{i\pi\alpha}(ST + S^{-1}T^{-1}) + e^{-i\pi\alpha}(ST^{-1} + S^{-1}T). \tag{17}$$

Consider the direct sums (particular cases of (4))

$$M_\alpha = \oplus_{\theta\in[0,1)} H_{\alpha,\theta}, \qquad \widetilde{M}_\alpha = \oplus_{\theta\in[0,1)} \widetilde{H}_{\alpha,\theta}. \tag{18}$$

We have

Theorem 3.1 *The operators $M_{2\alpha}$ and $\widetilde{M}_\alpha$ are unitarily equivalent. Namely, $\widetilde{M}_\alpha = QV^{-1}M_{2\alpha}VQ^{-1}$, where $V : \ell^2(\mathbb{Z}\times\mathbb{T}) \to \ell^2(2\mathbb{Z}\times\mathbb{T})$ is given by $(V\phi)(n,\theta) = \phi(2n,\theta)$, and*

$$Q = U_1 R U_{1/2}$$

in terms of the operators U_x, R given in (14), (15).

Proof. First, we need commutation relations between the operators (13)–(15). We have

$$\begin{aligned}(RS\phi)(n,\theta) &= \sum_{k\in\mathbb{Z}} e^{-2\pi ik(\theta+n\alpha)} \int_{\mathbb{T}} e^{-2\pi in\beta} e^{2\pi i(\beta+k\alpha)} \phi(k,\beta)d\beta \\ &= \sum_{k\in\mathbb{Z}} e^{-2\pi ik(\theta+(n-1)\alpha)} \int_{\mathbb{T}} e^{-2\pi i(n-1)\beta} \phi(k,\beta)d\beta = (T^{-1}R\phi)(n,\theta),\end{aligned}$$

so that

$$RS = T^{-1}R. \tag{19}$$

By taking the inverses and then multiplying by R from both sides we obtain

$$RS^{-1} = TR. \tag{20}$$

Similarly to (19), (20), we obtain

$$RT = SR, \qquad RT^{-1} = S^{-1}R. \tag{21}$$

Furthermore, we obtain in the same way as (19) that

$$TU_x = e^{i\pi x\alpha} S^x U_x T, \tag{22}$$

from which it follows that

$$U_x T = e^{-i\pi x\alpha} S^{-x} T U_x, \qquad U_x T^{-1} = e^{i\pi x\alpha} T^{-1} S^{x} U_x, \tag{23}$$

where the second relation is obtained by applying T^{-1} from both sides of (22).

Finally, it is easy to verify that

$$S^x T = e^{-2\pi i x\alpha} T S^x, \qquad T^{-1} S^{-x} = e^{2\pi i x\alpha} S^{-x} T^{-1}, \qquad U_x S^y = S^y U_x. \tag{24}$$

Using the above commutation relations we obtain that, with $Q = U_1 R U_{1/2}$,

$$QS = U_1 R S U_{1/2} = U_1 T^{-1} R U_{1/2} = e^{i\pi\alpha} T^{-1} S U_1 R U_{1/2} = e^{-i\pi\alpha} S T^{-1} Q, \tag{25}$$

i.e., $QS = e^{-i\pi\alpha} S T^{-1} Q$, which upon taking the inverses, then applying Q from both sides, and then using the commutation relation for TS^{-1} yields

$$QS^{-1} = e^{-i\pi\alpha} S^{-1} T Q. \tag{26}$$

Similarly to (25), (26), we obtain

$$QT^2 = e^{i\pi\alpha} S T Q, \qquad QT^{-2} = e^{i\pi\alpha} S^{-1} T^{-1} Q. \tag{27}$$

Collecting the last 4 relations together, we obtain

$$Q(T^2 + T^{-2} + S + S^{-1}) = (e^{i\pi\alpha}[ST + S^{-1}T^{-1}] + e^{-i\pi\alpha}[ST^{-1} + S^{-1}T])Q = \widetilde{M}_\alpha Q \tag{28}$$

Finally, observe that

$$V(T^2 + T^{-2} + S + S^{-1}) = M_{2\alpha} V, \tag{29}$$

and the statement of the theorem follows. □

4 Proof of Theorem 1.2

Let $N_\alpha(E), \rho_\alpha$ and $\widetilde{N}_\alpha(E), \widetilde{\rho}_\alpha$ denote the integrated densities of states and density of states measures of $H_{\alpha,\theta}$ and $\widetilde{H}_{\alpha,\theta}$, respectively, and $L_\alpha(E)$ and $\widetilde{L}_\alpha(E)$ denote the corresponding Lyapunov exponents. We then have

Theorem 4.1 $N_{2\alpha} = \widetilde{N}_\alpha$.

Proof. By Theorem 3.1, $M_{2\alpha} = U\widetilde{M}_\alpha U^{-1}$ for a unitary U such that $U\delta_0 = \delta_0$. Therefore, for any continuous function η we have

$$\begin{aligned} \int_{\mathbb{R}} \eta(E) d\widetilde{N}_\alpha(E) = \int_{\mathbb{T}} (\eta(\widetilde{H}_{\alpha,\theta})\delta_0, \delta_0)\ d\theta = (\eta(\widetilde{M}_\alpha)\delta_0, \delta_0) \\ = (\eta(U^{-1} M_{2\alpha} U)\delta_0, \delta_0) = \int_{\mathbb{T}} (U^{-1}\eta(H_{2\alpha,\theta}) U\delta_0, \delta_0)\ d\theta \\ = \int_{\mathbb{T}} (\eta(H_{2\alpha,\theta}) U\delta_0, U\delta_0)\ d\theta = \int_{\mathbb{R}} \eta(E) dN_{2\alpha}(E), \end{aligned} \tag{30}$$

and the result follows. □

Remark 4.2 Similar proofs have been used in [15, 17]. This also follows from Theorem 2 of [38].

For irrational α, $\sigma(M_{2\alpha}) = \operatorname{supp} \rho_{2\alpha} = \operatorname{supp} \widetilde{\rho}_\alpha = \sigma(\widetilde{M}_\alpha)$. Since for $E \in \sigma(M_{2\alpha})$ we have $L_{2\alpha}(E) = 0$ [11], by (8), (9), and Theorem 4.1 we also have $\widetilde{L}_\alpha(E) = 0$ for $E \in \sigma(\widetilde{M}_\alpha)$. The rest of the argument is the same as in Sec. 6.1 of [8]. Namely, since the Jacobi matrix defining (2) is singular, the absolutely continuous spectrum is empty [10]. However, by Kotani theory, it would not be empty if $|\{E \in \sigma(\widetilde{M}_\alpha) : \widetilde{L}_\alpha(E) = 0\}| > 0$.[10] Thus $|\sigma(M_\alpha)| = |\sigma(\widetilde{M}_{\alpha/2})| = 0$. □

5 Continuity of the spectrum

Consider the operator $H_{v,b,\alpha,\theta}$ given by (3) with $b(x), v(x) \in C^1(\mathbb{R})$, periodic of period 1. We further assume that $b(x)$ has at least one and at most a finite number of zeros on the period. Denote

$$b_n(\theta) = b(\theta + n\alpha), \qquad v_n(\theta) = v(\theta + n\alpha). \tag{31}$$

By the general theory, $\sigma(H_{v,b,\alpha,\theta})$ is purely essential and is a compact set; if $\alpha = p/q$ is rational, it consists of up to q intervals[11] separated by gaps; if α is irrational, it does not depend on θ and has no isolated points.

Consider also the half-line operator $H^+_{v,b,\alpha,\theta} = PH_{v,b,\alpha,\theta}P$, where $P\phi = (\ldots, 0, \phi(0), \phi(1), \ldots)$ is the projection onto $\ell^2(\mathbb{Z}_{\geq 0})$.

Let $\sigma_{ess}(H)$ denote the essential spectrum of H.

Lemma 5.1 *For any real α, θ, $\sigma_{ess}(H^+_{v,b,\alpha,\theta}) = \sigma(H_{v,b,\alpha,\theta})$. If $\alpha = p/q$ is rational, in addition to the essential spectrum, $\sigma(H^+_{v,b,p/q,\theta})$ may contain up to 2 eigenvalues inside each gap; and up to 1 in each of the infinite intervals above and below the essential spectrum.*

Proof. A perturbation of rank 2 of the form $\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$ with eigenvalues $\pm b$ which removes 2 symmetric off-diagonal elements splits $H_{v,b,\alpha,\theta}$ into a direct sum of 2 half-line operators.

Since finite-dimensional perturbations preserve the essential spectrum, $\sigma_{ess}(H^+_{v,b,\alpha,\theta}) \subset \sigma(H_{v,b,\alpha,\theta})$. The opposite inclusion is shown by the following argument which is an adaptation of a part of a much more general analysis by Last and Simon [36, 37] of essential spectra of Jacobi matrices. Let a sequence n_j be such that $\alpha n_j \mod 1 \to 0$. (In the rational

[10]The Kotani theory for non-singular Jacobi matrices is Theorem 5.17 in [44]. For singular matrices with $\ln|b(\theta)| \in L^1$, it is Theorem 8 in [8].

[11]Of positive length if $b(\theta + np/q) \neq 0$, $n = 1, 2, \ldots, q$. If one or several $b(\theta + np/q) = 0$, the spectrum is a finite number of points of infinite multiplicity.

case $\alpha = p/q$, we can take $n_j = qj$.) Then $b_{n_j+\ell}(\theta) \to b_\ell(\theta)$ and $v_{n_j+\ell}(\theta) \to v_\ell(\theta)$ for any given $\ell \in \mathbb{Z}$. If $E \in \sigma(H_{v,b,\alpha,\theta})$ and $\psi^{(m)}$ is a sequence of norm-1 trial functions with $\|(H_{v,b,\alpha,\theta} - E)\psi^{(m)}\| \to 0$ then for $j(m)$ which tends to infinity sufficiently fast with m, we have that $\|(H^+_{v,b,\alpha,\theta} - E)\psi^{(m)}(\cdot - n_{j(m)})\| \to 0$ and $\psi^{(m)}(\cdot - n_{j(m)}) \to 0$ weakly. Therefore $E \in \sigma_{ess}(H^+_{v,b,\alpha,\theta})$.

Finally, the statements about isolated points follow from the fact that the perturbation has at most 1 positive and 1 negative eigenvalue. □

We prove the following

Theorem 5.2 *Let $\alpha \in (0,1)$ be irrational. There exists a phase $\widetilde{\theta}$ and a subsequence of canonical approximants p_{n_j}/q_{n_j} to α such that for every $E \in \sigma_{ess}\left(H^+_{v,b,\alpha,\widetilde{\theta}}\right) \equiv S_{v,b,\alpha}$ there is $E' \in \sigma\left(H^+_{v,b,\frac{p_{n_j}}{q_{n_j}},\widetilde{\theta}}\right)$ with*

$$|E - E'| \le C\left|\alpha - \frac{p_{n_j}}{q_{n_j}}\right|\left|\ln\left|\alpha - \frac{p_{n_j}}{q_{n_j}}\right|\right|. \tag{32}$$

Remark 5.3 The constant $C = C(v,b) > 0$ in (32) depends only on the functions v and b.

Remark 5.4 The function $|\ln|x||$ here is sufficient for our purposes, but is not essential and can be replaced by a function growing slower as $x \to 0$ (in fact, arbitrarily slowly), with obvious modifications of the proof below. The corresponding subsequence of approximants may then be more rarified.

Proof. Let $\alpha \in (0,1)$ be irrational, $\theta \in [0,1)$, $x \in \mathbb{R}$, and ϕ be the corresponding formal solution of the equation $(H^+_{v,b,\alpha,\theta} - x)\phi = 0$, normalized by the condition $\phi(0) = 1$. Note that $\phi(n) \equiv \phi(n,x,\theta)$ are polynomials of degree n in x orthonormal w.r.t. the spectral measure μ_θ of $H^+_{v,b,\alpha,\theta}$ associated with the vector $e_0 = (1,0,0,\dots)$. In particular,

$$\int_{\mathbb{R}} |\phi(n,x,\theta)|^2 d\mu_\theta(x) = 1, \qquad n = 0,1,\dots. \tag{33}$$

It is immediate from the second order recurrence that for a fixed open bounded interval K containing the spectrum $S_{v,b,\alpha}$, there exists $0 < C_0 < \infty$ such that for all $x \in K$ we have, assuming $b_n(\theta) \neq 0$, $n = 0,1,\dots,m-1$,

$$|\phi(m,x,\theta)| \le C_0^m \prod_{n=0}^{m-1} \frac{1}{|b_n(\theta)|}, \tag{34}$$

and

$$\left|\frac{d}{dx}\phi(m,x,\theta)\right| \le C_0^m \prod_{n=0}^{m-1} \frac{1}{|b_n(\theta)|}. \tag{35}$$

Let $E \in S_{v,b,\alpha}$ and $\chi_{E,\varepsilon}(x)$ be the characteristic function of the interval $(E-\varepsilon, E+\varepsilon)$. Let $u_{E,\varepsilon}(x)$ be a continuous function such that $\chi_{E,\varepsilon/2}(x) \le u_{E,\varepsilon}(x) \le \chi_{E,\varepsilon}(x)$. Since for any $\theta \in [0,1)$ and any $\varepsilon > 0$, $\mu_\theta((E-\varepsilon, E+\varepsilon)) > 0$, we have for any θ,

$$\mu_\theta((E-\varepsilon, E+\varepsilon)) \ge (u_{E,\varepsilon}(H^+_{v,b,\alpha,\theta})e_0, e_0) \ge \mu_\theta((E-\varepsilon/2, E+\varepsilon/2)) > 0. \tag{36}$$

Since $(u_{E,\varepsilon}(H^+_{v,b,\alpha,\theta})e_0, e_0)$ is a continuous function of θ, we obtain that for $\varepsilon > 0$, there exists $f_E(\varepsilon) > 0$ such that $\inf_\theta \mu_\theta((E-\varepsilon, E+\varepsilon)) > f_E(\varepsilon)$. Since obviously $|E-E_0| < \varepsilon/2$, $E_0 \in S_{v,b,\alpha}$, implies $\mu_\theta((E-\varepsilon, E+\varepsilon)) > f_{E_0}(\varepsilon/2)$, we also have, by compactness of $S_{v,b,\alpha}$ and considering the cover $\cup_{E_0 \in S_{v,b,\alpha}}(E_0-\varepsilon/2, E_0+\varepsilon/2)$, a positive lower bound uniform in E. Thus there exists a function $f(x) > 0$, $x > 0$, $f(0) = 0$, which is (sufficiently slowly) strictly increasing from zero[12] such that

$$\liminf_{\varepsilon \to +0} \frac{\inf_{E \in S_{v,b,\alpha}} \inf_\theta \mu_\theta((E-\varepsilon, E+\varepsilon))}{f(2\varepsilon)} = +\infty. \tag{37}$$

We can assume $f(x)$ is continuous. (Indeed, as a monotone function, it is a sum of a continuous one and a jump function; the latter can be bounded from below by a nondecreasing continuous function.)[13] Then there is a function $g(x) > 0$, $x > 0$, $g(0) = 0$, which is (sufficiently fast) strictly increasing from zero, continuous, and such that

$$f(g(x)^2) \ge x, \qquad g(x) \ge x^{1/4}, \qquad x > 0. \tag{38}$$

Moreover, there is a function $h(x) > 0$, $x > 0$, which is (sufficiently fast) decreasing to zero as $x \to +\infty$ and such that

$$g(Bh(x)) \le \exp(-x), \qquad x > 0, \tag{39}$$

where $B = \max|db(x)/dx|$.

Let

$$\omega(n) = \frac{1}{q_n q_{n+1}}, \tag{40}$$

[12] There is always a point $E \in S_{v,b,\alpha}$ such that $\mu_\theta(\{E\}) = 0$ since the spectrum $S_{v,b,\alpha} = \sigma(H_{v,b,\alpha,\theta})$ has no isolated points.

[13] To define f, we could have used any positive constant instead of $+\infty$ on the r.h.s. of (37). We only need existence of such f in our proof. However, this suggests a question: can one obtain explicit estimates on f in terms of α, b, and v?

and note that by (11)

$$\frac{\omega(n)}{2} < \left|\alpha - \frac{p_n}{q_n}\right| < \omega(n), \qquad n = 1, 2, \ldots \tag{41}$$

We now choose a special value of θ to ensure that some off-diagonal elements b_n decrease to zero sufficiently fast along a certain sequence n_j. Without loss of generality, we assume that $b(0) = 0$. Since the number of zeros of $b(\theta)$ on the period is finite, there exists the largest nonnegative integer t such that $b(-t\alpha) = 0$. (For $b(\theta) = 2\sin 2\pi\theta$, $t = 0$.) Pick a large n_1 and take k_1 the smallest such that $k_1 \geq |\ln\omega(n_1)|$. Let $a_1(\theta) = C_0^{-(k_1+1)} \prod_{n=0}^{k_1-1} |b_n(\theta)|$ with C_0 from (35). The function $a_1(\theta)$ is continuous with $a_1(-(k_1+t)\alpha) > 0$. On the other hand, $g(|b_{k_1}(\theta)|)$ is a continuous function with $g(|b_{k_1}(-(k_1+t)\alpha)|) = 0$. Thus we can define a closed interval of positive length on the circle by[14] $I_1 \subset \{\theta : \|\theta + (k_1+t)\alpha\| \leq h(k_1)\}$ such that for $\theta \in I_1$ we have

$$g(|b_{k_1}(\theta)|) < a_1(\theta) = C_0^{-(k_1+1)} \prod_{n=0}^{k_1-1} |b_n(\theta)|. \tag{42}$$

We proceed by induction. Given $k_{j-1}, n_{j-1}, I_{j-1}$, we find a denominator $q_n > 3/|I_{j-1}|$, and then set n_j to be the smallest such that $\frac{1}{\omega(n_j)} > e^{q_n}$ and $\omega(n_j) < \omega(n_{j-1})^3$. As follows from the inequality $|\alpha - p_n/q_n| < 1/(q_n q_{n+1})$, for any interval I with $|I| > 3/q_n$, for every s, x, there is a $k \in \{s, s+1, \ldots, s+q_n-1\}$ with $x - k\alpha \mod 1 \in I$. Thus we can find $k_j \in [|\ln\omega(n_j)|, 2|\ln\omega(n_j)|]$ such that $-(k_j+t)\alpha \mod 1 \in I_{j-1}$. Note that $k_j > k_{j-1}$. Now, as above, define a closed interval of positive length $I_j \subset I_{j-1} \cap \{\theta : \|\theta + (k_j+t)\alpha\| \leq h(k_j)\}$ such that for $\theta \in I_j$, we have

$$g(|b_{k_j}(\theta)|) < C_0^{-(k_j+1)} \prod_{n=0}^{k_j-1} |b_n(\theta)|. \tag{43}$$

Therefore we have a nested sequence of closed intervals I_j of decreasing to zero length such that for $\theta \in I_j$

$$\|\theta + (k_j+t)\alpha\| \leq h(k_j), \qquad k_j \in [|\ln\omega(n_j)|, 2|\ln\omega(n_j)|], \tag{44}$$

and (43) holds. Let $\{\widetilde{\theta}\} = \cap_j I_j$. Then $\theta = \widetilde{\theta}$ satisfies (44), (43) for every $j \geq 1$.

Fix $E \in S_{v,b,\alpha}$. Define

$$\psi_j = (1, \phi(1, E, \widetilde{\theta}), \phi(2, E, \widetilde{\theta}), \ldots, \phi(k_j, E, \widetilde{\theta}), 0, 0, \ldots), \qquad j \geq 1, \tag{45}$$

that is the projection of the vector $(\phi(n, E, \widetilde{\theta}))_{n=0}^{\infty}$ onto the k_j+1-dimensional subspace with indices $0, \ldots, k_j$.

[14] As usual, $\|\theta\| = \operatorname{dist}(\theta, \mathbb{Z})$.

By the Weyl criterion, there exists E' in the spectrum of $H^+_{v,b,p_{n_j}/q_{n_j},\widetilde{\theta}}$ such that[15]

$$\begin{aligned}
|E'-E| &\le \|(H^+_{v,b,p_{n_j}/q_{n_j},\widetilde{\theta}}-E)\psi_j\|/\|\psi_j\| \\
&\le \|(H^+_{v,b,p_{n_j}/q_{n_j},\widetilde{\theta}}-H^+_{v,b,\alpha,\widetilde{\theta}})\psi_j\|/\|\psi_j\| + \|(H^+_{v,b,\alpha,\widetilde{\theta}}-E)\psi_j\|/\|\psi_j\| \\
&\le C|\alpha - p_{n_j}/q_{n_j}|k_j + \|H^+_{v,b,\alpha,\widetilde{\theta}}-E)\psi_j\|/\|\psi_j\|. \qquad (46)
\end{aligned}$$

Using (3) and the fact that $\|\psi_j\| > \phi(0) = 1$, we obtain from here that

$$|E'-E| \le C|\alpha - p_{n_j}/q_{n_j}|k_j + |b_{k_j}| + |b_{k_j}\phi(k_j+1,E,\widetilde{\theta})|, \qquad b_k \equiv b(\widetilde{\theta}+k\alpha). \qquad (47)$$

Note that by the definitions of $\widetilde{\theta}$, $b(x)$, and t,

$$|b_{k_j}| \le B\|\widetilde{\theta}+(k_j+t)\alpha\| \le Bh(k_j), \qquad j=1,2,\dots, \qquad (48)$$

where $B = \max|db(x)/dx|$. Since by (39), (44), (41), $0 \le Bh(k_j) \le \exp(-k_j) \le \omega(n_j) \le 2|\alpha - p_{n_j}/q_{n_j}|$, it remains to estimate $\phi(k_j+1,E,\widetilde{\theta})$. For that, we use the condition (33). Let

$$\zeta_j(x) = b_{k_j}\phi(k_j+1,x,\widetilde{\theta}), \qquad x \in \mathbb{R}. \qquad (49)$$

We have

Lemma 5.5 *With $g(x)$ defined in (38), $E \in S_{v,b,\alpha}$,*

$$|\zeta_j(E)| \le g(|b_{k_j}|), \qquad j=1,2,\dots \qquad (50)$$

Proof. Suppose that $|\zeta_j(E)| > g(|b_{k_j}|)$.

Then, by (35), $|\zeta_j(x)| \ge g(|b_{k_j}|)/2$ for all $x \in (E-\varepsilon,E+\varepsilon)$, where $\varepsilon>0$ is such that $(E-\varepsilon,E+\varepsilon) \subset K$ (expanding K if necessary) and

$$\varepsilon \ge \frac{g(|b_{k_j}|)/2}{\sup_{x\in K}\left|\frac{d}{dx}\zeta_j(x)\right|} \ge \frac{1}{2}g(|b_{k_j}|)C_0^{-(k_j+1)}\prod_{n=0}^{k_j-1}|b_n|. \qquad (51)$$

Using (43) we obtain

$$2\varepsilon \ge g(|b_{k_j}|)^2. \qquad (52)$$

We now apply the condition (33) to $\phi(k_j+1,x,\widetilde{\theta})$:

$$1 = \frac{1}{b_{k_j}^2}\int_{\mathbb{R}}\zeta_j^2(x)d\mu_{\widetilde{\theta}}(x) \ge \frac{1}{b_{k_j}^2}\int_{(E-\varepsilon,E+\varepsilon)}\zeta_j^2(x)d\mu_{\widetilde{\theta}}(x) \ge \frac{g(|b_{k_j}|)^2}{4b_{k_j}^2}\mu_{\widetilde{\theta}}((E-\varepsilon,E+\varepsilon)). \qquad (53)$$

[15] The value of the constant $C>0$ can be different in different formulae below.

By the definition of $f(x)$ in (37), and by choosing k_1 sufficiently large and ε sufficiently small (but such that (52) remains satisfied), we have, using (52) and (38),

$$\mu_{\widetilde{\theta}}((E-\varepsilon, E+\varepsilon)) \geq f(2\varepsilon) \geq f(g(|b_{k_j}|)^2) \geq |b_{k_j}|.$$

This implies by (53) that

$$1 \geq \frac{g(|b_{k_j}|)^2}{4|b_{k_j}|},$$

which is a contradiction as $g(x) \geq x^{1/4}$ by definition. The lemma is proved. □

Substituting (48) and (50) into (47), we obtain

$$|E' - E| \leq C|\alpha - p_{n_j}/q_{n_j}|k_j + Bh(k_j) + g(Bh(k_j)),$$

which implies (32) by the definition of $h(x)$ in (39), the inclusion $k_j \in [|\ln \omega(n_j)|, 2|\ln \omega(n_j)|]$, and the inequalities in (41).

□

6 Proof of Theorems 1.3, 1.1, 1.5.

The proof is a slightly modified argument of [34].

Denote the t-dimensional Hausdorff measure of a set A by

$$\mathrm{meas}_t(A) = \lim_{\delta \downarrow 0} \inf \left\{ \sum_{m=1}^{\infty} |w_m|^t : \cup_{m=1}^{\infty} w_m \text{ is a } \delta\text{-cover of } A \right\},$$

where the infimum is taken over all δ-covers of A by intervals: $A \subset \cup_{m=1}^{\infty} w_m$, where w_m is an interval with $|w_m| \leq \delta$. The Hausdorff dimension of A is then

$$\dim_{\mathrm{H}}(A) = \inf\{t > 0 : \mathrm{meas}_t(A) < \infty\}.$$

Let $\alpha \in (0,1)$ be an irrational and let p_{n_j}/q_{n_j} be the corresponding sequence of periodic approximants from Theorem 5.2. By arguments of [45, 6, 34] combined with Theorem 5.2,

$$|\sigma(M_{v,b,\alpha})| \geq \limsup_{j \to \infty} |\sigma(M_{v,b,p_{n_j}/q_{n_j}})|. \tag{54}$$

We now show that $|\sigma(M_{v,b,\alpha})| \leq \liminf_{j\to\infty} |\sigma(M_{v,b,p_{n_j}/q_{n_j}})|$. By Lemma 5.1, $\sigma\left(H^+_{v,b,p_{n_j}/q_{n_j},\widetilde{\theta}}\right)$ is a collection of up to q_{n_j} intervals which comprise $\sigma(H_{v,b,p_{n_j}/q_{n_j},\widetilde{\theta}})$ plus possibly $2q_{n_j}$ isolated eigenvalues.[16] Therefore we can write

$$\sigma\left(H^+_{v,b,p_{n_j}/q_{n_j},\widehat{\theta}}\right) = \cup_{m=1}^{q_{n_j}}[E_1^{j,m}, E_2^{j,m}] \cup \{E_3^{j,m}\} \cup \{E_4^{j,m}\}. \tag{55}$$

By Theorem 5.2 and (11), for large j,

$$\sigma(M_{v,b,\alpha}) \subset \cup_{m=1}^{q_{n_j}} w_m, \tag{56}$$

where

$$w_m = \left(E_1^{j,m} - C\frac{\ln q_{n_j}}{q_{n_j}^2}, E_2^{j,m} + C\frac{\ln q_{n_j}}{q_{n_j}^2}\right) \cup_{s=3,4} \left(E_s^{j,m} - C\frac{\ln q_{n_j}}{q_{n_j}^2}, E_s^{j,m} + C\frac{\ln q_{n_j}}{q_{n_j}^2}\right), \tag{57}$$

and therefore

$$|\sigma(M_{v,b,\alpha})| \leq \sum_{m=1}^{q_{n_j}} |w_m| \leq |\sigma(M_{v,b,p_{n_j}/q_{n_j}})| + C\frac{\ln q_{n_j}}{q_{n_j}}. \tag{58}$$

Thus $|\sigma(M_{v,b,\alpha})| \leq \liminf_{j\to\infty} |\sigma(M_{v,b,p_{n_j}/q_{n_j}})|$, which completes the proof of Theorem 1.3.

In the case of the critical almost Mathieu operator, where $v(\theta) = 0$, $b(\theta) = 2\sin 2\pi\theta$, and M_α is given by (18), in view of Theorem 3.1,

$$\sigma(M_{2\alpha}) \subset \cup_{m=1}^{q_{n_j}} w_m, \tag{59}$$

where by (6) and (57)

$$|\sigma(M_{2\alpha})| \leq \sum_{m=1}^{q_{n_j}} |w_m| \leq C\frac{\ln q_{n_j}}{q_{n_j}}. \tag{60}$$

By the Hölder inequality and (60), we obtain

$$\sum_{m=1}^{q_{n_j}} |w_m|^t \leq q_{n_j}^{1-t}\left(\sum_{m=1}^{q_{n_j}} |w_m|\right)^t \leq Cq_{n_j}^{1-2t}(\ln q_{n_j})^t, \tag{61}$$

which tends to zero as $j \to \infty$ for any $t > 1/2$. Therefore $\text{meas}_t(\sigma(M_{2\alpha})) = 0$ if $t > 1/2$. Hence $\dim_{\text{H}}(\sigma(M_{2\alpha})) \leq 1/2$, which proves Theorem 1.1.

As for Theorem 1.5, the proof closely follows the proof of Lemma 4.4 in [8], with Hölder-$\frac{1}{2}$ continuity replaced by Theorem 5.2 and with a modification as in the proof of Theorem 1.1. □

[16] For simplicity, we assume that there are $2q_{n_j}$ isolated eigenvalues. If there are less, the modification of the proof is obvious.

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