UPPER BOUNDS ON TRANSPORT EXPONENTS FOR LONG RANGE OPERATORS

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Abstract. We present a simple method, not based on the transfer matrices, to prove vanishing of dynamical transport exponents. The method is applied to long range quasiperiodic operators.

1. Introduction

Jean Bourgain, partially with collaborators, has developed a powerful method to prove Anderson localization for ergodic Schrödinger operators, see [1] and references therein. The method relies heavily, in both perturbative and nonperturbative settings, on the subharmonic function theory and the theory of semialgebraic sets and has turned out to be quite robust. While the precursor was the non-perturbative approach of [16] that initiated the emphasis on obtaining off-diagonal Green's function decay using bulk features rather than individual eigenfunctions, Bourgain's method has crystallized and developed the key ideas that did not require transfer matrices/nearest-neighbor Laplacians, thus allowing, in particular, the extension to Toeplitz matrices as well as multidimensional localization results.[1]

Discrete quasiperiodic operators with the Laplacian replaced by a Toeplitz operator appear naturally in the context of Aubry duality, and have been studied by several authors. Let $H_{\theta,\alpha,\epsilon}$, with $(\theta,\alpha)\in\mathbb{T}^2$, act on $\ell^2(\mathbb{Z})$ by

$$(H_{\theta,\alpha,\epsilon}u)_n := \epsilon\left(\sum_{k\in\mathbb{Z}} a_{n-k}u_k\right) + v(\theta+n\alpha)u_n. \tag{1}$$

where $|a_n| \le A_1 e^{-a|n|}$ for some $a, A_1 > 0$ and $a_{-n} = \overline{a_n}$. Bourgain's main localization result for the long-range case is

Theorem 1.1 ([1], Theorem 11.20). *If v is analytic non-constant on $\mathbb{T}$, then for $|\epsilon| \le \epsilon_0$, $\epsilon_0 = \epsilon_0(A_1, a, v)$, $H_{\theta,\alpha,\epsilon}$ satisfies Anderson localization for a full measure set of $(\theta,\alpha)\in\mathbb{T}^2$.*

[1] See also [12, 18] for streamlining and simplification of Bourgain's multidimensional method and the non-self-adjoint version.

We note that this theorem is *non-perturbative*, that is ϵ_0 does not depend on α. There is also a stronger, *arithmetic* (that is with an arithmetic full measure condition on the frequency and phase) localization result for $v(\theta) = \cos 2\pi\theta$ [2], and recently an arithmetic multidimensional result was obtained as a corollary of dual quantitative reducibility in [8], but for general function v Bourgain's non-arithmetic theorem 1.1 remains the strongest available. We note that the perturbative multidimensional version appears in [12]; however, in the multidimensional case there is no essential difference between the nearest neighbor and long-range Laplacians.

At the same time, Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) is extremely fragile. Indeed, it was shown by Gordon [9] and del Rio, Makarov and Simon [7] that a generic rank one perturbation of an operator with an interval in the spectrum even in the regime of *dynamical* localization leads to singular continuous spectrum, and therefore, by the RAGE theorem, growth of the moments. However, it was shown in [6] that, under the condition of SULE, present in many models, this growth can be at most logarithmic, and thus preserves vanishing of the dynamical exponents. Thus one can argue that it is vanishing of the dynamical exponents $\beta(q)$ (see (3) for the definition) that captures the physically relevant effect of localization.

Indeed, such localization-type results (vanishing of $\beta(q)$) have been obtained, in increasing generality, for random and quasiperiodic operators as a corollary of positive Lyapunov exponents in [3, 4, 13, 14], with [10] covering the entire class of ergodic operators with base dynamics of zero topological entropy, a class that includes shifts and skew-shifts on higher-dimensional tori. Clearly, those techniques are transfer-matrix based, thus don't extend to long-range operators.

In this note we present a very simple method to obtain such quantum dynamical upper bounds for the long-range case, and show that one part of Bourgain's localization proof can serve as an input to obtain an arithmetic result: vanishing of quantum dynamical exponents for all long-range quasiperiodic operators with Diophantine frequencies, all phases, and sufficiently large analytic potentials, see Corollary 1.6. This should be contrasted with the non-arithmetic result Theorem 1.1. We note also that Anderson localization for *all* phases does not even hold [11, 15]

Bourgain's method consists of multiple parts, and the one in question is establishing the sublinear bound (33) for the number of boxes of size N^c in a box of size N, that don't have the off-diagonal Green function decay. Our method requires only presence of *one* box of size N^c *with* the off-diagonal Green function decay, in a box of size N, thus Bourgain's sublinear bound is even an overkill for a needed input.

We note that, while suitable for long-range, our method is still one-dimensional, as only in dimension one does one box create a barrier and thus a good estimate for the Green's function in a bigger box. Yet it does provide the first departure from the Lyapunov exponent/transfer matrix based methods, and leads to a strong corollary. Also, it extends easily to the (not necessarily uniform) band, requiring only one "good box" to apply Theorem 1.4.

Let us now introduce the main concepts. We restrict here to dimension one, although many of the statements and definitions are easily extendable to higher dimensions. For a fixed self-adjoint operator H on $\ell^2(\mathbb{Z})$, $\phi \in \ell^2(\mathbb{Z})$ and $p, T > 0$, let

$$\langle |X|_\phi^p \rangle(T) = \frac{2}{T} \int_0^\infty e^{-2t/T} \sum_{n\in\mathbb{Z}} |n|^p |(e^{-itH}\phi, \delta_n)|^2 \tag{2}$$

The growth rate of $\langle |X|_\phi^p \rangle(T)$ characterizes how fast does $e^{-itH}\phi$ spread out. The power law bounds for $\langle |X|_\phi^p \rangle(T)$ are naturally characterized by the following upper transport exponents $\beta_\phi^+(p)$ defined as

$$\beta_\phi^+(p) = \limsup_{T\to\infty} \frac{\ln\langle |X|_\phi^p \rangle(T)}{p \ln T}. \tag{3}$$

Here we study Schrödinger operators on $\ell^2(\mathbb{Z})$ of the form,

$$H = A + V,$$

where $V = \{V_n\}_{n\in\mathbb{Z}}$ is real bounded and A is a long-range operator of the form

$$(Au)_n = \sum_{k\in\mathbb{Z}} a_{n-k} u_k,$$

where $|a_n| \le A_1 e^{-a|n|}$ for some $a, A_1 > 0$ and $a_{-n} = \overline{a_n}$.

More precisely,

$$(Hu)_n = \left(\sum_{k\in\mathbb{Z}} a_k u_{n-k} \right) + V_n u_n. \tag{4}$$

Just like Schrödinger operators, such operators admit a ballistic bound on the transport exponents

Theorem 1.2. *Let H be given by* (4)*. Assume ϕ is compactly supported. Then the upper transport exponent $\beta_\phi^+(q) \le 1$ for any $q > 0$.*

Remark 1.3. In fact, sufficiently fast decay works equally well, but we restrict in all results, to the compactly supported ϕ, for simplicity.

Theorem 1.2 is probably well known, but we didn't find the proof in the literature. The proof, following the ideas of [19, 20], is presented in the appendix.

Let R_Λ be the operator of restriction to $\Lambda \subset \mathbb{Z}$. Define the Green's function by

$$G_\Lambda(z) = (R_\Lambda(H - zI)R_\Lambda)^{-1}. \tag{5}$$

Set $G(z) = (H - zI)^{-1}$. Clearly, both $G_\Lambda(z)$ and $G(z)$ are always well defined for $z \in \mathbb{C}_+ \equiv \{z \in \mathbb{C} : \Im z > 0\}$. Sometimes, we drop the dependence on z for simplicity. Since the operator H given by (4) is bounded, there exists $K > 0$ such that $\sigma(H) \subset [-K+1, K-1]$. Our main general result is

Theorem 1.4. *Let H be given by (4). Suppose there exist $\delta > 0$ and $N_0 > 0$ such that the following is true. Let $z = E + i\varepsilon$ with $|E| \leq K$ and $0 < \varepsilon \leq \delta$. Suppose for $N > N_0$, there exists an interval $I \subset [-\frac{N}{2}, -\frac{N}{4}]$ or $I \subset [\frac{N}{4}, \frac{N}{2}]$ such that $|I| \geq N^\delta$ and for any $n, n' \in I$ and $|n - n'| \geq \frac{1}{20}|I|$, we have*

$$|G_I(z)(n; n')| \leq e^{-|I|^\delta}.$$

Assume ϕ is compactly supported. Then the upper transport exponent $\beta_\phi^+(p) = 0$ for any $p > 0$.

Remark 1.5. For the Schrödinger case, the existence of such interval I (in fact, a stronger statement, but this is not important) can be deduced from the positive Lyapunov exponents and Cramer's rule by the method going back to [17].

We say $\alpha \in \mathbb{R}$ is Diophantine if there exist κ and $\tau > 0$ such that for any $k \in \mathbb{Z}\backslash\{0\}$,

$$||k\alpha||_{\mathbb{R}/\mathbb{Z}} \geq \frac{\tau}{|k|^\kappa},$$

where $||x||_{\mathbb{R}/\mathbb{Z}} = \text{dist}(x, \mathbb{Z})$.

Let $H_{\alpha,\theta,\epsilon}$ be as in (1). Fixing α and ϵ, we denote the $\beta_\phi^+(p)$ for operator $H_{\alpha,\theta,\epsilon}$ by $\beta_{\phi,\theta}^+(p)$. Our main application is

Corollary 1.6. *There exists an $\epsilon_0 = \epsilon_0(v, A_1, a) > 0$ such that for any compactly supported ϕ and Diophantine α, $\beta_{\phi,\theta}^+(p) = 0$ for any $|\epsilon| \leq \epsilon_0$, any $\theta \in \mathbb{R}$ and $p > 0$.*

It immediately implies also

Corollary 1.7. *There exists an $\epsilon_0 = \epsilon_0(v, A_1, a) > 0$ such that for any $\phi \in \ell^2(\mathbb{Z})$ the spectral measure μ_ϕ of operator $H_{\theta,\alpha,\epsilon}$ is zero dimensional for any $\theta \in \mathbb{R}$, Diophantine α, and any $|\epsilon| \leq \epsilon_0$.*

2. Proof of Theorem 1.4

For the Schrödinger case, the proof would be just a double application of the resolvent identity:

$$\begin{aligned} G = & G_I + G_{I^c} - (G_I + G_{I^c})(H - H_I - H_{I^c})(G_I + G_{I^c}) \\ & + (G_I + G_{I^c})(H - H_I - H_{I^c})(G_I + G_{I^c})(H - H_I - H_{I^c})G \end{aligned}$$

ensuring the decay of $|G(0,n)|$ based on the "barrier" box I. The problem with the long-range case is that such expansion for $G(0,N)$ will contain terms all grouped nearby, thus neither incorporating the decay coming from the barrier box nor from $|a_n|$. In order to tackle this difficulty we introduce several extra steps, all involving applications of the resolvent identity, but with different boxes.

Since ϕ has a compact support, there exists K_1 such that $\phi(n) = 0$ for $|n| \geq K_1$.

Assume $T > \frac{1}{\delta}$. Fix $z = E + i\frac{1}{T}$ with $|E| \leq K$. Below, C (c) is a large (small) constant that may depend on δ, K, A_1, a, ϕ and $V = \{V_n\}$. Let $I = [b-\ell, b+\ell]$ with $\ell > 0$ and b such that $|b \pm \ell|$ is large. Suppose

$$|G_I(m,n)| \leq Ce^{-c\ell^c} \tag{6}$$

for any $m \in I, n \in I$ and $|m-n| \geq \frac{1}{20}\ell$.

Recall that if

$$\Lambda = \Lambda_1 \cup \Lambda_2, \Lambda_1 \cap \Lambda_2 = \emptyset,$$

then

$$G_\Lambda = G_{\Lambda_1} + G_{\Lambda_2} - (G_{\Lambda_1} + G_{\Lambda_2})(H_\Lambda - H_{\Lambda_1} - H_{\Lambda_2})G_\Lambda,$$

(provided the relevant matrices $R_\Lambda(H-zI)R_\Lambda$ and $R_{\Lambda_i}(H-zI)R_{\Lambda_i}$ are invertible) where $H_\Lambda = R_\Lambda H R_\Lambda$.

If $m \in \Lambda_1$ and $n \in \Lambda$, we have

$$G_\Lambda(m,n) = G_{\Lambda_1}(m,n)\chi_{\Lambda_1}(n) - \sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} G_{\Lambda_1}(m,n_1)a_{n_1-n_2}G_\Lambda(n_2,n). \tag{7}$$

Therefore,

$$|G_\Lambda(m,n)| \leq |G_{\Lambda_1}(m,n)\chi_{\Lambda_1}(n)| + C\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(m,n_1)|e^{-c|n_1-n_2|}|G_\Lambda(n_2,n)|. \tag{8}$$

Lemma 2.1. *Assume that for some interval $I = [b-\ell, b+\ell]$ and $z = E + \frac{i}{T}$, (6) holds. Then*

$$|G_\Lambda(m,n)| \leq CT^2e^{-c\ell^c} \tag{9}$$

for any $n \in I$, $m \in [b-\ell+\frac{\ell}{10}, b+\ell]$ and $|m-n| \geq \frac{1}{10}\ell$, where $\Lambda = (-\infty, b+\ell]$.

Proof. Let $\Lambda_1 = I = [b-\ell, b+\ell]$ and $\Lambda_2 = (-\infty, b-\ell-1]$. Clearly, $\Lambda = \Lambda_1 \cup \Lambda_2$. By (6) and (8), one has that

$$|G_\Lambda(m,n)| \leq Ce^{-c\ell^c} + C\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(m,n_1)|e^{-c|n_1-n_2|}|G_\Lambda(n_2,n)|. \tag{10}$$

It suffices to bound the second term on the right of (10).

For any $n_1 \in \Lambda_1$,

$$\sum_{n_2\in\Lambda_2} e^{-c|n_1-n_2|} \leq C. \tag{11}$$

If $n_1 \in [b-\ell, b-\ell+\frac{\ell}{20}]$, by the fact that $m \in [b-\ell+\frac{\ell}{10}, b+\ell]$ and (6), one has

$$|G_{\Lambda_1}(m,n_1)| \leq Ce^{-c\ell^c}. \tag{12}$$

If $n_1 \in [b-\ell+\frac{\ell}{20}, b+\ell]$, one has

$$\sum_{n_2\in\Lambda_2} e^{-c|n_1-n_2|} \leq Ce^{-c\ell}. \tag{13}$$

Since $\Im z = \frac{1}{T}$, one has that

$$|G_{\Lambda_1}(m,n_1)| \leq T, |G_{\Lambda}(n_2,n)| \leq T. \tag{14}$$

By (11), (12),(13) and (14), we have

$$\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(m,n_1)|e^{-c|n_1-n_2|}|G_{\Lambda}(n_2,n)| \leq CT^2e^{-c\ell^c}$$

□

Lemma 2.2. *Assume $b-\ell$ is large. Under the conditions of Lemma 2.1, we have that for any j with $|j| \leq K_1$ and $n \in [b+\ell-\frac{\ell}{10}, b+\ell]$,*

$$|G_{\Lambda}(j,n)| \leq CT^4e^{-c\ell^c}, \tag{15}$$

where $\Lambda = (-\infty, b+\ell]$.

Proof. Let $\Lambda_2 = [b-\ell, b+\ell]$, $\Lambda_1 = (-\infty, b-\ell-1]$ and $\Lambda = (-\infty, b+\ell]$. By (8), one has that for any j with $|j| \leq K_1$,

$$|G_{\Lambda}(j,n)| \leq C \sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(j,n_1)|e^{-c|n_1-n_2|}|G_{\Lambda}(n_2,n)|. \tag{16}$$

For any $n_2 \in \Lambda_2$,

$$\sum_{n_1\in\Lambda_1} e^{-c|n_1-n_2|} \leq C. \tag{17}$$

If $n_2 \in [b-\ell, b+\ell-\frac{\ell}{5}]$, by the fact that $n \in [b+\ell-\frac{\ell}{10}, b+\ell]$ and (9), one has

$$|G_{\Lambda}(n_2,n)| = |G_{\Lambda}(n,n_2)| \leq CT^2e^{-c\ell^c}. \tag{18}$$

If $n_2 \in [b+\ell-\frac{\ell}{5}, b+\ell]$, by the fact that $n_1 \leq b-\ell$, one has

$$\sum_{n_1\in\Lambda_1} e^{-c|n_1-n_2|} \leq Ce^{-c\ell}. \tag{19}$$

By (17), (18),(19) and (14), we have

$$\sum_{n_1\in\Lambda_1, n_2\in\Lambda_2} |G_{\Lambda_1}(j,n_1)|e^{-c|n_1-n_2|}|G_{\Lambda}(n_2,n)| \leq CT^4e^{-c\ell^c}.$$

This implies (15). □

Lemma 2.3. *Let* $z = E + \frac{i}{T}$*. Assume that* $\ell \geq |\tilde{N}|^\delta$ *and for some interval* $I = [b-\ell, b+\ell]$ *with* $I \subset \left[\frac{|\tilde{N}|}{4}, \frac{|\tilde{N}|}{2}\right]$ *or* $I \subset \left[-\frac{|\tilde{N}|}{2}, -\frac{|\tilde{N}|}{4}\right]$*, (6) holds. Then for any* j *with* $|j| \leq K_1$*,*

$$|G_\Lambda(j, \tilde{N})| \leq CT^6 e^{-c|\tilde{N}|^c}, \tag{20}$$

where $\Lambda = (-\infty, \infty)$*.*

Proof. Without loss of generality, assume $\tilde{N} > 0$. Let $\Lambda_1 = (-\infty, b+\ell]$, $\Lambda_2 = [b+\ell+1, \infty)$ and $\Lambda = (-\infty, \infty)$. By (8), one has

$$|G_\Lambda(j, \tilde{N})| \leq C \sum_{n_1 \in \Lambda_1, n_2 \in \Lambda_2} |G_{\Lambda_1}(j, n_1)| e^{-c|n_1 - n_2|} |G_\Lambda(n_2, N)|. \tag{21}$$

First, one has

$$\sum_{n_1 \in \Lambda_1, n_2 \in \Lambda_2} e^{-c|n_1 - n_2|} \leq C. \tag{22}$$

If $n_1 \in [b + \ell - \frac{\ell}{10}, b+\ell]$, By (15), one has

$$|G_{\Lambda_1}(j, n_1)| \leq CT^4 e^{-c\ell^c}. \tag{23}$$

If $n_1 \in (-\infty, b+\ell-\frac{\ell}{10}]$ and $n_2 \in \Lambda_2$, one has

$$e^{-c|n_1-n_2|} \leq Ce^{-c\ell}. \tag{24}$$

By (22), (23), (24) and (14), we have

$$\sum_{n_1 \in \Lambda_1, n_2 \in \Lambda_2} |G_{\Lambda_1}(j, n_1)| e^{-c|n_1 - n_2|} |G_\Lambda(n_2, \tilde{N})| \leq CT^6 e^{-c\ell^c}.$$

This implies (20). □

Proof of Theorem 1.4. This is standard. For any j with $|j| \leq K_1$, let

$$a(j, n, T) = \frac{2}{T} \int_0^\infty e^{-2t/T} |(e^{-itH}\delta_j, \delta_n)|^2 dt. \tag{25}$$

By the Parseval formula

$$a(j, n, T) = \frac{1}{T\pi} \int_{-\infty}^\infty |((H - E - \frac{i}{T})^{-1}\delta_j, \delta_n)|^2 dE. \tag{26}$$

Recall that $\sigma(H) \subset [-K+1, K-1]$. For any $E \in (-\infty, -K) \cup (K, \infty)$, $\eta = \text{dist}(E + \frac{i}{T}, \text{spec}(H)) \geq 1$. The well-known Combes-Thomas estimate yields for large n,

$$|((H - E - \frac{i}{T})^{-1}\delta_j, \delta_n)| \leq Ce^{-c|n|}. \tag{27}$$

By (26) and (27), one has that

$$a(j,n,T) \leq Ce^{-c|n|} + \frac{1}{T\pi}\int_{-K}^{K} |((H-E-\frac{i}{T})^{-1}\delta_j,\delta_n)|^2 dE. \tag{28}$$

By Lemma 2.3, we have for any $|E| \leq K$,

$$|((H-E-\frac{i}{T})^{-1}\delta_j,\delta_n)| \leq CT^6 e^{-c\ell^c} \leq CT^6 e^{-c|n|^c}. \tag{29}$$

By (28) and (29), one has that

$$a(j,n,T) \leq CT^{11} e^{-c|n|^c}. \tag{30}$$

Therefore,

$$\begin{aligned} \langle |X|_\phi^p \rangle (T) &\leq C \sum_{|j|\leq K_1} \sum_{n\in\mathbb{Z}} |n|^p a(j,n,T) \\ &\leq \sum_{n\in\mathbb{Z}} CT^{11}|n|^p e^{-c|n|^c} \\ &\leq CT^{11}. \end{aligned} \tag{31}$$

It implies

$$\beta_\phi^+(p) \leq \frac{11}{p}.$$

Since $\beta_\phi^+(p)$ are nondecreasing, we have that for every $p > 0$,

$$\beta_\phi^+(p) \leq \lim_{p\to\infty} \beta_\phi^+(p) = 0. \tag{32}$$

□

3. Proof of Corollary 1.6

Under the assumption of Corollary 1.6, one has that when $|\epsilon| \leq \epsilon_0$, the following holds for some $\delta_0 > 0$,

$$\#\{b \in \mathbb{Z} : |b| \leq N, I_b \text{ does not satisfy (6)}\} \leq N^{1-\delta_0}. \tag{33}$$

This was proved in Ch. 11 of [1] for $E \in \mathbb{R}$ and also holds for complex energies [18][2]. Therefore, Corollary 1.6 follows from Theorem 1.4. □

[2]This is mentioned already in [1]

Appendix A. Proof of Theorem 1.2

Proof. By an easy application of Hölder inequality, $\beta_\phi^+(q)$ is nondecreasing with respect to q. Therefore, it suffices to show that for any $N \in \mathbb{N}$, $\beta_\phi^+(2N) \leq 1$.

Define the free long range Schrödinger operator by

$$(H_0 u)_n = \sum_{k\in\mathbb{Z}} a_{n-k} u_k = \sum_{k\in\mathbb{Z}} a_k u_{n-k}.$$

For any sequence $\gamma = \{\gamma_k\}$ with $|\gamma_k| \leq Ce^{-c|k|}$, we define the momentum operator X_{2p}^γ:

$$(X_{2p}^\gamma u)_n = n^p \sum_k \gamma_k u_{n-k},$$

and $\hat{X}_{2p}^\gamma = -i[H_0, X_{2p}^\gamma]$, where $[B_1, B_2] = B_2B_1 - B_1B_2$.

Direct computations implies,

$$\begin{aligned}
(\hat{X}_{2p}^\gamma u)_n &= -i\left(n^p \sum_{k_1\in\mathbb{Z}} \gamma_{k_1}(H_0u)_{n-k_1} - \sum_{k\in\mathbb{Z}} a_k (X_{2p}^\gamma u)_{n-k}\right) \\
&= -i\left(n^p \sum_{k_1\in\mathbb{Z},k\in\mathbb{Z}} \gamma_{k_1} a_k u_{n-k_1-k} - \sum_{k\in\mathbb{Z},k_1\in\mathbb{Z}} a_k (n-k)^p \gamma_{k_1} u_{n-k-k_1}\right) \\
&= -i \sum_{k\in\mathbb{Z},k_1\in\mathbb{Z}} (n^p - (n-k)^p) a_k \gamma_{k_1} u_{n-k-k_1} \\
&= -i \sum_{m\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} (n^p - (n-k)^p) a_k \gamma_{m-k}\right) u_{n-m}
\end{aligned}$$

Therefore, $\hat{X}_{2p}^\gamma$ can be rewritten as

$$\hat{X}_{2p}^\gamma = \sum_{j=0}^{p-1} X_{2j}^{\gamma^j}, \tag{34}$$

for some new sequences $\{\gamma_k^j\}$ with $|\gamma_k^j| \leq C_j e^{-c_j|k|}$, $j = 0, 1, \cdots, p-1$.

Let

$$X_{2p}^\gamma(t) = e^{itH} X_{2p}^\gamma e^{-itH}, \hat{X}_{2p}^\gamma(t) = e^{itH} \hat{X}_{2p}^\gamma e^{-itH}.$$

Differentiating $X_{2p}^\gamma(t)$, one has that

$$\frac{dX_{2p}^\gamma(t)}{dt} = \hat{X}_{2p}^\gamma(t). \tag{35}$$

We will show inductively that

$$(X_{2N}^\gamma(t)\phi, X_{2N}^\gamma(t)\phi) \leq C_{\gamma,\phi,N} t^{2N} \text{ for large } t. \tag{36}$$

We first prove (36) for $N = 1$. Differentiating $X_2^\gamma(t)$, one has that

$$\frac{dX_2^\gamma(t)}{dt} = \hat{X}_2^\gamma(t), \tag{37}$$

where $\hat{X}_2^\gamma(t)$ is a bounded selfadjoint operator by (34). By (35), one has

$$X_2^\gamma(t) = X_2^\gamma + \int_0^t \hat{X}_2^\gamma(s)ds. \tag{38}$$

This implies

$$\begin{aligned}
&(X_2^\gamma(t)\phi, X_2^\gamma(t)\phi) \\
&= (X_2^\gamma\phi + \int_0^t \hat{X}_2^\gamma(s)\phi ds, X_2^\gamma\phi + \int_0^t \hat{X}_2^\gamma(s)\phi ds) \\
&\leq ||X_2^\gamma\phi||^2 + 2||X_2^\gamma\phi|| \int_0^t ||\hat{X}_2^\gamma(s)\phi||ds + \left(\int_0^t ||\hat{X}_2^\gamma(s)\phi||ds\right)^2 \\
&\leq C_{\gamma,\phi}t^2 + C_{\gamma,\phi}t + C_{\gamma,\phi},
\end{aligned}$$

since ϕ has compact support and $\hat{X}_2^\gamma(t)$ is bounded.

Assume that (12) holds for $p \leq N-1$. This means that for any sequence $\{\gamma_k\}$ and $p = 1, 2, \cdots N-1$,

$$(X_{2p}^\gamma(t)\phi, X_{2p}^\gamma(t)\phi) \leq C_{\gamma,\phi,p}t^{2p} \text{ for large } t. \tag{39}$$

By (35), one has

$$X_{2N}^\gamma(t) = X_{2N}^\gamma + \int_0^t \hat{X}_{2N}^\gamma(s)ds. \tag{40}$$

By (34) and (39), we have

$$||\hat{X}_{2N}^\gamma(t)\phi|| \leq C_{\gamma,\phi,N}t^{N-1} \text{ for large } t. \tag{41}$$

This implies, for large t,

$$\begin{aligned}
&(X_{2N}^\gamma(t)\phi, X_{2N}^\gamma(t)\phi) \\
&= \left(X_{2N}^\gamma\phi + \int_0^t \hat{X}_{2N}^\gamma(s)\phi ds, X_{2N}^\gamma\phi + \int_0^t \hat{X}_{2N}^\gamma(s)\phi ds\right) \\
&\leq ||X_{2N}^\gamma\phi||^2 + 2||X_{2N}^\gamma\phi|| \int_0^t ||\hat{X}_{2N}^\gamma(s)\phi||ds + \left(\int_0^t ||\hat{X}_{2N}^\gamma(s)\phi||ds\right)^2 \\
&\leq C_{\gamma,\phi,N}t^{2N}.
\end{aligned}$$

Let $\{\gamma_k\}$ be the sequence such that $\gamma_0 = 1$ and $\gamma_k = 0$ for $k \neq 0$. Therefore, one has

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |n|^{2N}|(e^{-itH}\phi,\delta_n)|^2 &= (X_{2N}^{\gamma}e^{-itH}\phi, X_{2N}^{\gamma}e^{-itH}\phi) \\
&= (e^{itH}X_{2N}^{\gamma}e^{-itH}\phi, e^{itH}X_{2N}^{\gamma}e^{-itH}\phi) \\
&= (X_{2N}^{\gamma}(t)^{\gamma}\phi, X_{2N}^{\gamma}(t)\phi) \\
&\leq C_{\phi,N}t^{2N}. \qquad (42)
\end{aligned}$$

By (2) and (42), one has

$$\begin{aligned}
\langle |\hat{X}|_{\phi}^{2N}\rangle(T) &\leq \frac{2}{T}\int_0^{\infty} e^{-2t/T}C_{\phi,N}t^{2N}dt \\
&\leq C_{\phi,N}T^{2N}.
\end{aligned}$$

Thus $\beta_{\phi}^{+}(q) \leq 1$ for any $q > 0$. □

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This paper is devoted to the memory of Jean Bourgain. Both authors have been profoundly influenced by Jean. S.J. was fortunate to experience direct influence, see [5]. W.L. believes he became a mathematician through detailed reading of [1].

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