1 Local estimates of exponential polynomials and their applications to inequalities of uncertainty principle type - part I

after F. L. Nazarov [3]
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Abstract

A classical result by Pál Turán, estimates the global behavior of an exponential polynomial on an interval by its supremum on any arbitrary subinterval. We discuss F. L. Nazarov's extension of this "global to local reduction" to arbitrary Borel sets of positive Lebesgue measure. More recently, an observation by O. Friedland and Y. Yomdin, enlarges the class of sets to encompass even discrete, in particular finite, sets of sufficient density.

1.1 Introduction

We consider an exponential polynomial, i.e. an expression of the form

$$p(t) = \sum_{k=1}^{n} c_k \mathrm{e}^{\lambda_k t} \ , \tag{1}$$

where both the *coefficients* c_k and the *frequencies* λ_k are complex. The number of non-vanishing coefficients defines its *order*. Following, depending on the context, μ denotes the Haar measure on $\mathbb{R}$ or $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ such that, respectively, $\mu(\mathbb{T}) = 1$ or $\mu([0, 1]) = 1$.

A classical Lemma due to Pál Turán [4] estimates the global behavior of (1) (order n) on an interval $I \subseteq \mathbb{R}$ by its supremum on any arbitrary sub-*interval* $E \subseteq I$:

$$\sup_{t \in I} |p(t)| \leq \mathrm{e}^{\mu(I) \cdot \max |\mathrm{Re} \lambda_k|} \cdot \left(\frac{A \mu(I)}{\mu(E)} \right)^{n-1} \sup_{t \in E} |p(t)| \ . \tag{2}$$

Here, $A > 0$ is an absolute constant, independent of n.

In particular, comparing (2) with an analogous result for algebraic polynomials dating back to Chebyshev, implies that exponential polynomials of order n behave like their algebraic counterparts of degree $n - 1$.

Following, we discuss the extension of Turán's Lemma to arbitrary Borel sets $E \subseteq I$, achieved by F. L. Nazarov in [3], Chapter 1 therein:

Theorem 1 ([3]). *Let $p(t)$ be an exponential polynomial of order n of the form given in (1). Turán's Lemma (2) holds for any Borel set $E \subseteq I$ with $\mu(E) > 0$.*

Thus, the above mentioned analogy between exponential and algebraic polynomials persists when considering arbitrary Borel sets, Theorem 1 then being paralleled by Remez type estimates. For an extensive review of available results for both algebraic and exponential polynomials, we refer to e.g. [2].

More recently, Theorem 1 was extended further to certain discrete, in particular finite, sets of sufficient density [1]. To this end, Friedland and Yomdin introduce the *metric span* of a set $E \subseteq I$ for a given pair (P, I):

Definition 2 ([1]). *Let $p(t)$ be an exponential polynomial of order n of the form given in (1) and I an interval. Set $m := \frac{n(n+1)}{2} + 1$, $C(n) := m(2m+1)^{2m}2^{2m^2}$ and $\lambda := \max|\mathrm{Im}\lambda_k|$. Letting $d := C(n)\mu(I)\lambda$, introduce the "frequency bound" $M(p, I) := \lfloor \frac{d}{2} \rfloor + 1$. The metric span of a set $E \subseteq \mathbb{I}$ is defined by*

$$\omega_{(p,I)}(E) := \sup_{\epsilon>0} \epsilon[M(\epsilon, E) - M(p, I)] \ , \tag{3}$$

where $M(\epsilon, E)$ is the ϵ-covering number of E [1].

Remark 3. *(i) Clearly, for any measurable E, $\omega_{(p,I)}(E) \geq \mu(E)$.*

(ii) $\omega_{(p,I)}(E) > 0$ if $[M(\epsilon, E) - M(p, I)] > 0$, for some $\epsilon > 0$. In particular, $\omega_{(p,I)}(E) > 0$, for discrete sets of sufficient density.

(iii) The number $M(p, I)$ characterizes the complexity of sub-level sets, $\{t \in I : |p(t)| \leq \delta\}$ (see also Lemma 8, below).

Theorem 4 ([1]). *Replacing $\mu(E)$ by $\omega_{(p,I)}(E)$, the statement of Theorem 1 holds for any $E \subseteq \mathbb{R}$ with $\omega_{(p,I)}(E) > 0$.*

We mention, that Theorem 4 was preceded by an analogous result for algebraic polynomials [5].

[1] We recall, that given a metric space (M, d) and $\epsilon > 0$, one defines the ϵ-covering number of a subset $X \subseteq M$ as the minimal number of ϵ-balls needed to cover X.

1.2 Nazarov's theorem

Following, we present the main ideas in the proof of Theorem 1. We will focus on the case when $p(t)$ has purely imaginary frequencies, $p(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$, $\lambda_1 < \cdots < \lambda_n$. Without loss of generality, we take $I = [-\frac{1}{2}, \frac{1}{2}]$.

Theorem 1 is based on two crucial Lemmas, one quantifying the number of zeros in a vertical strip (see Lemma 7, below), the other allowing reduction of order by a weak-type estimate of the logarithmic derivative (see Lemma 6, below).

1.2.1 Bernstein-type estimates and order reduction

The strategy of order reduction is most transparent when $p(t)$ is a trigonometric polynomial, i.e. $\lambda_k \in 2\pi\mathbb{Z}$. Substituting $z = e^{2\pi i t}$, consider $p(z) = \sum_{k=1}^{n} c_k z^{m_k}$ as a Laurent polynomial on the unit circle. We shall show:

Theorem 5 (see Theorem 1.4 in [3])**.** *Given $E \subset \mathbb{T}$, $\mu(E) > 0$, one has*

$$\|p\|_{\mathrm{W}} := \sum_{k=1}^{n} |c_k| \leq \left\{ \frac{16e}{\pi} \frac{1}{\mu(E)} \right\}^{n-1} \sup_{z \in E} |p(z)| \ . \tag{4}$$

To prove Theorem 5, one inductively reduces the order of $p(z)$ by constructing a sequence of Laurent polynomials $p = p_n, p_{n-1}, \ldots, p_1$ satisfying

(Ind1) $\mathrm{ord} p_k = k$

(Ind2) $\|p_k\|_{\mathrm{W}} \geq \frac{\pi}{16} \|p_{k-1}\|_{\mathrm{W}}$

such that

$$\mu \left\{ z \in \mathbb{T} : \left| \frac{p_{k-1}(z)}{p_k(z)} \right| > t \right\} \leq \frac{1}{t} \ , \tag{5}$$

for $2 \leq k \leq n$.

p_{k-1} is obtained from $p_k =: \sum_{s=1}^{k} d_s z^{r_s}$, $r_1 < \ldots r_k \in \mathbb{Z}$, choosing one of the following Laurent polynomials of order $k-1$

$$\underline{q}(z) := \frac{\mathrm{d}}{\mathrm{d}z}(z^{-r_1} p_k(z)) \text{ or } \overline{q}(z) := \frac{\mathrm{d}}{\mathrm{d}z}(z^{-r_k} p_k(z)) \ , \tag{6}$$

which guarantees a lower bound of $\|.\|_{\mathrm{W}}$ as indicated in (Ind2).

Thus, one can reduce the order of p from n to 1,

$$\left(\frac{\pi}{16}\right)^{n-1} \|p\|_{\mathrm{W}} \le \|p_1\| = |p_1(z_0)| = \left|\frac{p_1(z_0)}{p(z_0)}\right| \cdot |p(z_0)| \ , \tag{7}$$

arriving at (4), provided there exists some $z_0 \in E$ satisfying

$$\left|\frac{p_1(z_0)}{p(z_0)}\right| \le \left\{\frac{e}{\mu(E)}\right\}^{n-1} \ . \tag{8}$$

Existence of such z_0 follows from a measure estimate of the exceptional set where (8) is violated. Noticing that $|\frac{p_{k-1}(z)}{p_k(z)}|$ can be realized as a logarithmic derivative of an algebraic polynomial, such estimate is accomplished by the following Bernstein-type Lemma:

Lemma 6 (see Lemma 1.2 in [3])**.** *Let $g(z)$ be an algebraic polynomial of degree n. Then,*

$$\mu\left\{z \in \mathbb{T} : \left|\frac{g'(z)}{g(z)}\right| > y\right\} \le \frac{8n}{\pi y} \ . \tag{9}$$

1.2.2 The role of "zero counting"

Quantifying the distribution of zeros of exponential polynomials is a crucial ingredient for both the Theorems 1 and 4. Nazarov's argument is based on the Langer lemma:

Lemma 7 (see Lemma 1.3 in [3])**.** *Let $p(z) = \sum_{k=1}^{n} c_k \mathrm{e}^{i\lambda_k z}$, $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n =: \lambda$, be an exponential polynomial not vanishing identically. Then, the number of complex zeros of $p(z)$ in an open vertical strip $x_0 < \mathrm{Re} z < x_0 + \Delta$ of width $\Delta > 0$ does not exceed $(n-1) + \frac{\lambda\Delta}{2\pi}$.*

In particular, based on Lemma 7, one concludes that complex zeros $\{z_j\}$ of the given exponential polynomial $p(z)$ are sufficiently separated: Ordering z_j according to increasing $|\mathrm{Re} z_j|$, the inequality

$$|\mathrm{Re} z_j| \ge \pi \frac{j-(n-1)}{(n-1)} \ , \tag{10}$$

holds.

We employ the Hadamard factorization theorem,

$$p(z) = c\mathrm{e}^{az} \prod_{j=1}^{\kappa} (z - z_j) \prod_{j>\kappa} \left(1 - \frac{z}{z_j}\right) \mathrm{e}^{z/z_j} =: c\mathrm{e}^{az} Q(z) R(z) \ , \tag{11}$$

with κ chosen such that $Q(z)$ contains all zeros of $p(z)$ on $[-1/2, 1/2]$.

By (10), κ can be estimated depending on the relation between λ and $n-1$,

$$\kappa = \begin{cases} 2(n-1) & , \text{ if } \lambda \leq n-1 \ , \\ 2\lambda & , \text{ if } \lambda > n-1 \ . \end{cases} \tag{12}$$

Using order reduction similar to Sec. 1.2.1, it suffices to consider $\lambda \leq n-1$.

The argument principle allows to quantify the contribution of the zero-free part of $p(z)$,

$$\max_{z \in [-1/2, 1/2]} |c\mathrm{e}^{az} R(z)| \leq 3^{n-1} \min_{z \in [-1/2, 1/2]} |c\mathrm{e}^{az} R(z)| \ . \tag{13}$$

Finally, the polynomial $Q(z)$ is dealt with using a Cartan-like estimate: Given $0 < h < 1/8$, it is shown that for z outside an exceptional subset $\Omega_h \subseteq [-1/2, 1/2]$ with $\mu(\Omega_h) \leq 8h < 1 = \mu(I)$, one has

$$\frac{|Q(z)|}{\max\{|Q(t)| : t \in [-1/2, 1/2]\}} \geq \left\{ \frac{8h}{32\sqrt[3]{4}} \right\}^{n-1} \ . \tag{14}$$

Thus, letting $h = \mu(E)/8$, we may combine all the pieces to arrive at

$$\sup_{z \in I} |p(t)| \leq 3^{n-1} \inf_{z \in I} |c\mathrm{e}^{az} R(z)| \cdot \sup_{z \in I} |Q(z)| \tag{15}$$

$$\leq \left\{ \frac{96\sqrt[3]{4}}{\mu(E)} \right\}^{n-1} \inf_{z \in I} |c\mathrm{e}^{az} R(z)| \cdot \inf_{z \in E \setminus \Omega_h} |Q(z)| \tag{16}$$

$$\leq \left\{ \frac{96\sqrt[3]{4}}{\mu(E)} \right\}^{n-1} \sup_{z \in E} |p(z)| \ . \tag{17}$$

1.3 Extensions

We conclude by briefly commenting on Theorem 4. For $\delta := \sup_{t \in E} |p(t)|$, consider the sublevel set $V_\delta := \{t \in I : |p(t)| \leq \delta\}$. Clearly, one has $E \subseteq$

V_δ. The main idea in [1] is to reduce Theorem 4 to Theorem 1 by showing $\omega_{(p,I)}(E) \leq \mu(V_\delta)$ (see Lemma 2.3 in [1]).

Again, "counting zeros" provides the key ingredient and also explains the definition of $M_{(p,I)}$:

Lemma 8 (see Lemma 2.2 in [1])**.** *For $p(t)$ as in Theorem 4, given $\eta > 0$ the number of non-degenerate solutions of the equation $|p(t)|^2 = \eta$ in the interval I does not exceed $d = C(n)\mu(I)\lambda$. Here, $C(n)$ and λ are defined as in Definition 2.*

Lemma 8 allows to estimate from above the ϵ-covering number of V_δ, which in turn yields $\omega_{(p,I)}(E) \leq \mu(V_\delta)$, as claimed.

References

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