On point spectrum of critical almost Mathieu operators

S. Jitomirskaya[1]

Abstract. There isn't any.

1 Introduction

The critical almost Mathieu operator, that is

$$(H_{\alpha,\theta}\phi)(n) = \phi(n-1) + \phi(n+1) + 2\cos 2\pi(\alpha n + \theta)\phi(n), \tag{1}$$

acting on $\ell^2(\mathbb{Z})$, is a model important in several physics contexts (see e.g. [7] and references therein) and a subject to significant numerical/heuristic studies, demonstrating a host of remarkable features [22]. For E in the spectrum, its transfer-matrix cocycle is also critical in the sense of Avila's global theory [3], thus is not amenable to either super-critical (localization) or sub-critical (reducibility) methods. In fact, operators (1) serve as a boundary between two by now decently understood and very different regimes: sub-critical (that is with cos in (1) replaced by $\lambda\cos,\ \lambda < 1,$) and super-critical (the same with $\lambda > 1$. $H_{\alpha,\theta}$ has been long (albeit not from the very beginning [23][2]) conjectured to have purely singular continuous spectrum for every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and every θ. Since the spectrum (which is θ-independent for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ [9]) has Lebesgue measure zero [4], the problem boils down to the proof of absence of eigenvalues, see e.g. problem 7 in [17]. This simple question has a surprisingly rich (and dramatic) history.

Aside from the results on topologically generic absence of point spectrum [8, 20] that hold in a far greater generailty, all the proofs were, in one way or another, based on the Aubry duality [1], a Fourier-type transform for which the family $\{H_{\alpha,\theta}\}_\theta$ is a fixed point. One manifestation of the Aubry

[1]Department of Mathematics, UC Irvine, USA

[2]It is the paper where the name *almost Mathieu* was introduced.

duality is: if $u_n \in \ell^2(\mathbb{Z})$ solves the eigenvalue equation $H_{\alpha,\theta}u = Eu$, then $v_n^x := e^{2\pi in\theta}\hat{u}(x+n\alpha)$ solves

$$H_{\alpha,x}v_n^x = Ev_n^x \tag{2}$$

for a.e. x, where $\hat{u}(x) = \sum e^{2\pi inx}u_n$ is the Fourier transform of u. This led Delyon [13] to prove that there are no ℓ^1 solutions of $H_{\alpha,\theta}u = Eu$, for otherwise (7) would hold also for $x = \theta$, leading to a contradiction. Thus any potential eigenfunctions must be decaying slowly. Chojnacki [11] used duality-based C^*-algebraic methods to prove the existence of some continuous component, but without ruling out the point spectrum. [14] gave a duality-based argument for no point spectrum for a.e. θ, but it had a gap, as it was based on the validity of Deift-Simon's [12] theorem on a.e. mutual singularity of singular spectral measures, which is only proved in [12] in the hyperbolic case, and is still open in the regime of zero Lyapunov exponents. Avila and Krikorian (see [2]) used convergence of renormalization [5] and non-perturbative reducibility [10] to show that for every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, eigenvalues may only occur for countably many θ. Then Avila [2] found a simple proof of the latter fact, also characterizing this potentially exceptional set of phases explicitly: these are θ that are α-rational, i.e. $2\theta + k\alpha \in \mathbb{Z}$, for some k. The argument of [2] was incorporated in [6], where it was developed to prove a.e. absence of point spectrum for the extended Harper's model (EHM) in the entire critical region (the EHM result was later further improved by Han [15]). The proof in [2, 6] has as a starting point the dynamical formulation of the Aubry duality: if v_n^x solves the eigenvalue equation $H_{\alpha,x}v = Ev$, then so does its complex conjugate $\bar{v}_n^x$, and this can be used to construct an L^2-reducibility of the transfer-matrix cocycles to the rotation by θ, given independence of v and $\bar{v}$. Unfortunately those vectors are always linearly dependent if θ is α-rational. Thus the argument hopelessly breaks down for $2\theta + k\alpha \in \mathbb{Z}$.

Moreover, it was noted in [6] that in the bulk of the critical region, for α-rational phases θ, the extended Harper's operator actually does have eigenvalues. Also, supercritical almost Mathieu (that is with cos in (1) replaced by $\lambda\cos$, $\lambda > 1$,) with Diophantine α, has eigenvalues (with exponentially decaying eigenfunctions) for α-rational phases as well [18]. All this increased the uncertainty about whether eigenvalues may exist for the α-rational phases also for the critical almost Mathieu.

In this note we prove

Theorem 1.1. *$H_{\alpha,\theta}$ does not have eigenvalues for any α, θ (and thus has purely singular-continuous spectrum for all $\alpha \notin \mathbb{Q}$).*

In our proof we replace the Aubry duality by a new transform, inspired by the chiral gauge transform of [19]. The proof is fully self-contained.

2 Proof of Theorem 1.1.

Given $u_n \in \ell^2(\mathbb{Z})$, set

$$u(x) = \sum_{n=-\infty}^{\infty} u_n e^{\pi i n(\theta + n\alpha - 2x)} \tag{3}$$

and

$$u_n^x = u(x + n\alpha) e^{\pi i n(x + \frac{n\alpha - 3\theta}{2})} \tag{4}$$

where u_n^x is defined for a.e. x.

Let $\tilde{H}_\alpha^x : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}, x \in \mathbb{R}/2\mathbb{Z}$, be given by

$$(\tilde{H}_\alpha^x v)_n = 2\cos\pi(x + n\alpha) v_{n-1} + 2\cos\pi(x + (n+1)\alpha) v_{n+1} \tag{5}$$

Lemma 2.1. *If $u \in \ell^2(\mathbb{Z})$ solves $H_{\alpha,\theta} u = Eu$, then $u^x \in \mathbb{R}^{\mathbb{Z}}$ is a formal solution of the difference equation*

$$\tilde{H}_\alpha^{x + \frac{\theta - \alpha}{2}} u^x = E u^x \tag{6}$$

for a.e. x.

Proof. If $(Tu)_n := u_{n+1} + u_{n-1}$, and $(Su)_n := \cos 2\pi(\theta + n\alpha) u_n$, we obtain $(Tu)(x) = u(x - \alpha) e^{\pi i(\theta + \alpha - 2x)} + u(x + \alpha) e^{\pi i(-\theta + \alpha + 2x)}$ and $(Su)(x) = u(x - \alpha) e^{2\pi i\theta} + u(x + \alpha) e^{-2\pi i\theta}$, leading, by a straightforward computation, to $((T + S)u)_n^x = \tilde{H}_\alpha^{x + \frac{\theta - \alpha}{2}} u_n^x$. □

We note that the family $\{\tilde{H}_\alpha^x\}_{x \in \mathbb{R}/2\mathbb{Z}}$ is self-dual with respect to the Aubry-type duality. Namely, the following holds. For $x \in \mathbb{R}/2\mathbb{Z}, v \in \ell^2(\mathbb{Z})$ for a.e. β, we can define $w^\beta \in \mathbb{R}^{\mathbb{Z}}$ by

$$w_n^\beta = \hat{v}(\frac{\beta + n\alpha}{2}) e^{\pi i n(x + \frac{\alpha}{2})}. \tag{7}$$

Lemma 2.2. *If $v \in \ell^2(\mathbb{Z})$ solves $\tilde{H}_\alpha^x v = Ev$, then, for a.e. β, $w^\beta \in \mathbb{R}^{\mathbb{Z}}$ is a formal solution of the difference equation*

$$\tilde{H}_\alpha^{\beta-\frac{\alpha}{2}} w^\beta = E w^\beta. \tag{8}$$

Proof. A similar direct computation. □

Let now $u \in \ell^2(\mathbb{Z})$ with $\|u\|_2 = 1$ be a solution of $H_{\alpha,\theta} u = Eu$. By Lemma 2.1, (6) holds, which implies that we also have, for a.e. x,

$$\tilde{H}_\alpha^{x+\frac{\theta-\alpha}{2}} \bar{u}^x = E \bar{u}^x \tag{9}$$

thus the Wronskian of u^x and $\bar{u}^x$ is constant in n. That is

$$\cos \pi(x+n\alpha) \mathrm{Im}\, (u(x+n\alpha)\bar{u}(x+(n-1)\alpha) e^{\pi i (x+n\alpha+ia(\alpha,\theta))}) = c(x) \tag{10}$$

for some $c(x)$, all n and a.e. x. Here and below $a(\alpha,\theta)$ stands for (an explicit) real-valued function that does not depend on n, x. Its exact form is not important. $a(\alpha,\theta)$ may stand for different such functions in different expressions.

By ergodicity, this implies that, for a.e. x and some constant c,

$$\cos \pi x (u(x)\bar{u}(x-\alpha) e^{\pi i x + ia(\alpha,\theta)} - u(x-\alpha)\bar{u}(x) e^{-\pi i x - ia(\alpha,\theta)}) = c. \tag{11}$$

As in [15], it follows by Cauchy-Schwarz that $u(x)\bar{u}(x-\alpha)e^{\pi i x + ia(\alpha,\theta)} \in L^1$, which implies that $c = 0$, so

$$u(x)\bar{u}(x-\alpha) e^{\pi i x + ia(\alpha,\theta)} - u(x-\alpha)\bar{u}(x) e^{-\pi i x - ia(\alpha,\theta)} = 0 \tag{12}$$

for a.e. x.

Lemma 2.3. *For a.e. x, we have $u(x) \neq 0$.*

Proof. Indeed, otherwise there would exist (in fact, a full measure of, but it is not important) x such that u_n^x solves (6) and $u_n^x = 0$ for infinitely many n (in fact, only four such n suffice for the argument). Let $n_i < n_{i+1} - 1, i \in \mathbb{Z}$, be the labeling of zeros of such u_n^x. Clearly, if $v \in \mathbb{R}^{\mathbb{Z}}$ is a solution of (5) with $v_n = v_m = 0$, we have that $v_{[n,m]} \in \ell^2(\mathbb{Z})$ defined by $(v_{[n,m]})_k = \begin{cases} v_k, k \in [n+1, m-1] \\ 0, \text{otherwise} \end{cases}$ is also a solution of (6). Set $v^{x,i} := u^x_{[n_i,n_{i+1}]}$.

Clearly, for any $I \subset \mathbb{Z}$ the collection $\{v^{x,i}\}_{i\in I}$ is linearly independent in $\ell^2(\mathbb{Z})$. This implies that the corresponding Aubry dual collection $\{w^{x,i,\beta}\}_{i\in I}$ constructed by (7) from $\{w^{x,i}\}_{i\in I}$, is linearly independent in $\mathbb{R}^{\mathbb{Z}}$. Thus, by Lemmas 2.1,2.2 we obtain, for a.e. β, infinitely many linearly independent $w^{x,i,\beta} \in \mathbb{R}^{\mathbb{Z}}$, that all solve (8). This is in contradiction with the fact that the space of solutions of (8) is two-dimensional for a.e. β. □

Therefore we can define for a.e. x, a unimodular measurable function on $\mathbb{R}/2\mathbb{Z}$

$$\phi(x) := \frac{u(x)}{\bar{u}(x)} e^{\pi i x + i a(\alpha,\theta)} \tag{13}$$

By (12), (13) we have that, for a.e. x,

$$\phi(x) = \phi(x-\alpha) e^{-2\pi i x + i a(\alpha,\theta)}, \tag{14}$$

and expanding $\phi(x)$ into the Fourier series, $\phi(x) = \sum_{k=-\infty}^{\infty} a_k e^{\pi i k x}$, we obtain $|a_{k+2}| = |a_k|$, a contradiction. □

Acknowledgment

We are grateful to Rui Han and Barry Simon for helpful comments on an earlier version. This research was supported by NSF DMS-1901462.

References

[1] S. Aubry, G. André: Analyticity breaking and Anderson localisation in incommensurate lattices. Ann. Israel Phys. Soc. **3**, 133–164 (1980).

[2] A. Avila. On point spectrum at critical coupling. Preprint 2007.

[3] A. Avila. Global theory of one-frequency Schrödinger operators. Acta Math. **215**, 1–54 (2015).

[4] A. Avila, R. Krikorian. Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles. Ann. of Math. **164**, 911–940 (2006).

[5] A. Avila, R. Krikorian, Quasiperiodic SL(2, R) cocycles. In preparation.

[6] A. Avila, S. Jitomirskaya and C. Marx. Spectral theory of Extended Harpers Model and a question by Erdös and Szekeres. Invent. Math. **210.1**, 283–339 (2017)

[7] J. E. Avron, Topological quantum states and the 2016 Nobel prize, Bull. IAMP, July 2017, 3-22.

[8] Avron, J., Simon, B.: Singular continuous spectrum for a class of almost periodic Jacobi matrices. Bull. AMS **6**, 81-85 (1982).

[9] J.E. Avron, B. Simon: Almost periodic operators, II. The density of states. Duke Math J. **50**,369–391 (1983)

[10] Bourgain, J.; Jitomirskaya, S. Absolutely continuous spectrum for 1D quasiperiodic operators. Invent. Math. 148 (2002), no. 3, 453–463.

[11] W. Chojnacki, A generalized spectral duality theorem. Comm. Math. Phys., 143 (1992), 527–544.

[12] P. Deift, B. Simon. Almost periodic Schrödinger operators III. The absolutely continuous spectrum in one dimension. Commun. Math. Phys. **90**, 389–411 (1983).

[13] F. Delyon, Absence of localisation in the almost Mathieu equation. J. Phys. A 20 (1987), no. 1, L21–L23.

[14] A.Y. Gordon, S. Jitomirskaya, Y. Last and B. Simon. Duality and singular continuous spectrum in the almost Mathieu equation, Acta Mathematica **178**, 169–183 (1997).

[15] R. Han, Absence of point spectrum for the self-dual extended Harper's model International Mathematics Research Notices, **9**, 2801–2809 (2018)

[16] S. Ya. Jitomirskaya: Metal-insulator transition for the almost Mathieu Operator. Ann. Math. **150** 1159–1175 (1999)

[17] Ergodic Schrödinger operators (on one foot). Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, 613-647, Proc. Sympos. Pure Math., 76, Part 2, Amer. Math. Soc., Providence, RI, 2007.

[18] S. Jitomirskaya, D.A. Koslover, M.S. Schulteis, Localization for a family of one-dimensional quasiperiodic operators of magnetic origin. Ann. Henri Poincare 6 (2005), 103–124.

[19] S. Jitomirskaya and I. Krasovsky. Critical almost Mathieu operator: hidden singularity, gap continuity, and the Hausdorff dimension of the spectrum. Preprint 2019

[20] S. Jitomirskaya and B. Simon. Operators with singular continuous spectrum. III. Almost periodic Schrödinger operators. Comm. Math. Phys., 165(1):201-205, 1994.

[21] V. A. Mandelshtam, S. Ya. Zhitomirskaya. 1D-quasiperiodic operators. Latent symmetries. Commun. Math. Phys. **139**, 589–604 (1991).

[22] I. Satija and D. Hofstadter, Butterfly in the quantum world: the story of the most fascinating quantum fractal. IOP Science, 2016, 1–619.

[23] B. Simon, Almost periodic Schrödinger operators: A review, Adv. Appl. Math. 3 (1982), 463-490