BOUNDS ON THE DENSITY OF STATES FOR SCHRÖDINGER OPERATORS

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Abstract. We establish bounds on the density of states of Schrödinger operators. These are deterministic results that do not require the existence of the integrated density of states. The results are stated in terms of a "density of states outer-measure" that always exists. We prove log-Hölder continuity for the density of states in one, two, and three dimensions for Schrödinger operators, and in any dimension for discrete Schrödinger operators.

1. Introduction

We study the density of states of the Schrödinger operator

$$H = -\Delta + V \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{1.1}$$

where Δ is the Laplacian operator and V is a bounded potential. The density of states measure of an interval "gives the number of states per unit volume" with energy in the interval; its cumulative distribution function is the integrated density of states. Finite volume density of states measures, i.e., density of states measures for restrictions of the Schrödinger operator to finite volumes, are always well defined. The density of states measure is given by appropriate limits of finite volume density of states measures, when such limits exist. These limits are known to exist for Schrödinger operators where the potential V is in some sense uniform in space (e.g., periodic potentials, ergodic Schrödinger operators), but not for general Schrödinger operators; the density of states measure and the corresponding integrated density of states cannot be defined for general Schrödinger operators. For this reason we introduce the density of states outer-measure, which always exists, and provide an upper bound for the density of states measure, when it exists. We prove upper bounds on the density of states outer-measure of small intervals, establishing log-Hölder continuity in one, two, and three dimensions for Schrödinger operators, and in any dimension for discrete Schrödinger operators.

We let

$$\Lambda_L(x) := x + \left]-\tfrac{L}{2}, \tfrac{L}{2}\right[^d = \left\{y \in \mathbb{R}^d;\ |y-x|_\infty < \tfrac{L}{2}\right\} \tag{1.2}$$

denote the (open) box of side L centered at $x \in \mathbb{R}^d$. By a box Λ_L we will mean a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. We write $\|\psi\| = \|\psi\|_2$ for $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ or $\psi \in \mathrm{L}^2(\Lambda)$. We set $V_\infty = \|V\|_\infty$, the norm of the bounded potential V. By χ_B we denote the characteristic function of the set B. Constants such as $C_{a,b,\ldots}$ will always be finite and depending only on the parameters or quantities $a, b, \ldots$; they will be independent of other parameters or quantities in the equation. Note that $C_{a,b,\ldots}$ may stand for different constants in different sides of the same inequality.

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Given a finite box $\Lambda \subset \mathbb{R}^d$, we let $H_\Lambda^\sharp$ and $\Delta_\Lambda^\sharp$ be the restriction of H and Δ to $\mathrm{L}^2(\Lambda)$ with $\sharp$ boundary condition, where $\sharp = D$ (Dirichlet), N (Neumann), or P (periodic). We define finite volume density of states measures $\eta_{\Lambda,\sharp}$ on Borel subsets B of $\mathbb{R}^d$ by

$$\begin{aligned} \eta_{\Lambda,\sharp}(B) &:= \tfrac{1}{|\Lambda|} \operatorname{tr} \left\{ \chi_B(H_\Lambda^\sharp) \right\} \quad \text{for} \quad \sharp = D, N, P, \\ \eta_{\Lambda,\infty}(B) &:= \tfrac{1}{|\Lambda|} \operatorname{tr} \left\{ \chi_B(H)\chi_\Lambda \right\}. \end{aligned} \tag{1.3}$$

Note that for for all Borel subsets $B \subset]-\infty, E]$ we have

$$\eta_{\Lambda,\sharp}(B) \le C_{d,V_\infty,E} < \infty \quad \text{for} \quad \sharp = \infty, D, N, P. \tag{1.4}$$

Moreover, given $f \in C_c(\mathbb{R})$ and $\delta > 0$, there exists $L(d, V_\infty, \delta, f)$ such that for all $L \ge L(d, V_\infty, \delta, f)$ and $x_0 \in \mathbb{R}^d$ we have

$$\left| \eta_{\Lambda_L(x_0),\sharp_1}(f) - \eta_{\Lambda_L(x_0),\sharp_2}(f) \right| \le \delta \quad \text{for} \quad \sharp_1, \sharp_2 = \infty, D, N, P. \tag{1.5}$$

(This can be extracted from [DoIM, see Theorem 3.6, Theorem 6.2, and their proofs].) The finite volume integrated density of states are the corresponding cumulative distribution functions:

$$N_{\Lambda,\sharp}(E) := \eta_{\Lambda,\sharp}(]-\infty, E]). \tag{1.6}$$

For periodic and ergodic Schrödinger operators, density of states measures $\eta_\sharp$ can be defined as weak limits of the finite volume density of states measures $\eta_{\Lambda,\sharp}$ for sequences of boxes $\Lambda \to \mathbb{R}^d$ in an appropriate sense. In this case, the integrated density of states $N_\sharp(E) := \eta_\sharp(]-\infty, E])$ satisfies $N_\sharp(E) = \lim_{\Lambda \to \mathbb{R}^d} N_{\Lambda,\sharp}(E)$ except for a countable set of energies. Moreover, they all coincide, so we define the density of states measure η and the integrated density of states $N(E)$ by $\eta(B) := \eta_\sharp(B)$ and $N(E) := N_\sharp(E)$ for $\sharp = \infty, D, N, P$. (See [KM, PF, CL, DoIM, N].)

Since infinite volume density of states measures and integrated density of states cannot be defined for general Schrödinger operators, we define density of states outer-measures on Borel subsets B of $\mathbb{R}^d$ by

$$\left.\begin{aligned} \eta_{L,\sharp}^*(B) &:= \sup_{x \in \mathbb{R}^d} \eta_{\Lambda_L(x),\sharp}(B) \\ \eta_\sharp^*(B) &:= \limsup_{L \to \infty} \eta_{L,\sharp}^*(B) \end{aligned}\right\} , \qquad \sharp = \infty, D, N, P. \tag{1.7}$$

These are always finite on bounded sets in view of (1.4). (They are indeed outer-measures, so we call them outer-measures for lack of a better name.) Moreover, it follows from (1.5) that for all $E_1, E_2 \in \mathbb{R}$, $E_1 \le E_2$, and $\delta > 0$ we have

$$\eta_{\sharp_1}^*([E_1, E_2]) \le \eta_{\sharp_2}^*([E_1 - \delta, E_2 + \delta]) \quad \text{for all} \quad \sharp_1, \sharp_2 = \infty, D, N, P. \tag{1.8}$$

We will say that we have continuity of the density of states outer-measures $\eta_\sharp^*$ if

$$\lim_{\varepsilon \to 0} \eta_\sharp^*([E - \varepsilon, E + \varepsilon]) = 0 \quad \text{for all} \quad E \in \mathbb{R}. \tag{1.9}$$

In view of (1.8), continuity of $\eta_\sharp^*$ for some value of $\sharp$ implies

$$\eta_\infty^*([E_1, E_2]) = \eta_D^*([E_1, E_2]) = \eta_N^*([E_1, E_2]) = \eta_P^*([E_1, E_2]) \tag{1.10}$$

for all $E_1, E_2 \in \mathbb{R}$, $E_1 \le E_2$. In this case we will say that we have continuity of the density of states outer-measures, and set

$$\eta^*([E_1, E_2]) := \eta_\sharp^*([E_1, E_2]) \quad \text{for} \quad \sharp = \infty, D, N, P. \tag{1.11}$$

We are ready state our main result. Note that if the density of states measure $\eta_\sharp$ exists, we always have

$$\eta_\sharp(B) \le \eta_\sharp^*(B) \quad \text{for all Borel sets} \quad B \subset \mathbb{R}^d, \tag{1.12}$$

and continuity of the density of states outer-measures implies continuity of the integrated density of states

Theorem 1.1. *Let H be a Schrödinger operator as in* (1.1)*, where $d = 1, 2, 3$. Then we have continuity of the density of states outer-measures. Moreover, given $E_0 \in \mathbb{R}$, for all $E \le E_0$ and $\varepsilon \le \frac{1}{2}$ we have*

$$\eta^*\left([E, E+\varepsilon]\right) \le \frac{C_{d,V_\infty,E_0}}{\left(\log \frac{1}{\varepsilon}\right)^{\kappa_d}}, \quad \textit{where} \quad \kappa_1 = 1,\ \kappa_2 = \tfrac{1}{4},\ \kappa_3 = \tfrac{1}{8}. \tag{1.13}$$

We also prove a similar result for discrete Schrödinger operators, i.e., for

$$H = -\Delta + V \quad \text{on} \quad \ell^2(\mathbb{Z}^d), \tag{1.14}$$

where V is a bounded potential and Δ is the centered discrete Laplacian,

$$\Delta\psi(x) = \sum_{y \in \mathbb{Z}^d;\ |x-y|=1} \psi(y) \quad \text{for} \quad x \in \mathbb{Z}^d. \tag{1.15}$$

(Our results are still valid if we take Δ to be any translation invariant finite range self-adjoint operator on $\ell^2(\mathbb{Z}^d)$.) In $\mathbb{Z}^d$ we define the box of side L centered at $x \in \mathbb{Z}^d$ by

$$\Lambda = \Lambda_L(x) = \left\{y \in \mathbb{Z}^d;\ |y - x|_\infty \le \tfrac{L}{2}\right\}, \tag{1.16}$$

and define finite volume operators $H_\Lambda^\sharp$ and $\Delta_\Lambda^\sharp$ as the restriction of H and Δ to $\ell^2(\Lambda)$ with $\sharp$ boundary condition, where $\sharp = D$ (Dirichlet, i.e., simple boundary condition) or P (periodic). We define finite volume density of states measures $\eta_{\Lambda,\sharp}$ as in (1.3) and density of states outer-measures $\eta_{L,\sharp}^*, \eta_\sharp^*$ as in (1.7) for $\sharp = \infty, D, P$. In the discrete case it is easy to see that we also have (1.8), and hence continuity of $\eta_\sharp^*$ for some value of $\sharp$ implies (1.10), in which case we define η^* as in (1.11).

Theorem 1.2. *Let H be a discrete Schrödinger operator as in* (1.14)*. Then for all $d = 1, 2, \ldots$ we have continuity of the density of states outer-measures, and for all $E \in \mathbb{R}$ and $\varepsilon \le \frac{1}{2}$ we have*

$$\eta^*\left([E, E+\varepsilon]\right) \le \frac{C_{d,V_\infty}}{\log \frac{1}{\varepsilon}}. \tag{1.17}$$

We are not aware of previous results in the generality of Theorems 1.1 and 1.2. Published results appear to be restricted to cases where we have existence of the integrated density of states. For periodic potentials, continuity of the the integrated density of states is equivalent to the nonexistence of eigenvalues, a nontrivial result proved by Thomas [T]. For ergodic Schrödinger operators, continuity of the the integrated density of states is equivalent to the nonexistence of energies that are eigenvalues of infinite multiplicity with probability one (see [CL, Lemma V.2.1]). Although Schrödinger operators can have eigenvalues of infinite multiplicity (see [ThE]), it is hard to imagine how a fixed energy can be an eigenvalue of infinite multiplicity for almost all realizations of an ergodic Schrödinger operator.

Craig and Simon proved log-Hölder continuity (with exponent 1) of the integrated density of states for one-dimensional ergodic Schrödinger operators [CrS1] and for ergodic discrete Schrödinger operators in any dimension [CrS2]. Delyon and Souillard [DS] provided a simple proof of continuity of the integrated density of states in the discrete case. But continuity of the the integrated density of states for multi-dimensional (continuous) ergodic Schrödinger operators, albeit expected, has been hard to prove in full generality. It is Problem 14 in [S2], where it was called (in 2000) a 15 year old open problem.

For random Schrödinger operators continuity of the integrated density of states follows from a suitable Wegner estimate; the most general result is due to Combes, Hislop and Klopp [CoHK] that proved that for the Anderson model, both continuous and discrete, we always have continuity of the integrated density of states if the single-site probability distribution has no atoms. (They show that the integrated density of states has as much regularity as the concentration function of the single-site probability distribution.) Germinet and Klein [GK2] proved log-Hölder continuity of the integrated density of states for the continuous Anderson model with arbitrary single-site probability distribution (e.g., Bernouilli) in the region of localization. (More precisely, in the region of applicability of the multiscale analysis; the log-Hölder continuity of the integrated density of states is derived from the conclusions of the multiscale analysis.)

The cases $d = 1$ and $d = 2, 3$ of Theorem 1.1 have separate proofs, the proof for $d = 1$ being similar to the proof of Theorem 1.2. Note that it suffices to establish (1.13) and (1.17) with Dirichlet boundary condition ($\sharp = D$), since we would then have (1.11). Thus in the following sections we assume Dirichlet boundary condition and drop it from the notation.

Theorem 1.2 and the $d = 1$ case of Theorem 1.1 are proved in Section 2; they are immediate consequences of Theorems 2.2 and 2.3, respectively.

Section 3 is devoted to multi-dimensional Schrödinger operators. We start by studying the local behavior of approximate solutions of the stationary Schrödinger equation in Subsection 3.1; see Theorem 3.1. Solutions of the stationary Schrödinger equation admit a local decomposition into a homogeneous harmonic polynomial and a lower order term [HW, B]; in Lemma 3.2 we establish a quantitative version of this decomposition with explicit estimates of the lower order term. This result is extended to approximate solutions in Lemma 3.3, implying Theorem 3.1. We then state and prove Theorem 3.4, a version of Bourgain and Kenig's quantitative unique continuation principle [BoK, Lemma 3.10], in which we make explicit the dependence on the parameters relevant to this article. Finally, in Subsection 3.3 we prove Theorem 3.7, which implies the $d = 2, 3$ cases of Theorem 1.1.

The restriction to $d = 1, 2, 3$ in Theorem 1.1 is due to the present form of the quantitative unique continuation principle (Theorem 3.4), where there is a term $Q^{\frac{4}{3}}$ in the exponent on the left hand side of (3.59). If we had Q^{β} in (3.59), we would be able to prove Theorem 3.7, and hence Theorem 1.1, for dimensions $d < \frac{\beta}{\beta-1}$. Since $\beta = \frac{4}{3}$, we get $d < 4$. It is reasonable to expect that something like Theorem 3.4 holds with $\beta = 1+$ (there are no counterexamples for real potentials), in which case Theorem 1.1 would hold for all d, with $\kappa_d = \frac{\beta - d(\beta-1)}{2\beta} = \frac{1}{2}-$ for $d \geq 2$ in (1.13).

2. Discrete and one-dimensional Schrödinger operators

To prove Theorem 1.2 and the $d = 1$ case of Theorem 1.1, we will select a class of approximate eigenfunctions for which we can establish a global upper bound, and use Lemma 2.1 to pick an approximate eigenfunction for which we have a lower bound for the global upper bound.

2.1. **A lower bound for the global upper bound.**

Lemma 2.1. *Let $\mathcal{V}$ be a finite dimensional linear subspace of $\mathrm{L}^\infty(\Omega,\mathbb{P})$, where $(\Omega,\mathbb{P})$ is a probability space. Then there exists $\psi \in \mathcal{V}$ with $\|\psi\|_2 = 1$ such that*

$$\|\psi\|_\infty \geq \sqrt{\dim \mathcal{V}}. \tag{2.1}$$

This lemma is known to follow immediately from the theory of absolutely summing operators (e.g., [DiJA]). Denote by $\mathcal{V}_p$ the linear space $\mathcal{V}$ viewed as subspace of L^p and let $I^{p,q}$ be the identity map from $\mathcal{V}_p$ to $\mathcal{V}_q$, with $\pi_2(I^{p,q})$ being its 2-summing norm. Then $\pi_2(I^{2,2}) = \sqrt{\dim \mathcal{V}}$, since it is the same as the Hilbert-Schmidt norm of $I^{2,2}$. Factor $I^{2,2} = I^{2,\infty} I^{\infty,2}$, so $\pi_2(I^{2,2}) \leq \left\|I^{2,\infty}\right\| \pi_2(I^{\infty,2})$ by the the ideal property. Since $\pi_2(I^{\infty,2}) \leq 1$ [DiJA, Example 2.9(d)], we have $\left\|I^{2,\infty}\right\| \geq \sqrt{\dim \mathcal{V}}$, and the lemma follows.

This lemma can also be proved by a direct argument, as follows.

Proof of Lemma 2.1 . Using the the Gelfand-Neumark Theorem we can assume, without loss of generality, that Ω is a compact Hausdorff space and $\mathrm{L}^\infty(\Omega,\mathbb{P}) = C(\Omega)$. Thus $\mathcal{V}$ is a finite dimensional linear subspace of $C(\Omega) \subset \mathrm{L}^2(\Omega,\mathbb{P})$. Let $N = \dim \mathcal{V}$, and pick an orthonormal basis $\{\phi_j\}_{j=1}^N$ for $\mathcal{V}$. In particular,

$$\phi(x,y) := \sum_{j=1}^N \overline{\phi_j(x)}\phi_j(y) \in C(\Omega^2), \tag{2.2}$$

and we have

$$N = \int_\Omega \phi(x,x)\,\mathbb{P}(\mathrm{d}x) = \int_\Omega \left\{\frac{\phi(x,x)}{\sqrt{\phi(x,x)}}\right\}^2 \mathbb{P}(\mathrm{d}x) \leq \int_\Omega \max_{y\in\Omega} \left\{\frac{\phi(x,y)}{\sqrt{\phi(x,x)}}\right\}^2 \mathbb{P}(\mathrm{d}x). \tag{2.3}$$

Since $\mathbb{P}$ is a probability measure, there exists $x_0 \in \Omega$ such that

$$\max_{y\in\Omega} \frac{\phi(x_0,y)}{\sqrt{\phi(x_0,x_0)}} \geq \sqrt{N}. \tag{2.4}$$

Setting

$$\psi = \frac{\phi(x_0,\cdot)}{\sqrt{\phi(x_0,x_0)}} = \frac{1}{\sqrt{\phi(x_0,x_0)}} \sum_{j=1}^N \overline{\phi_j(x_0)}\phi_j, \tag{2.5}$$

we have $\psi \in \mathcal{V}$, $\|\psi\|_2 = 1$, and $\|\psi\|_\infty \geq \sqrt{N}$. □

2.2. **Discrete Schrödinger operators.**

Theorem 1.2 is an immediate consequence of the following theorem.

Theorem 2.2. *Let H be a discrete Schrödinger operator as in* (1.14). *Let $E \in \mathbb{R}$ and $0 < \varepsilon \leq \frac{1}{2}$. Then for all boxes $\Lambda = \Lambda_L$ with $L \geq L_{d,V_\infty} \log \frac{1}{\varepsilon}$ we have*

$$\eta_\Lambda\left([E, E+\varepsilon]\right) \leq \frac{C_{d,V_\infty}}{\log \frac{1}{\varepsilon}}. \tag{2.6}$$

Proof. Let $\Lambda_L = \Lambda_L(x_0)$ for some $x_0 \in \mathbb{Z}^d$, $E \in \mathbb{R}$, $\varepsilon \in]0, \frac{1}{2}]$. We set $P = \chi_{[E,E+\varepsilon]}(H_{\Lambda_L})$, and note that

$$\|(H_{\Lambda_L} - E)\,\psi\|_\infty \leq \|(H_{\Lambda_L} - E)\,\psi\| \leq \varepsilon\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P, \tag{2.7}$$

since we have $\|\psi\|_\infty \leq \|\psi\|$ for all $\psi \in \ell^2(\Lambda)$.

Suppose

$$\eta_{\Lambda_L}\left([E,E+\varepsilon]\right)=\tfrac{1}{|\Lambda_L|}\operatorname{tr}P\geq\rho>0. \tag{2.8}$$

We fix $R\in 2\mathbb{N}$, $R<L$, to be selected later, and pick $\mathcal{G}\subset\Lambda_L$ such that

$$\Lambda_L=\bigcup_{y\in\mathcal{G}}\Lambda_R(y)\quad\text{and}\quad\tfrac{|\Lambda_L|}{|\Lambda_R|}\leq\#\mathcal{G}\leq 2^d\tfrac{|\Lambda_L|}{|\Lambda_R|}. \tag{2.9}$$

Note that $(L-1)^d<|\Lambda_L|=\left(2\left\lfloor\frac{L}{2}\right\rfloor+1\right)^d\leq(L+1)^d$, and $|\Lambda_R|=(R+1)^d$. We set

$$\partial_2\Lambda_R(y)=\left\{x\in\Lambda_R(y);\ |x-y|_\infty\in\left\{\tfrac{R}{2},\tfrac{R}{2}-1\right\}\right\}, \tag{2.10}$$

and let

$$\partial_R\Lambda_L=\bigcup_{y\in\mathcal{G}}\partial_2\Lambda_R(y). \tag{2.11}$$

We have

$$|\partial_2\Lambda_R(y)|\leq c_dR^{d-1},\quad\text{so}\quad|\partial_R\Lambda_L|\leq 2^dc_dR^{d-1}\tfrac{|\Lambda_L|}{|\Lambda_R|}\leq 2^dc_d\tfrac{|\Lambda_L|}{R}. \tag{2.12}$$

We now consider the vector space

$$\mathcal{F}=\left\{\psi\in\operatorname{Ran}P;\ \psi(x)=0\quad\text{for all}\quad x\in\partial_R\Lambda_L\right\}. \tag{2.13}$$

Taking (and just writing c_d for $2^{d+1}c_d$)

$$R\in\left[\tfrac{c_d}{\rho},\tfrac{c_d}{\rho}+2\right)\cap 2\mathbb{N}, \tag{2.14}$$

we guarantee

$$\dim\mathcal{F}\geq\rho\,|\Lambda_L|-|\partial_R\Lambda_L|\geq\tfrac{1}{2}\rho\,|\Lambda_L|. \tag{2.15}$$

Let $\psi\in\mathcal{F}$ with $\|\psi\|=1$ and $y\in\mathcal{G}$. It follows from (1.14), (1.15), and (2.7) that if we know that $|\psi(x)|\leq C$ for all x with $|x-y|_\infty=k+1,k+2$, then we must have $|\psi(x)|\leq C(2d-1+\|V-E\|_\infty)+\varepsilon$ for $|x-y|_\infty=k$. Since $\psi(x)=0$ if $|x-y|_\infty=\frac{R}{2},\frac{R}{2}-1$, we conclude that

$$|\psi(x)|\leq\varepsilon A^{\frac{R}{2}-1-|x-y|_\infty}\leq\varepsilon A^{\frac{R}{2}}\quad\text{for all}\quad x\in\Lambda_R(y), \tag{2.16}$$

where $A=2d-1+\|V-E\|_\infty$. It follows, using (2.9), that

$$\|\psi\|_\infty\leq\varepsilon A^{\frac{R}{2}}\quad\text{for all}\quad x\in\Lambda. \tag{2.17}$$

We now use Lemma 2.1, obtaining $\psi_0\in\mathcal{F}$, $\|\psi_0\|=1$, such that

$$\|\psi_0\|_\infty\geq\sqrt{\frac{\dim\mathcal{F}}{|\Lambda_L|}}\geq\sqrt{\tfrac{1}{2}\rho}. \tag{2.18}$$

Combining (2.17), (2.18), and (2.14) we get

$$\sqrt{\tfrac{1}{2}\rho}\leq\varepsilon A^{\frac{R}{2}}\leq\varepsilon A^{\frac{c_d}{2\rho}+1}, \tag{2.19}$$

which implies

$$\rho\leq\frac{C_{d,\|V-E\|_\infty}}{\log\frac{1}{\varepsilon}}, \tag{2.20}$$

as long as L is sufficiently large, namely $L\geq C_dR\geq\frac{C_d}{\rho}$.

Since $\sigma\left(H_{\Lambda_L}\right)\subset[-2d-V_\infty,2d+V_\infty]$, we have $\eta_{\Lambda_L}\left([E,E+\varepsilon]\right)=0$ unless $|E|\leq 2d+V_\infty+\frac{1}{2}$, so we get (2.6) if $L\geq L_{d,V_\infty}\log\frac{1}{\varepsilon}$. □

2.3. **One-dimensional Schrödinger operators.** The case $d = 1$ of Theorem 1.1 is an immediate consequence of the following theorem. Note that one dimensional boxes are intervals.

Theorem 2.3. *Let H be a Schrödinger operator as in (1.1) with $d = 1$. Given $E_0 \in \mathbb{R}$, there exists L_{V_∞,E_0} such that for all $0 < \varepsilon \le \frac{1}{2}$, open intervals $\Lambda = \Lambda_L$ with $L \ge L_{V_\infty,E_0} \log \frac{1}{\varepsilon}$, and energies $E \le E_0$, we have*

$$\eta_\Lambda\left([E, E+\varepsilon]\right) \le \frac{C_{V_\infty,E_0}}{\log \frac{1}{\varepsilon}}. \tag{2.21}$$

Proof. Let $\Lambda = \Lambda_L =]a_0, a_0 + L[$, $E \in \mathbb{R}$, $\varepsilon \in]0, \frac{1}{2}]$. We set $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$. Recall that $\operatorname{Ran} P\chi_\Lambda \subset \mathcal{D}(\Delta_\Lambda) \subset C^1(\Lambda)$, and note that we have

$$\|(H_\Lambda - E)\,\psi\| \le \varepsilon\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P. \tag{2.22}$$

Given $0 < R < L$, set $a_j = a_0 + jR$ for $j = 1, 2, \ldots, \left\lceil \frac{L}{R} \right\rceil - 1$. We introduce the vector space

$$\mathcal{F}_R := \left\{ \psi \in \operatorname{Ran} P;\ \ \psi(a_j) = \psi'(a_j) = 0 \text{ for } j = 1, 2, \ldots, \left\lceil \frac{L}{R} \right\rceil - 1 \right\}. \tag{2.23}$$

Given $\psi \in \mathcal{F}_R$ and $j = 1, \ldots, \left\lceil \frac{L}{R} \right\rceil - 1$, it follows from Gronwall's inequality (see [Ho]), $\psi(a_j) = \psi'(a_j) = 0$, and (2.22) that for all $x \in]a_j - R, a_j + R[\cap\Lambda$ we have

$$|\psi(x)| \le \mathrm{e}^{K|x-a_j|} \left| \int_{a_j}^{x} \mathrm{e}^{-K|y-a_j|} \left|(H_\Lambda - E)\,\psi(y)\right| \mathrm{d}y \right| \le (2K)^{-\frac{1}{2}}\, \mathrm{e}^{KR} \varepsilon\, \|\psi\|, \tag{2.24}$$

where $K = 1 + \|V - E\|_\infty$. Since Λ is the union of these intervals, we conclude that

$$\|\psi\|_\infty \le (2K)^{-\frac{1}{2}}\, \mathrm{e}^{KR} \varepsilon\, \|\psi\| \quad \text{for all} \quad \psi \in \mathcal{F}_R. \tag{2.25}$$

We now assume that

$$\eta_\Lambda\left([E, E+\varepsilon]\right) = \tfrac{1}{L} \operatorname{tr} P \ge \rho > 0. \tag{2.26}$$

If $R \ge \frac{4}{\rho}$, it follows from (2.26) that

$$\dim \mathcal{F}_R \ge \rho L - 2\left(\left\lceil \tfrac{L}{R} \right\rceil - 1\right) \ge \rho L - 2\tfrac{L}{R} \ge \tfrac{1}{2}\rho L. \tag{2.27}$$

Applying Lemma 2.1, we obtain $\psi_0 \in \mathcal{F}_R$, $\psi_0 \ne 0$, such that

$$\|\psi_0\|_\infty \ge \sqrt{\frac{\dim \mathcal{F}_R}{L}}\, \|\psi_0\| \ge \sqrt{\tfrac{1}{2}\rho}\, \|\psi_0\|. \tag{2.28}$$

Taking $R = \frac{4}{\rho}$, it follows from (2.25) and (2.28) that

$$\sqrt{\tfrac{1}{2}\rho} \le (2K)^{-\frac{1}{2}}\, \mathrm{e}^{KR} \varepsilon = (2K)^{-\frac{1}{2}}\, \mathrm{e}^{\frac{4K}{\rho}}\, \varepsilon. \tag{2.29}$$

Thus, we get

$$\rho \le \frac{8K}{\log \frac{1}{\varepsilon}}, \tag{2.30}$$

if L is sufficiently large, namely $L \ge CR = \frac{4C}{\rho}$.

Since $\sigma(H_\Lambda) \subset [-V_\infty, \infty[$, we have $\eta_\Lambda([E, E+\varepsilon]) = 0$ unless $E \ge -V_\infty - \frac{1}{2}$. Thus, given $E_0 \in \mathbb{R}$, there exists L_{V_∞,E_0} such that, for all $0 < \varepsilon \le \frac{1}{2}$, open intervals $\Lambda = \Lambda_L$ with $L \ge L_{V_\infty,E_0} \log \frac{1}{\varepsilon}$, and energies $E \le E_0$, we have (2.21). □

3. Multi-dimensional Schrödinger operators

To prove Theorem 1.1 for $d = 2, 3$, we will select a class of approximate eigenfunctions for which we can establish local upper bounds, pick an approximate eigenfunction for which we have a global lower bound for the global upper bound. The local upper bounds will come from the local behavior of approximate solutions of the stationary Schrödinger equation; the global upper bound will come from the quantitative unique continuation principle.

Given $x \in \mathbb{R}^d$ and $\delta > 0$, we set $B(x, \delta) := \{y \in \mathbb{R}^d;\ |y - x| < \delta\}$.

3.1. **Local behavior of approximate solutions of the stationary Schrödinger equation.**

Theorem 3.1. *Let $\Omega = B(x_0, r_0)$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$, where $d = 2, 3, \ldots$, and fix a real valued function $W \in \mathrm{L}^\infty(\Omega)$; $W_\infty = \|W\|_{\mathrm{L}^\infty(\Omega)}$. Let $\mathcal{F}$ denote a linear subspace of $\mathcal{H}^2(\Omega)$ with the following property:*

$$\|(-\Delta + W)\,\psi\|_{\mathrm{L}^\infty(\Omega)} \le C_{\mathcal{F}}\,\|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{for all} \quad \psi \in \mathcal{F}. \tag{3.1}$$

Then there exists a constant $\gamma_d > 0$ and $0 < r_1 = r_1\,(d, W_\infty) < r_0$, with the property that for all $N \in \mathbb{N}$ there is a linear subspace $\mathcal{F}_N$ of $\mathcal{F}$, with

$$\dim \mathcal{F}_N \ge \dim \mathcal{F} - \gamma_d N^{d-1}, \tag{3.2}$$

such that for all $\psi \in \mathcal{F}_N$ we have

$$|\psi(x)| \le \left(C_{d,W_\infty,r_1}^{N^2}\,|x - x_0|^{N+1} + C_{\mathcal{F}}\right)\|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{for} \quad x \in B(x_0, r_1). \tag{3.3}$$

We take $d = 2, 3, \ldots$, and set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We consider sites $x \in \mathbb{R}^d$, partial derivatives $\partial_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2, \ldots, d$, multi-indices $\alpha \in \mathbb{N}_0^d$, and set

$$x^\alpha = \prod_{j=1}^d x_j^{\alpha_j}, \quad D^\alpha = \prod_{j=1}^d \partial_j^{\alpha_j}, \quad |\alpha| = \sum_{j=1}^d |\alpha_j|\,, \quad \alpha! = \prod_{j=1}^d \alpha_j!. \tag{3.4}$$

We let $\mathcal{H}_m^{(d)} = \mathcal{H}_m(\mathbb{R}^d)$ denote the vector space of homogenous harmonic polynomials on $\mathbb{R}^d$ of degree $m \in \mathbb{N}_0$, and recall that [ABR, Proposition 5.8 and exercises] we have $\dim \mathcal{H}_0^{(d)} = 1$, $\dim \mathcal{H}_1^{(d)} = d$, and, for $m = 2, 3, \ldots$,

$$\dim \mathcal{H}_m^{(d)} = \binom{d + m - 1}{d - 1} - \binom{d + m - 3}{d - 1}. \tag{3.5}$$

In particular, we have

$$\dim \mathcal{H}_m^{(2)} = 2 \quad \text{and} \quad \dim \mathcal{H}_m^{(3)} = 2m + 1 \quad \text{for} \quad m = 2, 3, \ldots, \tag{3.6}$$

and $\dim \mathcal{H}_m^{(d)} < \dim \mathcal{H}_{m+1}^{(d)}$ for $d > 2$. Moreover

$$\lim_{m \to \infty} \frac{\dim \mathcal{H}_m^{(d)}}{m^{d-2}} = \frac{2}{(d-2)!} \quad \text{for} \quad d \ge 2. \tag{3.7}$$

We also define $\mathcal{H}_{\le N}^{(d)} = \bigoplus_{m=0}^N \mathcal{H}_m^{(d)}$, the vector space of harmonic polynomials on $\mathbb{R}^d$ of degree $\le N$. It follows from (3.7) that for $d = 2, 3, \ldots$ there exists a constant $\gamma_d > 0$ such that

$$\dim \mathcal{H}_{\le N}^{(d)} = \sum_{m=0}^N \dim \mathcal{H}_m^{(d)} \le \gamma_d N^{d-1} \quad \text{for all} \quad N \in \mathbb{N}. \tag{3.8}$$

We let

$$\mathbf{\Phi}(x) = \mathbf{\Phi}_d(x) := \begin{cases} (d(d-2)\omega_d)^{-1} |x|^{-d+2} & \text{if} \quad d = 3, 4, \dots \\ -\frac{1}{2\pi} \log |x| & \text{if} \quad d = 2 \end{cases} \tag{3.9}$$

be the fundamental solution to Laplace's equation; ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. In particular,

$$-\Delta \mathbf{\Phi}(x) = \delta(x) \quad \text{on} \quad \mathbb{R}^d, \tag{3.10}$$

and

$$|D^\alpha \mathbf{\Phi}(x)| \leq C_{d,|\alpha|} |x|^{-d+2-|\alpha|}. \tag{3.11}$$

We fix a real valued function $W \in \mathrm{L}^\infty$, set $W_\infty = \|W\|_{\mathrm{L}^\infty(\Omega)}$, and consider the the stationary Schrödinger equation

$$-\Delta \phi + W\phi = 0 \quad \text{a.e. on} \quad \Omega = B(x_0, 2r_0). \tag{3.12}$$

We let $\mathcal{E}_0(\Omega) = \mathcal{E}_0(\Omega, W)$ denote the vector subspace formed by solutions $\phi \in \mathcal{H}^2(\Omega)$. We define linear subspaces

$$\mathcal{E}_N(\Omega) = \left\{ \phi \in \mathcal{E}_0(\Omega); \ \limsup_{x \to x_0} \tfrac{|\phi(x)|}{|x-x_0|^N} < \infty \right\} \quad \text{for} \quad N \in \mathbb{N}. \tag{3.13}$$

Note that $\mathcal{E}_1(\Omega) = \{\phi \in \mathcal{E}_0(\Omega); \ \phi(x_0) = 0\}$, $\mathcal{E}_N(\Omega) \supset \mathcal{E}_{N+1}(\Omega)$ for all $N \in \mathbb{N}_0$, and $\cap_{N=0}^\infty \mathcal{E}_N(\Omega) = \{0\}$ by the unique continuation principle.

A solution of the equation (3.12) admits a local decomposition into a homogeneous harmonic polynomial and a lower order term [HW, B]. The following lemma is a quantitative version of this decomposition; it gives an explicit estimate of the lower order term.

Lemma 3.2. *Let $\Omega = B(x_0, 2r_0)$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$, $d = 2, 3, \dots$, and fix a real valued function $W \in \mathrm{L}^\infty(\Omega)$. For all $N \in \mathbb{N}_0$ there exists a linear map $Y_N^{(\Omega)} \colon \mathcal{E}_N(\Omega) \to \mathcal{H}_N^{(d)}$ such that for all $\phi \in \mathcal{E}_N(\Omega)$ we have*

$$\begin{aligned} &\left|\phi(x) - \left(Y_N^{(\Omega)}\phi\right)(x - x_0)\right| \\ &\quad \leq r_0^{-\frac{d}{2}} \left(r_0^{-1} \widetilde{C}_{d, r_0^2 W_\infty}\right)^{N+1} \left(\tfrac{16}{3}\right)^{\frac{(N+1)(N+2)}{2}} ((N+1)!)^{d-2} |x - x_0|^{N+1} \|\phi\|_{\mathrm{L}^2(\Omega)} \end{aligned} \tag{3.14}$$

for all $x \in \overline{B}(x_0, \frac{r_0}{2})$. As a consequence, for all $N \in \mathbb{N}_0$ we have

$$\mathcal{E}_{N+1}(\Omega) = \ker Y_N^{(\Omega)} \quad \textit{and} \quad \dim \mathcal{E}_{N+1}(\Omega) \geq \dim \mathcal{E}_N(\Omega) - \dim \mathcal{H}_N^{(d)}. \tag{3.15}$$

In particular, if $\mathcal{J}$ is a vector subspace of $\mathcal{E}_0(\Omega)$ we have

$$\dim \mathcal{J} \cap \mathcal{E}_{N+1}(\Omega) \geq \dim \mathcal{J} - \gamma_d N^{d-1} \quad \textit{for all} \quad N \in \mathbb{N}, \tag{3.16}$$

where γ_d is the constant in (3.8).

Proof. We prove the lemma for $\Omega = B(0, 2)$; the general case then follows by translating and dilating. We will generally drop Ω from the notation. We recall that $\phi \in \mathcal{E}_0$ satisfies the following elliptic regularity estimates ([GiT, Theorem 8.17], [GiT, Theorem 8.32]):

$$\sup_{x \in B(0, \frac{3}{2})} |\phi(x)| \leq C_{d, W_\infty} \|\phi\|_{\mathrm{L}^2(\Omega)}, \tag{3.17}$$

$$\sup_{y \in B(0,1)} |\nabla \phi(y)| \leq C_{d, W_\infty} \sup_{x \in B(0, \frac{3}{2})} |\phi(x)|. \tag{3.18}$$

Given $\phi \in \mathcal{E}_0$ we consider its Newtonian potential given by

$$\psi(x) = -\int_{\Omega'} W(y)\phi(y)\mathbf{\Phi}(x-y)\,\mathrm{d}y \quad \text{for} \quad x \in \mathbb{R}^d. \tag{3.19}$$

In view of (3.17), for all $x \in B(0, \frac{3}{2})$ we have

$$|\psi(x)| \le W_\infty \|\phi\|_{\mathrm{L}^\infty(B(0,\frac{3}{2}))} \|\mathbf{\Phi}\|_{\mathrm{L}^1(B(0,3))} \le C_{d,W_\infty} W_\infty \|\phi\|_{\mathrm{L}^2(\Omega)}\,. \tag{3.20}$$

It follows from (3.10) that $\Delta\psi = W\phi$ weakly in Ω. Thus, letting $h = \phi - \psi$ we have $\Delta h = 0$ weakly in Ω, so we conclude that h is a harmonic function in $\Omega \supset \overline{B}(0,1)$. In particular (see [ABR, Corollary 5.34 and its proof]), h is real analytic in Ω and

$$h(x) = \sum_{m=0}^{\infty} p_m(x) \quad \text{for all} \quad x \in B(0,1), \tag{3.21}$$

where $p_m \in \mathcal{H}_m^{(d)}$ for all $m = 0, 1, \ldots$, and for $m = 1, 2, \ldots$ we have

$$|p_m(x)| \le C_d\, m^{d-2} |x|^m \sup_{y \in \partial B(0,1)} |h(y)| \quad \text{for all} \quad x \in B(0,1). \tag{3.22}$$

In addition, it follows from the mean value property that for all $y \in \partial B(0,1)$ we have

$$|h(y)| \le \tfrac{1}{|B(y,\frac{1}{2})|} \int_{B(y,\frac{1}{2})} |h(y')|\,\mathrm{d}y' \le C_{d,W_\infty} \|\phi\|_{\mathrm{L}^2(\Omega)}\,, \tag{3.23}$$

ising (3.17) and (3.20). Thus, for all $m = 1, 2, \ldots$ it follows from (3.22) that

$$|p_m(x)| \le C_{d,W_\infty} m^{d-2} \|\phi\|_{\mathrm{L}^2(\Omega)} |x|^m \quad \text{for all} \quad x \in B(0,1). \tag{3.24}$$

Setting $h_N = \sum_{m=0}^{N} p_m(x) \in \mathcal{H}_{\le N}^{(d)}$, it follows that

$$|h(x) - h_N(x)| \le C_{d,W_\infty} \|\phi\|_{\mathrm{L}^2(\Omega)} (N+1)^{d-2} |x|^{N+1} \quad \text{for all} \quad x \in \overline{B}(0,\tfrac{1}{2}). \tag{3.25}$$

For each $y \in \mathbb{R}^d \setminus \{0\}$ we consider $\mathbf{\Phi}_y(x) = \mathbf{\Phi}(x-y)$, a harmonic function on $\mathbb{R}^d \setminus \{y\}$. In particular, $\mathbf{\Phi}_y(x)$ is real analytic in $B(0,|y|)$, so, defining

$$J_m(x,y) = \sum_{\alpha \in \mathbb{N}_0^d,\, |\alpha| = m} \tfrac{1}{\alpha!} D^\alpha \mathbf{\Phi}(y) x^\alpha \quad \text{for} \quad x \in \mathbb{R}^d, \tag{3.26}$$

we have (see [ABR])

$$\mathbf{\Phi}(x-y) = \mathbf{\Phi}_y(x) = \sum_{m=0}^{\infty} J_m(x,y) \quad \text{for all} \quad x \in B(0,|y|), \tag{3.27}$$

the series converging absolutely and uniformly on compact subsets of $B(0,|y|)$. Moreover, $J_m(\cdot,y) \in \mathcal{H}_m^{(d)}$, and for all $y \in \mathbb{Z}^d$ and $m = 1, 2, \ldots$ we have (see [ABR, Corollary 5.34 and its proof]) that

$$|J_m(x,y)| \le C_d\, m^{d-2} \left(\tfrac{4|x|}{3|y|}\right)^m \sup_{x' \in \partial B\left(0,\frac{3}{4}|y|\right)} |\mathbf{\Phi}_y(x')| \le C_d\, m^{d-2} \left(\tfrac{4|x|}{3|y|}\right)^m \mathbf{\Phi}(\tfrac{y}{4}), \tag{3.28}$$

for all $x \in B\left(0, \frac{3}{4}|y|\right)$. Setting $\mathbf{\Phi}_{y,N}(x) = \sum_{m=0}^{N} J_m(x,y) \in \mathcal{H}_{\le N}^{(d)}$, it follows that for $x \in \overline{B}(0, \frac{1}{2}|y|)$ we have

$$|\mathbf{\Phi}_y(x) - \mathbf{\Phi}_{y,N}(x)| \le C_d (N+1)^{d-2} \left(\tfrac{4|x|}{3|y|}\right)^{N+1} \mathbf{\Phi}(\tfrac{y}{4}). \tag{3.29}$$

We now proceed by induction. We define $Y_0 \colon \mathcal{E}_0 \to \mathcal{H}_0^{(d)}$ by $Y_0\phi = \phi(0)$. Given $\phi \in \mathcal{E}_0$, it follows from the mean value theorem and the elliptic regularity estimates (3.17) and (3.18) that

$$|\phi(x) - \phi(0)| \le \sup_{y \in B(0,1)} |\nabla\phi(y)|\,|x| \le C_{d,W_\infty} \|\phi\|_{\mathrm{L}^2(\Omega)}\,|x| \quad \text{for} \quad x \in \overline{B}(0,1). \tag{3.30}$$

Thus the lemma holds for $N = 0$.

We now let $N \in \mathbb{N}$ and suppose that the lemma is valid for $N-1$. If $\phi \in \mathcal{E}_N$, it follows that $\phi \in \mathcal{E}_{N-1}$ with $Y_{N-1}\phi = 0$, so by the induction hypothesis

$$|\phi(x)| \le C_N \|\phi\|_{\mathrm{L}^2(\Omega)}\,|x|^N \quad \text{for all} \quad x \in \overline{B}(0, \tfrac{1}{2}), \tag{3.31}$$

where

$$C_N = \widetilde{C}_{d,W_\infty}^N \left(\tfrac{16}{3}\right)^{\frac{N(N+1)}{2}} (N!)^{d-2}. \tag{3.32}$$

Using (3.28) and (3.31), we define

$$\psi_N(x) = -\int_{\Omega'} W(y)\phi(y)\mathbf{\Phi}_{y,N}(x)\,\mathrm{d}y \in \mathcal{H}_{\le N}^{(d)}. \tag{3.33}$$

We fix $x \in \overline{B}(0, \frac{1}{2})$ and estimate

$$|\psi(x) - \psi_N(x)| \le W_\infty \int_{\Omega'} |\phi(y)|\,|\mathbf{\Phi}_{y,>N}(x)|\,\mathrm{d}y, \tag{3.34}$$

where $\mathbf{\Phi}_{y,>N}(x) = \mathbf{\Phi}_y(x) - \mathbf{\Phi}_{y,N}(x)$. Appealing to (3.29) and (3.31), we get

$$\int_{\overline{B}(0,\frac{1}{2}) \setminus B(0,2|x|)} |\phi(y)|\,|\mathbf{\Phi}_{y,>N}(x)|\,\mathrm{d}y \le C_d C_N \|\phi\|_2 (N+1)^{d-2} \left(\tfrac{4}{3}\right)^{N+1} |x|^{N+1}. \tag{3.35}$$

If $y \notin B(0, 2|x|)$ we have $|y| \ge 2|x| \ge 1$, and hence, using (3.29),

$$\begin{aligned} &\int_{\Omega' \setminus \left(B(0,2|x|) \cup \overline{B}(0,\frac{1}{2})\right)} |\phi(y)|\,|\mathbf{\Phi}_{y,>N}(x)|\,\mathrm{d}y \\ &\qquad \le C_d (N+1)^{d-2} \left(\tfrac{4}{3}\right)^{N+1} \mathbf{\Phi}(\tfrac{1}{4})\,|x|^{N+1} \int_{\Omega'} |\phi(y)|\,\mathrm{d}y \\ &\qquad \le C_d (N+1)^{d-2} \left(\tfrac{4}{3}\right)^{N+1} |x|^{N+1} \|\phi\|_{\mathrm{L}^2(\Omega)}. \end{aligned} \tag{3.36}$$

Using (3.28) and (3.31), we get

$$\begin{aligned}
&\int_{\overline{B}(0,\frac{1}{2})\cap B(0,2|x|)} |\phi(y)|\,|\mathbf{\Phi}_{y,>N}(x)|\;\mathrm{d}y \qquad (3.37)\\
&\quad\le C_N\|\phi\|_{\mathrm{L}^2(\Omega)}\int_{\overline{B}(0,\frac{1}{2})\cap B(0,2|x|)} |y|^N\,|\mathbf{\Phi}_{y,>N}(x)|\;\mathrm{d}y\\
&\quad\le C_N\|\phi\|_{\mathrm{L}^2(\Omega)}\int_{\overline{B}(0,\frac{1}{2})\cap B(0,2|x|)} |y|^N\,|\mathbf{\Phi}(x-y)|\;\mathrm{d}y\\
&\qquad + C_dC_N\|\phi\|_{\mathrm{L}^2(\Omega)}\sum_{m=0}^{N} m^{d-2}\left(\tfrac{4}{3}|x|\right)^m\int_{\overline{B}(0,\frac{1}{2})\cap B(0,2|x|)} |y|^{N-m}\left|\mathbf{\Phi}(\tfrac{y}{4})\right|\;\mathrm{d}y\\
&\quad\le C_dC_N\|\phi\|_{\mathrm{L}^2(\Omega)}\left(1+N^{d-2}\left(\tfrac{4}{3}\right)^{N+1}\right)|x|^{N+1},
\end{aligned}$$

where we used $|x-y|\le 3|x|$ for $y\in B(0,2|x|)$. (Note that we get $|x|^{N+2}$ if $d\ge 3$ and $|x|^{(N+2)-}$ if $d=2$.) Also using (3.28), we get

$$\begin{aligned}
\int_{\Omega'\setminus\overline{B}(0,\frac{1}{2})} |\phi(y)|\,|\mathbf{\Phi}_{y,>N}(x)|\;\mathrm{d}y &\le \int_{\Omega'\setminus\overline{B}(0,\frac{1}{2})} |\phi(y)|\,|\mathbf{\Phi}(x-y)|\;\mathrm{d}y \qquad (3.38)\\
&\quad + C_d\sum_{m=0}^{N} m^{d-2}\left(\tfrac{4}{3}|x|\right)^m\int_{\Omega'\setminus\overline{B}(0,\frac{1}{2})} |\phi(y)|\,|y|^{-m}\left|\mathbf{\Phi}(\tfrac{y}{4})\right|\;\mathrm{d}y\\
&\le C_d\|\phi\|_{\mathrm{L}^2(\Omega)}\left(1+N^{d-2}\left(\tfrac{4}{3}\right)^{N+1}\right),
\end{aligned}$$

where we used $|x|\le\frac{1}{2}$. Since $|x|>\frac{1}{4}$ if $y\in B(0,2|x|)\setminus\overline{B}(0,\frac{1}{2})$, we obtain

$$\begin{aligned}
&\int_{(\Omega'\cap B(0,2|x|))\setminus\overline{B}(0,\frac{1}{2})} |\phi(y)|\,|\mathbf{\Phi}_{y,>N}(x)|\;\mathrm{d}y \qquad (3.39)\\
&\qquad\qquad\le C_d\|\phi\|_{\mathrm{L}^2(\Omega)}\left(1+N^{d-2}\left(\tfrac{16}{3}\right)^{N+1}\right)|x|^{N+1}.
\end{aligned}$$

Putting together (3.34), (3.35), (3.36), (3.37), and (3.39), we conclude that for all $x\in\overline{B}(0,\frac{1}{2})$ we have ($C_N\ge 1$)

$$|\psi(x)-\psi_N(x)|\le C_dC_NW_\infty(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}. \qquad (3.40)$$

We now define $Y_N\phi=h_N+\psi_N\in\mathcal{H}_N^{(d)}$. Since $\phi=h+\psi$, for all $x\in\overline{B}(0,\frac{1}{2})$ it follows from (3.25), (3.40), and (3.32), that

$$\begin{aligned}
&|\phi(x)-(Y_N\phi)(x)|\le|h(x)-h_N(x)|+|\psi(x)-\psi_N(x)| \qquad (3.41)\\
&\quad\le\left(C_{d,W_\infty}+C_dW_\infty C_N\right)(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}\\
&\quad\le\widetilde{C}_{d,W_\infty}C_N(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}\\
&\quad\le\widetilde{C}_{d,W_\infty}\left(\widetilde{C}_{d,W_\infty}^N\left(\tfrac{16}{3}\right)^{\frac{N(N+1)}{2}}(N!)^{d-2}\right)(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}\\
&\quad\le\widetilde{C}_{d,W_\infty}^{N+1}\left(\tfrac{16}{3}\right)^{\frac{(N+1)(N+2)}{2}}((N+1)!)^{d-2}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)},
\end{aligned}$$

by choosing the constant $\widetilde{C}_{d,W_\infty}$ in (3.32) large enough. This completes the induction.

The lemma is proven, as (3.15) is an immediate consequence of (3.14), and (3.16) follows from (3.15) and (3.8). □

Theorem 3.1 is an immediate consequence from the following lemma.

Lemma 3.3. *Let $\Omega = B(x_0, r_1)$ for some $x_0 \in \mathbb{R}^d$ and $r_1 > 0$, and fix a real valued function $W \in \mathrm{L}^\infty(\Omega)$. Let $\mathcal{F}$ denote a linear subspace of $\mathcal{H}^2(\Omega)$ with the following property:*

$$\|(-\Delta + W)\,\psi\|_{\mathrm{L}^\infty(\Omega)} \leq C_{\mathcal{F}}\,\|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{for all} \quad \psi \in \mathcal{F}. \tag{3.42}$$

Then there exists $0 < r_2 = r_2\,(d, W_\infty) < r_1$, where $W_\infty = \|W\|_{\mathrm{L}^\infty(\Omega)}$, with the property that for all $r \in]0, r_2]$ there is a linear map $Z_r \colon \mathcal{F} \to \mathcal{E}_0(B(x_0, r))$ such that

$$\|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B(x_0,r))} \leq C_{d,r} C_{\mathcal{F}}\,\|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{where} \quad \lim_{r\to 0} C_{d,r} = 0. \tag{3.43}$$

As a consequence, for all $N \in \mathbb{N}$ there is a vector subspace $\mathcal{F}_N$ of $\mathcal{F}$, with

$$\dim \mathcal{F}_N \geq \dim \mathcal{F} - \gamma_d N^{d-1}, \tag{3.44}$$

where γ_d is the constant in (3.8), such that for all $\psi \in \mathcal{F}_N$ we have

$$\begin{aligned} |\psi(x)| &\leq \left(\widehat{C}^{N+1}_{d,W_\infty,r_1}((N+1)!)^{d-2}\, 3^{N^2}\, |x - x_0|^{N+1} + C_{\mathcal{F}}\right) \|\psi\|_{\mathrm{L}^2(\Omega)} \\ &\leq \left(C^{N^2}_{d,W_\infty,r_1}\, |x - x_0|^{N+1} + C_{\mathcal{F}}\right) \|\psi\|_{\mathrm{L}^2(\Omega)} \end{aligned} \tag{3.45}$$

for all $x \in \overline{B}(x_0, \frac{r_2}{4})$.

Proof. It suffices to consider $x_0 = 0$. We set $B_r = B(0, r)$. Given $0 < r < r_1$ and $\psi \in \mathcal{F}$, we define $Z_r\psi \in \mathcal{E}_0(B_r)$ as the unique solution $\phi \in \mathcal{H}^2(B_r)$ to the Dirichlet problem on B_r given by

$$\begin{cases} -\Delta\phi + W\phi = 0 \quad \text{on} \quad B_r \\ \phi = \psi \quad \text{on} \quad \partial B_r \end{cases}. \tag{3.46}$$

This map is well defined in view of [GiT, Theorem 8.3] and is clearly a linear map.

To prove (3.43) we will use the Green's function $G_r(x, y)$ for the ball B_r. We recall that, abusing the notation by writing $\mathbf{\Phi}(|x|)$ instead of $\mathbf{\Phi}(x)$ (see [GiT, Section 2.5]; note that with our definition $\mathbf{\Phi}(x) = -\Gamma(|x|)$),

$$G_r(x, y) = \begin{cases} \mathbf{\Phi}(|x - y|) - \mathbf{\Phi}\left(\frac{|y|}{r}\left|x - \frac{r^2}{|y|^2}y\right|\right) & \text{if} \quad y \neq 0 \\ \mathbf{\Phi}(|x|) - \mathbf{\Phi}(r) & \text{if} \quad y = 0 \end{cases}. \tag{3.47}$$

Using Green's representation formula [GiT, Eq. (2.21)] for ψ and $Z_r\psi$, for all $x \in B_r$ we have

$$\begin{aligned} \psi(x) = &- \int_{\partial B_r} \psi(\zeta)\partial_\nu G_r(x, \zeta)\mathrm{d}S(\zeta) - \int_{B_r} W(y)\psi(y)G_r(x, y)\mathrm{d}y \\ &+ \int_{B_r} (-\Delta + W(y))\psi(y)G_r(x, y)\mathrm{d}y, \end{aligned} \tag{3.48}$$

$$Z_r\psi(x) = -\int_{\partial B_r} \psi(\zeta)\partial_\nu G_r(x, \zeta)\mathrm{d}S(\zeta) - \int_{B_r} W(y)Z_r\psi(y)G_r(x, y)\mathrm{d}y, \tag{3.49}$$

where $\mathrm{d}S$ denotes the surface measure and ∂_ν is the normal derivative. Since by an explicit calculation we have, with $p_2 = 2$ and $p_d = \frac{d-1}{d-2}$ for $d \geq 3$, that for all $x \in B_r$

$$\|G_r(x, \cdot)\|_{\mathrm{L}^1(B_r)} \leq C'_d r^{\frac{d(p_d-1)}{p_d}} \; \|G_r(x, \cdot)\|_{\mathrm{L}^{p_d}(B_r)} \leq C_d r^{\frac{d(p_d-1)}{p_d}}, \tag{3.50}$$

it follows that

$$\begin{aligned} \|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B_r)} & \qquad\qquad (3.51)\\ \le C_d r^{\frac{d(p_d-1)}{p_d}} \left(W_\infty \|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B_r)} + \|(-\Delta + W)\psi\|_{\mathrm{L}^\infty(B_r)} \right). \end{aligned}$$

Selecting $r_2 \in]0, r_1[$ such that $C_d r_2^{\frac{d(p_d-1)}{p_d}} (1 + W_\infty) \le \frac{1}{2}$, and using (3.42), we get (3.43).

Now let $\mathcal{J} = \operatorname{Ran} Z_{r_2}$, a linear subspace of $\mathcal{E}_0(B_{r_2})$; note that

$$\dim \mathcal{J} + \dim \ker Z_{r_2} = \dim \mathcal{F}. \qquad (3.52)$$

We set $\mathcal{J}_N = \mathcal{J} \cap \mathcal{E}_{N+1}(B_{r_2})$ and $\mathcal{F}_N = Z_{r_2}^{-1}(\mathcal{J}_N)$. It follows from (3.16) and (3.52) that

$$\dim \mathcal{F}_N = \dim \ker Z_{r_2} + \dim \mathcal{J}_N \ge \dim \mathcal{F} - \gamma_d N^{d-1}. \qquad (3.53)$$

If $\psi \in \mathcal{F}_N$, we have $Z_{r_2}\psi \in \mathcal{E}_{N+1}(B_{r_2})$,

$$\|\psi\|_{\mathrm{L}^\infty(B_{r_2})} \le \|\psi - Z_{r_2}\psi\|_{\mathrm{L}^\infty(B_{r_2})} + \|Z_{r_2}, \psi\|_{\mathrm{L}^\infty(B_{r_2})} \qquad (3.54)$$

and hence (3.45) follows from (3.43) and (3.14). □

3.2. **A quantitative unique continuation principle for approximate solutions of the stationary Schrödinger equation.** We state and prove a a version of Bourgain and Kenig's quantitative unique continuation principle [BoK, Lemma 3.10], in which we make explicit the dependence on the parameters relevant to this article. We give a proof following [GK2, Theorem A.1].

Given subsets A and B of $\mathbb{R}^d$, and a function φ on set B, we set $\varphi_A := \varphi\chi_{A\cap B}$. In particular, given $x \in \mathbb{R}^d$ and $\delta > 0$ we write $\varphi_{x,\delta} := \varphi_{B(x,\delta)}$.

Theorem 3.4. *Let Ω be an open subset of $\mathbb{R}^d$ and consider a real measurable function V on Ω with $\|V\|_\infty \le K < \infty$. Let $\psi \in \mathrm{H}^2(\Omega)$ be real valued and let $\zeta \in \mathrm{L}^2(\Omega)$ be defined by*

$$-\Delta\psi + V\psi = \zeta \quad \text{a.e. on} \quad \Omega. \qquad (3.55)$$

Let $\Theta \subset \Omega$ be a bounded measurable set where $\|\psi_\Theta\|_2 > 0$. Set

$$Q(x, \Theta) := \sup_{y\in\Theta} |y - x| \quad \text{for} \quad x \in \Omega. \qquad (3.56)$$

Consider $x_0 \in \Omega \setminus \overline{\Theta}$ such that

$$Q = Q(x_0, \Theta) \ge 1 \quad \text{and} \quad B(x_0, 6Q + 2) \subset \Omega. \qquad (3.57)$$

Then, given

$$0 < \delta \le \min\left\{\operatorname{dist}(x_0, \Theta), \tfrac{1}{24}\right\}, \qquad (3.58)$$

we have

$$\left(\frac{\delta}{Q}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}} + \log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)} \|\psi_\Theta\|_2^2 \le \|\psi_{x_0,\delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2, \qquad (3.59)$$

where $m > 0$ is a constant depending only on d.

We we will apply this theorem with $\delta \ll 1 \ll Q$.

The proof of this theorem is based on the the Carleman-type inequality estimate given in [BoK, Lemma 3.15], [EV, Theorem 2], We state it as in [GK2, Lemma A.5].

Lemma 3.5. *Given $\varrho > 0$, the function $w_\varrho(x) = \varphi(\frac{1}{\varrho}|x|)$ on $\mathbb{R}^d$, where $\varphi(s) := s\,\mathrm{e}^{-\int_0^s \frac{1-\mathrm{e}^{-t}}{t}\,\mathrm{d}t}$, is a strictly increasing continuous function on $[0,\infty[$, C^∞ on $]0,\infty[$, satisfying*

$$\tfrac{1}{C_1\varrho}|x| \le w_\varrho(x) \le \tfrac{1}{\varrho}|x| \quad \text{for} \quad x \in B(0,\varrho), \quad \text{where} \quad C_1 = \varphi(1)^{-1} \in]2,3[. \tag{3.60}$$

Moreover, there exist positive constants C_2 and C_3, depending only on d, such that for all $\alpha \ge C_2$ and all real valued functions $f \in \mathrm{H}^2(B(0,\varrho))$ with $\operatorname{supp} f \subset B(0,\varrho)\setminus\{0\}$ we have

$$\alpha^3 \int_{\mathbb{R}^d} w_\varrho^{-1-2\alpha} f^2\,\mathrm{d}x \le C_3\,\varrho^4 \int_{\mathbb{R}^d} w_\varrho^{2-2\alpha}(\Delta f)^2\,\mathrm{d}x. \tag{3.61}$$

Proof of Theorem 3.4. Let $x_0 \in \Omega \setminus \overline{\Theta}$ satisfy (3.57), where C_1 is defined in (3.60). For convenience we may assume $x_0 = 0$, in which case $\Theta \subset B(0, 2C_1Q)$, and take $\Omega = B(0,\varrho)$, where $\varrho = 2C_1Q + 2$.

Let δ be as in (3.58), and fix a function $\eta \in C_c^\infty(\mathbb{R}^d)$ given by $\eta(x) = \xi(|x|)$, where ξ is an even C^∞ function on $\mathbb{R}$, $0 \le \xi \le 1$, such that

$$\begin{aligned} &\xi(s) = 1 \text{ if } \tfrac{3\delta}{4} \le |s| \le 2C_1Q, \quad \xi(s) = 0 \text{ if } |s| \le \tfrac{\delta}{4} \text{ or } |s| \ge 2C_1Q+1, \\ &\left|\xi^{(j)}(s)\right| \le \left(\tfrac{4}{\delta}\right)^j \text{ if } |s| \le \tfrac{3\delta}{4}, \quad \left|\xi^{(j)}(s)\right| \le 2^j \text{ if } |s| \ge 2C_1Q, \quad j = 1,2. \end{aligned} \tag{3.62}$$

Note that $|\nabla\eta(x)| \le \sqrt{d}\,|\xi'(|x|)|$ and $|\Delta\eta(x)| \le d\,|\xi''(|x|)|$.

We will now apply Lemma 3.5 to the function $\eta\psi$. In what follows C_1, C_2, C_3 are the constants of Lemma 3.5, which depend only on d. By C_j, $j = 4,5,\ldots$, we will always denote an appropriate nonzero constant depending only on d.

Given $\alpha \ge C_2 > 1$ (without loss of generality we take $C_2 > 1$), it follows from (3.61) that

$$\begin{aligned} \frac{\alpha^3}{3C_3\varrho^4} \int_{\mathbb{R}^d} w_\varrho^{-1-2\alpha}\eta^2\psi^2\,\mathrm{d}x &\le \tfrac{1}{3}\int_{\mathbb{R}^d} w_\varrho^{2-2\alpha}(\Delta(\eta\psi))^2\,\mathrm{d}x \le \int_{\mathbb{R}^d} w_\varrho^{2-2\alpha}\eta^2(\Delta\psi)^2\,\mathrm{d}x \\ &+ 4\int_{\operatorname{supp}\nabla\eta} w_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2\,\mathrm{d}x + \int_{\operatorname{supp}\nabla\eta} w_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2\,\mathrm{d}x, \end{aligned} \tag{3.63}$$

where $\operatorname{supp}\nabla\eta \subset \left\{\frac{\delta}{4} \le |x| \le \frac{3\delta}{4}\right\} \cup \{2C_1Q \le |x| \le 2C_1Q+1\}$.

Using (3.55), recalling $\|V\|_\infty \le K$, and noting that $w_\varrho \le 1$ on $\operatorname{supp}\eta$, we get

$$\int_{\mathbb{R}^d} w_\varrho^{2-2\alpha}\eta^2(\Delta\psi)^2\,\mathrm{d}x \le 2K^2\int_{\mathbb{R}^d} w_\varrho^{-1-2\alpha}\eta^2\psi^2\,\mathrm{d}x + 2\int_{\mathbb{R}^d} w_\varrho^{2-2\alpha}\eta^2\zeta^2\,\mathrm{d}x. \tag{3.64}$$

We take

$$\alpha_0 := \alpha\rho^{-\frac{4}{3}} \ge C_4\left(1 + K^{\frac{2}{3}}\right), \tag{3.65}$$

ensuring $\alpha > C_2$ and

$$\frac{\alpha^3}{3C_3\varrho^4} = \frac{\alpha_0^3}{3C_3} \ge 6K^2. \tag{3.66}$$

As a consequence, using (3.60) and recalling (3.56), we obtain

$$\int_{\mathbb{R}^d} w_\varrho^{-1-2\alpha}\eta^2\psi^2\,\mathrm{d}x \ge \left(\frac{\varrho}{Q}\right)^{1+2\alpha}\|\psi_\Theta\|_2^2 \ge (2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2. \tag{3.67}$$

Combining (3.63), (3.64), (3.66), and (3.67), we conclude that

$$\frac{2\alpha_0^3}{9C_3}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \leq 4\int_{\operatorname{supp}\nabla\eta} w_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2\,\mathrm{d}x \\ +\int_{\operatorname{supp}\nabla\eta} w_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2\,\mathrm{d}x + 2\int_{\operatorname{supp}\eta} w_\varrho^{2-2\alpha}\eta^2\zeta^2\,\mathrm{d}x. \tag{3.68}$$

We have

$$\int_{\{2C_1Q\leq|x|\leq 2C_1Q+1\}} w_\varrho^{2-2\alpha}\left(4|\nabla\eta|^2|\nabla\psi|^2+(\Delta\eta)^2\psi^2\right)\mathrm{d}x \\ \leq 4d^2\left(\frac{C_1\varrho}{2C_1Q}\right)^{2\alpha-2}\int_{\{2C_1Q\leq|x|\leq 2C_1Q+1\}}\left(4|\nabla\psi|^2+\psi^2\right)\mathrm{d}x \\ \leq C_5\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}\int_{\{2C_1Q-1\leq|x|\leq 2C_1Q+2\}}\left(\zeta^2+(1+K)\psi^2\right)\mathrm{d}x \\ \leq C_5\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}\left(\|\zeta_\Omega\|_2^2+(1+K)\|\psi_\Omega\|_2^2\right), \tag{3.69}$$

where we used an interior estimate (e.g., [GK1, Lemma A.2]). Similarly,

$$\int_{\left\{\frac{\delta}{4}\leq|x|\leq\frac{3\delta}{4}\right\}} w_\varrho^{2-2\alpha}\left(4|\nabla\eta|^2|\nabla\psi|^2+(\Delta\eta)^2\psi^2\right)\mathrm{d}x \\ \leq 16d^2\delta^{-2}\left(4\delta^{-1}C_1\varrho\right)^{2\alpha-2}\int_{\left\{\frac{\delta}{4}\leq|x|\leq\frac{3\delta}{4}\right\}}\left(4|\nabla\psi|^2+\psi^2\right)\mathrm{d}x \\ \leq C_6\delta^{-2}\left(4\delta^{-1}C_1\varrho\right)^{2\alpha-2}\int_{\{|x|\leq\delta\}}\left(\zeta^2+(K+\delta^{-2})\psi^2\right)\mathrm{d}x \\ \leq C_6\delta^{-2}\left(16\delta^{-1}C_1^2Q\right)^{2\alpha-2}\left(\|\zeta_\Omega\|_2^2+(K+\delta^{-2})\|\psi_{0,\delta}\|_2^2\right). \tag{3.70}$$

In addition,

$$\int_{\operatorname{supp}\eta} w_\varrho^{2-2\alpha}\eta^2\zeta^2\,\mathrm{d}x \leq \left(4\delta^{-1}C_1\varrho\right)^{2\alpha-2}\|\zeta_\Omega\|_2^2 \leq \left(16\delta^{-1}C_1^2Q\right)^{2\alpha-2}\|\zeta_\Omega\|_2^2. \tag{3.71}$$

Thus, if we have

$$\alpha_0^3\left(\tfrac{8}{5}\right)^{2\alpha}\|\psi_\Theta\|_2^2 \geq C_7(1+K)\|\psi_\Omega\|_2^2, \tag{3.72}$$

we obtain

$$C_5\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}(1+K)\|\psi_\Omega\|_2^2 \leq \tfrac{1}{2}\frac{2\alpha_0^3}{9C_3}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2, \tag{3.73}$$

so we conclude that

$$\frac{\alpha_0^3}{9C_3}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \leq C_8\delta^{-2}\left(16\delta^{-1}C_1^2Q\right)^{2\alpha-2}\left((K+\delta^{-2})\|\psi_{0,\delta}\|_2^2+\|\zeta_\Omega\|_2^2\right), \tag{3.74}$$

where we used (3.58). Thus,

$$\alpha_0^3Q^2\left((8C_1Q)^{-1}\delta\right)^{2\alpha}\|\psi_\Theta\|_2^2 \leq C_9\left((K+\delta^{-2})\|\psi_{0,\delta}\|_2^2+\|\zeta_\Omega\|_2^2\right), \tag{3.75}$$

which implies

$$\alpha_0^3Q^4\left(\frac{\delta}{Q}\right)^{4\alpha+4}\|\psi_\Theta\|_2^2 \leq C_{10}\left((1+K)\|\psi_{0,\delta}\|_2^2+\delta^2\|\zeta_\Omega\|_2^2\right), \tag{3.76}$$

since $\frac{\delta}{Q} \leq \frac{1}{24} < \frac{1}{8C_1}$ by (3.58).

We now choose α. Requiring (3.65), to satisfy (3.72) it suffices to also require

$$\alpha \geq C_{11}\left(1+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right). \tag{3.77}$$

Thus we can satisfy (3.65) and (3.72) by taking

$$\alpha = C_{12}\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right). \tag{3.78}$$

Combining with (3.76), and recalling $Q \geq 1$, we get

$$\begin{aligned}\left(1+K^{\frac{2}{3}}\right)^3\left(\frac{\delta}{Q}\right)^{C_{13}\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)} & \|\psi_\Theta\|_2^2 \\ & \leq C_{14}\left((1+K)\|\psi_{0,\delta}\|_2^2+\delta^2\|\zeta_\Omega\|_2^2\right),\end{aligned} \tag{3.79}$$

and hence,

$$\left(\frac{\delta}{Q}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)}\|\psi_\Theta\|_2^2 \leq \|\psi_{0,\delta}\|_2^2+\delta^2\|\zeta_\Omega\|_2^2, \tag{3.80}$$

where $m > 0$ is a constant depending only on d. □

We will apply Theorem 3.4 to approximate eigenfunctions of Schrödinger operators defined on a box Λ with Dirichlet boundary condition. In this case the second condition in (3.57) seems to restrict the application of Theorem 3.4 to sites $x_0 \in \Lambda$ sufficiently far away from the boundary of Λ. But, as noted in [GK2, Corollary A.2], in this case Theorem 3.4 can be extended to sites near the boundary of Λ as in the following corollary.

Corollary 3.6. *Consider the Schrödinger operator* $H_\Lambda := -\Delta_\Lambda + V$ *on* $\mathrm{L}^2(\Lambda)$*, where* $\Lambda = \Lambda_L(x_0)$ *is the open box of side* $L > 0$ *centered at* $x_0 \in \mathbb{R}^d$*,* Δ_Λ *is the Laplacian with either Dirichlet or periodic boundary condition on* Λ*, and* V *a is bounded potential on* Λ *with* $\|V\|_\infty \leq K < \infty$*. Let* $\psi \in \mathcal{D}(\Delta_\Lambda)$ *and fix a bounded measurable set* $\Theta \subset \Lambda$ *where* $\|\psi_\Theta\|_2 > 0$*. Set* $Q(x,\Theta) := \sup_{y\in\Theta}|y-x|$ *for* $x \in \Lambda$*, and consider* $x_0 \in \Lambda \setminus \overline{\Theta}$ *such that* $Q = Q(x_0,\Theta) \geq 1$*. Then, given* $0 < \delta \leq \min\left\{\operatorname{dist}(x_0,\Theta), \frac{1}{24}\right\}$ *such that* $B(x_0,\delta) \subset \Lambda$*, we have*

$$\left(\frac{\delta}{Q}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi\|_2}{\|\psi_\Theta\|_2}\right)}\|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2+\delta^2\|H_\Lambda\psi\|_2^2, \tag{3.81}$$

where $m > 0$ *is a constant depending only on* d*.*

This corollary is proved exactly as [GK2, Corollary A.2].

3.3. **Two and three dimensional Schrödinger operators.**

Theorem 3.7. *Let* H *be a Schrödinger operator as in* (1.1)*, where* $d = 2,3$*. Given* $E_0 \in \mathbb{R}$*, there exists* L_{d,V_∞,E_0} *such that for all* $0 < \varepsilon \leq \frac{1}{2}$*, open boxes* $\Lambda = \Lambda_L$ *with* $L \geq L_{d,V_\infty,E_0}\left(\log\frac{1}{\varepsilon}\right)^{\frac{3}{8}}$*, and energies* $E \leq E_0$*, we have*

$$\eta_\Lambda([E,E+\varepsilon]) \leq \frac{C_{d,V_\infty,E_0}}{\left(\log\frac{1}{\varepsilon}\right)^{\frac{4-d}{8}}}. \tag{3.82}$$

Proof. Let $\Lambda = \Lambda_L(x_0)$ for some $x_0 \in \mathbb{R}^d$ and $\varepsilon \in]0, \frac{1}{2}]$. Since $\sigma(H_\Lambda) \subset [-V_\infty, \infty[$, it suffices to consider $E_0 \geq -V_\infty - 1$ and $E \in [-V_\infty - 1, E_0]$. We set $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$; note that $\operatorname{Ran} P \subset \mathcal{D}(\Delta_\Lambda) \subset \mathcal{H}^2(\Lambda)$ and

$$\|(H_\Lambda - E)\,\psi\| \leq \varepsilon\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P. \tag{3.83}$$

Moreover, for $\psi \in \operatorname{Ran} P$ we have

$$\begin{aligned} \|\psi\|_\infty &= \left\| \mathrm{e}^{-(H_\Lambda + V_\infty)} \mathrm{e}^{(H_\Lambda + V_\infty)} \psi \right\|_\infty \\ &\leq \left\| \mathrm{e}^{-(H_\Lambda + V_\infty)} \right\|_{\mathrm{L}^2(\Lambda) \to \mathrm{L}^\infty(\Lambda)} \left\| \mathrm{e}^{(H_\Lambda + V_\infty)} \psi \right\| \leq C_d \mathrm{e}^{E_0 + V_\infty + 1} \|\psi\|, \end{aligned} \tag{3.84}$$

where we used that for $t > 0$

$$\left\| \mathrm{e}^{-t(H_\Lambda + V_\infty)} \right\|_{\mathrm{L}^2(\Lambda) \to \mathrm{L}^\infty(\Lambda)} \leq \left\| \mathrm{e}^{t\Delta_\Lambda} \right\|_{\mathrm{L}^2(\Lambda) \to \mathrm{L}^\infty(\Lambda)} \leq \left\| \mathrm{e}^{t\Delta} \right\|_{\mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^\infty(\mathbb{R}^d)} < \infty. \tag{3.85}$$

Since $P\,(H_\Lambda - E)\,\psi = (H_\Lambda - E)\,P\psi = (H_\Lambda - E)\,\psi$ for $\psi \in \operatorname{Ran} P$, we conclude that

$$\|(H_\Lambda - E)\,\psi\|_\infty \leq \varepsilon\, C_{d,V_\infty,E_0}\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P. \tag{3.86}$$

Suppose now that

$$\eta_\Lambda\left([E, E + \varepsilon]\right) = \tfrac{1}{L^d} \operatorname{tr} P \geq \rho > 0. \tag{3.87}$$

We recall that $\operatorname{tr} P \leq C_{d,V_\infty,E_0} L^d$, and hence we must have

$$\rho \leq C_{d,V_\infty,E_0}. \tag{3.88}$$

We fix $0 < R < L$, to be selected later, and pick $\mathcal{G} \subset \Lambda$ such that

$$\overline{\Lambda} = \bigcup_{y \in \mathcal{G}} \overline{\Lambda}_R(y) \quad \text{and} \quad \#\mathcal{G} \in \left[\left(\tfrac{L}{R}\right)^d, \left(\tfrac{2L}{R}\right)^d \right] \cap \mathbb{N}. \tag{3.89}$$

We take

$$N = \left\lfloor 2 \left(\tfrac{1}{\gamma_d} \rho \right)^{\frac{1}{d-1}} R^{\frac{d}{d-1}} \right\rfloor \quad \text{with } \gamma_d \text{ as in (3.8)}, \tag{3.90}$$

where R (and L) are large enough so $N \geq 1$. (Since we want to prove (3.82), we may assume a lower bound for ρ, say $\rho \geq \log \frac{1}{\varepsilon}$.) Using (3.87), (3.86), and applying Theorem 3.1 in succession at the sites in $\mathcal{G}$, we conclude that there exists a vector subspace $\mathcal{F}_R$ of $\operatorname{Ran} P$ and $r_0 = r_0(d, V_\infty, E_0) > 0$, such that

$$\dim \mathcal{F}_R \geq \rho L^d - \gamma_d N^{d-1} \left(\tfrac{2L}{R}\right)^d \geq \tfrac{1}{2} \rho L^d, \tag{3.91}$$

and for all $\psi \in \mathcal{F}_R$ and $y \in \mathcal{G}$ we have

$$|\psi(y + x)| \leq \left(C_{d,V_\infty,E_0}^{N+1} \left((N+1)!\right)^{d-2} |x|^{N+1} + \varepsilon C_{d,V_\infty,E_0} \right) \|\psi\| \quad \text{if} \quad |x| < r_0. \tag{3.92}$$

We let Q_R denote the orthogonal projection onto $\mathcal{F}_R$. Since $\operatorname{tr} Q_R = \dim \mathcal{F}_R$, it follows from (3.91) that that we can find a box $\Lambda_1 = \Lambda_1(x_1) \subset \Lambda$ such that $\operatorname{tr} Q_R \chi_{\Lambda_1} \geq \frac{1}{3}\rho$ (we can take $\frac{1}{2}\rho$ if $L \in \mathbb{N}$). But $Q_R = Q_R P = P Q_R$ since $\mathcal{F}_R \subset \operatorname{Ran} P$, and hence

$$\tfrac{1}{3}\rho \leq \operatorname{tr} Q_R \chi_{\Lambda_1} = \operatorname{tr} \chi_{\Lambda_1} P Q_R \chi_{\Lambda_1} \leq \left\| \chi_{\Lambda_1} P \right\|_1 \left\| Q_R \chi_{\Lambda_1} \right\|. \tag{3.93}$$

We now recall that $\left\| \chi_{\Lambda_1} P \right\|_1 = \left\| P \chi_{\Lambda_1} \right\|_1 \leq C_{d,V_\infty,E_0}$ for all Λ. (This is [S1, Theorem B.9.2] when $\Lambda = \mathbb{R}^d$. But by an argument similar to (3.85) the crucial estimate [S1, Eq. (B11)] holds on finite boxes Λ with constants uniform in Λ, so a careful reading of the proof of [S1, Theorem B.9.2] shows that the result holds on finite boxes Λ with constants uniform in Λ.). We thus conclude that

$$\|Q_R \chi_{\Lambda_1}\| = \|Q_R \chi_{\Lambda_1}\| \geq C'_{d,V_\infty,E_0}\rho, \tag{3.94}$$

so there exists $\psi_0 = Q_R\psi_0$ with $\|\psi_0\| = 1$ such that

$$\|\chi_{\Lambda_1}\psi_0\| \geq \gamma\rho, \quad \text{where} \quad \gamma = \tfrac{1}{2}C'_{d,V_\infty,E_0} > 0. \tag{3.95}$$

(Note that $\gamma\rho < \|\psi\| = 1$.)

We then pick $y_0 \in \mathcal{G}$ such that

$$\tfrac{R}{2} \leq \operatorname{dist}(y_0, \Lambda_1(x_1)) \leq R\sqrt{d} \tag{3.96}$$

Taking $0 < \delta < \frac{1}{24}$, it follows from Corollary 3.6 that

$$\left(\tfrac{\delta}{\sqrt{d}R}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(R^{\frac{4}{3}}-\log\left\|\psi_0\chi_{\Lambda_1(x_1)}\right\|_2\right)} \left\|\psi_0\chi_{\Lambda_1(x_1)}\right\|_2^2 \leq \left\|\psi_0\chi_{B(y_0,\delta)}\right\|_2^2 + \delta^2\varepsilon^2, \tag{3.97}$$

with a constant $m = m_d > 0$ and $K = \|V - E\|_\infty$. Making $\delta \leq r_0$ and using (3.92) and (3.95), we get

$$\left(\tfrac{\delta}{\sqrt{d}R}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(R^{\frac{4}{3}}-\log(\gamma\rho)\right)} (\gamma\rho)^2 \leq C_d C_{d,V_\infty,E_0}^{N^2}\delta^{2(N+1)+d} + C_{d,V_\infty,E_0}\,\varepsilon^2. \tag{3.98}$$

Using $\rho R^d \geq 2^{-(d-1)}\gamma_d$ (it follows from (3.90) and $N \geq 1$), and noting that $\delta \leq d^{-\frac{1}{2}}R$, we get

$$\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \leq C_{d,V_\infty,E_0}^{N^2}\delta^{2N} + C_{d,V_\infty,E_0}\,\varepsilon^2, \tag{3.99}$$

with $M = M_{d,V_\infty,E} > 0$. We now choose δ by

$$C_{d,V_\infty,E_0}^{N}\delta^2 = \tfrac{\delta}{R}, \quad \text{i.e.,} \quad \delta = \left(C_{d,V_\infty,E_0}^{N}R\right)^{-1}, \tag{3.100}$$

obtaining

$$\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \leq \left(\tfrac{\delta}{R}\right)^{N} + C_{d,V_\infty,E_0}\,\varepsilon^2, \tag{3.101}$$

We now take $d = 2, 3$ and take R large enough so that

$$\left(\tfrac{\delta}{R}\right)^{N} \leq \tfrac{1}{2}\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}}, \tag{3.102}$$

which can always be done since for $d = 2, 3$ we have $\frac{4}{3} < \frac{d}{d-1}$, so

$$MR^{\frac{4}{3}} < N \leq \left\lfloor 2\left(\tfrac{1}{\gamma_d}\rho\right)^{\frac{1}{d-1}} R^{\frac{d}{d-1}}\right\rfloor \quad \text{if} \quad \rho > C_{d,V_\infty,E_0}R^{\frac{d-4}{3}}. \tag{3.103}$$

(Here we need $d < 4$, that is, for $d = 2$ and $d = 3$ since we assumed $d \geq 2$.)

It follows from (3.101) and (3.102) that taking R large enough we have

$$\tfrac{1}{2}\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \leq C_{d,V_\infty,E_0}\,\varepsilon^2. \tag{3.104}$$

Recalling (3.100),(3.90), and (3.88), we get

$$\left(C_{d,V_\infty,E_0}^{N}R^2\right)^{-MR^{\frac{4}{3}}} \leq 2C_{d,V_\infty,E_0}\,\varepsilon^2. \tag{3.105}$$

We now choose R by

$$\rho = c_{d,V_\infty,E_0}R^{\frac{d-4}{3}}, \tag{3.106}$$

where the constant c_{d,V_∞,E_0} is chosen so (3.103) holds. Combining (3.90), (3.105), and (3.106), we get

$$\mathrm{e}^{-M'R^{\frac{8}{3}}} = \mathrm{e}^{-M'R^{\frac{d-4}{3(d-1)}+\frac{d}{d-1}+\frac{4}{3}}} \le C_{d,V_\infty,E_0}\,\varepsilon^2, \tag{3.107}$$

where $M' = M'_{d,V_\infty,E_0}$. Thus

$$\log\tfrac{1}{\varepsilon} \le C_{d,V_\infty,E_0} R^{\frac{8}{3}} = \frac{C'_{d,V_\infty,E_0}}{\rho^{\frac{8}{4-d}}}, \tag{3.108}$$

and hence

$$\rho \le C_{d,V_\infty,E_0}\left(\log\tfrac{1}{\varepsilon}\right)^{-\frac{4-d}{8}}, \tag{3.109}$$

as long as L is large enough, namely $L \ge L_{d,V_\infty,E_0}\left(\log\tfrac{1}{\varepsilon}\right)^{\frac{3}{8}}$. □

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