# Chapter 8

# Logarithms and Exponentials: $\log x$ and $e^x$

These two functions are ones with which you already have some familiarity. Both are introduced in many high school curricula, as they have widespread applications in both the scientific and financial worlds. In fact, as recently as 50 years ago, many high school mathematics curricula included considerable study of "Tables of the Logarithm Function" ("log tables"), because this was prior to the invention of the hand-held calculator. During the Great Depression of the 1930's, many out-of-work mathematicians and scientists were employed as "calculators" or "computers" to develop these tables by hand, laboriously using difference equations, entry by entry! Here, we are going to use our knowledge of the Fundamental Theorem of Calculus and the Inverse Function Theorem to develop the properties of the Logarithm Function and Exponential Function. Of course, we don't need tables of these functions any more because it is possible to buy a hand-held electronic calculator for as little as $10.00, which will compute any value of these functions to 10 decimal places or more!

## 8.1 The Logarithm Function

Define $\log(x)$ (which we shall be thinking of as the natural logarithm) by the following:

**Definition 8.1**

$$\log(x) = \int_1^x \frac{1}{t} dt \quad \textit{for } x > 0.$$

**Theorem 8.1** *$\log x$ is defined for all $x > 0$. It is everywhere differentiable, hence continuous, and is a 1-1 function. The Range of $\log x$ is $(-\infty, \infty)$.*

Proof: Note that for $x > 0$, $\log x$ is well-defined, because $1/t$ is continuous on the interval $[1, x]$ (if $x > 1$) or $[x, 1]$ (if $0 < x < 1$). Since continuous functions on closed, bounded intervals are integrable, the integral of $1/t$ over $[1, x]$ or over $[x, 1]$ is well-defined and finite. Next, by the Fundamental Theorem of Calculus in Chapter 6,

$$\frac{d}{dx} \log x = 1/x > 0,$$

so $\log x$ is increasing (Why?).

We postpone the proof of the statement about the Range of $\log x$ until a bit later.

**Theorem 8.2 (Laws of Logarithms)** *(from which we shall subsequently derive the famous "Laws of Exponents"): For all positive $x, y$,*

1. $\log xy = \log x + \log y$
2. $\log 1/x = -\log x$
3. $\log x^r = r \log x$ for rational $r$.
4. $\log \dfrac{x}{y} = \log x - \log y$.

Proof: To prove (1), fix $y$ and compute

$$\frac{d}{dx} \log xy = \frac{1}{xy} \frac{d}{dx}(xy) = \frac{1}{xy} y = \frac{1}{x} = \frac{d}{dx} \log x.$$

Then $\log xy$ and $\log x$ have the same derivative, from which it follows by the Corollary to the Mean Value Theorem that these two functions differ by a constant:

$$\log xy = \log x + c.$$

To evaluate $c$, let $x = 1$. Since $\log 1 = 0$, (why?) $c = \log y$, which proves (1).

To prove (2), we use the same idea:

$$\frac{d}{dx}\log\frac{1}{x} = \frac{1}{1/x}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{1/x}(-1/x^2) = -\frac{1}{x} = -\frac{d}{dx}\log x,$$

from which it follows (why?) that

$$\log\frac{1}{x} = -\log x + c.$$

Again, to evaluate $c$, let $x = 1$, and observe that $c = 0$, which proves (2).

To prove (3),

$$\frac{d}{dx}\log x^r = \frac{1}{x^r}\frac{d}{dx}x^r = \frac{1}{x^r}rx^{r-1} = \frac{r}{x} = r\frac{d}{dx}\log x.$$

It follows that

$$\log x^r = r\log x + c,$$

and letting $x = 1$, we observe $c = 0$, which proves (3). (Why did we need to require $r$ to be rational? Why didn't this prove the theorem for all real $r$?)[1]

(4) Follows from (1) and (2).

**Theorem 8.3 (Postponed Theorem)** *Range*$(\log x) = (-\infty, \infty)$.

Proof: First observe that $1/2 < \log 2 < 1$. This follows from the fact that $1/2 < 1/t < 1$ on $(1, 2)$, so

$$\frac{1}{2} = \int_1^2 \frac{1}{2}dt < \int_1^2 \frac{1}{t}dt < \int_1^2 1dt = 1.$$

Now observe that since $\log x$ is monotone increasing in $x$, to compute $\lim_{x\to 0^+}\log x$ it suffices[2] to compute the limit along a subsequence of $x$'s of our choice, and we choose $x_n = 1/2^n$, $n = 1, 2, \ldots$ .

$$\lim_{x\to 0^+}\log x = \lim_{n\to\infty}\log\left(\frac{1}{2^n}\right) = \lim_{n\to\infty} -n\log 2 = -\infty.$$

[1] Because the corollary after Chain Rule only proved differentiation for rational exponents. After we develop properties of the Exponential Function we will be able to extend (3) to arbitrary real numbers $r$.

[2] See Exercise 1.

Similarly,

$$\lim_{x\to\infty} \log x = \lim_{n\to\infty} \log 2^n = \lim_{n\to\infty} n \log 2 = +\infty.$$

**Exercise 1** a. Prove: if $f$ is monotonic, then $\lim_{x\to a^+} f(x)$ exists if and only if $\lim_{n\to\infty} f(a+1/2^n)$ exists, and if either of these limits exists,

$$\lim_{x\to a^+} f(x) = \lim_{n\to\infty} f(a + 1/2^n).$$

b. What if $f$ is not monotonic? Construct a counter-example: find a continuous function $f$ on $(0, 1)$ such that $\lim_{n\to\infty} f(1/2^n)$ exists, but $\lim_{x\to 0^+} f(x)$ does not.

**Exercise 2** a. Evaluate $\displaystyle\int \frac{2x}{1+x^2} dx$.

b. Evaluate $\displaystyle\int \frac{f'(x)}{f(x)} dx$.

## 8.2 The Exponential Function

### Definition of the Exponential Function

We define the Exponential Function $\exp(x)$ as the **inverse** of the Logarithm:

**Definition 8.2**

$$y = \exp(x) \quad \textit{if and only if} \quad x = \log y.$$

From this definition, we can see that exp is defined for *all real* $x$ since Domain(exp) = Range(log) $= (-\infty, \infty)$. Since Range(exp)=Domain(log)$= (0, +\infty)$, it follows that $exp(x) > 0$ for all $x$.

Since exp is the inverse of log, it follows that

$$\exp(\log x) = x \quad \text{for all } x > 0$$

and

$$\log(\exp(x)) = x \quad \text{for all } x.$$

Further, note that since $\log x$ is continuous on $(0, \infty)$ and strictly increasing, it follows (from an exercise near the end of Chapter 6) that $\exp(x)$ is continuous and strictly increasing.

Let $e = \exp(1)$, and note that for $r$ rational,

$$\log(e^r) = r \log e = r \log(\exp(1)) = r = \log(\exp(r)).$$

But $\log x$ is a 1-1 function, so

$$\log(e^r) = \log(\exp(r)) \implies e^r = \exp(r),$$

for all rational $r$.

Further, since $\exp(x)$ is continuous, it makes sense to extend the definition of $e^x$ to

$$e^x = \exp(x)$$

for all $x$. Note that before this, we did not have a meaning for, e.g.,

$$e^{\sqrt{2}}.$$

We can now also define $a^b$ for any $a > 0$ : Whatever $a^b$ is, it must obey the Law of Logarithms,

$$\log(a^b) = b \log a.$$

But exp and log are inverses, so

$$a^b = \exp(\log a^b) = e^{\log a^b} = e^{b \log a},$$

which we take to be our definition of $a^b$ :

**Definition 8.3** $a^b = e^{b \log a}$, *for all* $a > 0$ *and all real* $b$.

**Example 1**

$$\sqrt{2}^{\sqrt{2}} = e^{\sqrt{2} \log \sqrt{2}} = e^{\frac{\sqrt{2}}{2} \log 2}.$$

**Exercise 3** Prove there is only **one** continuous function, up to multiplicative constant, that obeys the Laws of Logarithms, by showing the following:

1. Suppose $f(x)$ is continuous and obeys the Laws of Logarithms. Let $\lambda = f(e)$. Then prove
$$f(e^r) = \lambda r$$
for $r$ rational.

2. Prove that $\{e^r : r \in \mathcal{Q}^1\}$ is a dense set[3] in $(0, \infty)$

3. Conclude that $f(x) = \lambda \log(x)$ for all $x \in (0, \infty)$.

**Laws of Exponents**

Since $e^x$ and $\log x$ are inverses to one another, the Laws of Logarithms will translate to Laws of Exponents:

**Theorem 8.4 (Laws of Exponents)** *If $x$ and $y$ are real numbers, and $r$ is rational, then*

$$\begin{array}{ll} 1. & e^{xy} = e^x e^y \\ 2. & e^{-x} = \dfrac{1}{e^x} \\ 3. & (e^x)^r = e^{xr} \ \textit{for rational } r. \end{array}$$

Proof: This is left as Exercise 4.

**Exercise 4** Prove the Laws of Exponents. Hint: make use of the fact that $\log x$ is a 1-1 function.

**Derivatives of the Exponential Function**

We already know, from the Inverse Function Theorem, that $e^x$ is differentiable for every $x$. To compute its derivative, write
$$y = e^x$$

[3]Recall, a set $C$ of real numbers is **dense** in $(0, \infty)$ if for every $a, b$ such that $0 < a < b$ there exists $c \in C$ such that $a < c < b$.

and take inverses:

$$x = \log y.$$

Now differentiate both sides with respect to $x$, using Chain Rule on the right-hand side:

$$1 = \frac{d}{dx}(x) = \frac{d}{dx}(\log y) = \frac{1}{y}y',$$

and therefore

$$y' = y = e^x.$$

In other words,

$$\frac{d}{dx}e^x = e^x.$$

One conclusion of this is that since the derivative of $e^x$ is $e^x$, an everywhere differentiable function, it follows that $e^x$ is **infinitely differentiable**, that is, it has a $n$-th derivative, for every $n$.

### Taylor's Theorem and the Exponential Function

Now that we can differentiate $e^x$, we can compute a **Taylor Series expansion** for $e^x$ about $x = 0$, as follows:

Note that $e^0 = 1$, so that

From the differentiability of $f(x) = e^x$, we can compute first a **linear approximation** to $e^x$ by

$$f(x) \approx f(0) + f'(0)x,$$

which we saw already from Chapter 6. Applying that here, we obtain

$$e^x \approx e^0 + e^0 x = 1 + x. \tag{8.1}$$

Another way of obtaining equation (1) is by l'Hopital's rule:

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1,$$

so

$$e^x - 1 \approx^4 x,$$

or

$$e^x \approx 1 + x.$$

Now, apply l'Hopital's rule to $e^x - (1+x)$:

$$\lim_{x \to 0} \frac{e^x - (1+x)}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2},$$

so

$$e^x - (1+x) \approx \frac{x^2}{2},$$

or

$$e^x \approx 1 + x + \frac{x^2}{2}.$$

Because $e^x$ is infinitely differentiable, we can continue this process indefinitely, and obtain

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots . \tag{8.2}$$

Later, in Chapter 8 on Taylor Series and Power Series, we shall see that "$\approx$" can actually be replaced by "$=$" in (8.2) above, and the resulting equation is true *for all $x$:*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots . \tag{8.3}$$

**Exercise 5** Assume equation (8.3) holds with $x$ everywhere replaced with $z$, and $z$ understood to be an arbitrary *complex* number. Recall that for a complex number $z = x + iy$, where $x$ and $y$ are real numbers, $\overline{z} = x - iy$ is the complex conjugate of $z$, and that $z\overline{z} = x^2 - i^2y^2 = x^2 + y^2 = |z|^2$.

a. Prove $|e^{i\theta}| = 1$ for all real numbers $\theta$.

[4] We say "$f(x) \approx g(x)$ for $x$ near 0", if

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1.$$

b. Use the formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and equation (8.3) above to derive the formulas

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \dots.$$

Hint: consider $e^{i\theta} + e^{-i\theta}$.

## 8.3 Rates of Growth

One very useful characteristic of a function $f(x)$ defined at least for $(a, \infty)$ is to study how fast $f(x)$ grows as $x \to \infty$. This is only interesting in the case where $\lim_{x\to+\infty} f(x) = \infty$.

Def: Suppose $f(x)$ and $g(x)$ are both defined on some interval $(a, \infty)$. We say $f$ **grows faster than** $g$ (equivalently: $g$ **grows slower than** $f$) if

$$\lim_{x\to+\infty} \frac{g(x)}{f(x)} = 0.$$

In this case, we use the notation

$$g(x) \ll f(x) \quad \text{as} \quad x \to \infty.$$

We say $f$ **and** $g$ **are of the same order of magnitude** if

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L,$$

where $0 < L < \infty$. In this case we write

$$f(x) \approx g(x).$$

Note: First, observe that if $p(x)$ and $q(x)$ are two polynomials of degree $n$, then $p(x) \approx q(x)$. This follows from the fact that $p(x) = a_n x^n +$ terms of lower degree and $q(x) = b_n x^n +$

terms of lower degree, where both $a_n$ and $b_n$ are not 0 (for otherwise the polynomials would not be of degree $n$.) Then[5]

$$\lim_{x\to\infty}\frac{p(x)}{q(x)}=\lim_{x\to\infty}\frac{a_nx^n+\dots}{b_nx^n+\dots}=\lim_{x\to\infty}\frac{a_n+\mathcal{O}(1/x)}{b_n+\mathcal{O}(1/x)}=\frac{a_n}{b_n}.$$

Now we can discuss the relative rates of growth of the logarithm, the exponential, and polynomials. In essence, the result which we state and prove, is that log grows more slowly than any polynomial, or indeed, any positive fractional power of $x$, and $e^x$ grows faster than any polynomial.

**Theorem 8.5** *Let $p(x)$ be any polynomial. Then*

$$\lim_{x\to+\infty}\frac{p(x)}{e^{\alpha x}}=0$$

*for any $\alpha>0$, i.e. $e^{\alpha x}$ grows faster than any polynomial.*

Proof: If $p(x)$ is of degree $n$, apply l'Hopital's rule $n+1$ times.

**Theorem 8.6** *Let $\alpha>0$. Then*

$$\lim_{x\to+\infty}\frac{\log x}{x^\alpha}=0,$$

*i.e.* $\log x$ *grows slower than any positive fractional power of $x$.*

Proof: Apply l'Hopital's rule.

**Corollary 8.7** $\log x$ *grows more slowly than any polynomial $p(x)$.*

Proof: See Exercise 7.

**Exercise 6** Prove that $\approx$ is an equivalence relation[6] on the set of all polynomials. What are the equivalence classes into which the relation partitions the set?

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[5] We say a function $f(x)$ is $\mathcal{O}(g(x))$ as $x\to\infty$ if there is an $L$ and an $M$ such that $|f(x)/g(x)|\le L$ for all $x\ge M$. In other words, $f$ is of at most the order of $g$.

[6] An equivalence relation on a set $A$ is a subset $R$ of the cartesian product, $A\times A$, with the following properties: (We write "$xRy$ for $(x,y)\in R$.")

**Exercise 7** Prove that $\log x$ grows more slowly than any polynomial $p(x)$.

**Exercise 8** a. Prove that the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous for all $x$. Hint: handle the cases $x = 0$, $x \neq 0$, separately. We shall refer henceforth to this extended function as $e^{-1/x^2}$.

b. Prove that for any positive integer exponent $n$,

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^n} = 0.$$

(Hint: a change of variables will prove helpful.) Then use this to prove

c. $e^{-1/x^2} \in C^\infty(-\infty, \infty)$. Hint: handle the cases $x = 0$, $x \neq 0$ separately.

d. What is the value of the $n$-th derivative of $e^{-1/x^2}$ at $x = 0$?

**Exercise 9** Compute each of the following limits:

a. $\lim\limits_{n \to \infty} (1 + \frac{1}{n})^n$

b. $\lim\limits_{n \to \infty} (1 + \frac{r}{n})^n$

c. $\lim\limits_{n \to \infty} (1 + \frac{r}{n})^{nt}$

**Exercise 10** Prove: If $a_n \to 0$ and $a_n b_n \to \lambda$, then $\lim\limits_{n \to \infty} (1 + a_n)^{b_n} \to e^\lambda$. Hint: prove first that for every $|x| < 1$, there exists a $c$, $0 < |c| < |x|$ such that

$$\log(1 + x) = \frac{x}{1 + c}.$$

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1. $xRx$ for every $x \in A$. (reflexivity)
2. $xRy \Longrightarrow yRx$. (symmetry)
3. $xRy$ and $yRz$ implies $xRz$. (transitivity)

The **equivalence classes** are the *disjoint sets* that $A$ is divided into: An equivalence class $C$ is a non- empty subset of $A$ with the property that for any $c \in C$ and all $a \in A$, $a \in C$ if and only if $cRa$.

## 8.4 Compound Interest and Exponential Growth

In this section we relate the issue of **compound interest** to exponential growth. Suppose that we deposit $M$ dollars in the bank, which agrees to pay interest at the rate $r$ per time period (think months or years, perhaps). Then the amount that the deposit will earn during the period is $Mr$, and hence at the end of the interest period, the account will be worth

$$M(1+r)$$

dollars. Now suppose that we leave the money in the account for a second interest period, and the interest rate is again $r$. Then the $M(1+r)$ dollars will earn $M(1+r)r$ interest, so at the end of the second interest period, the total value of the account will be

$$M(1+r)+M(1+r)r = M(1+r)^2$$

dollars. This is the principal of **compound interest**. In general, after $t$ interest periods, the account value will be

$$M(1+r)^t.$$

Now, suppose that we shorten the interest periods, and increase their number, so that, for example, if the interest rate is $r$ per year, but we compound monthly, an initial deposit of $M$ dollars will be subjected to 12 compounding periods (months), with an interest rate of $r/12$ for each period, and hence at the end of one year $M$ dollars will then be worth

$$M(1+\frac{r}{12})^{12}.$$

We can imagine continuing to shorten the interest period, and increase their number, letting the number of interest periods tend to $\infty$ :

$$\lim_{n\to\infty} M(1+\frac{r}{n})^n = M \lim_{n\to\infty}(1+\frac{r}{n})^n.$$

To evaluate this limit, we use l'Hopital's rule: Take logs, then limits, evaluate, and then exponentiate:

$$\lim_{n\to\infty}\log((1+\frac{r}{n})^n) = \lim_{n\to\infty} n\log(1+\frac{r}{n}) = \lim_{n\to\infty}\frac{\log(1+\frac{r}{n})}{\frac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{(\frac{1}{1+\frac{r}{n}})(-\frac{r}{n^2})}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{r}{1+\frac{r}{n}} = r,$$

so

$$\lim_{n\to\infty}(1+\frac{r}{n})^n = \lim_{n\to\infty} e^{\log((1+\frac{r}{n})^n)} = e^{\lim_{n\to\infty}\log((1+\frac{r}{n})^n)} = e^r.$$

The value of a deposit of $M$ dollars, at the end of one interest period, continuously compounded for the period at the rate of $r$ per period is therefore

$$Me^r,$$

and hence for $t$ periods, the value is

$$Me^{rt}.$$

To reiterate, if we lend $M$ dollars at the interest rate of $r$ per unit time, for $t$ time units, at the end, our $M$ dollars are now worth

$$Me^{rt}.$$

Now, we invert the process. If we ask, how much must we deposit *today* so that at the end of $t$ time units, our money will be worth $M$ dollars, clearly, we must deposit

$$Me^{-rt}.$$

We therefore define, the **net present value** of $M$ dollars at time $t$ (in the future), discounted to today, as

$$Me^{-rt},$$

assuming that the interest rate is constant at $r$.

### 8.4.1 Mortgage payments

Now we are in a position to determine mortgage payments: If we want to borrow (or lend) \$100,000, say, with the payments to be made monthly, say, for the next 30 years (a typical length of mortgage), at the interest rate $r$ per year (typically, today, $r = 8\% = 0.08$), we suppose that the monthly payment is $M$ dollars, and note that we are going to pay (or receive) $M$ at the end of each month, for 360 months. The net present value of the first month's payment, discounted to today, at the interest rate of $r$ per year, or $\frac{r}{12}$ per month, is

$$Me^{-\frac{r}{12}},$$

and the net present value of the second month's payment, discounted to today is

$$Me^{-\frac{2r}{12}},$$

etc. The net present value of the last (i.e. 360-th) payment is

$$Me^{-\frac{360r}{12}}.$$

Therefore the net present value of *all* the 360 payments is

$$\sum_{k=1}^{360} Me^{-\frac{kr}{12}} = M\sum_{k=1}^{360} e^{-\frac{kr}{12}} = Me^{-\frac{r}{12}}\sum_{k=0}^{359} e^{-\frac{kr}{12}} = Me^{-\frac{r}{12}}\frac{1-e^{-\frac{360r}{12}}}{1-e^{-\frac{r}{12}}}$$

When we equate this expression with the amount of the loan

$$L = Me^{-\frac{r}{12}}\frac{1-e^{-\frac{360r}{12}}}{1-e^{-\frac{r}{12}}},$$

and solve for $M$, we have determined the monthly payment, principal and interest. So much for complicated "mortgage rate calculators!"

**Exercise 11** Suppose I borrow \$100,000 in the form of a mortage at 8% for 30 years.

a. What are the monthly payments? [\$735.63]

b. What is the total amount of interest I will have paid? [\$164,826]

c. Answer the same questions, (a) and (b), if the time is shortened to 15 years instead. [\$957.20; \$72,295] Is there a moral to this?

### 8.4.2 The Differential Equation of Exponential Growth: $y' = ky$

Suppose we have a population (of bacteria, of $C^{14}$ radioactive isotopes, of money,...) which changes over time, and which, therefore, we shall regard as a function of time, $P(t)$. And suppose further that **the population grows (or shrinks) in direct proportion to its size**. Such a population satisfies a differential equation of the form

$$P'(t) = kP(t)$$

where $k$ is the **growth rate constant**, and determines the rate at which the population will grow (shrink). We can solve the differential equation by the method of separation of variables,

$$\frac{P'(t)}{P(t)} = k$$

which, on integrating both sides with respect to $t$, yields

$$\log P(t) = kt + c$$

or

$$P(t) = e^{kt+c} = P_0 e^{kt},$$

where $P_0 = P(0)$. This is the equation, or function, of **uninhibited growth**, and equally well descibes, at least over some time interval $[t_0, t_1]$, the weight of bacteria in a petrie dish, money in a bank account, value of a stock option, or weight of $C^{14}$, the radioactive isotope of carbon.

**Exercise 12** Use the method of the section to solve the differential equation for Newton's Law of Cooling:

$$P'(t) = k[P(t) - C]$$

where k and C are some constants.