Chapter 9

The Topology of Metric Spaces

9.1 Abstract Topological Spaces

We undertake a study of metric spaces because we wish to study, among other things the set of continuous functions defined on $\mathcal{R}^n$, and $\mathcal{R}^n$ is a simple instance of a metric space, which we shall shortly define. The topology of a space is of particular interest to us, because the topology of the space and the set of continuous function defined on that space are intimately connected, as we shall soon see.

Definition 9.1 *Let X be a set, and $\mathcal{C}$ a collection of subsets of X. Then $(X, \mathcal{C})$ is called a* **topological space***, and the elements of $\mathcal{C}$ are called the* **open sets** *of X, provided the following hold:*

1. $\emptyset \in \mathcal{C}$.

2. $X \in \mathcal{C}$.

3. *If $X, Y \in \mathcal{C}$ then $X \cap Y \in \mathcal{C}$.*

4. *The union of any number (i.e. finite or infinite number) of elements of $\mathcal{C}$ is again an element of $\mathcal{C}$.*

Note that by induction, (3) implies that the intersection of any *finite* number of elements of $\mathcal{C}$ is an element of $\mathcal{C}$, or said another way: $\mathcal{C}$ is closed under finite intersection, and from (4), $\mathcal{C}$ is closed under arbitrary union.

Often, having exhibited the topological space $(X, \mathcal{C})$, we will often refer to "an open set O in the topological space X", understanding that that means $O \in \mathcal{C}$.

Example 1 Let $X = \{0, 1\}$, that is, a set consisting of two elements. Then if we let $\mathcal{C} = \{\{0\}, \{1\}, X, \emptyset\}$, then $(X, \mathcal{C})$ is a topological space. This is true because (1) and (2) can be verified by inspection. (3) and (4) require that *certain* subsets of X are elements of $\mathcal{C}$, but we have chosen $\mathcal{C}$ to be *all* subsets of $\mathcal{C}$, which make (3) and (4) hold automatically. This reasoning generalizes to the following example:

Example 2 Let X be an arbitrary set, and $\mathcal{C}$ be the set of *all* subsets of X, including both $\emptyset$ and X. Then according to the reasoning of Example 1, $(X, \mathcal{C})$ is a topological space. This is called **the discrete topology for** X**.**

Example 3 Let X be arbitrary, and let $\mathcal{C} = \{\emptyset, X\}$. Then $(X, \mathcal{C})$ is a topological space, and the topology is called **the trivial topology**.

Example 4 [**The Usual Topology for** $\mathcal{R}^1$**.**] Let $X = (-\infty, \infty)$, and let $\mathcal{C}$ consist of all intervals of the form (a, b), the arbitrary union of such intervals, and the intersection of any finite number of elements of $\mathcal{C}$. Then $(X, \mathcal{C})$ is a topological space, and the open sets are just the open sets we studied in Chapter 1. To see this, first note that since $(1, 2) \in \mathcal{C}$ and $(3, 4) \in \mathcal{C}$, $\emptyset = (1, 2) \cap (3, 4) \in \mathcal{C}$. Further,

$$\bigcup_{x \in X} (x - 1, x + 1) \in \mathcal{C}$$

which implies that $X \in \mathcal{C}$. To prove (3), suppose (a, b) and (c, d) are elements of $\mathcal{C}$. Then if $(a, b) \cap (c, d) \neq \emptyset$, either $(a, b) \subset (c, d)$ or $(c, d) \subset (a, b)$ or $a < c < b < d$ or $c < a < d < b$. In each of these four cases the intersection is an interval.[1] The intersection of two arbitrary

[1] In Exercise 1 you are asked to compute the intersection explicitly, for each of the four cases.

unions of intervals is again a union of such intervals, and hence so is the intersection of finitely many such unions.

Example 5 [The Right Order Topology] Let $X = (-\infty, \infty)$, and let $\mathcal{C}$ consist of all intervals of the form $[a, \infty)$, arbitrary unions of such intervals, the empty set, and X. Then $(X, \mathcal{C})$ is a topological space.

We conclude the introduction with a simple and yet powerful theorem about open sets:

Theorem 9.1 *Let $(X, \mathcal{C})$ be a topological space. Then $O \subset X$ is an open set, that is, $O \in \mathcal{C}$, if and only if for every $x \in O$ there exists a $B \in \mathcal{C}$ such that $x \in B \subset O$.*

Proof: If $O \in \mathcal{C}$, then for every $x \in O$, there exists a $B \in \mathcal{C}$ (O will play the role of B) so the theorem is trivially true.

Conversely, if for every $x \in O$ there exists a $B \in \mathcal{C}$ such that $x \in B \subset O$, then indicating the dependence of B upon x by B_x,

$$O \subset \bigcup_{\{x : x \in O\}} B_x \subset O$$

and therefore

$$O = \bigcup_{\{x : x \in O\}} B_x.$$

Since arbitrary unions of sets in $\mathcal{C}$ are again in $\mathcal{C}$, it follows that $O \in \mathcal{C}$.

Continuous Functions on an Arbitrary Topological Space

Definition 9.2 *Let $(X, \mathcal{C})$ and $(Y, \mathcal{C}')$ be two topological spaces. Suppose f is a function whose domain is X and whose range is contained in Y. Then f is* **continuous** *if and only if the following condition is met:*

For every open set O in the topological space $(Y, \mathcal{C}')$, the set $f^{-1}(O)$ is open in the topological space $(X, \mathcal{C})$.

Informally, we say: The inverse image under f of every open set in Y is an open set in X.

If $f : X \to Y$ is continuous we occasionally call f a **mapping from X to Y**.

Note that whether or not a particular function f is continuous depends upon the topologies, that is, what the open sets are, of both the domain and range. This is an important property of continuity.

Later, when we specialize our study of topological spaces to Metric Spaces, we shall see that our $\epsilon - \delta$ definition of continuity and the topological definition of continuity are the same.

A useful way to test continuity of a function is given by the following theorem.

Theorem 9.2 *$f : (X, \mathcal{C}) \to (Y, \mathcal{C}')$ is continuous on X if and only if for every $x \in X$ and every open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset V$.*

Proof: If $f : X \to Y$ is continuous on X, then since the inverse of every open set in Y is open in X, for any $V \in \mathcal{C}'$, $f^{-1}(V)$ is open in X and provides the U. Thus the necessity of the condition is proved.

Conversely, suppose that V is open in Y, and $y \in V$. Then for any $x \in X$ such that $f(x) = y$, by hypothesis there exists an open set U in X, containing x, and such that $f(U) \subset V$. Label the above U by U_x to indicate its dependence upon x, and let

$$O = \bigcup_{\{x : f(x) \in V\}} U_x.$$

Then O is open because it is the union of open sets, and clearly[2] $O = f^{-1}(V)$.

Corollary 9.3 *Let $f : \mathcal{R}^1 \to \mathcal{R}^1$ be any function where $\mathcal{R}^1 = (-\infty, \infty)$ with the usual topology (see Example 4), that is, the open sets are open intervals (a, b) and their arbitrary unions. Then in $\mathcal{R}^1$, f is continuous in the $\epsilon - \delta$ sense if and only if f is continuous in the topological sense.*

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[2] Provide the details. See Exercise 2.

Proof: Suppose f is $\epsilon - \delta$ continuous, and $x_0 \in \mathcal{R}^1$. Let V be an open set in $\mathcal{R}^1$, and suppose $v = f(x_0) \in V$. Since V is open, it is the union of open intervals, and hence $v \in (a, b)$ for some a, b. Then let $\epsilon = \min\{v - a, b - v\}$, and note that $(v - \epsilon, v + \epsilon) \subset (a, b)$. Since f is assumed continuous in the $\epsilon - \delta$ sense, there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$ implies $x \in V$. Letting $U = (x_0 - \delta, x_0 + \delta)$ in Theorem 9.2 shows that f is continuous in the topological sense.

Conversely, suppose f is continuous in the topological sense. Let x be arbitrary, and $y = f(x)$. Let $\epsilon > 0$. Then since $V = (y - \epsilon, y + \epsilon)$ is an open set in the range, by hypothesis there exists an open set U, containing x, in the domain of f, and such that $f(U) \subset V$. But U open implies there exists an interval (a, b), containing x, which is contained in U. That is, $z \in (a, b)$ implies $f(z) \in V$. Now let $\delta = \min\{x - a, b - x\}$ and observe that if $|z - x| < \delta$, then $z \in (a, b)$ and hence $f(z) \in V$. But $f(z) \in V$ implies $|y - f(z)| < \epsilon$. Thus, f is $\epsilon - \delta$ continuous at x.

Exercise 1 Compute the four intersections in Example 1.

The following two exercise indicate the degree with which continuity is connected to the *topology* of the spaces involved:

Exercise 2 In the proof of Theorem 9.2, why is $O = f^{-1}(V)$?

The following exercises show the intimate connection between topology and continuity.

Exercise 3 Let X be an arbitrary set with the discrete topology: $\mathcal{C}$ is the set of all subsets of X. Let $(Y, \mathcal{C}')$ be an arbitrary topological space. Then $f : (X, \mathcal{C}) \to (Y, \mathcal{C}')$ is continuous, for any f !

Exercise 4 Let Y be an arbitrary set endowed with the trivial topology: $\mathcal{C}' = \{\emptyset, Y\}$, and $(X, \mathcal{C})$ an arbitrary topological space. Then $f : (X, \mathcal{C}) \to (Y, \mathcal{C}')$ is continuous, for any f.

Exercise 5 Let $X = (-\infty, \infty)$ and the open sets of X be half-open, half-closed intervals of the form $[a, b)$, for $a < b$, and their arbitrary unions. Let $Y = (-\infty, \infty)$, and the open sets of Y be *arbitrary*. Then prove that $f(x) = [x]$ is continuous(!) [Hint: what does the range of f consist of?]

Limit Points and the Derived Set

Definition 9.3 *Let $(X, \mathcal{C})$ be a topological space, and $A \subset X$. Then $x \in X$ is called a* **limit point** *of the set A provided every open set O containing x also contains at least one point $a \in A$, with $a \neq x$.*

Definition 9.4 *Let $(X, \mathcal{C})$ be a topological space, and $A \subset X$. The* **derived set** *of A, denoted A', is the set of all limit points of A.*

Exercise 6 Let $(X, \mathcal{C})$ be $\mathcal{R}^1$ with the usual topology. Then prove that for any $a < b$, the set of limit points of the interval (a, b) is the interval $[a, b]$.

Exercise 7 Let $(X, \mathcal{C})$ be $\mathcal{R}^1$ with the right order topology, as defined in Example 5. Show that the derived set of the set consisting of the single point a, $\{a\}$, is $(-\infty, a)$. Note that a is not a limit point of $\{a\}$.

Exercise 8 Let $(X, \mathcal{C})$ be $(-\infty, \infty)$ with the trivial topology. Compute the derived set of $\{a\}$.

Exercise 9 Let $(X, \mathcal{C})$ be $(-\infty, \infty)$ with the discrete topology. Compute the derived set of $\{a\}$.

The Closure of a Set; Closed Sets

Definition 9.5 *Let $(X, \mathcal{C})$ be a topological space. Let $A \subset X$. The* **closure** *of A, denoted $\overline{A}$, is defined as the union of A and its derived set, A':*

$$\overline{A} = A \cup A'.$$

Definition 9.6 *Let $(X, \mathcal{C})$ be a topological space. Let $A \subset X$. We say A is* **closed** *if it contains all its limit points.*

The major theorem relating closed sets and open sets is the following:

Theorem 9.4 *A set A in a topological space $(X, \mathcal{C})$ is closed if and only if its complement, A^c, is open.*

Proof: Suppose A is closed, and $x \in A^c$. Then since A contains all its limit points, x is not a limit point of A, that is, there exists an open set O containing x, such that $O \cap A = \emptyset$. Then $x \in O \subset A^c$, and by Theorem 9.1 A^c is open.

Conversely, suppose A^c is open. If $x \in A^c$, then since $A^c \cap A = \emptyset$, x cannot be a limit point of A. Therefore, all limit points of A are contained in A, that is, A is closed.

Theorem 9.5 *The closure of A is closed, for any set A.*

Proof: We prove that the complement is open, which in light of Theorem 9.4, is equivalent. Suppose $x \in (\overline{A})^c = A^c \cap (A')^c$. $x \notin A' \Rightarrow x \in O, \quad O \cap A = \emptyset, \quad O$ some open set. If $O \cap A' \neq \emptyset$, then there exists some limit point a' of A, $a' \in O$. But a' a limit point of A means that every open set containing a' has non-empty intersection with A. But O is an open set that contains a', and does not intersect A, a contradiction. Hence $O \cap A' = \emptyset$. But then $x \in O \subset (\overline{A})^c$, and hence $(\overline{A})^c$ is open.

9.2 Metric Spaces:

9.2.1 Definition of a Metric:

We think of a **metric** as a way of measuring distance between points in a topological space. A metric has certain properties, which we elaborate below. If X is a set and $d(x, y)$ is a metric on X, then the pair (X, d) is called a **metric space**.

Definition 9.7 *A* **metric** *d on a space X is a function $d : X \times X \to [0, \infty)$ with the following properties:*

1. *$d(x, y) \geq 0$ for all $x, y \in X$.*

2. *$d(x, y) = 0$ if and only if $x = y$.*

3. *$d(x, y) = d(y, x)$.*

4. *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (Triangle inequality)*

Note: We shall often write $d(x, y)$ as $||x - y||$ or occasionally as $|x - y|$, when doing so will cause no confusion.

Example 6 $X = \mathcal{R}^1$, and $d(x, y) = |x - y|$, the usual absolute value on $\mathcal{R}^1$. This is the "euclidean metric" on $\mathcal{R}^1$. It is easy to check that it satisfies all four properties of a metric.

Example 7 X is an arbitrary non-empty set, and

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

It is not difficult[3] to verify that this is a metric! In this metric, all points are "far apart."

Example 8 $X = C[0, 1]$, the set of continuous functions on $[0, 1]$, and the metric d is the "sup-norm":

$$d(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)| = ||f - g||_{[0,1]}.$$

To verify that this is a metric, properties (1) and (3) are immediate. Property (2) holds because

$$||f - g||_A = 0 \Longleftrightarrow \sup_{x \in A} |f(x) - g(x)| = 0 \Longleftrightarrow f(x) = g(x) \text{ for all } x \in A \Longleftrightarrow f(x) \equiv g(x).$$

Property (4) was proved in Chapter **.

Exercise 10 Prove that the distance function of Example 7 above is a metric.

[3] See Exercise 10.

9.2.2 The Topology of Metric Space

Definition 9.8 *Let* (X, d) *be a metric space. Let* $a \in X$*. The set*

$$B(a, r) = \{x \in X : d(a, x) < r\} = \{x \in X : ||x - a|| < r\}$$

is called **the ball about** a **of radius** r*. Informally,* $B(a, r)$ *is the set of all points in* X *which are at distance less than* r *from* a*.*

Definition 9.9 *Suppose* (X, d) *is a metric space. Let* $x \in X$*. Then a* **neighborhood of** x*,* N_x *is any set containing* $B(x, \epsilon)$*, for some* $\epsilon > 0$.

Next, we shall show that the *metric of the space induces a topology on the space* so that the metric space (X, d) is also a topological space $(X, \mathcal{C})$, where the elements of $\mathcal{C}$ are determined by the balls of X:

Definition 9.10 *Let* (X, d) *be a metric space. Define a family* $\mathcal{C}$ *of subsets of* X *as follows: A set* $O \subset X$ *is an element of* $\mathcal{C}$ *(we will be thinking of such an* O *as "open") if, for every* $x \in O$ *there exists an* $\epsilon > 0$ *such that* $B(x, \epsilon) \subset O$*. (Note that in general,* ϵ *will depend on* x*.)*

Theorem 9.6 (Metric space is a topological space) *Let* (X, d) *be a metric space. The family* $\mathcal{C}$ *of subsets of* (X, d) *defined in Definition 9.10 above satisfies the following four properties, and hence* $(X, \mathcal{C})$ *is a topological space. The open sets of* (X, d) *are the elements of* $\mathcal{C}$*. We therefore refer to the metric space* (X, d) *as the topological space* (X, d) *as well, understanding the open sets are those generated by the metric* d*.*

1. $\emptyset \in \mathcal{C}$.

2. $X \in \mathcal{C}$.

3. *If* $A, B \in \mathcal{C}$*, then* $A \cap B \in \mathcal{C}$.

4. If $\{O_\alpha : \alpha \in A\}$ is a family of sets in $\mathcal{C}$ indexed by some index set A, then

$$\bigcup_{\alpha \in A} O_\alpha \in \mathcal{C}.$$

Informally, (3) and (4) say, respectively, that $\mathcal{C}$ is closed under finite intersection and arbitrary union.

Exercise 11 Prove Theorem 9.6.

Theorem 9.7 (The ball in metric space is an open set.) *Let (X, d) be a metric space. Then for any $x \in X$ and any $r > 0$, the ball $B(x, r)$ is open.*

Proof: Let $x \in X$ and $r > 0$. Let $y \in B(x, r)$, and let $r_1 = r - ||x - y||$. Since $||x - y|| < r$, $r_1 > 0$. We claim that $B(y, r_1) \subset B(x, r)$. To see this, let $z \in B(y, r_1)$. Then $||z - y|| < r_1 = r - ||x - y||$, so $||x - y|| + ||y - z|| < r$. By Triangle inequality for metric spaces, $||x - z|| \leq ||x - y|| + ||y - z|| < r$. But then $z \in B(x, r)$. Thus, $B(y, r_1) \subset B(x, r)$, so by Theorem 9.1, $B(x, r)$ is open.

Continuous Functions on a Metric Space

We remarked earlier, that our notion of a continuous function from $\mathcal{R}^1$ to $\mathcal{R}^1$ in terms of $\epsilon - \delta$ would carry over to metric space, and now we are in a position to state and prove the theorem:

Theorem 9.8 *Suppose $f : (X, d) \to (Y, d')$ is a function from one metric space to another. Then f is continuous in the topological sense if and only if for every $x \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$f(B(x, \delta)) \subset B(f(x), \epsilon).$$

Proof: Suppose f is continuous in the topological sense. Let $x \in X$ and $\epsilon > 0$. Let $V = B(f(x), \epsilon)$. By Theorem 9.2, since V is open in Y, there exists a U open in X such that

$x \in U$ and $f(U) \subset V$. Since U is open in X and $x \in U$ there exists a ball centered at x and contained in U. Suppose the radius of this ball is δ. That is, $B(x, \delta) \subset U$. But then

$$f(B(x, \delta)) \subset f(U) \subset V = B(f(x), \epsilon),$$

which proves the theorem in the forward direction.[4]

Conversely, suppose O is an open set in (Y, d'), and let $x \in f^{-1}(O)$. Since O is open, there exists a ball $B(f(x), \epsilon) \subset O$ for some $\epsilon > 0$. By assumption, there exists a $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon) \subset O$. So, $B(x, \delta) \subset f^{-1}(O)$, which shows that $f^{-1}(O)$ is open in (X, d).

Exercise 12 Let X be arbitrary and the metric d be that of Example 7 above:

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Determine the family $\mathcal{C}$ of open sets of (X, d).

Definition 9.11 *Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence of elements of X. We say* **the sequence $\{x_n\}$ converges to** *x, and write*

$$\lim_{n \to \infty} x_n = x$$

or

$$x_n \to x$$

if and only if

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} ||x_n - x|| = 0,$$

that is,

$$||x_n - x|| \to 0.$$

[4] It is interesting to note that the translation of this last statement is: "for all y, if $||y - x|| < \delta$ then $||f(y) - f(x)|| < \epsilon$."

Definition 9.12 *Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence of elements of X. We say the sequence $\{x_n\}$ is a* **Cauchy sequence** *if and only if*

For every $\epsilon > 0$ there exists N such that for all $m, n \geq N$,

$$d(x_n, x_m) < \epsilon.$$

Informally, "the sequence is Cauchy in the metric d."

Exercise 13 Suppose $f : (X, d) \to (Y, d')$ is a function from one metric space into another metric space. Then f is continuous if and only if for every $x \in X$, if $x_n \to x$, then $f(x_n) \to f(x)$. That is,

$$\lim_{n\to\infty} x_n = x \Longrightarrow \lim_{n\to\infty} f(x_n) = f(x).$$

Hint: it is easiest to prove the contrapositive in each direction. Use Theorem 9.8.

In the following exercises, we are assuming that X is a metric space, with metric d:

Exercise 14 Let $A \subset X$. Prove $x \in X$ is a limit point of A if and only if for every $\epsilon > 0$ the ball $B(x, \epsilon)$ contains infinitely many points of A.

Exercise 15 Let $A \subset X$. Prove $x \in X$ is a limit point of A if and only if there exists a sequence $\{x_n\}$, $x_n \in A$, $x_n \neq x$, and $||x_n - x|| \to 0$.

Exercise 16 Prove that in $\mathcal{R}^2$, the closure of the ball

$$B((0, 0), 1) = \{(x, y) : x^2 + y^2 < 1\}$$

is the set

$$\{(x, y) : x^2 + y^2 \leq 1\}.$$

Exercise 17 For the topological space (X, d) of Example 7, determine all the limit points of X. (Answer: none). Alternatively, prove X has no limit points.

Exercise 18 Let $(\mathcal{R}^1, d)$ be the real numbers with the "usual" distance metric. Determine all the limit points of the set of rationals, $\mathcal{Q}$.

Exercise 19 Prove that in $\mathcal{R}^2$ there exists a countable family $\mathcal{B}$ of open balls which form a basis for all the open sets in the topology of the space: every open set O in the topology of $\mathcal{R}^2$ can be written as the (necessarily countable) union of sets from $\mathcal{B}$. Hint: prove that the set of all balls whose centers have rational coordinates, and whose radii are rational, meet the requirements.

A topological space with such a countable family $\mathcal{B}$ of open sets is called **second countable**. (Note that your proof would generalize to $\mathcal{R}^n$ as well, and hence $\mathcal{R}^n$ is second countable as well.)

Theorem 9.9 *If A and B are closed, then $A \cup B$ is closed. Hence the union of any finite number of closed sets is closed.*

Exercise 20 Prove Theorem 9.9.

Theorem 9.10 *If A is an index set[5], and $\{G_\alpha : \alpha \in A\}$ is a family of closed sets, then the intersection*

$$\bigcap_{\alpha \in A} G_\alpha$$

is closed. Restated: the intersection of an arbitrary number of closed sets is closed.

Proof:

$$\left\{ \bigcap_{\alpha \in A} G_\alpha \right\}^c = \bigcup_{\alpha \in A} (G_\alpha)^c,$$

which, by Theorem 9.4, is open. QED.

Definition 9.13 *The* **interior** *of A, denoted A^0 is defined as follows:*

$$A^0 = \{a \in A : B(x, \epsilon) \subset A \;\; \textit{for some } \epsilon > 0\}.$$

[5] Just think of A as a set of indices, such as (in the simple case) $A = \{1, 2, 3, \ldots\}$.

Exercise 21 In $\mathcal{R}^1$ with the usual topology, what is the interior of $[0,1]$? What is the interior of $\mathcal{Q}$?

Definition 9.14 *The* **exterior** *of A is defined as the interior of A^c.*

Definition 9.15 *The* **boundary** *of A is the set of points $x \in X$ which lie in neither A^0 nor the exterior of A. It is denoted ∂A.*

Theorem 9.11 *The boundary of A is the set of $x \in X$ for which every open set containing x contains both points of A and points of A^c.*

Proof: See Exercise 23.

Example 9 In $\mathcal{R}^2$, let D be the "closed" unit disc[6],

$$D = \{(x,y) : x^2 + y^2 \leq 1\}.$$

Then D^0 is the "open" disc $B((0,0),1)$:

$$D^0 = \{(x,y) : x^2 + y^2 < 1\}.$$

Also, the exterior of D,

$$(D^c)^0 = \{(x,y) : x^2 + y^2 > 1\}.$$

Finally, the boundary of D,

$$\partial D = \{(x,y) : x^2 + y^2 = 1\}.$$

Proof: Suppose $(x,y) \in D$. If $x^2+y^2 < 1$ (so that $(x,y) \in B(0,1)$), by Theorem 9.7 ("The ball is open"), there exists[7] an $\epsilon > 0$ such that $B((x,y),\epsilon) \subset B(0,1)$. Hence $B(0,1) \subset D^0$.

On the other hand, if $x^2 + y^2 = 1$ and $0 < \epsilon < 1/2$,

$$\left(\frac{x}{1-\epsilon}, \frac{y}{1-\epsilon}\right) \in \{B(0,1)\}^c \cap B((x,y),2\epsilon)$$

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[6]Exercise 16 justifies the name.

[7]Challenge: can you construct ϵ?

while

$$((1-\epsilon)x, (1-\epsilon)y) \in B(0,1) \cap B((x,y), 2\epsilon)$$

which shows[8] that $(x, y) \in \partial D$.

Definition 9.16 *A is* **dense** *in B if* $B \subset \overline{A}$. *If A is dense in X (the entire space), we say A is* **dense**.

In each of the following assume (X, d) is a metric space.

Exercise 22 Give two proofs that $x_n \to x \in O$ and O open imply there exists N so that $n \geq N \Longrightarrow x_n \in O$.

Exercise 23 Prove Theorem 9.11.

Exercise 24 Give an example of two *closed* sets A and B in $\mathcal{R}^2$ such that the distance between A and B,

$$d(A, B) = \inf\{d(a, b) : a \in A, \quad b \in B\} = 0,$$

but

$$A \cap B = \emptyset.$$

9.2.3 Continuous Functions on a Metric Space: Three Theorems

One of our major purposes for studying Topology is to understand the deep connection between continuous functions and the topological structure of the space. In fact, we proved three major theorems about continuous functions in Chapter 3, the Boundedness Theorem, the Extreme Value Theorem, and the Intermediate Value Theorem. In this Chapter we shall show that these are really properties about continuous functions on an *arbitrary* metric space, by proving these three theorems in this setting. We proceed with our study.

[8]Work out the details to verify these two statments. Mathematics can be alot of hard work.

Exercise 25 Prove that in a metric space, the distance function is continuous: Let (X, d) be a metric space, and $a \in X$. Define $f(x) = d(a, x) = ||x - a||$. Then prove that f is continuous: $X \to \mathcal{R}^1$.

Example 10 [Closed Graph Theorem] Let f be a continuous real-valued function defined on some interval $[a, b]$. Then in $\mathcal{R}^2$, the graph of f is a closed set.

Proof: We show that the complement is open. Let (x_0, y_0) be in the complement of the graph of f. Then $y_0 \neq f(x_0)$, and since f is continuous on $[a.b]$, there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \dfrac{|y_0 - f(x_0)|}{2}$. Then the set

$$\left\{(x, y) : |x - x_0| < \delta \text{ and } |y - y_0| < \frac{|y_0 - f(x_0)|}{2}\right\}$$

contains no points of the graph of f. This set is a rectangle centered at (x_0, y_0). Thus there is a ball centered at (x_0, y_0) contained in the rectangle, which therefore is in the complement of the graph of f, and hence the graph of f is closed.

Euclidean Space: $\mathcal{R}^n$

Let $\mathcal{R}^n = \mathcal{R}^1 \times \mathcal{R}^1 \times \ldots \times \mathcal{R}^1$, where we take the Cartesian product of $\mathcal{R}^1$ with itself n times. The objects $x \in \mathcal{R}^n$ are n-tuples: $x = (x_1, x_2, \ldots, x_n)$ where each $x_i \in \mathcal{R}^1$.

The Euclidean distance between points in $\mathcal{R}^n$ is given by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}.$$

It can be verified that this distance function is a norm. Triangle inequality follows from an inequality which we won't prove here, called the Cauchy-Schwartz inequality.

Complete Metric Spaces

Definition 9.17 *We say a metric space (X, d) is* **complete** *if every Cauchy sequence (in the metric d) converges to some element of the space.*

Compactness; Compact Sets

Definition 9.18 *Let A be a set in a topological space X. A* **cover of** *A is a family $\mathcal{F}$ of subsets of X with the property that the union of the sets in $\mathcal{F}$ contains A:*

$$A \subset \bigcup_{B \in \mathcal{F}} B.$$

A cover is called an **open cover** *if every set B in the family $\mathcal{F}$ is an open set.*

Definition 9.19 *Suppose $\mathcal{F}$ is a cover of the set A. A* **subcover** *of $\mathcal{F}$ is a family $\mathcal{F}' \subset \mathcal{F}$, which is also a cover of A. If, in addition, $\mathcal{F}'$ contains only finitely many sets, we call it a* **finite subcover***.*

Definition 9.20 *A set A in a topological space $(X, \mathcal{C})$ is called* **compact** *if every open cover of A has a finite subcover. If X itself is compact, we say X* **is a compact space.**

Example 11 $\mathcal{N}$ is not compact in $\mathcal{R}^1$. To see this, let $O_n = (n - 1/4, n + 1/4)$, for $n = 1, 2, \ldots$. Then $\mathcal{N} \subset \cup_{n=1}^{\infty} O_n$, since $n \in O_n$ for every $n \in \mathcal{N}$. But if $\mathcal{N}$ were compact, finitely many of the O_n would cover $\mathcal{N}$. But each O_n contains only one integer! Hence the natural numbers would be finite, a contradiction.

Example 12 A finite set in $\mathcal{R}^1$ is compact.

Suppose the finite set $E = \{x_1, \ldots, x_k\}$ is covered by $\cup_\alpha O_\alpha$. Then $x_i \in O_{\alpha_i}, 1 \leq i \leq k$, and hence the finite collection $\{O_{\alpha_1}, \ldots, O_{\alpha_k}\}$ covers E.

Theorem 9.12 *If f is a continuous mapping from the topological space $(X, \mathcal{C})$ to the topological space $(Y, \mathcal{C}')$ and $A \subset X$ is compact, then $f(A)$ is compact in $(Y, \mathcal{C}')$.*

Proof: To show $f(A)$ is compact, suppose $\mathcal{O}$ is an open cover of $f(A)$. We need to produce a finite subcover. Since each set $B \in \mathcal{O}$ is open, $f^{-1}(B)$ is open in X. Then

$$A \subset \bigcup_{B \in \mathcal{O}} f^{-1}(B) \quad \text{(why?)},$$

so $\bigcup_{B \in \mathcal{O}} f^{-1}(B)$ is an open cover of A. But A is compact, so finitely many of

$$\{f^{-1}(B) : B \in \mathcal{O}\},$$

call them $f^{-1}(B_1), f^{-1}(B_2), \ldots, f^{-1}(B_n)$, form an open cover of A. But then $B_1, B_2, \ldots, B_n$ form an open cover of $f(A)$.

We continue to explore the properties of continuous functions in metric space. We shall show that certain properties of sets and spaces are inherited under mapping by a continuous function. The idea is explained in the following definition:

Definition 9.21 *Suppose f is a continuous function from one topological space X to another, Y. If it is the case that some property possessed by a set A is then possessed by the (continuous) image of A under the continuous function f, we say that the property is a* **continuous invariant***.*

In the terminology just introduced, the previous theorem can be restated:

Theorem *Compactness is a continuous invariant.*

Definition 9.22 *A set A in a metric space (X, d) is* **bounded** *if there exists an $x_0 \in X$ and $r > 0$ such that $A \subset B(x_0, r)$.*

Theorem 9.13 *Let $f : (X, \mathcal{C}) \to (Y, d)$ be a continuous function from a topological space into a metric space. Let A be a compact set in X. Then $f(A)$ is bounded in (Y, d).*

Exercise 26 Prove Theorem 9.13.

Corollary 9.14 *In a metric space, a compact set is bounded.*

Proof: Let A be compact in (X, d). Note that the identity map $i : (X, d) \to (X, d)$ is continuous. (Why?)[9] Hence A is bounded in (X, d).

[9] See Exercise 27 to see that this is not "obvious".

Exercise 27 Let X be an arbitrary set and $\mathcal{C}$ be the set of all subsets of X. Let $\mathcal{C}' = \{\emptyset, X\}$. Then $(X, \mathcal{C})$ is the discrete topology, while $(X, \mathcal{C}')$ is the trivial topology. Prove that if X consists of at least two points, then the identity map $i : (X, \mathcal{C}') \to (X, \mathcal{C})$ is NOT continuous.

Theorem 9.15 *In a metric space, a compact set is closed.*

Proof: Let A be compact, and $a \notin A$. We show that a is not a limit point of A, and hence A is closed. Let

$$E_n = \left\{ x \in X : d(a,x) > \frac{1}{n} \right\} \quad n = 1, 2, \ldots$$

and observe that each E_n is open and

$$A \subset \cup_{n=1}^{\infty} E_n.$$

But A compact and the E_n's are *nested* implies that

$$A \subset E_{n_0},$$

for some n_0. Since every point of A is at distance greater than $1/n_0$ from a, $B(a, 1/n)$ is a ball about a which does not intersect A. Hence a is not a limit point of A.

Theorem 9.16 *In the usual topology on* $\mathcal{R}^1$*, a closed bounded interval* $[a,b]$ *is compact.*

Proof: Suppose we have proven that $[0,1]$ is compact in the usual topology on $\mathcal{R}^1$. Then the map $f(x) = a + (b-a)x$ is continuous (for any number of reasons), and hence the image of $[0,1]$, namely $[a,b]$, is compact in the usual topology on $\mathcal{R}^1$.

We are thus reduced to considering the case of $[0,1]$. Let $\bigcup_{\alpha} O_\alpha$ be an open cover of $[0,1]$, and suppose there exists no finite subcover. Let

$$E = \{x \in [0,1] : \text{ the closed interval } [0,x] \text{ can be covered by a finite number of the open sets } O_\alpha\}$$

and note that $0 \in E$. Let $s = \sup E$. It follows from the Approximation Theorem for sup's that for any $x < s$, the closed interval $[0,x]$ can be covered by a finite number of the open sets. But $s \in [0,1)$ implies that $s \in O_{\alpha_0}$ for some α_0, and since $\mathcal{R}^1$ is a metric space, $s \in O_{\alpha_0}$

implies that for some $\epsilon > 0$, $(s-\epsilon, s+\epsilon) \subset O_{\alpha_0}$. But then from the remarks about the Approximation Theorem, it follows that $[0, s-\epsilon/2]$ can be covered by a finite number of the open sets in the cover, which, together with O_{α_0}, provide a finite subcover of $[0, s+\epsilon]$, a contradiction.

Theorem 9.17 (Heine-Borel Theorem) *A set in $\mathcal{R}^1$ is compact if and only if it is closed and bounded.*

Proof: We have already proved that a compact set is closed and bounded. The converse will be proved if we prove the following theorem, since a closed and bounded set in $\mathcal{R}^1$ is a closed subset of an interval $[a, b]$.

Theorem 9.18 *A closed subset of a compact set is compact.*

Proof: Let G be a closed subset of the compact set C. Let $\{O_\alpha\}$ be an open cover of G. Then the family $\{O_\alpha\}$, together with G^c, form an open cover of X, and hence an open cover of C. Hence finitely many elements of the cover cover C. Since G is a subset of C, this finite subcover also covers G. G^c is not needed in the subcover to cover G since it is disjoint from G. Hence finitely many of the $\{O_\alpha\}$ cover G.

Exercise 28 In the proof of Theorem 9.16, what happens to the argument if $s = 0$? Why must $s > 0$?

Theorem 9.19 (Bolzano-Weierstrass) *Let A be a compact subset of a metric space X. Then every sequence of elements of A has a convergent subsequence (to an element of A).*

Proof: Let A be compact and $\{x_n\}$ be a sequence in A.

Case I: Suppose there exists an $a \in A$ such that for every $r > 0$, $B(a, r)$ contains x_n for infinitely many n. Then for each $k \in \mathcal{N}$ there exists[10] an n_k so that $n_1 < n_2 < \ldots < n_k$ and

$$|a - x_{n_k}| < \frac{1}{k}.$$

[10] Provide the details. Some care needs to be taken so that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Then $x_{n_k} \to a$.

<u>Case II</u>: If no such $a \in A$ exists, for every $a \in A$ there exists an $r_a > 0$ such that $B(a, r_a)$ contains x_n for only finitely many n. In this case

$$\bigcup_{a \in A} B(a, r_a) \quad \text{covers } A$$

and A is compact, so finitely many of these open balls cover A:

$$A \subset \bigcup_{i=1}^{m} B(a_i, r_{a_i}).$$

But each of these balls contains x_n for only finitely many n, which implies that $\mathcal{N}$ is finite, a contradiction.

Theorem 9.20 (Extreme Value Theorem) *Let f be a continuous function from (X, d) to $\mathcal{R}^1$. If A is a compact subset of X, then $f(A)$ has a maximum: that is, there exists an $a \in A$ such that $f(x) \le f(a)$ for all $x \in A$.*

Proof: By Theorem 9.13, $f(A)$ is bounded, so $t = \sup_{x \in A} f(x) < \infty$. For every $n \in \mathcal{N}$ there exists an $x_n \in A$ such that $f(x_n) > t - 1/n$. Hence, by Bolzano-Weierstrass, there exists a convergent subsequence $x_{n_k} \to x_0$. But

$$t - \frac{1}{n_k} < f(x_{n_k}) \le t, \quad k = 1, 2, \ldots$$

and the Sandwich Theorem imply[11]

$$f(x_0) = t.$$

Exercise 29 Give an alternate proof that a compact set in a metric space is bounded, using an open cover with balls.

Countable Compactness

Definition 9.23 *A space X is called* **countably compact** *if every infinite subset has a limit point.*

[11] Provide the details.

Theorem 9.21 *If X is compact, it is countably compact.*

Proof: Suppose A is an infinite subset of X with no limit point. Let $\{x_n\}$ be an infinite set of distinct elements of A. Since A has no limit point, it follows that $\{x_n\}$ has no limit point, as well. Hence, for each n, there exists an $r_n > 0$ such that $B(x_n, r_n)$ contains no points of $\{x_n\}$ other than x_n itself. For any other $y \in X - \{x_n\}$, there is also a $r_y > 0$ such that $B(y, r_y)$ contains no points of $\{x_n\}$. Let

$$O = \bigcup_{y \in X - \{x_n\}} B(y, r_y).$$

Then

$$X = \bigcup_{n=1}^{\infty} B(x_n, r_n) \cup O.$$

But X is compact, so finitely many of the above open sets cover X. But each contains at most one point of $\{x_n\}$, and there were infinitely many distinct x_n's, a contradiction.

The following theorem shows that completeness is a consequence of compactness. In fact, the completeness of $\mathcal{R}^1$ is "due" to the fact that each point in $\mathcal{R}^1$ has a compact neighborhood (called **local compactness**).

Theorem 9.22 *If X is a compact metric space, then X is complete: every Cauchy sequence converges.*

Proof: Let $\{x_n\}$ be a Cauchy sequence. Suppose first that there is some x_0 such that for infinitely many n, $x_n = x_0$. Then it is not difficult to conclude[12] that $x_n \to x_0$, so the Cauchy sequence converges.

On the other hand, if there is no such x_0, there are infinitely many different x_n, and by the preceding theorem that a compact space is countably compact, the set $\{x_n\}$ has a limit point, call it a. It is not difficult to conclude[13] that in this case, $x_n \to a$.

Exercise 30 Complete the proof that if $\{x_n\}$ is a Cauchy sequence for which infinitely many elements are equal to x_0, then $x_n \to x_0$.

[12] As usual, provide the details. See Exercise 30.

[13] Exercise 31.

Exercise 31 Complete the proof that if a is a limit point of the Cauchy sequence $\{x_n\}$, then $x_n \to a$.

Example 13 A subset A of a metric space which is closed and bounded but not compact.

We show this by showing there is a bounded sequence which contains no convergent subsequence. Hence it fails to have the Bolzano-Weierstrass property, and hence cannot be compact.

Let X be the set of all sequences of real numbers $\{a_n\}$ for which $\sum(a_n)^2 < \infty$. This set is known as ℓ^2, (pronounced: "little L-2"), the space of all square-summable sequences. We take as the metric on ℓ^2:

$$d(x, y) = \left(\sum(x_i - y_i)^2\right)^{1/2}.$$

It can be shown that d satisfies triangle inequality (but we won't do that here.)

Now let $x_n \in \ell^2$ be the sequence $< 0, 0, 0, \ldots, 0, 1, 0, \ldots >$ where all the terms in the sequence are 0 except for the n-th term, which is 1. We compute

$$d(x_m, x_n) = \begin{cases} \sqrt{2} & \text{if } m \neq n \\ 0 & \text{if } m = n \end{cases}$$

Now we consider the set A of sequences $\{x_n\}$. Since the distance between any two terms in the sequence is $\sqrt{2}$, there cannot be any convergent subsequence. Hence the set A cannot be compact. If a were a limit point of A, then every ball $B(a, \epsilon)$ would contain infinitely many points of A, which would imply by triangle inequality that for two different x_m and x_n, the distance between them was less than 2ϵ. But we already observed that the distance between any two points of A is $\sqrt{2}$, and if we choose $\epsilon = 1$, say, we obtain a contradiction. Thus, there are no limit points of A, and hence A is closed. Further, since $||x_n|| = 1$, it follows that $A \subset B(0, 2)$, hence A is bounded.

Connectedness; The Intermediate Value Theorem

Definition 9.24 *We say a set C in a topological space X is* **connected** *if $C = A \cup B$, A, B open in X, and $A \cup B = \emptyset$ imply that either $A = \emptyset$ or $B = \emptyset$. That is, C is connected if* *it cannot be written as the disjoint union of two non-empty open sets.*

Theorem 9.23 *In $\mathcal{R}^1$, any interval $[a, b]$, with $a < b$, is connected.*

This requires proof. Not suprisingly, in light of our remarks above, the proof will contain elements similar to the proof of the Intermediate Value Theorem, in Chapter 3.

Proof: Suppose $I = [a, b]$ and $I = A \cup B$, A and B open and disjoint. We reason to a contradiction: Suppose by symmetry[14] that $b \in B$. Since B is open, there is an interval $(b - \epsilon, b + \epsilon) \subset B$, for some $\epsilon > 0$. Now let $t = \sup A$, and observe that $a < t < b$ (why cannot $t = a$?). If $t \in A$, then since A is open, there exists an interval $(t - \delta, t + \delta) \subset A$, contradicting $t = \sup A$. Hence $t \in B$. But B is open, and $t \in B$ implies $(t - \delta, t + \delta) \subset B$ for some $\delta > 0$. Then, the Approximation Property for sup's provides an increasing sequence $\{a_n\} \to t$, $a_n \in A$. This contradicts the fact that the interval $(t - \delta, t + \delta)$ is a subset of B.

Theorem 9.24 (Connectedness is a continuous invariant) *If f is a continuous function, $f : X \to Y$, and X is connected, then the image $f(X)$ will be connected in Y.*

Proof: Suppose not. Then $f(Y)$ can be written as the disjoint union of two non-empty open sets:

$$f(Y) = A \cup B.$$

But A open in Y implies $f^{-1}(A)$ is open in X, and similarly for $f^{-1}(B)$. Then

$$X = f^{-1}(A) \cup f^{-1}(B),$$

which shows X fails to be connected as well.

Corollary 9.25 (Intermediate Value Theorem on $\mathcal{R}^1$) *If f is a real-valued continuous function on $[a, b]$ and $f(a) < f(b)$, then for any $c \in [f(a), f(b)]$ there exists an $x \in [a, b]$ such that*

$$f(x) = c.$$

[14] If $b \in A$, merely reverse the roles of A and B in the argument.

<u>Proof</u>: The closed interval $[a, b]$ is connected, by Theorem 9.23. Therefore, by Theorem 9.24 the image is connected. That is,

$$[f(a), f(b)] \subset Range(f).$$