# 1 Fractions from an advanced point of view

We are going to study fractions from the viewpoint of "modern algebra", or "abstract algebra". Our goal is to develop a deeper understanding of what

$$\frac{a}{b}$$

and

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

mean. One consequence of our deeper understanding will be why the rule: "invert and multiply" is correct, when we write

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}.$$

Let's start with the problem of understanding what the fraction $\dfrac{3}{4}$ "means". Often, we teach this by introducing the idea of a "pie" which we divide into 4 equal parts, which we call "quarters" of the pie. Then the fraction $\dfrac{3}{4}$ means three quarters (of the pie), which we can represent symbolically as

$$\frac{3}{4} = 3 \times \frac{1}{4}.$$

This equation is not an empty phrase! We are saying that the fraction "three-fourths" is three of something, the something is that number which when you have four of them, you have 1, or said another way, is that number $x$ which solves the equation

$$4x = 1.$$

Just exactly how we solve this equation will be discussed in some detail, below. Of course, we already "know" how to solve it, just divide both sides by 4. But that begs the question, because if we knew how to divide both sides by 4, we would know how to divide 3 by 4, and thus what the meaning of 3/4 is. That's what we are going to address.

The set of real numbers forms a **field**. The **rational numbers** are a subset of the real numbers, which subset itself is a field. It is the field properties which allow division to take place, and tell us (via the axioms for a field) what division means, and thus what 3/4 means.

The axioms for a field $\mathcal{F}$ are listed at the end of this section. There are too many to talk about right here. That's better left for a graduate course in Modern Algebra, or Abstract Algebra. We'll just talk about the axioms we need for our work now.

Axiom of Commutativity

When we multiply two (or several) numbers together, the **order** in which we multiply them doesn't matter. That is

$$xy = yx$$

for every $x$ and $y$ in $\mathcal{F}$.

Axiom of Multiplicative Identity

There exists an element $e$ in $\mathcal{F}$ such that for all $x$ in $\mathcal{F}$,

$$ex = x.$$

We call such an element a **multiplicative identity**. This element can be proven to be unique[1], that is, there is only one such element in a field $\mathcal{F}$. We usually denote the multiplicative identity by the symbol "1".

Axiom of Multiplicative Inverse

For every $x$ in $\mathcal{F}$, if $x \neq 0$, then there exists a $y$ in $\mathcal{F}$ such that

$$xy = 1.$$

We call such an element $y$ a **multiplicative inverse** of $x$, and usually denote it by

$$x^{-1} \quad \text{or} \quad \frac{1}{x}.$$

It is important to note that "-1" here is not an exponent. It is merely a symbol that is conveniently chosen. We could have chosen $x^*$ as the symbol for inverse, instead. When we write $x^{-1}$ or $\frac{1}{x}$ we understand the symbol stands for **that number which when multiplied by $x$, yields 1.**

To return to our earlier consideration, we were interested in the $x$ which solved the equation

$$4x = 1.$$

Now we know that such a $x$ is the multiplicative inverse of 4, namely $4^{-1}$, or $1/4$.

So, then, the fraction $\frac{3}{4}$ stands for $3 \times 4^{-1}$, where by the axioms, there is a number $4^{-1}$ which is the multiplicative inverse of 4. In short,

$$\frac{3}{4} = 3 \times \left(\frac{1}{4}\right).$$

[1] A proof appears in the section dealing with the Axioms.

Now we can understand what we mean by division. It is really *multiplication by the multiplicative inverse.*

### 1.0.1 $\left(\frac{c}{d}\right)^{-1} = \left(\frac{d}{c}\right)$

Let's see what $x^{-1}$ is, when $x$ is a fraction, say $(c/d)$:

$\left(\frac{c}{d}\right)^{-1}$ is that number which when multiplied by $\frac{c}{d}$ equals 1. But $\frac{d}{c}$ is such a number:

$$\begin{aligned}
\left(\frac{c}{d}\right)\left(\frac{d}{c}\right) &= \left(c\frac{1}{d}\right)\left(d\frac{1}{c}\right) = \left(cd^{-1}\right)\left(dc^{-1}\right) && (1)\\
&= c\left(d^{-1}d\right)c^{-1} && (2)\\
&= c\left(dd^{-1}\right)c^{-1} && (3)\\
&= c\,(1)\,c^{-1} && (4)\\
&= cc^{-1} = 1, && (5)
\end{aligned}$$

which means that $\left(\frac{d}{c}\right)$ is the[2] multiplicative inverse of $\left(\frac{c}{d}\right)$, that is,

$$\left(\frac{c}{d}\right)^{-1} = \frac{1}{\frac{c}{d}} = \frac{d}{c}.$$

To justify the steps in equations (1) to (5), above: In equation (1), we used what a fraction $c/d$ stands for. In going from (1) to (2), we used the *associative property* of multiplication, namely, that we can re-group the factors as we like. This is Multiplication Axiom 3, below. In going from (2) to (3) we used the *commutative property* of multiplication, namely that $ab = ba$, Multiplication Axiom 2. Then going from (3) to (4) we used the definition of multiplicative identity, that $dd^{-1} = 1$, and finally in going from (4) to (5), we used the property of multiplicative identity, and the associativity of multiplication twice.

Recall, we said that division was really multiplication by the multiplicative inverse:

$$\frac{3}{4} = 3 \times \frac{1}{4} = 3 \times 4^{-1}.$$

The same holds true for fractions as well:

[2] Actually, even this needs to be proved, that there is only *one* multiplicative inverse for an element in a field. That requires that the group multiplication be commutative. Otherwise, it is possible to have many different **left-inverses**, and many different **right-inverses**. That's another topic for another time. See Section 1.3 below for a proof.

$$\left(\frac{\frac{a}{b}}{\frac{c}{d}}\right) = \frac{a}{b} \times \frac{1}{\frac{c}{d}} = \frac{a}{b}\left(\frac{c}{d}\right)^{-1} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right),$$

that is, the rule for fractions is "invert and multiply"!

While we're at it, let's justify another rule for evaluating $\frac{\frac{a}{b}}{\frac{c}{d}}$:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \times 1 = \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{a}{b}\frac{d}{c}}{1} = \frac{a}{b}\frac{d}{c}.$$

The reason this works is because we are multiplying both the numerator and denominator of the big fraction by the *multiplicative inverse of the denominator*, so automatically the denominator becomes 1.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{1}{\frac{c}{d}} = \frac{a}{b}\left(\frac{c}{d}\right)^{-1} = \frac{a}{b}\frac{d}{c}.$$

### 1.0.2 $\frac{a}{b}\frac{d}{c} = \frac{ad}{bc}$

The justification for this step is quite a bit of work: First let's observe that the multiplicative inverse of $bc$ is $c^{-1}b^{-1}$. That's not a given! To see this, we have to show that when we multiply $bc$ by $c^{-1}b^{-1}$, we get 1:

$$\begin{aligned}
\left(c^{-1}b^{-1}\right)(bc) &= c^{-1}\left(b^{-1}b\right)c && (6)\\
&= c^{-1}(1)\,c && (7)\\
&= c^{-1}c && (8)\\
&= 1. && (9)
\end{aligned}$$

You might want to take a moment and see if you can think of the justifications necessary in Equation (6), and then in going from (6) to (7), to (8), etc. The answers are at the end of this handout.

Consequently, from the definition of fractions,

$$\begin{aligned}
\frac{ad}{bc} &= (ad)\,(bc)^{-1} && (10)\\
&= (ad)\left(c^{-1}b^{-1}\right) && (11)\\
&= \left(ab^{-1}\right)\left(dc^{-1}\right) && (12)\\
&= \left(\frac{a}{b}\right)\left(\frac{d}{c}\right), && (13)
\end{aligned}$$

where we used associativity and commutativity several times in going from (11) to (12).

## 1.1 Axioms for a Field $\mathcal{F}$

We are given a set $\mathcal{F}$, together with two operators on $\mathcal{F}$ which we shall think of as addition and multiplication, and so we shall label them as " + " and " $\times$ ". The elements of $\mathcal{F}$ will satisfy the following axioms:

### 1.1.1 Axioms for addition

There is a binary operation on the set $\mathcal{F}$, called "+", with the following properties:

1. For every $x, y$ in $\mathcal{F}$, $x + y$ is also an element of $\mathcal{F}$. (Axiom of Closure.)

2. For every $x, y$ in $\mathcal{F}$,
$$x + y = y + x.$$
(Commutativity of addition).

3. For every $x, y, z$ in $\mathcal{F}$,
$$(x + y) + z = x + (y + z).$$
(Associativity of addition).

4. There exists an element $e$ in $\mathcal{F}$ with the property that
$$e + x = x$$
for every $x$ in $\mathcal{F}$. We call this element an **additive identity for** $\mathcal{F}$. (Notice that we haven't assumed that there is only one such additive identity. That will follow from the axioms[3].) We shall call the additive identity "0".

[3] Here is a proof: Suppose that there were two additive identities, that is, two elements, 0 and $0^*$, such that for all $x$, $0 + x = x$, and $0^* + x = x$. Then
$$0 + 0^* = 0^*$$
since 0 is an additive identity, but
$$0^* + 0 = 0$$
since $0^*$ is also an additive identity. Since addition is commutative,
$$0^* + 0 = 0 + 0^*$$
from which it then follows that
$$0 = 0^*.$$

5. For every element $x$ in $\mathcal{F}$ there exists an element $y$ in $\mathcal{F}$ such that

$$x + y = 0.$$

   $y$ is called an **additive inverse** for $x$, and it will also be shown later, as with identity, that the additive inverse of an element is unique (that is, there are not two different additive inverses for an element.) We shall denote the additive inverse of the element $x$ by "$-x$", where we understand this symbol "-" as the so-called unary minus sign. Nowadays it is fashionable to call $-x$ "negative $x$" rather than "minus $x$" because we understand "minus" to stand for subtraction.

Any set $\mathcal{F}$ together with an operator "+" which satisfies the above 5 axioms is called a **group**. Being an additive commutative group is part of the requirement for being a field.

### 1.1.2 Axioms for multiplication

Now we continue on to the remainder of the definition of a field. In addition to being a group under addition, we require that the elements of $\mathcal{F}$ which are **non-zero** form a *group under multiplication*, that is, the axioms (1)-(5) above are repeated, this time with the operation being multiplication rather than addition, after we exclude 0:

Let $\mathcal{F}^*$ be the subset of $\mathcal{F}$ which is all of $\mathcal{F}$ except 0.

1. For every $x, y$ in $\mathcal{F}^*$, $x \times y$ is also an element of $\mathcal{F}^*$. (Axiom of Closure.)

2. For every $x, y$ in $\mathcal{F}^*$,

$$x \times y = y \times x.$$

   (Commutativity of multiplication).

3. For every $x, y, z$ in $\mathcal{F}^*$,

$$(x \times y) \times z = x \times (y \times z).$$

   (Associativity of multiplication).

4. There exists an element $e$ in $\mathcal{F}^*$ with the property that

$$e \times x = x$$

for every $x$ in $\mathcal{F}^*$. We call this element a **multiplicative identity for** $\mathcal{F}^*$. (Notice that we haven't assumed that there is only one such multiplicative identity. That will follow from the axioms[4] .) We shall call the multiplicative identity "1".

5. For every element $x$ in $\mathcal{F}^*$ there exists an element $y$ in $\mathcal{F}^*$ such that

$$x \times y = 1.$$

$y$ is called a **multiplicative inverse** for $x$, and it is also shown later, as with identity, that the multiplicative inverse of an element is unique (that is, there are not two different inverses for an element.) We shall denote the multiplicative inverse of the element $x$ by "$x^{-1}$", or also by "$1/x$". Further, we shall write

$$\frac{x}{y}$$

to stand for

$$x \times y^{-1}.$$

We have been denoting multiplication by " $\times$" but now we will agree to indicate multiplication in the usual way, by writing $xy$ to stand for $x \times y$.

### 1.1.3 Axiom of Distributivity (of Multiplication over Addition.)

For every $x, y, z$ in $\mathcal{F}$,

$$x(y + z) = xy + xz.$$

[4]Here is a proof that the multiplicative identity is unique. It is essentially identical to the proof that we gave earlier that the additive identity is unique: Suppose there were two multiplicative identities, 1 and $1'$. Then

$$1 \times 1' = 1'$$

since 1 is a multiplicative identity. But

$$1' \times 1 = 1$$

since $1'$ is a multiplicative identity. But multiplication is commutative, so

$$1 \times 1' = 1' \times 1,$$

which then shows that

$$1 = 1'.$$

This important axiom underlies all our usual algorithms for **multiplication with carries**:

$$\begin{aligned} 3 \times 24 &= 3(2 \cdot 10 + 4) \\ &= 6 \cdot 10 + 3 \cdot 4 \\ &= 6 \cdot 10 + 12 \\ &= 6 \cdot 10 + 1 \cdot 10 + 2 \\ &= 7 \cdot 10 + 2 \\ &= 72. \end{aligned}$$

## 1.2 The Additive Inverse of an Element is Unique

Suppose that both $y$ and $z$ were additive inverses to $x$:

$$x + y = x + z = 0.$$

By the definition of additive identity, the asssociative property of addition, and the commutativity of addition,

$$\begin{aligned} y &= 0 + y && (14) \\ &= (x + z) + y && (15) \\ &= (z + x) + y && (16) \\ &= z + (x + y) && (17) \\ &= z + 0 && (18) \\ &= z. && (19) \end{aligned}$$

So $y = z$, and hence there is only one additive inverse to $x$.

Exercise: For each of the steps in (14) to (19) above, state which axiom justifies the step.

## 1.3 The Multiplicative Inverse of an Element is Unique

Suppose that both $y$ and $z$ were multiplicative inverses to $x$:

$$xy = xz = 1.$$

Then

$$y = 1y = (xz)y = (zx)y = z(xy) = z1 = z, \tag{20}$$

by the existence of multiplicative identity, commutativity and associativity of multiplication. But then

$$y = z.$$

So, there is only one inverse to $x$.

Exercise In (20) above, justify every step.

## 1.4 Why is a negative number times a negative number positive?

In order to answer this question, we need first to agree that what "negative" means is being the *additive inverse* of a positive number, and the positive numbers we shall take as the counting numbers: 1,2,3, ..., and the quotients $p/q$ of such counting numbers.

If $x$ is a positive number, we agree to call $-x$ a negative number, recalling that by $-x$ we mean the **additive inverse of** $x$:

$$x + (-x) = 0.$$

Let's take a moment to prove an "obvious" fact: For every $x$ in $\mathcal{F}$ ,

$$0x = 0.$$

Since 0 is the additive identity,

$$0 = 0 + 0.$$

But then

$$0x = 0x + 0x.$$

When we add the additive inverse of $0x$ to both sides, we get

$$\begin{aligned} 0 = 0x + (-0x) &= (0x + 0x) + (-0x) \\ &= 0x + (0x + (-0x)) \\ &= 0x + 0 \\ &= 0x, \end{aligned}$$

i.e. $0x = 0$.

Now let us consider the product of two negative numbers:

$$(-x)(-y),$$

where each of these is, respectively, the additive inverse of a positive number $x$ and $y$. Observe that

$$0 = 0y = (x + (-x))y = xy + (-x)y,$$

which means that $(-x)y$ is the additive inverse of $xy$.

Also

$$\begin{aligned} 0 = 0(-x) &= (-x)0 \\ &= (-x)(y + (-y)) \\ &= (-x)y + (-x)(-y), \end{aligned}$$

which makes $(-x)y$ the additive inverse of $(-x)(-y)$. But we said that $(-x)y$ was the additive inverse of $xy$, and *additive inverses are unique!* Therefore,

$$(-x)(-y) = xy,$$

i.e. the product of two negative numbers is positive!

## 1.5 In addition of fractions, why do we need a common denominator?

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Do you recall the old saying, "You can't add apples and oranges?" Well, it's true, in this context, at least. If we ask, how much (volume) is 2 gallons and 2 cups, we need to find a way to express cups and gallons in common terms before we can add them. While it is true that we could use "cups" as the common term, let's pretend for a moment that we don't know how many cups there are in a gallon. In this case, what can we do? We can find a common unit of measure, in which we can express both cups and gallons. The obvious choice (but by no means only choice) is ounces. A cup is 8 ounces, and a gallon is 128 ounces. So, 2 gallons and 2 cups is $2 \cdot 128 + 2 \cdot 8 = 272$ ounces. (Of course another solution would be $2.125 = 2\ 1/8$ gallons.) The idea here, as with fractions, is that in order to carry out addition, the items have to be in **common terms**. If we wish to add $a/b$ and $c/d$, recalling that $a/b$ means $a$ things, each of which is $b$-inverse, in order to perform the addition, just as with cups and gallons, we need to find a common term in which to express $b$-inverse and $d$-inverse. When $b$ and $d$ are integers,

$$b^{-1} = \frac{1}{b} = \frac{d}{bd}$$

so $\frac{1}{b}$ is a multiple of $\frac{1}{bd}$, and, similarly,

$$d^{-1} = \frac{1}{d} = \frac{b}{bd}.$$

Our "common term" becomes

$$\frac{1}{bd}.$$

Now addition of fractions becomes easy to understand:

$$\begin{aligned}
\frac{a}{b} + \frac{c}{d} &= a\frac{1}{b} + c\frac{1}{d} && (21)\\
&= a\frac{d}{bd} + c\frac{b}{bd} && (22)\\
&= \frac{ad}{bd} + \frac{bc}{bd} && (23)\\
&= \frac{ad + bc}{bd}. && (24)
\end{aligned}$$

## 1.6 Justifications for Equations (6) to (9)

We used the axiom of associativity of multiplication to re-write the parentheses in Equation (6) and again in Equation (7), then the Axiom of Identity in going from (7) to (8) and the definition of multiplicative inverse to go from (8) to (9).