CHAPTER 4

RIEMANN'S EXISTENCE THEOREM

This chapter introduces the foundation of the book: The construction of all compact Riemann surfaces through Riemann's classification of the branched covers of the sphere (Thm. 2.6). Still, one cover at a time, won't give us much useful information. We need to know the nature of families of related covers. The Existence Theorem serves well, though it takes additional ideas to find a useful naming scheme for the families. This chapter's nontraditional treatment of modular curves motivates many general ideas in Chap. 5.

1. Presentations of fundamental groups of Riemann surfaces

Our command of Riemann's Existence Theorem requires combinatorial ability to list finite quotients of the fundamental group of $U_{\boldsymbol{z}}$. Thm. 1.8 tells us $\pi_1(U_{\boldsymbol{z}})$ is a free group on $r-1$ generators (with $r=|\boldsymbol{z}|$) and more. It is the basis for describing families of covers (Chap. 5) of $\mathbb{P}^1_z$. Our main computational tools for this are Hurwitz monodromy actions. These are on explicit sets running from types of *Nielsen classes* (§3.2) to special fundamental group generators of Riemann surfaces defined by Nielsen classes (§9.2).

1.1. Presentations and free products. Most fundamental groups appear as quotients of free groups. Further, we define the kernel of that quotient by listing specific *relation* elements in the kernel. We recognize the smallest normal subgroup containing these relations as the kernel. A *presentation*, however, doesn't list all relations from this normal subgroup condition. Presenting groups as quotients of free groups this way is convenient for forming their quotients. To see whether a group G is a quotient of some fundamental group, we need only check if specific generators of G satisfy a tiny list of relations. This suits how we form compact Riemann surfaces from unramified covers of $U_{\boldsymbol{z}}$. Still, this often leaves a tough problem. How to check if an expression from the free group is in that kernel.

For S a set, we first define the group $F(S)$ that S *freely generates*. The following construction is of a free group with relations. Generalizing this to groups generated freely by subgroups is a categorical rather than quotient construction.

For $s\in S$ and $n\in\mathbb{Z}$, use the symbol s^n to denote the pair (s,n). If $t\in S$ and $m\in\mathbb{Z}$ then $s^n=t^m$ if and only if $s=t, n=m$.

Elements of $F(S)$ are (finite) sequences $\boldsymbol{s}^{\boldsymbol{n}}=(s_1^{n_1},\cdots,s_k^{n_k})$ satisfying

$$k\in\mathbb{N}; s_1,\cdots,s_k\in S;\ n_1,\cdots,n_k\in\mathbb{Z}\setminus\{0\};\text{ and } s_i\neq s_{i+1},\ i=1,\cdots,k-1.$$

Regard the sequence $\emptyset$ with no elements as an element of $F(S)$. Denote $(t_1^{m_1},\cdots,t_\ell^{m_\ell})\in F(S)$ by $\boldsymbol{t}^{\boldsymbol{m}}$. Define the product of $\boldsymbol{s}^{\boldsymbol{n}}$ and $\boldsymbol{t}^{\boldsymbol{m}}$ by cancellation to be the elimination of any consecutive terms of the form tt^{-1}. Formally, Find the smallest integer u with this property: $t_u^{-m_u} \neq s_{k-u+1}^{n_{k-u+1}}$; but $t_i^{-m_i} = s_{k-i+1}^{n_{k-i+1}}$, $i = 1, \cdots, u-1$. Then

$$\begin{aligned} \boldsymbol{s^n t^m} = & \quad (s_1^{n_1}, \cdots, s_{k-u}^{n_{k-u}}, \alpha, t_{u+1}^{m_{u+1}}, \cdots, t_\ell^{m_\ell}) \\ \text{where} & \quad \alpha = \begin{cases} (s_{k-u+1}^{n_{k-u+1}}, t_u^{m_u}) & \text{for } t_u \neq s_{k-u+1} \\ t_u^{n_{k-u+1}+m_u} & \text{for } t_u = s_{k-u+1}. \end{cases} \end{aligned} \tag{1.1}$$

With this multiplication $F(S)$ is a group with $\emptyset$ the identity. For example, an induction on the length of the sequence of the middle term, in a product of 3 terms, suffices to establish the associative law. The inverse of $\boldsymbol{s^n}$ is $(s_k^{-n_k}, \cdots, s_1^{-n_1})$.

For a group G and a subset S of G, denote by $\langle S \rangle$ the subgroup of G that S generates. The elements $\boldsymbol{s^n} \in F(S)$ for which $s_1^{n_1} \cdots s_k^{n_k}$ is the identity in G form a subset $\bar{R}(S)$ called the *relations satisfied by* S. It is a normal subgroup of $F(S)$.

DEFINITION 1.1. Let S be a set of generators of a group G. A sequence $\{r_1, r_2, \dots\}$ of $F(S)$ is a *presentation* of G if $\bar{R}(S)$ is the smallest *normal* subgroup of $F(S)$ containing $\{r_1, r_2, \dots\}$. We say $\{r_1, r_2, \dots\}$ generates $\bar{R}(S)$. A presentation is *finite* if both S and $\{r_1, r_2, \dots\}$ are finite sets.

It is standard to denote $(s_1^{n_1}, \dots s_k^{n_k}) = \boldsymbol{s^n} \in F(S)$ by $s_1^{n_1} \cdots s_k^{n_k}$ when this symbol could not be confused with the product in another group.

EXAMPLE 1.2. Let $G = \mathbb{Z}^2$, the additive group of integer pairs. Let $s_1 = (1, 0)$ and $s_2 = (0, 1)$. Take for S the set $\{s_1, s_2\}$. Then $\{s_1 s_2 s_1^{-1} s_2^{-1}\}$ is a presentation of G. Indeed, $\bar{R}(S) = [F(S), F(S)]$, the commutator subgroup of $F(S)$. [11.7c]

EXAMPLE 1.3. Take for S the set $\{s_1, s_2, \dots, s_r\}$. From now on we denote $F(S)$ by F_r. There is a natural map from F_r to F_{r-1} that maps s_i to itself, $i = 1, \dots, r-1$, and s_r to $s_{r-1}^{-1} s_{r-2}^{-1} \cdots s_1^{-1}$. A nonidentity element of $\bar{R}(S)$ becomes 1 when you make the above substitution for s_r. Therefore such an element involves s_r, and $\{s_1 \cdots s_r\}$ gives a presentation of F_{r-1}.

The following treatment on *free products* of groups, from [**Wae48**], appears also in [**Ma67**, p. 97-100]. Let $G_1, \dots, G_t$ be groups. We define their free product G by its properties. There are homomorphisms $\alpha_i : G_i \to G$, $i = 1, \dots, t$, satisfying this condition: For any group H and homomorphisms $\beta_i : G_i \to H, i = 1, \cdots, t$, there exists a unique homomorphism $\beta : G \to H$ with

$$\beta \circ \alpha_i = \beta_i, \ i = 1, \dots, t. \tag{1.2}$$

Modern terminology might suggest the term *free sum* or pushout; it generalizes for arbitrary groups the direct sum of abelian groups[11.10a]. We now show a free product exists. From the definition it is unique up to isomorphism.

Define $T(\boldsymbol{G}) = T(G_1, \cdots, G_t)$ as those (finite) sequences $(x_1, \cdots, x_n)$ where each x_k is a nonidentity element of one of the groups G_i, and where consecutive terms of the sequence are in different groups. Each $g \in G_i$ acts faithfully on the right of $T(\boldsymbol{G})$ as a permutation $\alpha_i(g)$ given by the following formula. For $g \in G_i$ and $(x_1, \dots, x_n) \in T(\boldsymbol{G})$, $\alpha_i(g)$ maps $(x_1, \dots, x_n)$ to this element:

(1.3a) $(x_1, \dots, x_n g)$ if $x_n \in G_i$ and $x_n g \neq 1_{G_i}$;
(1.3b) $(x_1, \dots, x_{n-1})$ if $x_n \in G_i$ and $x_n = g^{-1}$;
(1.3c) $(x_1, \dots, x_n)$ if $x_n \notin G_i$ and $g = 1_{G_i}$;
(1.3d) $(x_1, \dots, x_n, g)$ if $x_n \notin G_i$ and $g \notin 1_{G_i}$; and
(1.3e) (g) if $(x_1, \dots, x_n) = \emptyset$.

Let $\mathrm{Per}(T(\boldsymbol{G}))$ be the group of (right action) permutations of $T(\boldsymbol{G})$. Then G is the subgroup of $\mathrm{Per}(T(\boldsymbol{G}))$ that the images of the G_i under the homomorphisms α_i, $i = 1, \dots, t$, generate.

LEMMA 1.4. *The group G just defined is a free product of $G_1, \dots, G_t$.*

PROOF. Express a given nonidentity element γ of G (in *reduced form*) as

$$\alpha_{i_1}(g_{i_1}) \cdots \alpha_{i_n}(g_{i_n})$$

where g_{i_k} is a nonidentity element of G_{i_k} and $i_k \neq i_{k+1}$, $i = 1, \dots, n-1$. This expression is unique. Apply γ to $\emptyset$ (as in (1.3e)) to get $(g_{i_1}, \dots, g_{i_n})$.

Suppose $\beta_i : G_i \to H$, $i = 1, \dots, t$, is any collection of homomorphisms. Define $\beta : G \to H$ as follows: $\beta(\alpha_{i_1}(g_{i_1}) \cdots \alpha_{i_n}(g_{i_n}))$ is equal to $\beta_{i_1}(g_{i_1}) \cdots \beta_{i_n}(g_{i_n})$. Induction on the lengths of the reduced forms of two elements of G shows that β is a homomorphism. Clearly β is the unique homomorphism satisfying (1.2). □

1.2. Fundamental groups of unions of spaces. Let X be a connected union of finitely many differentiable manifolds. Suppose U and V are open subsets of X with $U \cup V = X$, and U, V and $U \cap V$ nonempty and *connected*. For topological spaces Y and Z with Y a subspace of Z and $y_0 \in Y$, denote the induced homomorphism $\pi_1(Y, y_0) \to \pi_1(Z, y_0)$ by $i(Y, Z)_*$.

THEOREM 1.5 (Seifert-van Kampen). *Let $x_0 \in U \cap V$. For H a group, let $\beta(U) : \pi_1(U, x_0) \to H$ and $\beta(V) : \pi_1(V, x_0) \to H$ be two homomorphisms for which*

$$\beta(U) \circ i(U \cap V, U)_* = \beta(V) \circ i(U \cap V, V)_*. \tag{1.4}$$

Then, there is a unique homomorphism $\beta(X) : \pi_1(X, x_0) \to H$ with

$$\beta(U) = \beta(X) \circ i(U, X)_* \text{ and } \beta(V) = \beta(X) \circ i(V, X)_*. \tag{1.5}$$

In using Thm. 1.5, don't forget $U \cap V$ must be connected. Neglecting this would lead to concluding the torus has trivial fundamental group (Fig. 1).

REMARK 1.6. Commutativity of this diagram characterizes $\pi_1(X, x_0)$:

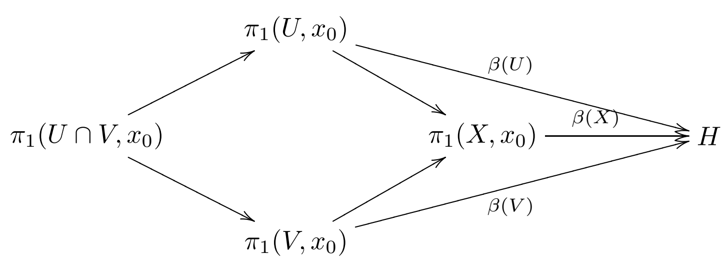
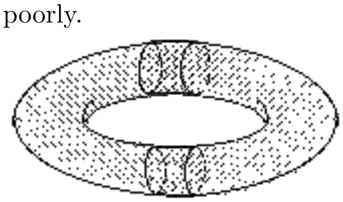


FIGURE 1. Two cylinders try to share the fundamental group of a torus, but they connect poorly.



1.3. Proof of Seifert-van Kampen, Thm. 1.5. This is a special case of [**Ma67**, p.114-22]. We give the proof in four subsections.

1.3.1. *$\pi_1(U,x_0)$ and $\pi_1(V,x_0)$ generate $\pi_1(X,x_0)$.* Let $\gamma:[a,b]\to X$ represent an element of $\pi_1(X,x_0)$. Find $t_0=a<t_1<\cdots<t_n=b$ so either U or V entirely contains the image of $\gamma_{|_{[t_i,t_{i+1}]}}$, $i=1,\dots,n$. Let U_i be either U or V, so U_i contains the image of $\gamma_{|_{[t_i,t_{i+1}]}}$. As $\gamma(t_i)$ lies in both U_{i-1} and U_i there is a path γ_i in $U_{i-1}\cap U_i$ joining x_0 to $\gamma(t_i)$, $i=1,\dots,n-1$. Then each of the following closed paths is in U_i for the corresponding value of i:

$$\gamma'_0=\gamma_{|_{[t_0,t_1]}}\gamma_1^{-1},\ i=0,\qquad \gamma'_i=\gamma_i\gamma_{|_{[t_i,t_{i+1}]}}\gamma_{i+1}^{-1},\ i=1,\dots,n-2,$$
$$\text{and } \gamma'_{n-1}=\gamma_{n-1}\gamma_{|_{[t_{n-1},t_n]}}.$$

The product $\gamma'_0\cdots\gamma'_{n-1}$ is equivalent to γ. Write $[\gamma]$ as

$$i(U_0,X)_*([\gamma'_0])i(U_1,X)_*([\gamma'_1])\cdots i(U_{n-1},X)_*([\gamma'_{n-1}]), \tag{1.6}$$

a product of paths, each from $\pi_1(U,x_0)$ or $\pi_1(V,x_0)$.

1.3.2. *Condition for existence of β.* It is natural to define $\beta([\gamma])$ from (1.6):

$$\beta(U_0)([\gamma'_0])\beta(U_1)([\gamma'_1])\cdots\beta(U_{n-1})([\gamma'_{n-1}]). \tag{1.7}$$

We show, if (1.6) is the identity, then so is (1.7); β is well-defined.

Let $F:[a,b]\times[0,1]\to X$ be a homotopy between γ and the constant path:

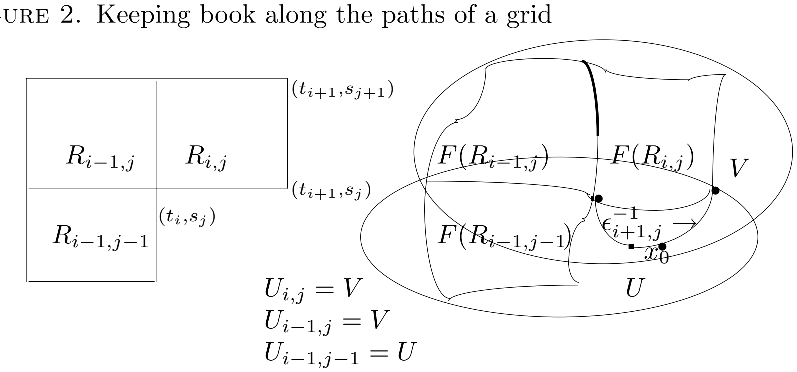
$$F(t,s)=\gamma_s(t),\ \gamma_0(t)=\gamma(t),\ \text{and } \gamma_1(t)=x_0.$$

Refine the subdivision $t_0=a<t_1<\cdots<t_n=b$ to find $s_0=0<\cdots<s_m=1$ so $U_{i,j}$, one of U or V, contains the image under F of each rectangle

$$R_{i,j}=\{(t,s)\mid s_j\le s\le s_{j+1},\ t_i\le t\le t_{i+1}\}.$$

Let $V_{i,j}$ be the intersection of $U_{i-1,j},U_{i-1,j-1}$ and $U_{i,j}$. This refinement doesn't change the value of (1.7). Choose a path $\epsilon_{i,j}:[a,b]\to V_{i,j}$ with initial point x_0 and end point $\gamma_{s_j}(t_i)=F(t_i,s_j)$. When $F(t_i,s_j)=x_0$, choose $\epsilon_{i,j}$ to be the constant path, and choose $\epsilon_{i,0}$ to be γ_i (as in §1.3.1), $i=1,\dots,n-1$.

FIGURE 2. Keeping book along the paths of a grid



1.3.3. *Grid following paths.* Denote the path $t \in [t_i, t_{i+1}] \mapsto F(t, s_j)$ (resp., $s \in [s_j, s_{j+1}] \mapsto F(t_i, s))$ by $F_{|_{[t_i,t_{i+1}]\times s_j}}$ (resp., $F_{|_{t_i\times[s_j,s_{j+1}]}}$). Let $\gamma_{i,j}$ be the path $\epsilon_{i,j}(F_{|_{[t_i,t_{i+1}]\times s_j}})(\epsilon_{i+1,j})^{-1}$. Let $\delta_{i,j}$ be the path $\epsilon_{i,j}(F_{|_{t_i\times[s_j,s_{j+1}]}})(\epsilon_{i,j+1})^{-1}$. Define $g_{i,j}$ to be the image under $\beta(U_{i,j})$ of the homotopy class of $\gamma_{i,j}$ in $\pi_1(U_{i,j}, x_0)$, $i = 0, \dots, n-1$; $j = 0, \dots, m-1$. Note: (1.4) implies $g_{i,j}$ is also the image under $\beta(U_{i,j-1})$ of the class of $\gamma_{i,j}$ in $\pi_1(U_{i,j-1}, x_0)$, $i = 0, \dots, n-1$; $j = 1, \dots, m$. So, we consistently define $g_{i,m}$ to be $\beta(U_{i,m-1})$, the image of $\gamma_{i,m}$ in $\pi_1(U_{i,m-1}, x_0)$, $i = 0, \dots, n-1$. Similarly, $\delta_{i,j}$ gives $h_{i,j} \in H$, $i = 0, \dots, n$; $j = 0, \dots, m-1$.

Since the boundary of $R_{i,j}$ (traversed clockwise) is homotopic to a constant path in $R_{i,j}$, its image under F is homotopic to a constant path in $U_{i,j}$. Therefore

$$(F_{|_{t_i\times[s_j,s_{j+1}]}})(F_{|_{[t_i,t_{i+1}]\times s_{j+1}}}) \text{ is homotopic to } (F_{|_{[t_i,t_{i+1}]\times s_j}})(F_{|_{t_{i+1}\times[s_j,s_{j+1}]}})$$

in $U_{i,j}$. Conclude:

(1.8) $$\gamma_{i,j}\delta_{i+1,j} \text{ is homotopic to } \delta_{i,j}\gamma_{i,j+1} \text{ in } U_{i,j}.$$

Denote the identity in H by 1_H. An application of $\beta(U_{i,j})$ gives

(1.9a) $g_{i,j}h_{i+1,j} = h_{i,j}g_{i,j+1}$, $i = 0, \dots, n-1$; $j = 0, \dots, m-1$.

(1.9b) As a consequence of $F(t, 1) = F(a, s) = F(b, s) = x_0$:

$$g_{i,m} = 1_H,\ i = 0, \dots, n-1;\ h_{0,j} = h_{n,j} = e_H,\ j = 0, \dots, m-1.$$

Finally, (1.7) is the same as

(1.10) $$g_{0,0}g_{1,0}\cdots g_{n-1,0}.$$

1.3.4. (1.9a) *and* (1.9b) *imply* (1.10) *is* 1_H. From (1.9b), $g_{0,0}\cdots g_{n-1,0}h_{n,0}$ equals (1.10). From (1.9a), this is $g_{0,0}\cdots h_{n-1,0}g_{n-1,1}$. Repeat using (1.9a) and (1.9b) to see (1.10) is $g_{0,1}g_{1,1}\cdots g_{n-1,1}$. Inductively: (1.10) is $g_{0,j}g_{1,j}\cdots g_{n-1,j}$ for each j. With $j = m$, (1.9b) shows this is 1_H.

Since $\pi_1(X, x_0)$ is a pushout of the homomorphisms $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$, this uniquely defines $\pi_1(X, x_0)$ [11.10b] and concludes the proof.

1.4. Classical generators on an r-punctured sphere. Let Y be a subspace of a space X. Then Y is a *retract* of X if there is a continuous map $f : X \to Y$ such that $f(y) = y$ for $y \in Y$. The sequence of maps

$$Y \xrightarrow{i(Y,X)} X \xrightarrow{f} Y$$

induces the sequence of homomorphisms of groups

$$\pi_1(Y, y_0) \xrightarrow{i(Y,X)_*} \pi_1(X, y_0) \xrightarrow{f_*} \pi_1(Y, y_0)$$

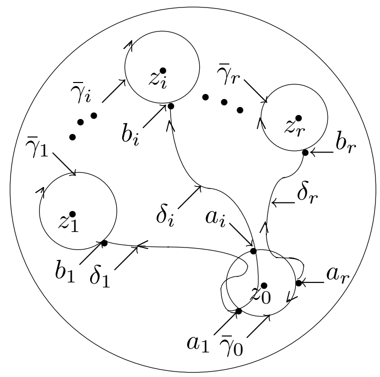
where $f_* \circ i(Y, X)_*$ is the identity. This splitting of the sequence of groups means $\pi_1(X, y_0)$ is the direct product of $\pi_1(Y, y_0)$ and the kernel of f_*.

DEFINITION 1.7. A retract Y of X is a *deformation retract* of X if there exists a continuous map $F : X \times [0, 1] \to X$ for which $F(x, 0) = x$ and $F(x, 1) = f(x)$ for $x \in X$, and $F(y, s) = y$ for $y \in Y$, $s \in [0, 1]$.

For each $s \in [0, 1]$ the map F, restricted to $X \times s$, induces a continuous map $\pi_1(X, y_0) \to \pi_1(X, y_0)$. (Regard these fundamental groups as topological spaces with the discrete topology.) Such a map is clearly independent of s. For $s = 0$ this map is the identity, and for $s = 1$ the image of this map identifies with $\pi_1(Y, y_0)$. So, f_* identifies the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.

1.4.1. *Defining classical generators.* Chap. 2 §1.1 introduced the *r-punctured sphere*: $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\} \stackrel{\text{def}}{=} U_{\boldsymbol{z}}$, r distinct points $z_1, \dots, z_r$ removed from $\mathbb{P}^1_z$.

FIGURE 3. Example classical generators based at z_0



Let z_0 be a point on $U_{\boldsymbol{z}}$. Let D_i be a disc with center z_i, $i = 1, \dots, r$. Assume these discs are disjoint and each excludes z_0. Let b_i be a point on the boundary of D_i. Regard this boundary, oriented clockwise, as a path $\bar{\gamma}_i$ with initial and end point b_i. Finally, let δ_i be a simple *simplicial* (Chap. 2 Def. 2.1) path with initial point z_0 and end point b_i. Assume, also, that δ_i meets none of $\bar{\gamma}_1, \dots, \bar{\gamma}_{i-1}, \bar{\gamma}_{i+1}, \dots, \bar{\gamma}_r$, and it meets $\bar{\gamma}_i$ only at its endpoint.

With D_0 a disc with center z_0 and disjoint from each of the discs $D_1, \dots, D_r$, consider the first point of intersection of δ_i and the boundary $\bar{\gamma}_0$ of D_0. Call this point a_i. Suppose $\delta_1, \dots, \delta_r$ satisfy two further conditions:

(1.11a) they are pairwise nonintersecting, excluding their initial point z_0; and
(1.11b) $a_1, \dots, a_r$ appear in order clockwise around $\bar{\gamma}_0$.

Since the paths are simplicial this last condition is independent of the choice of D_0, at least for D_0 sufficiently small.

With these conditions, the ordered collection of closed paths $\delta_i \bar{\gamma}_i \delta_i^{-1} = \gamma_i$, $i = 1, \dots, r$, in Fig. 3 are *classical generators* (for $\boldsymbol{z}$) based at z_0. We say γ_i is a *classical loop around* z_i. In our case this has a precise meaning.

1.4.2. *Main Theorem for classical generators of* $\pi_1(U_{\boldsymbol{z}}, z_0)$. Chap. 5 deforms classical generators compatible with deformations of the set $\{\boldsymbol{z}\} = \{z_1, \dots, z_r\}$. Such deformations produce very complicated sets of classical generators. Thus the generality of our next result.

THEOREM 1.8. *Let* $(\gamma_1, \dots, \gamma_r)$ *by any collection of classical generators for* $\boldsymbol{z} = (z_1, \dots, z_r)$ *based at* z_0 *on the* r*-punctured sphere* $\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}$. *Then the homotopy classes* $[\gamma_1] = s_1, \dots, [\gamma_r] = s_r$ *generate* $\pi_1(\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}, z_0)$ *with the one relation* $s_1 \cdots s_r$*: The Product-One condition. So,* $\pi_1(\mathbb{P}^1_z \setminus \{\boldsymbol{z}\}, z_0)$ *is isomorphic to* F_{r-1} *through the presentation* $\{s_1 \cdots s_r\}$ *(Ex. 1.3).*

If $[\gamma'_1] = s'_1, \dots, [\gamma'_r] = s'_r$ *is another collection of classical generators, then there is a* $\pi \in S_r$ *so that* s'_i *is conjugate to* $s_{(i)\pi}$, $i = 1, \dots, r$.

1.5. Proof of classical generators Thm. 1.8. For the statement on the presentation of $\pi_1(U_{\boldsymbol{z}}, z_0)$, induct on r. For $r = 0$, write $\mathbb{P}^1$ as the union of

$$\mathbb{P}^1 \setminus \{\infty\} = U_1 \text{ and } \mathbb{P}^1 \setminus \{0\} = U_2$$

as in Chap. 3 Ex. 3.2.1. Apply Thm. 1.5 (just §1.3.1). For $r \geq 1$ we show $\pi(U_{\boldsymbol{z}}, z_0)$:

(1.12a) $\gamma_1 \cdots \gamma_r$ is homotopic (on $U_{\boldsymbol{z}}$) to the identity.

(1.12b) $[\gamma_1], \dots, [\gamma_{r-1}]$ are free generators of the fundamental group.

These suffice to show the statement gives a correct presentation of $\pi_1(U_{\boldsymbol{z}}, z_0)$ if we show any relation among $s_1, \dots, s_r$ is in the group generated by products of conjugates of the product-one condition. Hints: Do an induction starting with a nontrivial relation containing no subproduct conjugate to the product one relation, and having a minimal number of appearances of s_r. No appearances of s_r is impossible from (1.12b); by conjugating shift the any one appearance of s_r to the far right. We divide the proof of (1.12) into 4 parts to separate the conceptual proof from a technical preliminary.

1.5.1. *Polygonal paths.* We show the set of paths $\gamma_1, \dots, \gamma_r$ is (simultaneously) homotopic to a set of simple polygonal paths based at z_0, intersecting only at z_0; and that $\gamma_1 \cdots \gamma_r$ is homotopic to a simple polygonal path based at z_0.

Choose D_0 so a_i is the only intersection of δ_i and $\bar{\gamma}_0$, $i = 1, \dots, r$. This is possible because $\delta_1, \dots, \delta_r$ are simplicial. For an integer $n > 2$, let $\bar{\gamma}_i^*$ be the regular n-gon inscribed in $\bar{\gamma}_i$ as a clockwise path from the vertex b_i. Chap. 2 Lem. 4.3 allows replacing each δ_i by a polygonal path homotopic to δ_i (with its endpoints fixed), so as to assume our classical generators are polygonal paths.

We explain the formation of the shaded region around the polygonal path δ_i in Fig. 4. The points b_i' and b_i'' are the vertices of $\bar{\gamma}_i^*$ next to b_i. Draw the lines through b_i' and b_i'' parallel to the last segment of δ_i, and let $d = d_n$ be the maximum of the distances between these lines and the last segment. Now continue drawing the lines at a distance d parallel to each segment of δ_i. For large n: the lines parallel to the last segment meet $\bar{\gamma}_0$ at points a_i' and a_i''; the paths δ_i^* and δ_i^{**} traced by these lines on either side of δ_i are simple and have segments corresponding one-one with the segments of δ_i. The shaded region (bounded by δ_i^*, δ_i^{**}, the two sides of $\bar{\gamma}_i^*$ next to b_i, and the line segments a_i to a_i' and a_i to a_i'') meets none of the corresponding shaded regions around δ_j for $j \neq i$. In addition, the path going from a_i to a_i', then along δ_i^*, and then from b_i' to b_i is homotopic (with a_i and b_i fixed) to δ_i through a homotopy of simple polygonal paths that stay within the shaded region and, until the end, do not meet δ_i.

Indeed, with a few choices of lines separating the *elbows* and *ends* of the shaded region from the intermediate stretches — this may require a larger value of n — we can make the homotopy canonical. To illustrate, consider the *elbow* of the last two segments of δ_i. The lines ℓ' and ℓ'' (perpendicular, respectively, to the last and second last segments of δ_i) that meet at P outline this elbow in Fig. 4. In this region the homotopy takes points along the projection from P. In general, the homotopy carries points of δ_i^* along the perpendicular to the corresponding segment of δ_i.

Let λ_i^* (resp., λ_i^{**}) be a path tracing the ray from z_0 to a_i' (resp., z_0 to a_i''). Finally, let γ_i^* be the part of $\bar{\gamma}_i^*$ with initial point b_i' and end point b_i''. Then,

$$\gamma_i' = \lambda_i^* \delta_i^* \gamma_i^* (\delta_i^{**})^{-1} (\lambda_i^{**})^{-1}, \; i = 1, \dots, r,$$

are simple, polygonal, pairwise nonintersecting (except at z_0) paths that are respectively homotopic to $\gamma_1, \dots, \gamma_r$ on $U_{\boldsymbol{z}}$.

Let $\bar{a}_i$ be the midpoint of the arc from a_i'' to a_{i+1}', $i = 1, \ldots, r-1$. Denote the path along the two straight line segments from a_i'' to $\bar{a}_i$, and then from $\bar{a}_i$ to a_{i+1}' by ϵ_i^*. Then the following simple polygonal path, γ', is homotopic on $U_{\boldsymbol{z}}$ to $\gamma_1' \cdots \gamma_r'$, and thus to $\gamma_1 \cdots \gamma_r$:

$$\lambda_1^* \delta_1^* \gamma_1^* (\delta_1^{**})^{-1} \epsilon_1^* \delta_2^* \gamma_2^* (\delta_2^{**})^{-1} \epsilon_2^* \cdots \epsilon_{r-1}^* \delta_r^* \gamma_r^* (\delta_r^{**})^{-1} (\lambda_r^{**})^{-1}. \tag{1.13}$$

This homotopy shows the interior of a polygonal sector of the disc (marked off clockwise) from a_1' to a_r'', together with the shaded regions of Fig. 4 and the interiors of $\bar{\gamma}_1^*, \ldots, \bar{\gamma}_r^*$ to be part of one connected component of $\mathbb{P}^1 \setminus \gamma'$.

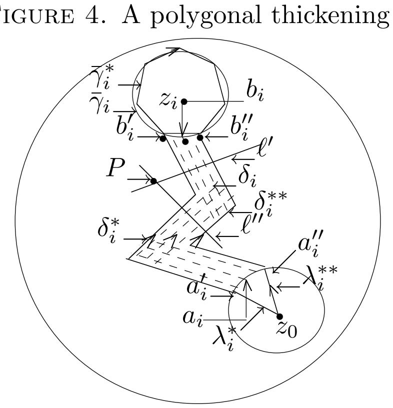
1.5.2. *Homotopy of $\gamma_1 \cdots \gamma_r$ to 1.* It suffices to show γ' is homotopic to the identity. The *Jordan curve theorem* says the complement of the simple closed path γ' on $\mathbb{P}^1$ consists of two components. For a polygonal path, however, this is fairly easy ([**He62**, p. 146] or [11.3a]). The *Schwartz-Christoffel transformation* ([**He66**, p.351-3] or §6.6) gives a one-one continuous map φ' from the closed upper hemisphere on $\mathbb{P}^1$ to $\mathbb{P}^1$, analytic on the open hemisphere, that maps the equator onto γ'. With no loss we assume φ' maps onto the component excluding z_r. From the last line of §1.5.1, none of $z_1, \ldots, z_r$ are in the image of φ'. Since the closed upper hemisphere is simply connected, so is the image of the one-one map φ' on $U_{\boldsymbol{z}}$. Thus, γ' is homotopic to the identity (see §6.6).

1.5.3. *Retraction of $U_{\boldsymbol{z}}$ onto $\gamma_1' \cup \cdots \cup \gamma_{r-1}'$.* To simplify our discussion, identity a simple path with its collection of image points. Notice this further use of the Jordan curve theorem (for polygonal paths). The path λ_{r-1}^{**} divides the interior W of γ' into two parts. The collection of points $\{z_1, \ldots, z_{r-1}\}$ is accessible from one side of λ_{r-1}^{**}, and z_r from the other. So, $\{z_1, \ldots z_{r-1}\}$ and $\{z_r\}$ lie in distinct components of $W \setminus \lambda_{r-1}^{**}$ [11.3b]. In the above replace γ' with following path:

$$\gamma'' = \lambda_1^* \delta_1^* \gamma_1^* (\delta_1^{**})^{-1} \epsilon_1^* \cdots \epsilon_{r-2}^* \delta_{r-1}^* \gamma_{r-1}^* (\delta_{r-1}^{**})^{-1} (\lambda_{r-1}^{**})^{-1}. \tag{1.14}$$

§1.5.2 shows there is a continuous one-one map φ'' from the upper hemisphere mapping the equator onto the path γ''; and mapping onto the component of $\mathbb{P}^1 \setminus \gamma''$ that includes z_r, but excludes $\{z_1, \ldots, z_{r-1}\}$.

FIGURE 4. A polygonal thickening of δ_i



The upper hemisphere minus $(\varphi'')^{-1}(z_r)$ clearly retracts to the equator. Therefore the closure of the component of $\mathbb{P}^1 \setminus \gamma''$ containing z_r, with z_r removed, retracts to γ''. Denote the closure of the other component by X''. Similarly, denote the closure of the component of $\mathbb{P}^1 \setminus \gamma_i'$ containing z_i by X_i, $i = 1, \dots, r-1$. Let Y_i be the quadrilateral with vertices a_i'', $\bar{a}_i$, a_{i+1}' and z_0, $i = 1, \dots, r-2$. Retract Y_i onto the union of the two sides defined by $\{a_i', z_0\}$ and $\{a_i'', z_0\}$. Since

$$X'' = X_1 \cup \dots \cup X_{r-1} \cup Y_1 \cup \dots \cup Y_{r-2},$$

this retracts X'' onto $X_1 \cup \dots \cup X_{r-1}$. Apply the Schwartz-Christoffel transformation to retract $X_i \setminus \{z_i\}$ onto γ_i', $i = 1, \dots, r-1$. This retracts $U_{\boldsymbol{z}}$ onto $\gamma_1' \cup \dots \cup \gamma_{r-1}'$.

1.5.4. $[\gamma_1], \dots, [\gamma_{r-1}]$ *generate* $\pi_1(\mathbb{P}^1 \setminus \{\boldsymbol{z}\}, z_0)$ *freely.* The retraction of §1.5.3 reduces this to showing $[\gamma_1'], \dots, [\gamma_{r-1}']$ generate $\pi_1(\lambda_1' \cup \dots \cup \gamma_{r-1}', z_0)$ freely.

Let c_i be a vertex of γ_i^* different from b_i' or b_i'' (Fig. 4), $i = 1, \dots, r-1$. Take U to be $\gamma_1' \cup \dots \cup \gamma_{r-1}' \setminus \{c_{r-1}\}$ and V to be $\gamma_1' \cup \dots \cup \gamma_{r-1}' \setminus \{c_1, \dots, c_{r-2}\}$. Then $\gamma_1' \cup \dots \cup \gamma_{r-2}'$ is a deformation retract of U; γ_{r-1}' is a deformation retract of V; and $\{z_0\}$ is a deformation retract of $U \cap V$. From Thm. 1.5, $\pi_1(\gamma_1' \cup \dots \cup \gamma_{r-1}', z_0)$ is a free product of $\pi_1(U, z_0)$ and $\pi_1(V, z_0)$.

To complete the proof of the theorem, consider another r-tuple of classical generators: $[\gamma_1'] = s_1', \dots, [\gamma_r'] = s_r'$. Identify the point around which s_i' loops as the unique point $z' \in \boldsymbol{z}$ for which $s_i' \mapsto 1$ in the natural map $\pi_1(U_{\boldsymbol{z}}, z_0) \to \pi_1(U_{\boldsymbol{z}'}, z_0)$ where $\boldsymbol{z}' \dot{\cup} \{z'\} = \boldsymbol{z}$. So, there is a $\pi \in S_r$ for which s_i' loops around $z_{(i)\pi}$. An easy homotopy of both $\gamma_{(i)\pi}$ and γ_i' has these properties.

- It moves only points on these paths within the outermost of $\bar{\gamma}_{(i)\pi}$ and $\bar{\gamma}_i'$.
- The homotopies end so the respective bounding path to the discs about $z_{(i)\pi}$ are the same.

At time t in the homotopy of $\gamma_{(i)\pi}$ denote the resulting path by $\gamma_{(i)\pi,t}$. In Fig. 5: γ_i' remains constant in the homotopy; $\bar{\gamma}_{(i)\pi,1}$ is $\bar{\gamma}_i'$; and only the end portion of $\delta_{(i)\pi,t}$ varies in the homotopy. With $\gamma_{(i)\pi,1}$ replacing $\gamma_{(i)\pi} = \gamma_{(i)\pi,0}$ (and the other $r-1$ paths fixed), the equivalence classes in $\pi(U_{\boldsymbol{z}}, z_0)$ give the same elements $s_1, \dots, s_r$. With no loss, as in Fig. 5, assume $\gamma_{(i)\pi}$ and γ_i' are respectively $\delta_{(i)\pi}\bar{\gamma}_{(i)\pi}(\delta_{(i)\pi})^{-1}$ and $\delta_i'\bar{\gamma}_i'(\delta_i')^{-1}$. The homotopy class of $\delta_{(i)\pi,1}(\delta_i')^{-1}$ conjugates the former to the latter. That completes the proof of the theorem.

REMARK 1.9. Massey notes [**Ma67**, p. 125]:

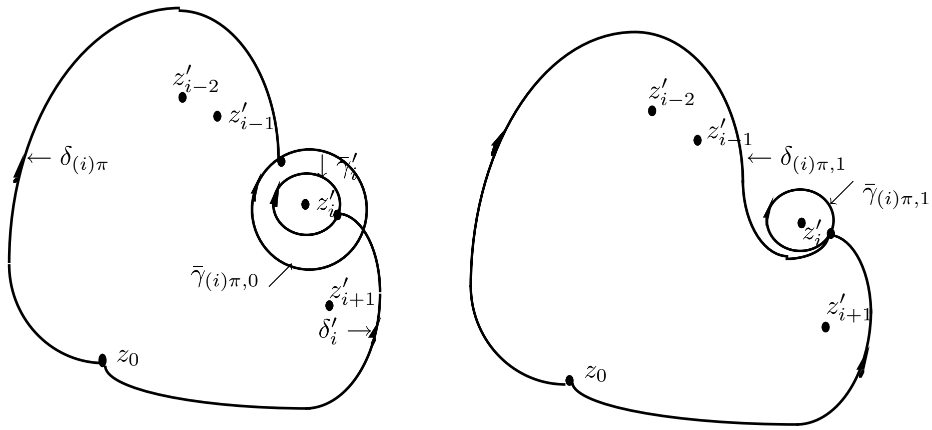
> To actually apply the Seifert-van Kampen Theorem, it is usually necessary to use the properties of deformation retracts.

2. Ramified covers from the Existence Theorem

Return to the notation of §2.1. Let $\psi : Y \to X$ be a nonconstant analytic map between two connected compact Riemann surfaces. The first part of the Existence Theorem is a combinatorial formula for contructing such ramified covers ψ.

2.1. Nonconstant maps of Riemann surfaces. Let $\psi : Y \to X$ be a nonconstant analytic map of *compact* connected Riemann surfaces. For any subset V of X denote $\psi^{-1}(V)$ by Y_V. If V is a point $x \in X$, simplify Y_V to be Y_x, the *fiber* over x. Recall the definition of unramified cover from Chap. 3 Def. 7.12.

FIGURE 5. Comparing two loops around $z_{(i)\pi}$



2.1.1. *The divisor of ramification.* We first attach a multiplicity to a point in a fiber. The outcome is that all fibers of ψ will have the same degree.

LEMMA 2.1. *The map ψ is open and so is surjective. Two analytic functions $\psi_i : Y \to \mathbb{P}^1_z$, $i = 1, 2$, with exactly the same zeros and poles (with multiplicity) on X differ by multiplication by a constant.*

For some integer n, $|Y_x| = n$ for all but finitely many $x \in X$. For $x \in X$, $|Y_x| \le n$. Let $D(\psi)$ be those x with $|Y_x| < n$. Then $Y_{X\setminus D(\psi)} \to X \setminus D(\psi)$ is an unramified cover.

Representing restriction of ψ around any point y_0 by an analytic function in a disk allows assigning a multiplicity e_{y_0} to y_0 in $Y_{\psi(y_0)}$. This gives a degree of the fiber Y_x by $\deg(Y_x) \stackrel{\text{def}}{=} \sum_{y\in Y_x} e_y$ and all fibers of ψ have degree n.

If $X = \mathbb{P}^1_z$, then the divisor (ψ) of the meromorphic function ψ has degree 0. Any meromorphic function on Y comes from an analytic map where $X = \mathbb{P}^1_z$.

PROOF. If ψ maps open sets to open sets, then the range of ψ is open. Since X is compact, the range of ψ is also closed. As X is connected, that means the range is the only possible nontrivial open and closed set, X. The statement that ψ is open is local: We have only to show it maps small open sets to small open sets. [**Ahl79**, p. 131] (as it is used below) shows ψ is locally an open map. Apply this by considering two analytic functions $\psi_i : Y \to \mathbb{P}^1_z$, $i = 1, 2$, with the same divisor of zeros and poles on Y. Then, the ratio ψ_1/ψ_2 has no zeros, and no poles. It gives an analytic map to $\mathbb{P}^1_z$ missing ∞ for example. So, it must be constant.

Let f be a nonconstant analytic function on an open connected subset U on $\mathbf{C}$, and let $z_0 \in U$. There is a neighborhood V of z_0 on which f is one-one if and only if $\frac{df}{dz}(z_0) \ne 0$ [**Ahl79**, p. 131]. Suppose $\frac{df}{dz}(z_0) \ne 0$. Then there is a neighborhood U_{z_0} of z_0 for which $\frac{df}{dz}$ is not 0 and f restricted to U_{z_0} is one-one. Let $\{(U^Y_\alpha, \varphi^Y_\alpha)\}_{\alpha\in I}$ (resp., $\{(U^X_\beta, \varphi^X_\beta)\}_{\beta\in J}$) be an atlas for the manifold Y (resp. X).

Consider the set R of $y \in Y$ with

$$\frac{d}{dz}(\varphi^X_\beta \circ \psi \circ (\varphi^Y_\alpha)^{-1})(\varphi^Y_\alpha(y)) = 0 \tag{2.1}$$

for some $\alpha \in I$, $\beta \in J$ with $y \in U_\alpha^Y \cap \psi^{-1}(U_\beta^X)$. The condition is independent of the choice of α and β (as in Chap. 3 Lem. 5.2). If R is infinite, then R has a limit point y_0. We show this leads to a contradiction.

There exists $\alpha \in I$ and $\beta \in I$ with $y_0 \in U_\alpha^Y$ and $\psi(y_0) \in U_\beta^X$. The zeros of $\frac{d}{dz}(\varphi_\beta^X \circ \psi \circ (\varphi_\alpha^Y)^{-1})$ have limit point $\varphi_\alpha^Y(y_0)$. So $\varphi_\beta^X \circ \psi \circ (\varphi_\alpha^Y)^{-1}$ is constant in a neighborhood of $\varphi_\alpha^Y(y_0)$ [**Ahl79**, p. 127], and ψ is constant in a neighborhood of y_0. The points of Y with a neighborhood on which ψ is constant is an open set contained in R. Any accumulation point of it is therefore an accumulation point of R. The above argument shows this set is closed. Since Y is connected, the existence of y_0 shows ψ is constant on all of Y, contrary to assumption. So R is finite.

Each $y \in Y \setminus R$ has a connected neighborhood U_y of y to which the restriction of ψ is a one-one function. Let $x \in X \setminus \psi(R)$. For each $y \in R$, let U_y be a neighborhood of y with $x \notin \psi(U_y)$. As ψ is one-one on U_y, U_y contains at most one point of Y_x. The cover $\{U_y\}_{y\in Y}$ of the compact space Y contains a finite subcover. Therefore Y_x is finite. Now consider neighborhoods of points of Y_x.

Let V_x be a connected neighborhood of x contained in $\psi(U_y)$ for each $y \in Y_x$. Then the connected components of Y_{V_x} are $\{U_y \cap Y_{V_x}\}_{y\in Y_x}$, and the restriction of ψ to each of these is one-one. From Chap. 3 Def. 7.12, ψ restricted to $Y_{X\setminus\psi(R)}$ is a cover, and the fibers have constant cardinality (Chap. 3 [9.21b]).

Now consider a fiber Y_x with $x \in D(\psi)$. Expression (2.1) generalizes. Any point $y \in Y_x$ gives a well-defined integer e_y: The minimal $e \ge 1$ with

$$\frac{d^e}{dz^e}(\varphi_\beta^X \circ \psi \circ (\varphi_\alpha^Y)^{-1})(\varphi_\alpha^Y(y)) \neq 0.$$

This is the *ramification index* of ψ at y (Chap. 2 Def. 7.6). Suppose $|Y_x| = t$. [**Ahl79**, p. 131] shows $f = \varphi_\beta^X \circ \psi \circ (\varphi_\alpha^Y)^{-1}$ is e to 1 in a neighborhood of $\varphi_\alpha^Y(y)$ with y removed. So, in some small punctured neighborhood $V_x^0 = V_x \setminus \{x\}$ of x, the punctured neighborhoods $U_1^0, \dots, U_t^0$ above V_x^0 have this property: $\psi_{U_i^0} : U_i^0 \to V^0$ is everywhere e_i to 1. Since the degree of each fiber over $x \in V_{x_0}$ is n, conclude $\sum_{y\in Y_x} e_y = n$. This is the formula stated in the lemma.

Now assume $X = \mathbb{P}_z^1$. So, Chap. 4 §5.3.1 assigns to ψ a well-defined divisor: $Y_0 - Y_\infty$. Its degree is $\deg(Y_0) - \deg(Y_\infty) = n - n = 0$. Finally, let f be any global meromorphic function on Y. Then, locally f is a ratio of two holomorphic functions on a disk. At each point of the disk this defines a map to $\mathbb{P}_z^1$ which is analytic, even at the zeros of the denominator (Chap. 2 §4.6). So, f is an analytic map to $\mathbb{P}_z^1$. □

We often refer to a cover $\psi : Y \to X$ by the pair (Y, ψ). With the hypotheses of Lem. 2.1, call (Y, ψ) a *ramified cover of X of degree n*: $\deg(\psi) = n$. Then $D(\psi)$ consists of the *branch points* of ψ.

DEFINITION 2.2. Let $\psi : Y \to X$ be an analytic map of 1-dimensional complex manifolds (not necessarily compact or connected). If $(\psi)^{-1}(K)$ is compact for each compact subset K of X and $|(\psi)^{-1}(x)| = n$ for all but a discrete subset of points $x \in X$, then (Y, ψ) is a *finite* ramified cover of degree n. Denote the set $\{x \mid |Y_x| \neq n\}$ by $D(\psi)$.

2.1.2. *s-equivalence of covers.* Let $\psi^i : Y^i \to X$, $i = 1, 2$, be two finite ramified covers of X. Then (Y^1, ψ^1) and (Y^2, ψ^2) are *s(trong)-equivalent* (as ramified covers of X) if there is a one-one and onto continuous map $\psi : Y^1 \to Y^2$ for which $\psi^2 \circ \psi = \psi^1$. Colloquially: There is an isomorphism that commutes with the projection maps to the base. In §3.2.2 this corresponds to the notion of *absolute* s-equivalence; there is no extra condition on the s-equivalence of these covers.

Then, ψ is automatically an analytic isomorphism [11.2]. Clearly $D(\psi^1) = D(\psi^2)$. Using the phrase s-equivalence differentiates this from other equivalences of covers that appear later. The compactification process for covers of complex manifolds in higher dimensions does not necessarily produce a manifold, as it does in dimension 1 (Thm. 2.6). Still, the notion of s-equivalence makes sense and we extend its use to many situations.

Let D be a finite subset of the connected 1-dimensional compact complex manifold X. Cor. 2.9 classifies s-equivalence classes of finite ramified covers $\psi : Y \to X$ with $D(\psi) \subseteq D$. Restricting ψ to $Y_{X\setminus D(\psi)}$ gives an unramified cover. Therefore explicitly completing such a classification requires explicitly presenting the fundamental group $\pi_1(X \setminus D, x_0)$ for $x_0 \in X \setminus D$.

2.2. Constructing ramified covers. Now take X to be the Riemann sphere, $\mathbb{P}^1 = \mathbb{P}^1_z$. Versions of these results work in the general case [11.11].

2.2.1. *Product-One Condition.* Label points of $D(\psi)$ as $\{\boldsymbol{z}\} = \{z_1, \dots, z_r\}$. Let $z_0 \in \mathbb{P}^1 \setminus D(\psi) = U_{\boldsymbol{z}}$. Let $(\gamma_1, \dots, \gamma_r) = \boldsymbol{\gamma}$ be *classical generators* for $\pi_1(U_{\boldsymbol{z}}, z_0)$. A labeling $\boldsymbol{y} = (y_1, \dots, y_n)$ of the points of Y lying over z_0 determines a transitive permutation representation $T(\boldsymbol{y})$ of $\pi_1(U_{\boldsymbol{z}}, z_0)$ of degree n. This is as in Chap. 3 Thm. 7.16, except we now have additional information. Denote $T(\boldsymbol{y})([\gamma_i])$ by $g_i \in S_n$, $i = 1, \dots, r$, and let $G(\boldsymbol{g})$ be the subgroup of S_n the g_i s generate.

LEMMA 2.3. *With the hypotheses above, $g_1 \cdots g_r = 1$. Conversely, given elements $g_i \in S_n$, $i = 1, \dots, r$ satisfying $g_1 \cdots g_r = 1$, there exists a unique homomorphism $\psi_* : \pi_1(U_{\boldsymbol{z}}, z_0) \to S_n$ mapping γ_i to g_i, $i = 1, \dots, r$. This canonically produces a(n unramified) cover $\psi : Y^0 \to U_{\boldsymbol{z}}$ whose components correspond one-one to the orbits of $G(\boldsymbol{g})$ on $\{1, \dots, n\}$.*

PROOF. Thm. 1.8 says $\pi_1(U_{\boldsymbol{z}}, z_0)$ is a free group on $\boldsymbol{\gamma}$ modulo the product one relation $[\gamma_1 \cdots \gamma_r] = 1$ in the fundamental group. This implies the quotient relation

$$[\gamma_1 \cdots \gamma_r] = [\gamma_1] \cdots [\gamma_r] = g_1 \cdots g_r = 1.$$

Conversely, the product-one relation on the g_i s implies there is a homomorphism having the desired properties. The corresponding permutation representations on the orbits of $G(\boldsymbol{g})$ correspond to connected covers of $U_{\boldsymbol{z}}$. □

DEFINITION 2.4. We call the r-tuple $\boldsymbol{g} = (g_1, \dots, g_r)$ in Lem. 2.3 a *branch cycle description of the cover* $\psi : Y \to \mathbb{P}^1$ with respect to $\boldsymbol{\gamma}$.

The group $G(\boldsymbol{g})$ is the *monodromy group* of the ramified cover (Y, ψ) (with respect to $\boldsymbol{y}$). Refer to an r-tuple $\boldsymbol{g}' \in S_n^r$ for which there is β in S_n with $\beta^{-1} g_i \beta = g'_i$, $i = 1, \dots, r$, as *absolutely equivalent* to $\boldsymbol{g}$.

2.2.2. *Compactification of unramified Riemann surface covers.* The first part of Riemann's Existence Theorem, the part so technically useful, is that there is a unique compactification of any finite cover $\psi^0 : Y^0 \to U_{\boldsymbol{z}}$ to a cover $\psi : Y \to \mathbb{P}^1_z$ of compact Riemann surfaces. We now show this.

Let D_i be the disc about z_i in Fig. 3, $i = 1, \dots, r$. Consider $Y_{D_i} \to D_i$, the restriction of ψ over D_i. Then, $\bar{\gamma}_i$ generates $\pi_1(D_i \setminus \{z_i\}, b_i)$ which maps naturally to $\pi_1(\mathbb{P}^1 \setminus D(\psi), b_i)$. Identify $\pi_1(\mathbb{P}^1 \setminus D(\psi), b_i)$ with $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ using the path δ_i (of Fig. 3). Apply unique pathlifting along δ_i (Chap. 3 Lem. 7.13). So, the labeling on $\boldsymbol{y}$ uniquely labels points of Y over b_i.

With this, the permutation from $\bar{\gamma}_i$ on the fiber over b_i is g_i. Write $Y_{D_i\setminus\{z_i\}}$ as a disjoint union of connected components $\cup_{j=1}^{t_i} M_{i,j}$. Up to s-equivalence as a cover of $D_i \setminus \{z_i\}$, each $M_{i,j}$ corresponds to an orbit of $\pi_1(D_i \setminus \{z_i\}, b_i)$ on $\{1, 2, \dots, n\}$. Disjoint cycles in the decomposition of the generator g_i determine the orbits (Chap. 2 Prop. 7.4). The degree of $M_{i,j}$ as a cover is the length of its corresponding cycle, $i = 1, \dots, t$. Thus, g_i determines the covers $M_{i,1}, \dots, M_{i,t_i}$ (and their degrees).

Suppose $z_i = 0$ and D_0 is a disc about the origin in $\mathbb{C}$. Then, for each integer $e > 0$, the s-equivalence class of the connected cover of degree e is represented by

$$M' = \{(w, z) \in \mathbb{C} \times \mathbb{C} \mid w^e = z\}_{D_0\setminus\{0\}} \xrightarrow{\text{proj. on } z} D_0 \setminus \{0\}.$$

For each $z \in D_0 \setminus \{0\}$, let D_z be a disc about z contained in $D_0 \setminus \{0\}$. The components of M'_{D_z}, with their projections to $D_0 - \{0\}$, give an atlas on M'.

LEMMA 2.5. *The space $M' \cup \{(0, 0)\} = M$ has a complex manifold structure (extending that of M') that makes it a ramified cover of D_0 with exactly one point over* 0. *Indeed, M is analytically isomorphic to a disc.*

PROOF. The mapping $(w, z) \mapsto w$ gives a homeomorphism of $M' \cup \{(0, 0)\}$ to the subset of $\mathbb{C}$ that lies over D_0 via the map $w \mapsto w^e$. This subset is a disc around the origin, so it is complex analytically isomorphic to D_0. With this identification of M with D_0, add it to the atlas to conclude the manifold property. Compactness of the inverse image of a compact subset of D_0 follows easily (Def. 2.2). □

2.2.3. *From unramified to ramified covers.* Now for **Riemann's Existence Theorem**: Equivalence classes of ramified covers $\psi : Y \to X$ with $D(\psi)$ contained in a given set D' correspond exactly to classes of permutation representations of $\pi_1(X \setminus D', z_0)$ (Chap. 3 §7.2.2). Our next two results give formal restatements.

THEOREM 2.6. *Let $\boldsymbol{z} = \{z_1, \dots, z_r\}$ be a collection of r distinct points of $\mathbb{P}^1_z$. There is a one-one correspondence between connected unramified covers of $U_{\boldsymbol{z}}$ and connected covers of $\mathbb{P}^1$ ramified over a subset of $\boldsymbol{z}$.*

PROOF. From the opening remarks of this subsection we must show that if $\psi' : Y' \to \mathbb{P}^1 \setminus D'$ is an unramified cover, then there exists a unique ramified cover $\psi : Y \to \mathbb{P}^1$ such that $Y_{\mathbb{P}^1\setminus D'}$ is equivalent to (Y', ψ').

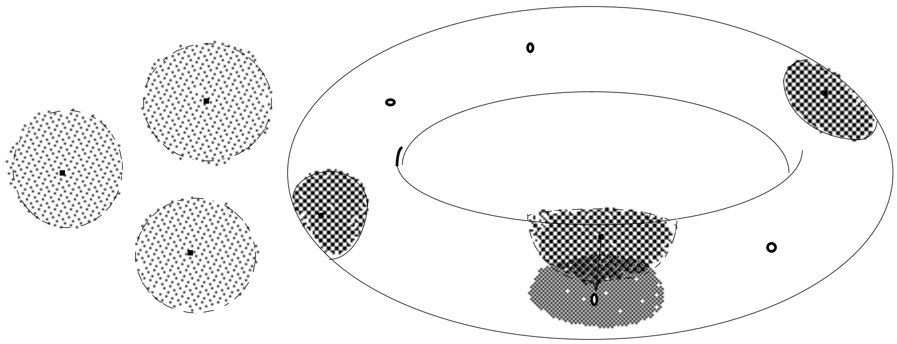
Use the notation prior to Lem. 2.5. For each $i = 1, \dots, r$, it shows how to add just one point $m_{i,j}$ to each component $M_{i,j}$, $j = 1, \dots, t_i$, of $Y'_{D_i\setminus\{z_i\}}$ to obtain a disjoint union $\cup_{j=1}^{t_i} \bar{M}_{i,j} = Y_i$ of manifolds with these properties.

(2.2a) There is a ramified covering map $\psi_i : Y_i \to D_i$.
(2.2b) $\psi_i^{-1}(D_i \setminus \{z_i\})$ is equivalent to $Y'_{D_i\setminus\{z_i\}}$.
(2.2c) $\bar{M}_{i,j}$ is analytically isomorphic to a disc.

The identification of $\bar{M}_{i,j}$ with a disc in (2.2c), $j = 1, \dots, t_i$; $i = 1, \dots, r$, added to an atlas for Y' gives an atlas for $Y = Y' \bigcup_{i,j}\{m_{i,j}\}$. Extend ψ' to $\psi : Y \to \mathbb{P}^1$ by mapping $m_{i,j}$ to z_i, $j = 1, \dots, t_i$; $i = 1, \dots, r$. Then Y_{D_i} is equivalent to Y_i, $i = 1, \dots, r$. Now we show Y is a compact manifold.

Since Y has an atlas, it is a manifold if it is Hausdorff. But $\mathbb{P}^1$ is Hausdorff. Thus if $y_1, y_2 \in Y$ with $\psi(y_1) \neq \psi(y_2)$, then we get $\psi^{-1}(U_1)$ and $\psi^{-1}(U_2)$, disjoint open sets, respectively, containing y_1 and y_2, by taking U_1 and U_2 to be disjoint open sets of $\mathbb{P}^1$, respectively, containing $\psi(y_1)$ and $\psi(y_2)$. Also, Y' is a manifold. Thus we only need consider $y_1, y_2 \in Y$ distinct points with $\psi(y_1) = \psi(y_2) = z_i$ for

FIGURE 6. Virtual neighborhoods awaiting a disc call—see Fig. 7



some $i = 1, \dots, r$. Therefore $y_1 = m_{i,\ell}$ and $y_2 = m_{i,k}$ for some $\ell \neq k$ between 1 and t_i. In particular, $\bar{M}_{i,\ell}$ and $\bar{M}_{i,k}$ are disjoint open sets, respectively, containing y_1 and y_2. The Hausdorff property follows.

For $z \in \mathbb{P}^1$ let D_z be a disc neighborhood of z. If $D_z \setminus \{z\}$ contains no points of D', then each component of Y_{D_z} contains a point of $\psi^{-1}(z)$. Thus the open sets Y_{D_z} form a neighborhood base for Y_z. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of Y. The fiber Y_z is contained in a finite union U_z of the sets U_α, so $Y_{D_z} \subset U_z$ for some choice of D_z. Since $\mathbb{P}^1$ is compact, $\mathbb{P}^1 = \cup_{i=1}^t D_{z_i}$ and $Y = \cup_{i=1}^t U_{z_i}$. Thus $\mathcal{U}$ has a finite subcover, and Y is compact.

The theorem is complete if we show $\psi : Y \to \mathbb{P}^1_z$ is unique. Let $\psi_1 : Y^1 \to \mathbb{P}^1$ be a ramified cover with $Y^1_{\mathbb{P}^1 \setminus D'}$ equivalent to (Y', ψ'), and therefore to $Y_{\mathbb{P}^1 \setminus D'}$. Thus there is an analytic isomorphism $\varphi : Y^1_{\mathbb{P}^1 \setminus D'} \to Y_{\mathbb{P}^1 \setminus D'}$. If φ extends to Y^1 then Lem. 2.1 shows Y^1 and Y are analytically isomorphic. Let $y \in (\psi_1)^{-1}(z_i)$ for some $i = 1, \dots, r$. Let U be a connected open neighborhood of y contained in some coordinate neighborhood with $\psi_1(U)$ contained in D_i. Since U is connected, φ maps $U \setminus \psi_1^{-1}(z_i)$ into $\bar{M}_{i,j}$ for some j. Riemann's removable singularities theorem extends φ to y uniquely [11.2b]. □

Conspicuous among covers of $U_{\boldsymbol{z}}$ that now compactify to a manifold are those from an algebraic function $f(z) \in \mathcal{E}(U_{\boldsymbol{z}}, z_0)$, labeled as X_f^0 in Chap. 3 Prop. 3.12.

DEFINITION 2.7. Call the manifold compactification X_f of X_f^0 (or more sloppily, of f) from Thm. 2.6 its *rs-compactification*. This theorem says any manifold compactification of X_f^0 will have a unique complex extending structure. Still, this notation differentiates X_f from a different compactification that might not have a manifold structure (as in Chap. 3 §4.2).

2.3. Combinatorial RET, algebraic and abelian covers. Let $\varphi : X \to \mathbb{P}^1_z$ be an analytic map of compact Riemann surfaces with $\boldsymbol{z}$ the branch points of φ. For $z' \in \mathbb{P}^1_z$ consider $D_{\varphi,z'} = D_{z'}$, the divisor of $\varphi - z'$ on X (Chap. 3 §5.3.1). For $z' = \infty$, interpret $D_{\varphi,\infty}$, the *polar* divisor, as counting (with multiplicity) points on X over ∞.

2.3.1. *An atlas from a compact cover.* For $z' \notin \boldsymbol{z} \cup \{\infty\}$, and $D_{z'} = \sum_{j=1}^n x_j$, choose a neighborhood $U_{z'}$ of z' and U_{x_i} so φ is invertible on U_{x_i}. As in Chap. 3 Prop. 3.12, use (U_{x_i}, φ) as a coordinate chart around x_i as $\varphi \stackrel{\text{def}}{=} w_{x_i} : U_{x_i} \to U_{z'} \subset$

$U_{\boldsymbol{z}} \setminus \{\infty\} \subset \mathbb{C}_z$. We extend this around ramified points (when $z' \in \boldsymbol{z}$) and the possibility $z' = \infty$, where e_i is the ramification index of x_i in the fiber $X_{z'}$ (§2.1), and $\{x_1, \dots, x_t\} = X_{z'}$. First, assume $z' \neq \infty$. As in applying (2.2), for some coordinate neighborhood (U_{x_i}, ψ_{x_i}) of x_i, (with $\varphi_{x_i}(x_i) = 0$) there is a branch of e_ith root of the function $\varphi \circ \psi_{x_i}^{-1} : \mathbb{C} \to \mathbb{C}$. So, there is a well defined function — designate it $w_{x_i} = \varphi^{1/e_i}$ — one-one in a neighborhood of x_i with w_{x_i} giving a coordinate chart about x_i. (Again select $U_{z'}$ to avoid ∞ and any other points of $\boldsymbol{z}$.) If $z' = \infty$, use $w_{x_i} = 1/\varphi^{1/e_i}$ instead.

Definition 2.8. Call $\{(U_x, w_x)\}_{x \in X}$ the *atlas for X from φ*. In basing a construction on this atlas, we must guarantee the result does not depend on the choice of branches of e_ith roots; we have made no canonical choice for these here.

2.3.2. *Algebraic and abelian covers of $\mathbb{P}^1_z$.* Combined with Nielsen classes (§3.2), Cor. 2.9 is the statement we use most often in describing types of covers.

Corollary 2.9. *Let $\boldsymbol{z} = \{z_1, \dots, z_r\}$ as in Thm. 2.6. Each set of classical generators $(\gamma_1, \dots, \gamma_r) = \boldsymbol{\gamma}$ for $\boldsymbol{z}$ based at $z_0 \in \mathbb{P}^1_z \setminus D'$ determines a one-one correspondence between equivalence classes of the following sets:*

(2.3a) *connected covers $\psi : Y \to \mathbb{P}^1_z$ with $D(\psi) \subseteq D'$ and $deg(\psi) = n$; and*
(2.3b) *r-tuples $\boldsymbol{g} = (g_1, \dots, g_r) \in S_n^r$ with $G(\boldsymbol{g})$ transitive, and $g_1 \cdots g_r = 1$.*

For a representative $\psi : Y \to \mathbb{P}^1_z$ of (2.3a) *and a labeling $\boldsymbol{y} = (y_1, \dots, y_n)$ of $\psi^{-1}(z_0)$, the correspondence produces a unique representative $\boldsymbol{g}$ of the class of* (2.3b)*; and the disjoint cycles of g_i identify with points of $\psi^{-1}(z_i)$, $i = 1, \dots, r$.*

Proof. From Thm. 2.6, elements of (2.3a) correspond to equivalence classes of unramified covers of $U_{\boldsymbol{z}}$. Excluding the last line, the corollary follows from the discussion prior to Def. 2.4. Given a representative $\psi : Y \to \mathbb{P}^1$ of a class of (2.3a), and a labeling $\boldsymbol{y}$ of $\psi^{-1}(z_0)$, the discussion following Def. 2.4 shows connected components of $Y_{D_i \setminus \{z_i\}}$ correspond uniquely to the disjoint cycles of g_i, $i = 1, \dots r$, in the correspondence of (2.3). Then, (2.2) gives a correspondence of the points of $\psi^{-1}(z_i)$ with the components of $Y_{D_i \setminus \{z_i\}}$, $i = 1, \dots, r$. This gives the corollary. □

Chap. 2 Thm. 8.8 describes all abelian algebraic functions of z. We compare that precise description with Cor. 2.9. An abelian cover $\varphi : X \to \mathbb{P}^1_z$ is one that is the compactification of a cover of $U_{\boldsymbol{z}}$ with abelian monodromy group. The same terminology is useful in describing *nilpotent* or *solvable* covers of any Riemann surface (or of any manifold if there is an appropriate construction of the compactification).

Definition 2.10 (Algebraic cover of $\mathbb{P}^1_z$). Call a cover of compact Riemann surfaces $\varphi : X \to \mathbb{P}^1_z$ *algebraic* if there is a second analytic map $f : X \to \mathbb{P}^1_w$ so that for some $z' \in U_{\boldsymbol{z}}$, f separates points in the fiber $X_{z'}$: $f(x') \neq f(x'')$ for distinct points $x', x'' \in X_{z'}$. Then, $\mathbb{C}(z, f) \stackrel{\text{def}}{=} \mathbb{C}(X)$ is the *field of functions* of X.

If $\varphi' : X' \to \mathbb{P}^1_z$ is s-equivalent to φ (§2.1.2), then φ is algebraic if and only if φ' is.

Proposition 2.11 (Algebraists' RET). *Every algebraic cover $\varphi : X \to \mathbb{P}^1_z$ is s-equivalent to to an rs-compactification (Def. 2.7) X_f of an algebraic function (*Chap. 3 *Prop. 3.12). The lattice of fields between $\mathbb{C}(z, f(z))$ and $\mathbb{C}(z)$ is dual to the lattice of covers $\varphi_Y : Y \to \mathbb{P}^1_z$ through which φ factors.*

Suppose $\hat{L}$ is the Galois closure of $\mathbb{C}(z, f(z)) = L$ over $\mathbb{C}(z)$, with branch points $\boldsymbol{z} = \{z_1, \dots, z_r\}$. Then a set of classical generators, $\gamma_1, \dots, \gamma_r$, for $\pi_1(U_{\boldsymbol{z}}, z_0)$ defines a set of embeddings $\psi_i : \hat{L} \to \mathcal{P}_{z_i, e_i}$ with e_i the ramification index of f over z_i.

Consider the restrictions $g_{z_i,\psi_i} \in G_f$ *of the canonical generator of* $G(\mathcal{P}_{z_i,e_i}/\mathcal{L}_{z_i})$ *to* $\hat{L}$, $i = 1,\dots,r$ *(*Chap. 2 *Lem. 7.9). Then* $(g_{z_1,\psi_1},\dots,g_{z_r,\psi_r}) = \boldsymbol{g}$ *generates* $G\hat{L}/\mathbb{C}(z)$ *and satifies the product-one condition.*

Any abelian cover of $\mathbb{P}^1_z$ *is the rs-compactification of an explicit algebraic function* f *from branches of log. So, each abelian cover of* $\mathbb{P}^1_z$ *is an algebraic cover.*

Proof. Consider the function $f : X \to \mathbb{P}^1_w$. As in Rem. 2.14, this produces an analytic structure on X. The phrase, f *is a meromorphic function on* X, means f and φ give same analytic structure on X.

As usual form $U_{\boldsymbol{z}} \subset \mathbb{P}^1_z$. Let V be an open set in $\varphi^{-1}(U_{\boldsymbol{z}})$ on which φ maps one-one to a disk D in $\mathbb{P}^1_z$. Use the notation φ_V^{-1} for the inverse map. Then, $f_D = f \circ \varphi_V^{-1} : D \to \mathbb{P}^1_w$ is meromorphic. Now we show the analytic continuations of f_D along paths in $U_{\boldsymbol{z}}$ satisfy Chap. 2 (1.1), properties. Chap. 2 Prop. 6.4 guarantee f_D is an algebraic function of z.

Let $z_0 \in D$, $x_1 \in V$ over z_0 and let $\gamma^* : [a,b] \to X$ be the unique lift to $\varphi^{-1}(U_{\boldsymbol{z}})$ starting at x_1. Consider analytic continuation of f_D along $\gamma \in \Pi_1(U_{\boldsymbol{z}}, z_0)$: $f_{D,\gamma}(t)$ is the function defined by $f \circ \gamma^*(t)$. This gives an analytic continuation according to Chap. 2 Def. 4.1. Further, analytic continuation gives only finitely many possible functions, the functions defined by f at the finite set of points above z_0. Similarly, test what happens as we approach the points $z' \in \boldsymbol{z}$. We evaluate f points with a limit on X. So the values remain bounded around a point of the range.

Now consider $\hat{L}$, the Galois closure of $\mathbb{C}(z, f(z)) = L$ over $\mathbb{C}(z)$, with branch points $\boldsymbol{z} = \{z_1,\dots,z_r\}$. First note that each element among the r classical generators $\gamma_1,\dots,\gamma_r$ defines an embedding of $\hat{L}$ in the corresponding $\mathcal{P}_{z_i,e_i}$. Write $\gamma_i = \delta_i\bar{\gamma}_i\delta_i^{-1}$ (as in Fig. 3), then δ_i gives an analytic contuation of f and all its conjugates to a disk neighborhood about z_i. Then, Chap. 2 Lem. 7.9, gives the desired embedding ψ_i. Generation and product-one conditions follow because they hold for the classical generators.

Finally consider when the cover φ has abelian monodromy. Chap. 2 (8.8) gives a branch cycle description with values in an abelian group. This was the hypothesis for producing an abelian function through branches of log. So, Chap. 2 Thm. 8.8 says branches of log display this unique cover (up to s-equivalence). □

2.3.3. *New covers from subfields of algebraic function fields.* Def. 3.5 explains normal fiber products of compact Riemann surface covers. This shows Prop. 2.11 directly gives many covers with nonabelian monodromy group as algebraic. §6 explains why any of the competing definitions of algebraic apply to algebraic covers.

Many uses of Riemann's Existence Theorem (including for the Inverse Galois Problem) require knowing covers are algebraic and more. Given f attesting to an algebraic cover, there is a unique $h(w) = w^n + \sum_{j=0}^{n-1} u_j(z)w^j \in \mathbb{C}(z)[w]$ (monic and irreducible in w) relating f to z in Prop. 2.11. We eventually need the minimal field (of *definition*) containing all coefficients (in z) of those u_j s, $j = 0,\dots,n-1$. We usually want the minimal such field as f varies. It is inefficient (sometimes hopeless), outside special cases, to compute f or h to find this out. There should be a good reason for doing such calculations. For example, theory might show there is a good choice of f, yet give reasons for looking more deeply at the algebraic relation. Our examples will show when theory is not yet sufficient to tell everything we want. Then, computing h may give us new clues about theory.

The best situation is that among these fields, as f varies, there is one that is minimal in that any nontrivial isomorphism of that field gives a new cover. This is the situation when the *field of moduli* is a field of definition (§6.2); §8.6 gives the first step in investigating this possibility and variant questions. *This is a question that tacitly assumes there is such an* f: One reason why Thm. 2.13 is so important.

Given that $\varphi : X \to \mathbb{P}^1_z$ is algebraic, we know that nonconsant elements of $\mathbb{C}(X)$ give all ways that X covers the Riemann sphere.

COROLLARY 2.12. *Each field L properly between $\mathbb{C}$ and $\mathbb{C}(X)$ corresponds to a cover $\psi : X \to Y$ with Y algebraic and the embedding $f \in \mathbb{C}(Y) \mapsto f \circ \psi \in \mathbb{C}(X)$ identifies $\mathbb{C}(Y)$ with L. Conversely, a cover ψ corresponds to subfield L.*

PROOF. For $w \in L$ nonconstant, $x \in X \mapsto w(x)$ gives a cover $\varphi_w : X \to \mathbb{P}^1_w$. Apply Thm. 2.11 to L between $\mathbb{C}(X)$ and $\mathbb{C}(w)$. Prop. 6.3 shows the converse. □

Though we do not complete showing all covers of compact Riemann surfaces are algebraic until Chap. 5 §**??**, we record that here. Examples in the remainder of this chapter emphasize aspects of applying Riemann's Existence Theorem. Several concentrate on showing the historical attention given to finding functions displaying covers as algebraic.

THEOREM 2.13. *Each cover $\varphi : X \to \mathbb{P}^1_z$ of compact Riemann surfaces is not only algebraic, it is $\mathbb{P}^1$-algebraic.*

REMARK 2.14 (Warning on constructing f in Def. 2.10). Suppose X is any compact Riemann surface and $f : X \to \mathbb{P}^1_w$ is any differentiable map with but finitely many points at which df is 0. There are many such maps. Thm. 2.6 says f induces a complex structure on X. Chances are, however, that complex structure will differ from that we started with. That is why it is difficult to construct an f that demonstrates a cover of $\mathbb{P}^1_z$ is algebraic.

2.4. Cuts and impossible pictures. Chap. 3 §7.2.3 discusses problems with traditional renderings of covers. Even the case when the degree n is 2, as in § 7.1. Assuming Y has a presentation as a sphere with g handles in $\mathbb{R}^3$, presenting the map ψ by a picture in $\mathbb{R}^3$ can be confusing. Still, something akin to Fig. 7 appears in many books; for example, [**Con78**, p. 243].

It includes all the usual elements, especially the *cuts*. We understand from [**Ne81**] that Gauss suggested cuts to Riemann (see §10.2). We don't rely on these cuts. Still, it will be valuable to see what they represent and how we can use symbols from them to draw pictures of the covers. The short and general description in §2.4.3 suffices for an alternate description of the manifold. The slower treatment in §2.4.1 establishes that the idea behind cuts is that covers are a locally constant structure.

2.4.1. *The simplest possible cuts.* The left of Fig. 7 represents a disc snipped and separated along a radius on the nonpositive real axis $\mathbb{R}^{\le 0} \stackrel{\text{def}}{=} \{x \le 0\} \dot\cup \{\infty\}$ from $-\infty$ to 0. Our perspective is taken from looking along the front edge. So the cut side that is on top has label T and the edge along the bottom has label B. The mathematical reality, however, is that (unlike the figure) we shouldn't separate the two sides of the cut (on either disk) by lifting one above the other. Rather, we intend just to remove the cut $\mathbb{R}^{\le 0}$, including the points (0 and ∞) at the ends of the cut. To continue the explanation, call the result of this $U_{z,l}$ and the corresponding figure on the right $U_{z,r}$. At each point z' of either of these two figures, the ring of functions we call analytic in a disc about z' (entirely within $U_{z,l}$ or $U_{z,r}$) identifies with the ring of analytic functions on $\mathbb{P}^1_z$ about that same disc regarded as on $\mathbb{P}^1_z$. Now let S be an open strip on $\mathbb{P}^1_z$ along the cut. Remove from S the negative real axis $\mathbb{R}^{<0} \stackrel{\text{def}}{=} \{x < 0\}\dot\cup\{\infty\}$ to leave two open substrips on each side of S.

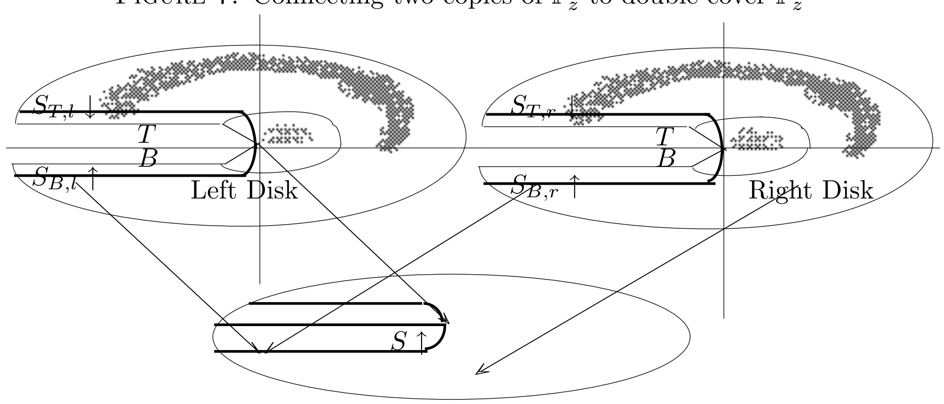
We want to consider the copies $S_{T,l}$ and $S_{B,l}$ (resp. $S_{T,r}$ and $S_{B,r}$) on $U_{z,l}$ (resp. $U_{z,r}$). These appear in Fig. 7. We also need two copies of S, S_l and S_r. Identify the analytic functions on each with those of S, exactly as you would expect from S being on $\mathbb{P}^1_z$.

The complex space X^0 we construct to cover $U_{0,\infty}$ consists of four pieces: $U_{z,l}$, $U_{z,r}$ and S_l and S_r. The map from all four pieces to $\mathbb{P}^1_z$ is exactly from the identification of each with a subset of $\mathbb{P}^1_z$. The only item left unsaid is the identification of points of $U_{z,l}$, $U_{z,r}$ and S_l and S_r between each of these four pieces. We don't want to identify these with points of $\mathbb{P}^1_z$ for this purpose. That would just give (two copies of) the manifold $U_{0,\infty}$ back. Here is the right final identification.

(2.4a) Points of $S_{T,l}$ identify with points of the corresponding strip on S_l, but $S_{B,l}$ identifies with the corresponding strip on S_r.
(2.4b) Identify points of $S_{T,r}$ with the points of the corresponding strip on S_r, but identify $S_{B,r}$ with the corresponding strip on S_l.
(2.4c) Make no further identifications.

Consider the path $\bar\gamma : [0,1] \to U_{0,\infty}$ by $t \in [0,1] \mapsto e^{-2\pi it}$ and let $\bar\gamma_1$ be its lift to X starting at $1 \in U_{z,l}$. We follow it to what happens as it gets to the different pieces. As t increases to $\frac{1}{2}$, within the points of $S_{B,l}$, switch to points we identify with them on S_r. Now cross $\mathbb{R}^{<0}$ on S_r, to get to points that identify with points on $S_{T,r}$. Finally, complete $\bar\gamma_1$ around to 1 on $U_{z,r}$. Total result: Traversing the unique lift of $\bar\gamma$ (a clockwise path) starting at $1 \in U_{z,l}$ ends at $1 \in U_{z,R}$. Exercise: Do the same with the lift of $\bar\gamma$ starting at $1 \in U_{z,R}$ to see it ends at $1 \in U_{z,l}$.

FIGURE 7. Connecting two copies of $\mathbb{P}^1_z$ to double cover $\mathbb{P}^1_z$



2.4.2. *Any cycle, and any one cut.* Instead of using the labels l(eft) and r(ight) in §2.4.1, we might have used x_1 and x_2 by associating everything on the left with the point $1 \in U_{z,l}$ renamed to x_1, and everything on the right with the point $1 \in U_{z,r}$ renamed to x_2. There was no loss in our picture of changing left to right to regard it as going from bottom to top. We generalize this in two steps. First: consider any integer n, two distinct points z'_1 and z'_2 on $\mathbb{P}^1_z$ and any simple, piecewise simplicial path $\delta_{z'_1,z'_2} : [a,b] \to \mathbb{P}^1_z$ starting at z'_1 and ending at z'_2.

Let $U_{\delta_{z'_1,z'_2},j}$ be a copy of $\mathbb{P}^1_z$ minus the range of $\delta_{z'_1,z'_2}$, $j = 1,\dots,n$. Think of these copies listed from left to right, according to their numbering ($U_{\delta_{z'_1,z'_2},j}$ on the far right). Let $S_{\delta_{z'_1,z'_2}}$ be a thin open strip playing the same role toward $\delta_{z'_1,z'_2}$ as did S toward $\mathbb{R}^{\le 0}$ (starting from ∞ and going toward 0 along the negative real axis). Then, consider copies of $S_{\delta_{z'_1,z'_2}}$, $S_{\delta_{z'_1,z'_2},j}$, $j = 1,\dots,n$, with their corresponding substrips $S_{\delta_{z'_1,z'_2},j,T}$ (on the left of $\delta_{z'_1,z'_2}$) and $S_{\delta_{z'_1,z'_2},j,B}$ (on the right of $\delta_{z'_1,z'_2}$). Let $\bar\gamma : [0,1] \to U_{z'_1,z'_2}$ by $t \mapsto z'_2 + r_0 e^{-2\pi i t + t_0}$ so $\bar\gamma$ meets $\delta_{z'_1,z'_2}$ precisely once and not when $t = 0$ ($\bar\gamma(0) = z_0 \notin \delta_{z'_1,z'_2}$). Label the point on $U_{\delta_{z'_1,z'_2},j}$ above z_0 as x_j, $j = 1,\dots,n$.

LEMMA 2.15. *Let $g \in S_n$. Then, there is a canonical equivalence on the union of the open sets $U_{\delta_{z'_1,z'_2},j}$ and $S_{\delta_{z'_1,z'_2},j}$, $j = 1,\dots,n$, so the following holds.*

(2.5a) *The resulting equivalence classes form a complex manifold X^0 giving an unramified cover $\varphi^0 : X^0 \to U_{z'_1,z'_2}$.*

(2.5b) *The unique lift of $\bar\gamma$ starting at x_j ends at $(j)g$, $j = 1,\dots,n$.*

So, $(g, \delta_{z'_1,z'_2})$ produces a canonical ramified cover $\varphi : X \to \mathbb{P}^1_z$ of compact Riemann surfaces, ramified only over z'_1 and z'_2, the completion of φ^0 from Thm. 2.6.

PROOF. We do the case $g = (1\,2\,\cdots\,n)$ and leave the adjustments for the general case as an exercise. Most of even this case imitates the case $n = 2$. To simplify notation, drop extra reference to the path $\delta_{z'_1,z'_2}$. The map of the union of the S_j s and U_j s to $U_{z'_1,z'_2}$ is by identifying the points (and the local complex functions) on these sets with those on $\mathbb{P}^1_z$. The only item left unsaid is the identification of points of the S_j s with corresponding points of the $S_{T,j}$ s and $S_{B,j}$ s.

(2.6a) Identify points of $S_{T,j}$ with the points of the corresponding strip on S_j, but identify $S_{B,j}$ with the corresponding strip on S_{j+1}.

(2.6b) Make no further identifications, except for $j = n$, we take $j + 1$ to be 1.

Do the rest of the lemma as [11.17a] requests. □

REMARK 2.16 (Locally constant structures). Chap. 3 Ex. 8.18 uses that degree n unramified covers are equivalent to locally constant bundles on $\{1,\dots,n\}$. Such structures, over $U_{\boldsymbol{z}}$ for example, are equivalent to looking at elements of $\mathrm{Hom}(\pi_1(U_{\boldsymbol{z}}), S_n)$. In Lem. 2.15, the sets S_j and U_j are simply connected. So above these sets, the cover consists of n connected copies of each of these sets. Using cuts is equivalent to explicitly laying out this locally constant structure.

2.4.3. *Any r rooted cuts.* Look again at the case of one cut. We may turn this into two rooted cuts by selecting any point z_0 along the cut. For simplicity assume for now it is not one of the endpoints of the cut. Now follow the procedure below.

Fig. 3 has the notation for the construction of classical generators. We show how the paths $\delta_1,\dots,\delta_r$ correspond one-one with r rooted cuts by the following simple device. Extend δ_i to a path $\bar\delta_i$ by adding the ray from b_i to z_i, $i = 1,\dots,r$. Thm. 1.8 says the *rooted bush* formed by the union of $\bar\delta_1,\dots,\bar\delta_r$ has simply connected complement, an essential property for having a collection of cuts on $U_{\boldsymbol{z}}$.

Any sequence of covers $Y \xrightarrow{\varphi_u} \mathbb{P}^1_u \xrightarrow{\varphi_{u,z}} \mathbb{P}^1_z$ gives three covers for which we would like an algorithm to precisely relate branch cycle descriptions. Especially, we have applications that allow computing classical generators for $\mathbb{P}^1_u$ automatically from classical generators for $\mathbb{P}^1_z$. This would allow constructing a branch cycle description for φ_u immediately from such a description for $\varphi_z = \varphi_{u,z} \circ \varphi_u$ (§6.3).

Suppose $\varphi : X \to \mathbb{P}^1_z$ is a ramified cover with branch cycles $\boldsymbol{g}$ from the classical generators that give $\bar{\delta}_1, \dots, \bar{\delta}_r$. This assumes a labeling $x_1, \dots, x_r$ of X_{z_0}. Then, we form a cover $\varphi_c : X_c \to \mathbb{P}^1_z$ from the cut construction canonically identifies with φ. Here are the ingredients.

(2.7a) Label copies of $\mathbb{P}^1_z$ as $\mathbb{P}^1_{z,j} = \mathbb{P}^1_j$, $j = 1, \dots, r$. On each remove the points labeled $z_0, z_1, \dots, z_r$ and call the result $\mathbb{P}_j$.

(2.7b) Use each element g_i and the cut $\bar{\delta}_i$ from z_0 to z_i to attach the $\mathbb{P}_j$ s along the lift of the ith cut. When done, compactify what we get.

We use the word *triangle* on a Riemann surface to mean a(clockwise oriented) boundary of a topological disk with the boundary divided into three oriented simplicial segments (edges) by three points called its vertices (Fig. 8). Call the triangle with its *interior* (which makes sense as the region to the right of the boundary) a (simplicial) *simplex*. The proof of Prop. 2.18 consists of describing these attachments and forming from them a natural triangulation of the result.

DEFINITION 2.17. A *triangulation* of a compact Riemann surface X is a cover of it by simplices satisfying these conditions. The simplex sides meet other simplices in their sides (in opposite orientation), and no two simplices have overlapping interiors. Let n_v (resp. n_e, n_s) be the number of vertices (resp. edges, simplices). The *Euler characteristic* of the triangulation is the alternating sum $n_v - n_e + n_s$.

Form a pre-manifold $\mathbb{P}^{\pm}_j$ (not Hausdorff) from $\mathbb{P}_j$ by replacing each point z along any one of the $\bar{\delta}_i$ s (minus its endpoints) by two points: z^+ and z^-. We put a new topology on a quotient relation on the union of $\{\mathbb{P}^{\pm}_j\}_{j=1}^n$. This uses an expected neighborhood basis at all points, except the pairs labeled z^+ and z^-: Disks not meeting any of the cuts $\bar{\delta}_1, \dots, \bar{\delta}_r$. The right neighborhood basis around z^+ and z^- on a cut use the following. Write $D_{j,z}$, a disk around z (on $\bar{\delta}_i$), as a union of $D^+_{j,z}$ and $D^-_{j,z}$: $D^+_{j,z}$ (resp. $D^-_{j,z}$) is all points on and to the left (resp. right) of $\bar{\delta}_i$.

PROPOSITION 2.18. *Compactifying X^0_c gives a cover $\varphi_x : X_c \to \mathbb{P}^1_z$ unramified over z_0. A map giving the equivalence to φ takes x_j to the point identified with z_0 on $\mathbb{P}^1_j$. Let t_i be the number of disjoint cycles in g_i, $i = 1, \dots, r$. The cuts from $\bar{\delta}_1, \dots, \bar{\delta}_r$ produce a triangulation of X_c with $n_s = 2nr$ simplices, $3nr$ sides and $2n + \sum_{i=1}^r t_i$ vertices. So the Euler characteristic of X_c is $2n + \sum_{i=1}^r t_i - nr$.*

PRECISE CUT PASTING. Form X^0_c as an equivalence relation on $\cup_{j=1}^n \mathbb{P}^{\pm}_j$. Suppose g_i maps k to l and z lies on $\bar{\delta}_i$. Then, identify $z^- \in \mathbb{P}^{\pm}_k$ with $z^+ \in \mathbb{P}^{\pm}_l$. In the resulting set, take a neighborhood of z^- to be $D^+_{l,z} \cup D^-_{k,z}$ identified along the part of $\bar{\delta}_i$ running through z.

Interpret the path $\delta_i \bar{\gamma}_i \delta_i^{-1} = \gamma_i$ in Fig. 3 as follows.

(2.8a) The lift of δ_i starting at z_0 on $\mathbb{P}^1_k$ rides along the right edge of the g_i-cut on $\mathbb{P}^1_k$ until it gets to $\bar{\gamma}_i$.

(2.8b) The initial point of $\bar{\gamma}_i$ is on the $^-$-edge of the g_i-cut on $\mathbb{P}^1_k$; it ends at the $^-$-edge of the g_i-cut on $\mathbb{P}^1_l$.

(2.8c) The lift of δ_i^{-1} starting at z_j on $\mathbb{P}^1_l$ rides along the $^-$ edge of g_i-cut on $\mathbb{P}^1_l$ until it gets to z_0.

So traversing the lift of γ_i from $z_0 \in \mathbb{P}^1_k$ will end at $z_0 \in \mathbb{P}^1_l$. Consider the small clockwise circle about z_0 denoted $\bar{\gamma}_0$ in Fig. 3. Our construction shows that traversing a lift of $\bar{\gamma}_0$ has the same effect on the points over z' in the range of $\bar{\gamma}_0$ as the product $\Pi(\boldsymbol{g}) = 1$ has on the integers $\{1, \dots, n\}$. It leaves them fixed. So, a deleted neighborhood of z_0 has above it n disjoint copies of that neighborhood on X^0_c. According to Lem. 2.5, the compactification does not ramify over z_0.

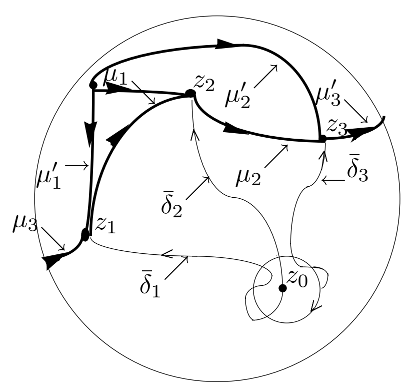
Triangulate $\mathbb{P}^1_z$ using the cuts $\bar{\delta}_1, \dots, \bar{\delta}_r$ and the proof of Thm. 1.8. Especially recall §1.5.2 showing the outside of the product of the classical generator paths bounds a disk. From this, draw paths μ_i from z_i to z_{i+1}, $i = 1, \dots, r-1$, and μ_r from z_r to z_1 with the following properties. The closed path $\mu_1 \cdot \mu_2 \cdots \mu_r$ bounds a closed (topological) disk $\bar{\Delta}_\infty$ that meets the $\bar{\delta}_i$ s only at the endpoint z_i s. From any point z_∞, interior to $\bar{\Delta}_\infty$, draw paths $\mu'_1, \dots, \mu'_r$, intersecting only at their beginning point, entirely in $\bar{\Delta}_\infty$ from z_∞ to the respective z_i s.

Triangulate $\mathbb{P}^1_j$ by listing the three ordered edges of the triangles:

$$
\begin{aligned}
&(\bar{\delta}_i, \mu_i, \bar{\delta}^{-1}_{i+1}),\ i = 1, \dots, r-1, \quad (\bar{\delta}_r, \mu_r, \bar{\delta}^{-1}_1), \\
&((\mu'_i)^{-1}, \mu'_{i+1}, \mu^{-1}_i),\ i = 1, \dots, r-1, \quad (\mu'_r)^{-1}, \mu'_1, \mu^{-1}_r).
\end{aligned} \tag{2.9}
$$

Now, triangulate X_c using the following simple principle. Each of the $2r$ triangles in (2.9) bounds a simplex with exactly two endpoints in $\boldsymbol{z}$. Let S be one of these. Remove the two points from $\boldsymbol{z}$ in the boundary; call this S^0. It is simply-connected, and $\varphi_c : X_c \to \mathbb{P}^1_z$ is unramified over it. So, S^0 has n connected components $S^0_1, \dots, S^0_n$ over it. With each take the closure in X_c (adding back points of X_c over $\boldsymbol{z}$). These simplices give the triangulation of X_c. Just count to get the statement of the proposition. □

FIGURE 8. Cuts for a triangulation of X_c when $r = 3$



REMARK 2.19. The expression for the Euler characteristic in Prop. 2.18 is $2 - 2g_{\boldsymbol{g}} = \chi_X$, appearing in Prop. 3.10. This shows, all triangulations of a compact Riemann surface X from presenting φ using cuts, have the same Euler characteristic. We leave the following observations to the many topology books that treat Euler characteristic in detail and generality. We will do exercises in that direction to illustrate how it works.

(2.10a) χ_X is an invariant of the homeomorphism class of the compact of X (whether from cuts or not).
(2.10b) If the Euler characteristic is 2 then X is topologically a sphere: genus 0.
(2.10c) If the Euler characteristic is 0 then X is topologically a torus (as in Chap. 3 Fig. 2): genus 1.

To conclude these results from a triangulation of X in either case requires only laying out on the sphere (resp. torus) an *equivalent triangulation* [11.6].

REMARK 2.20 (Using a branch point as a base point). The beginning literature on Riemann surfaces has figures with cuts. Often the cuts don't have an obvious base point z_0 attached to them. That early literature is usually about the nature of integrals of meromorphic differentials around closed paths. So the fundamental group action is through the first homology group $H_1(U_{\boldsymbol{z}})$. As in Lem. 7.1, analytic continuations of the primitive give a complicated analytic continuation action (of course, not through a finite group). Since this is about integration, [11.16b] explores how to use a branch point as a base point for the cuts.

2.5. Residues and traces. Cauchy's Residue Theorem (Chap. 2 §5.4.4) implies the sum of the residues of any meromorphic differential ω on $\mathbb{P}^1_z$ is 0. We prove the same holds on any compact Riemann surface X. Then we give Abel's famous necessary condition for a divisor on X to be the divisor of a meromophic function $\varphi : X \to \mathbb{P}^1_z$. That it is also sufficient is the cornerstone of the theory of Riemann surfaces (§7.6 for surfaces of genus 0 and 1, and Chap. 5 §**??** in general).

2.5.1. *Sum of the residues is 0.* Let $\omega \in \mathcal{M}^1(X)$ be a meromorphic differential on the compact Riemann surface X. Chap. 2 §4.3 has the definition of the residue of a meromorphic differential at $z_0 \in \mathbb{C}_z$. Since X is compact, ω has but finitely many poles (as in the argument for Lem. 2.1). So, it has only finitely many points at which there is a nonzero residue. There are two approaches to showing the sum of the residues of ω is 0. We use here Green's Theorem, to have available the exterior calculus for later. Another approach, reducing the sum of the residues to exactly Cauchy's Theorem in the plane comes from uniformization [11.11].

2.5.2. *Orientation and Green's Theorem.* When we say a path bounds a closed disk $\bar{D}'$ in X we mean here that the oriented path has the disc on its left. Suppose X is 2-dimensional differentiable manifold with atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$. Use (x_α, y_α) for the variables of the range of $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$. For $\alpha, \beta \in I$, use $F_{\beta,\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ for the transition function on $\varphi_\alpha(U_\alpha \cap U_\beta)$. A differential 2-form on X consists of giving $f_\alpha(x_\alpha, y_\alpha)\, dx_\alpha \wedge dy_\alpha$ for each $\alpha \in I$, satisfying these two conditions.

(2.11a) $f_\alpha(x_\alpha, y_\alpha) : \mathbb{R}^2 \to \mathbb{C}$ is differentiable on U_α.
(2.11b) $f_\beta(F_{\beta,\alpha}(x_\alpha, y_\alpha)) = \mathrm{Det}(J(F_{\beta,\alpha}(x_\alpha, y_\beta)))f_\alpha(x_\alpha, y_\alpha)$ where $J(F)$ denotes the Jacobian matrix as in Chap. 3 Lem. 3.2.

[11.4] reminds that 2-forms appear to form integrals over 2-dimensional subsets of X. The change of variables $(x_\alpha, y_\alpha) \mapsto (y_\alpha, x_\alpha)$ would change the sign of this 2-form. In the case $f_\alpha(x_\alpha, y_\alpha)$ is invariant under this transformation, the new contribution to integrating over U_α would subtract from, not add to, the integral. Fortunately, that is not an allowable transformation of coordinates on a 1-dimensional complex manifold. The (x_α, y_α) coordinates come from the complex coordinates $x_\alpha + iy_\alpha$. Any analytic change of $x_\alpha + iy_\alpha$ leaves the sign of the determinant positive [11.4a].

Definition 2.21 (Orientation). An *orientation* on a differentiable dimension 2 manifold is a choice of subatlas for which the determinant of the coordinate transformation Jacobian is always positive.

Proposition 2.22 (Green's Theorem). *Suppose ω is meromorphic 1-form in a domain $D \subset X$. The residue of ω has a well-defined meaning at each $x' \in D$. Denote the set of $x' \in D$ at which ω has a nonzero residue by $R_\omega(D)$. Let γ be a disjoint union $\gamma_1, \dots, \gamma_t$ of simple closed paths on D, where each γ_i is the counterclockwise boundary of a closed topological disk $\bar{D}_i \subset D$. Assume ω has no poles on γ and all its residues are in $\cup_{i+1}^t \bar{D}_i$. Then, $\frac{1}{2\pi i}\int_\gamma \omega = \sum_{x' \in R_\omega(D)} \mathrm{Res}_{x'}(\omega)$. In particular, if $D = X$, then $\frac{1}{2\pi i}\int_\gamma \omega = 0$.*

More generally, let ω' be any differentiable differental 1-form on the domain $D \setminus \cup_{i+1}^t \bar{D}_i$ as above. Then, there is a differential 2-form $d\omega'$ on D so that

$$\int_\gamma \omega = \int_{D \setminus \cup_{i=1}^t \bar{D}_i} d\omega. \tag{2.12}$$

Proof. We show the last paragraph first. Use the notation from above for a 2-dimensional differentiable manifold. Then, on $\varphi_\alpha(U_\alpha)$ from the coordinate chart $(U_\alpha, \varphi_\alpha)$, express ω' as $f_\alpha(x_\alpha, y_\alpha)\, dx_\alpha + g_\alpha(x_\alpha, y_\alpha)\, dy_\alpha$. The production of the differential 2-form from ω' comes from the exterior derivative:

$$\begin{aligned} d(\omega') = &\ df_\alpha \wedge dx_\alpha + dg_\alpha \wedge dy_\alpha \\ = &\ \tfrac{\partial f_\alpha}{\partial y_\alpha}\, dy_\alpha \wedge dx_\alpha + \tfrac{\partial g_\alpha}{\partial x_\alpha}\, dx_\alpha \wedge dy_\alpha = (\tfrac{\partial g_\alpha}{\partial x_\alpha} - \tfrac{\partial f_\alpha}{\partial y_\alpha})\, dx_\alpha \wedge dy_\alpha. \end{aligned} \tag{2.13}$$

We must establish this is a 2-form: (2.11b) holds [11.23b]. Then, the integration on the right of (2.12) is independent of the coordinate chart. We already know that is true of the integration on the left from Chap. 2 Lem. 2.3. Then, the conclusion is a consequence of Green's Theorem from vector calculus in the plane. While [**Rud76**, p. 272] has a complete treatment, as our paths are semi-simplicial, the case of bounding by rectangles suffices.

To apply the result to the first paragraph requires only noting that if we are on a 1-dimensional complex manifold, then locally an analytic differential has the form $f_\alpha(z_\alpha)\, dz_\alpha$. In that case the Cauchy-Riemann equations immediately imply $d(f_\alpha(z_\alpha)\, dz_\alpha) = 0$ [11.4b]. □

Remark 2.23. Apply Thm. 2.25 to the translate of φ by a constant, $\varphi - c$, $c \in \mathbb{C}$. Conclude $\deg(D_{z'})$ is constant running over all $z' \in \mathbb{P}^1_z$, a case of Lem. 2.1.

2.5.3. *Traces of differentials and functions.* Let $\varphi : X \to \mathbb{P}^1_z$ be an analytic map of compact Riemann surfaces. Use notation from the coordinate chart from φ (Def. 2.8). Denote meromorphic differentials on X by $\Gamma(X, \mathcal{M}^1)$. Suppose $\omega \in \Gamma(X, \mathcal{M}^1)$, and $\boldsymbol{z}$ is the branch point set of φ. For $z' \notin \boldsymbol{z}$, consider $D_{z'} = \sum_{j=1}^n x_j$. Since z' is not a branch point, there is a neighborhood $U_{z'}$ of z' and U_{x_i} so φ is invertible on U_{x_i}. To keep our neighborhoods straight, denote the inverse of φ on U_{x_i} by φ_i^{-1}. For each $i \in \{1, \dots, n\}$, $\varphi_i^{-1} : U_{z'} \to U_{x_i}$ is a section for φ. Denote the local variable on U_{x_i} by w_i. On $\varphi_i(U_{x_i})$ write ω as

$$h_i(w_i \circ \varphi_i^{-1}(z))\, d(w_i \circ \varphi_i^{-1}(z)).$$

Define $\mathrm{t}(\omega)$ on $U_{z'}$ as a differential in z by $\sum_{i=1}^n h_i(w_i \circ \varphi_i^{-1}(z))\, d(w_i \circ \varphi_i^{-1}(z))$.

We extend this around ramified points (when $z' \in \boldsymbol{z}$) where e_i is the ramification index of x_i in the fiber $X_{z'}$ and $\{x_1, \dots, x_t\} = X_{z'}$. Let $\zeta_{e_i} = e^{2\pi i/e_i}$, exactly as in Lem. 2.5. To simplify notation designate φ as z. The only extension of $\mathrm{t}(\omega)$ that gives the same values over a deleted neighborhood of $U_{z'}$ requires the expression $\sum_{i=1}^{t}\sum_{j=0}^{e_i-1} h_i(\zeta_{e_i}^j z^{1/e_i})d(\zeta_{e_i}^j z^{1/e_i})$ for $\mathrm{t}(\omega)$. Write $dz = e_i z^{\frac{e_i-1}{e_i}} dw_i$ to reexpress the contribution around x_i as

$$\sum_{j=0}^{e_i-1} \frac{h_i(\zeta_{e_i}^j z^{1/e_i})}{e_i z^{\frac{e_i-1}{e_i}}}\, dz. \tag{2.14}$$

So, (2.14) is a Laurent series in z^{1/e_i} times dz, symmetric in $\{\zeta_{e_i}^j z^{1/e_i}\}_{j=0}^{e_i-1}$, the conjugates of z^{1/e_i} over $\mathbb{C}\{\{z\}\}$. Conclude: Each term in $\mathrm{t}(\omega)$, the *trace* of ω is a Laurent series in z (times dz), and $\mathrm{t}(\omega)$ is a differential on $\mathbb{P}^1_z$.

REMARK 2.24. There is a similar definition of trace for meromorphic functions (elements of $\mathbb{C}(X)$) on X. Further, the following extensions are also easy: We may replace $\mathbb{P}^1_z$ by any Riemann surface Y (not necessarily compact) and $\varphi : X \to Y$ is a ramified cover. Recall: Regard meromorphic differentials (resp. functions) on Y as meromorphic differentials (resp. functions) on X by pullback (Chap. 3 §5.3.3).

THEOREM 2.25. *Given a ramified cover $\varphi : X \to Y$ of Riemann surfaces, the trace $\mathrm{t} = \mathrm{t}_{X/Y}$ from meromorphic differentials on X to those on Y is a $\mathbb{C}$-linear. It maps homolomorphic differentials to holomorphic differentials. In particular, if $Y = \mathbb{P}^1_z$, then the range of* t *on holomorphic differentials is 0.*

If $\omega \in \Gamma(Y, \mathcal{M}^1)$, then $\mathrm{t}(\varphi^*(\omega)) = \deg(X/Y)\omega$.

PROOF. Consider the statements on holomorphicity. If ω is holomorphic, each h_i above is holomorphic. From (2.14), $\mathrm{t}(\omega)$ has a pole of order no more than $\frac{e_i-1}{e_i}$ at z'. The order, however, of the pole must be an integer. That means it has no pole at z' and ω is holomorphic. As there are no holomorphic differentials on the sphere (Chap. 3 Ex. 5.17), $\mathrm{t}(\omega)$ vanishes.

More generally, if ω is any differential, then its trace has the same sum of residues as does ω. This comes back to the case the differential is locally dx_i/x_i with its trace locally reexpressed as $\sum_{j=0}^{e_i-1} h(\zeta_{e_i}^j z^{1/e_i})/e_i z^{\frac{e_i-1}{e_i}} dz$ with $h_i = 1/x_i$. The final equation is a consequence of the definitions and Rem. 2.24. $\square$

2.6. Abel's necessary condition. With X a compact Riemann surface, let $\Gamma(X, \Omega)$ be the vector space of global holomorphic differentials on X. We don't know its dimension yet, though Lem. 6.14 shows it is $g_X = g_{\boldsymbol{g}}$ (as in Thm. 3.10). Suppose $D^0 = \sum_{i=1}^n x_i^0$ and $D^\infty = \sum_{i=1}^n x_i^\infty$ are two degree n divisors on X. We allow some points repeated with multiplicity.

Consider those n-tuples of paths $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ for which there is $\sigma \in S_n$, with γ_i having beginning point x_i^0 and end point $x_{(i)\sigma}^\infty$, $i = 1, \dots, n$. Denote these by $\Pi_1(X, D^0, D^\infty)$. If σ is the appropriate permutation, define the endpoint evaluation map by $E_{D^0,D^\infty}(\boldsymbol{\gamma}) = \sigma$. When the support of D^∞ consists of distinct points, this defines π uniquely, otherwise it is a coset of the subgroup of permutations stabilizing the ordered set $(x_1^\infty, \dots, x_n^\infty)$.

2.6.1. *Integrating a basis of holomorphic differentials.* Abel's necessary condition tests for existence of a meromorphic degree n function on X whose divisor of zeros (resp. poles) is D^0 (resp. D^∞). It is tacit that D^0 and D^∞ have no common support and are both positive divisors. Lem. 2.1 says the divisor of zeros and poles determine a function up to multiplication by a constant.

The test on integrals is made efficient by using a basis $\mathcal{B} \stackrel{\text{def}}{=} (\omega_1, \dots, \omega_u)$ for $\Gamma(X, \Omega)$. We integrate the entries of $\mathcal{B}$ along elements of $\Pi_1(X, D^0, D^\infty)$. Such integrals are equivalent to evaluating analytic continuations of a branch of a primitive (Chap. 2 §4.3). So, the monodromy theorem says results will only depend on homotopy classes of such paths (with their endpoints fixed; Chap. 2 Thm. 8.3). Denote these $\pi_1(X, D^0, D^\infty)$. For these definitions we may allow common support to D^0 and D^∞. When, however, $D^0 = D^\infty$, write $\pi_1(X, D^0)$ for the homotopy classes of n-tuples of closed paths. In this case, the paths in an ordered n-tuples of paths may each have a different end point than beginning point. The case $D^0 = nx_0$ is allowed, to indicate an n-tuple of closed paths.

From Thm. 2.6, each meromorphic function on X gives an analytic map $\varphi : X \to \mathbb{P}^1_z$. This gives a map from $\boldsymbol{\gamma} \in \pi_1(X, D^0, D^\infty)$ to the integral of $\mathcal{B}$ over $\boldsymbol{\gamma}$:

$$\mathrm{Int}_{D^0,D^\infty} = \mathrm{Int}_{X,D^0,D^\infty}(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \int_{\boldsymbol{\gamma}} \mathcal{B} = (\sum_{j=1}^{n} \int_{\gamma_j} \varphi_1, \dots, \sum_{j=1}^{n} \int_{\gamma_j} \varphi_u).$$

THEOREM 2.26. *The range of* Int_{X,mx_0}*, for* $x_0 \in X$*, is an abelian subgroup* L_X *of* $\mathbb{C}^u$*, independent of either* z_0 *or* $m \geq 1$*. A change of basis for* $\Gamma(X, \Omega)$ *changes* L_X *by the action (on the left) of some element of* $\mathrm{GL}_n(\mathbb{C})$*.*

Suppose there is a nonconstant analytic map $\varphi : X \to \mathbb{P}^1_z$ *with* $D^0 = \varphi^{-1}(0)$ *and* $D^\infty = \varphi^{-1}(\infty)$*. Then,* $\ker(\mathrm{Int}_{D^0,D^\infty}) \neq \emptyset$ *and the range of* $\mathrm{Int}_{D^0,D^\infty}$ *is* L_X*.*

Let $\boldsymbol{z}$ *be the branch points of* φ*, and suppose* $0 \notin \boldsymbol{z}$ *(resp.* $\infty \notin \boldsymbol{z}$*). Then,* $\pi_1(U_{\boldsymbol{z}}, 0)$ *(resp.* $\pi_1(U_{\boldsymbol{z}}, \infty)$*) has a faithful left (resp. right) action on* $\ker(\mathrm{Int}_{D^0,D^\infty})$*. Therefore,* $\{E_{D^0,D^\infty}(\boldsymbol{\gamma})\}_{\boldsymbol{\gamma} \in \ker(\mathrm{Int}_{D^0,D^\infty})}$ *contains the monodromy group* G_φ *of* φ*. This holds even if* $0 \in \boldsymbol{z}$ *(resp.* $\infty \in \boldsymbol{z}$*) using a tangential base point at* 0 *(resp.* ∞*).*

2.6.2. *Proof of Thm. 2.26 and integrations along* $\gamma \in \pi_1(U_{\boldsymbol{z}}, z_0)$*.* Consider $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \pi_1(X, mx_0)$. Then, the component wise product $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}' = (\gamma_1 \cdot \gamma_1', \dots, \gamma_1 \cdot \gamma_1')$ is in $\pi_1(X, mx_0)$. Apply Int to these to see the range is independent of m and is an abelian group. Given another basis $\mathcal{B}'$, there exists $A \in \mathrm{GL}_n(\mathbb{C})$ so that $A(\mathcal{B}) = \mathcal{B}'$. Therefore $A(\int_\gamma(\mathcal{B})) = \int_\gamma A(\mathcal{B})$ has range in $A(L_X)$.

Now suppose φ exists. Start with the case D^0 and D^∞ have n distinct points in their support. Let $\gamma \in \pi_1(U_{\boldsymbol{z}}, 0, \infty)$, and define $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$ so γ_i is the unique lift of γ starting at x_i^0. Write $\gamma : [0,1] \to U_{\boldsymbol{z}}$ to define $(\gamma_1(t), \dots, \gamma_n(t))$ for $t \in [0,1]$, an ordering of $\varphi^{-1}(t)$.

Apply Thm. 2.25 to $\mathrm{Int}_{D^0,D^\infty}(\boldsymbol{\gamma})$ by designating the trace from φ by t_φ. Then, $\mathrm{Int}_{D^0,D^\infty}(\boldsymbol{\gamma})$ is just $(\int_\gamma \mathrm{t}_\varphi(\omega_1), \dots, \mathrm{t}_\varphi(\omega_u))$. Since each of the integrand entries is 0, this shows that any element of $\pi_1(U_{\boldsymbol{z}}, 0, \infty)$ defines an element of $\ker(\mathrm{Int}_{D^0,D^\infty})$.

If either D^0 or D^∞ has support with multiplicity, connect 0 and ∞ by paths $\gamma_{z'}$ and $\gamma_{z''}$ to respective points z' and z'' that lie (excluding endpoints) entirely in $U_{\boldsymbol{z}}$. Let γ denote a path in $U_{\boldsymbol{z}}$ connecting z' and z''. There is still an n-tuple of lifts of the path $\gamma_0 \cdot \gamma \cdot \gamma_\infty^{-1} : (0,1) \to X_0$ (avoiding endpoints). Now form the paths to replace $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ by taking the closure of these lifted paths in X. The integral is 0, again from Thm. 2.25.

Similarly, we may compose on the right of $\mathrm{Int}_{D^0,D^\infty}$ by $\pi_1(U_{\boldsymbol{z}}, \infty)$ so long as 0 and ∞ are not in $\boldsymbol{z}$. Suppose, however, $0 \in \boldsymbol{z}$ (the case for ∞ is analogous). Then, $\pi_1(U_{\boldsymbol{z}}, 0)$ doesn't make sense.

Chap. 2 §8.4 has the notion of a tangential base point. We need a convenient (nonempty) simply connected open set $D_{\boldsymbol{v}}$ tangent to 0 in $U_{\boldsymbol{z}}$. The choice there was a disk with 0 on the boundary, defined by a tangent vector $\boldsymbol{v}$ to 0. Let $\lambda : [0,1] \to \bar{D}_{\boldsymbol{v}}$ be a path with these properties: $\lambda(0) = 0$, restriction to $(0,1]$ has range in $D_{\boldsymbol{v}}$ and $\lambda(1) = z' \in D_{\boldsymbol{v}}$. Consider paths (minus beginning and endpoint) given by $\lambda_{(0,1]} \cdots \gamma \lambda^{-1}_{(0,1]}$, γ representing an element of $\pi_1(U_{\boldsymbol{z}}, D_{\boldsymbol{v}}, z')$. This has n distinct lifts to X. Their closures have their beginning and end points in D^0.

Up to homotopy, these paths don't depend on z' or λ (though it does on $D_{\boldsymbol{v}}$). So, up to homotopy, composition of these paths defines a group $\pi_1(U_{\boldsymbol{z}}, D_{\boldsymbol{v}})$ with an action on the left of $\ker(\mathrm{Int}_{D^0, D^\infty})$. The isomorphism class of the group is the same no matter the choice of $D_{\boldsymbol{v}}$. There is, however, no canonical isomorphism between the groups if you change $D_{\boldsymbol{v}}$ to another tangential disk [11.16a].

3. Nielsen classes and Hurwitz monodromy

This section introduces combinatorial group theory that helps display the myriad covers from Cor. 2.9. §4 uses this to illustrate Riemann's Existence Theorem. We suggest the reader go between the two sections on a first reading; we put many concepts together in this section. That includes interpretation of the genus of a compact surface, and the related fiber product and Galois closure of compact covers topics. Braid and Hurwitz monodromy representations are critical to this book. [**Ar25**], [**Ar47**], [**Bi75**], [**Boh47**], [**Ch47**], [**KMS66**], [**Ma34**], [**Mar45**] hint at early literature on the Braid group. None, however, of these sources apply these to the families of Riemann surface covers. Further, the Hurwitz monodromy group is a modest player in them though some of their combinatorics, especially [**Boh47**] and [**KMS66**], appears in our picture.

3.1. Artin Braids and Hurwitz monodromy. Let F_r be the free group on the elements of $S = \{s_1, \dots, s_r\}$. Since F_r is a free group, any r words $w_1, \dots, w_r$ in S determine a homomorphism of F_r into itself by mapping the ordered r-tuple $(s_1, \dots, s_r) = \boldsymbol{s}$ respectively to $(w_1, \dots, w_r)$. So, given $\boldsymbol{s}$, any other r-tuple, $(s'_1, \dots, s'_r)$, of generators of F_r determines an element of the automorphism group $\mathrm{Aut}(F_r)$ of F_r. Denote the set of (ordered) r-tuples of generators of F_r by $\mathcal{G}_{F_r}$.

3.1.1. *Automorphisms of* $\pi_1(U_{\boldsymbol{z}}, z_0)$ *permuting classical generators.* Certain automorphisms of $\pi_1(U_{\boldsymbol{z}}, z_0)$ play a big role from here on. Chap. 5 describes the geometry that produces them. Here they are a combinatorial tool.

Let Q_i be the permutation of $\mathcal{G}_{F_r}$ that sends entries of $(s_1, \dots, s_r) = \boldsymbol{s}$ (in order) to the new r-tuple of generators

$$(3.1) \qquad (s_1, \dots, s_{i-1}, s_i s_{i+1} s_i^{-1}, s_i, s_{i+2}, \dots, s_r),\ i = 1, \dots, r-1.$$

The *Artin braid group* (of degree r), is the subgroup of permutations of $\mathcal{G}_{F_r}$ that $Q_1, \dots, Q_{r-1}$ generate. We denote it B_r.

LEMMA 3.1. *Any* $\boldsymbol{s} \in \mathcal{G}_{F_r}$ *gives a faithful map* $\psi_{\boldsymbol{s}} : B_r \to Aut(F_r)$*:* $Q \in B_r$ *maps to the automorphism that takes* $\boldsymbol{s}$ *to* $(\boldsymbol{s})Q$*. Suppose* $Q \in B_r$ *and* $\alpha \in \mathrm{Aut}(F_r)$*. Then,* Q *acts on* α *by this formula:* $(\boldsymbol{s})\alpha^Q \stackrel{\text{def}}{=} (\boldsymbol{s})Q^{-1}\alpha Q$*. The action of* B_r *on inner automorphisms of* F_r *is trivial. Also,* $\psi_{\boldsymbol{s}}$ *is a 1-cocycle on the group* B_r*:* $(QQ')\psi_{\boldsymbol{s}} = ((Q')\psi_{\boldsymbol{s}})(Q)\psi_{\boldsymbol{s}}^{Q'}$.

PROOF. The effect of $Q \in B_r$ on any one $\boldsymbol{s} \in \mathcal{G}_{F_r}$ determines it. So, $\psi_{\boldsymbol{s}}$ is faithful. Notice that conjugation by w commutes with the action of Q_i on $\boldsymbol{s}$. As these are generators of B_r, this implies $\alpha^Q = \alpha$ for $Q \in B_r$ if α is conjugation by w. Check how both sides of the cocycle condition act on $\boldsymbol{s}$:

$$(\boldsymbol{s})QQ' = (\boldsymbol{s})(QQ')\psi_{\boldsymbol{s}} = ((\boldsymbol{s})Q)Q' = (((\boldsymbol{s})Q'Q'^{-1})(Q)\psi_{\boldsymbol{s}})Q' = ((\boldsymbol{s})(Q')\psi_{\boldsymbol{s}})(Q)\psi_{\boldsymbol{s}}^{Q'}.$$

This concludes the lemma. □

3.1.2. *Hurwitz monodromy quotient of the braids.* The word *cocycle* in Lem. 3.1 has a more complicated meaning than in Chap. 3 §5.4.1 where it was a condition on transition functions. This is a group cocycle, for a group acting on a nonabelian group (rather than on a module). Our emphasis is that $\psi_{\boldsymbol{s}}$ is a cocycle, not a homomorphism. The *Hurwitz monodromy group* (of degree r) is the quotient of B_r by the normal subgroup generated by

$$Q(r) = Q_1Q_2\cdots Q_{r-1}Q_{r-1}\cdots Q_2Q_1. \tag{3.2}$$

Denote this quotient group by H_r.

Observations from the following proposition will appear in examples of §4. It simplifies reading Chap. 5 to be already acquainted with these. Let $\bar{R}$ be the normal subgroup of F_r that $s_1\cdots s_r = u_{\boldsymbol{s}}$ generates (Ex. 1.3). Denote $F_r/\bar{R}$ by G_r.

PROPOSITION 3.2. *The following properties hold for B_r (acting on $\mathcal{G}_r$).*

(3.3a) *Each $Q \in B_r$ maps $s_1\cdots s_r$ to itself and s_i to a conjugate of s_j for some j (dependent on i). This induces a homomorphism $\Psi_{r,*} : B_r \to S_r$ (the Noether representation) mapping Q_i to $(i\,i{+}1) \in S_r$, $i = 1,\dots,r$.*
(3.3b) *The Q_i s have these relations: $Q_iQ_j = Q_jQ_i$, $1 \le i \le j \le r-1$; $j \ne i-1$ or $i+1$, and $Q_iQ_{i+1}Q_i = Q_{i+1}Q_iQ_{i+1}$, $i = 1,\dots,r-2$.*
(3.3c) *Elements of* $\ker(B_r \to H_r)$ *induce inner automorphisms of G_r.*

PROOF. Each formula is a simple computation on the effect of sides of the equation on elements of S. For example, since S is a set of generators, to see (3.3a) note that the result of applying any Q_i to S is another generating set. Then, induct on the length of a word in the Q_i s to conclude the result from the application of Q_i which maps s_i to a conjugate of s_{i+1} and s_{i+1} to s_i.

The first relation of (3.3b) is obvious, for Q_i and Q_j with i and j separated, move indices with no common support. The other formula follows from a renaming of the indices and showing that $Q_1Q_2Q_1 = Q_2Q_1Q_2$ in its application to (s_1, s_2, s_3):

$$\begin{aligned}(s_1,s_2,s_3)Q_1Q_2Q_1 = {}& (s_1s_2s_1^{-1}, s_1s_3s_1^{-1}, s_1)Q_1 = (s_1s_2s_3s_2^{-1}s_1^{-1}, s_1s_2s_1^{-1}, s_1)\\ (s_1,s_2,s_3)Q_2Q_1Q_2 = {}& (s_1s_2s_3s_2^{-1}s_1^{-1}, s_1, s_2)Q_2 = (s_1s_2s_3s_2^{-1}s_1^{-1}, s_1s_2s_1^{-1}, s_1).\end{aligned}$$

Finally, consider an extension of this computation.

$$\begin{aligned}(s_1,\dots,s_r)Q(r) = {}& (s_1s_2s_1^{-1}, s_1s_3s_1^{-1},\dots,s_1s_rs_1^{-1}, s_1)Q_{r-1}\cdots Q_1\\ = {}& (u_{\boldsymbol{s}}s_1u_{\boldsymbol{s}}^{-1}, s_1s_2s_1^{-1}, s_1s_3s_1^{-1},\dots,s_1s_rs_1^{-1}).\end{aligned}$$

As $u_{\boldsymbol{s}}$ has image the identity in the group G_r, $Q(r)$ induces conjugation by s_1^{-1} in G_r. If $Q \in B_r$ maps $(s_1,\dots,s_r)$ to $(s_1',\dots,s_r')$, then $QQ(r)Q^{-1}$ gives this chain of mappings: $\boldsymbol{s} \mapsto \boldsymbol{s}' \mapsto (s_1'\boldsymbol{s}'(s_1')^{-1})Q^{-1} = s_1'\boldsymbol{s}(s_1')^{-1}$. Everything in $\ker(B_r \to H_r)$ is a product of powers of elements of form $QQ(r)Q^{-1}$. So, this shows (3.3c). □

3.2. s-equivalences on Nielsen classes. The original definition of Nielsen class is from [**Fri77**]. Special cases appearing in [**Fri73**], and many illustrating examples related to elliptic curves in [**Fri78**]. They loom large in the books of Matzat-Malle and Voelklein. The former calls them *generating s-systems* [**MM95**, p. 26] (our r is their s) and the latter uses the name *ramification type* [**Vö96**, p. 37] for the most closely related definition.

An old literature on simple branched covers influenced classical geometers ([**Cl1872**], [**Hu1891**]). This continued through papers of Lefschetz, Segre and Zariski. Simple branched covers apply to the moduli space of genus g curves, knot types and Lefschetz pencils (of surfaces). Our interest came through complex multiplication and modular curves. We found every finite group produces a modular curve-like setup (Chap. 5 §**??**). S(trong)-equivalences and r(educed)-equivalences classes on elements of Nielsen classes give geometric meanings to some valuable group properties. These showed the Inverse Galois Problem fit very generally with many classical problems. A reader will require time to acclimate to these.

3.2.1. *Setup for Nielsen classes.* Consider any cover of compact connected Riemann surfaces $\varphi : X \to \mathbb{P}^1_z$ with r branch points $\boldsymbol{z}$. Denote the degree of the cover by n. Thm. 2.6 shows one way to picture how that cover arises. Choose an ordered r-tuple of classical generators $\boldsymbol{s}$ for $\pi_1(U_{\boldsymbol{z}}, z_0)$. Then φ and an ordering of the points of X over z_0 determines the image of the entries of $\boldsymbol{s}$ in the monodromy group G of the cover: Each s_i in $\boldsymbol{s}$ maps to some $g_i \in G$.

Conversely, given $\boldsymbol{s}$ and $\boldsymbol{g} = (g_1, \dots, g_r)$ generators of G satisfying the product-one condition $g_1 \cdots g_r = 1$, interpreting $\boldsymbol{s}$ as cuts (§2.4.3) attached according to the branch cycle description $\boldsymbol{g}$ produces φ (Def. 2.4).

As $\boldsymbol{s}$ runs over all classical generators, Thm. 1.8 gives this data attached to φ:

(3.4a) an associated group $G = G(\boldsymbol{g})$;
(3.4b) a permutation representation $T : G \to S_n$; and
(3.4c) conjugacy classes $\mathbf{C} = (\mathrm{C}_1, \dots, \mathrm{C}_r)$ of G into which entries of $\boldsymbol{g}$ fall in some order (denoted $\boldsymbol{g} \in \mathbf{C}$).

Further, running over all possible classical generators $\boldsymbol{s}$, the collection of images of $\boldsymbol{s}$ (branch cycle descriptions $\boldsymbol{g}$) that correspond to φ all fall in this set:

$$\mathrm{Ni}(G, \mathbf{C}, T) = \{(g_1, \dots, g_r) \mid \prod_{i=1}^{r} g_i = 1,\ G(\boldsymbol{g}) = G \le S_n,\ \boldsymbol{g} \in \mathbf{C}\}. \tag{3.5}$$

We often use $\Pi(\boldsymbol{g})$ in place of $\prod_{i=1}^{r} g_i$. Then, (3.5) is the Nielsen class of (r-tuples in G) corresponding to $(G, \mathbf{C}, T)$. Elements in this set are *Nielsen class representatives*.

3.2.2. *The s(trong)-equivalences on* $\mathrm{Ni}(G, \mathbf{C}, T)$. Consider the subgroup of S_n that normalizes G and permutes entries of $\mathbf{C}$. Denote this $N_{S_n}(G, \mathbf{C}) = N_T(G, \mathbf{C})$. For convenience we list some equivalences on a Nielsen class that will appear later. For N any group between G and $N_T(G, \mathbf{C})$, let $n \in N$ act on $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}, T)$ by

$$\boldsymbol{g} \mapsto n\boldsymbol{g}n^{-1} \stackrel{\text{def}}{=} (ng_1n^{-1}, \dots, ng_rn^{-1}).$$

Denote the orbits for this action by $\mathrm{Ni}(G, \mathbf{C}, T)/N$.

We reserve a special notation, for two cases:

(3.6a) $\mathrm{Ni}(G, \mathbf{C}, T)^{\mathrm{abs}}$ when $N = N_T(G, \mathbf{C})$ for *absolute s-equivalence classes* (of Nielsen class representatives); and

(3.6b) $\text{Ni}(G, \mathbf{C}, T)^{\text{in}}$ when $N = G$ and T is the regular representation (acting on cosets of the identity subgroup), for *inner s-equivalence classes*.

In applyings Prop. 3.2, for an element $Q \in B_r$, when possible use the notation q for its image in H_r. For all s-equivalences, Prop. 3.2 gives an action of H_r that preserves these equivalence classes. Here is how the generator $q_i \in H_r$ acts on $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C}, T)/N$, corresponding to (3.1):

$$(3.7) \qquad (\boldsymbol{g})q_i \stackrel{\text{def}}{=} (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_r),\ i = 1, \dots, r-1.$$

As in Chap. 3 §7.1.2 denote the elements $g \in G$ with $(1)T(g) = 1$ by $G(T, 1)$.

Suppose G is abelian. Then, the action of $q \in H_r$ permutes the entries of $\boldsymbol{g}$ according to $\Psi_{r,*}(q) \in S_r$. This holds for inner classes. We give some standard situations that generalize this using the commutator notation $(g_1, g_2) \stackrel{\text{def}}{=} g_1 g_2 g_1^{-1} g_2^{-1}$. Let $G^* = (G, G) = \langle (g_1, g_2) \mid g_1, g_2 \in G \rangle$ be the commutator subgroup of G.

For G any finite group and $H \triangleleft G$, suppose T is a transitive permutation representation of G and $T^{G/H}$ is the induced representation of G/H from the cosets of $G(T, 1)/(G(T, 1) \cap H \equiv G(T, 1) \cdot H/H$. The next lemma follows from the definitions. We will see this situation come up often. We do not assume $\mathbf{C}$ is a set of conjugacy classes whose elements lie outside H. So it is possible some entries of $\mathbf{C}$ will become trivial mod H.

LEMMA 3.3. *Mapping $\boldsymbol{g} \in \text{Ni}(G, \mathbf{C}, T_G)$ to the r-tuple with entries reduced modulo H produces a natural map $\psi_{G,\mathbf{C},T_G;H} : \text{Ni}(G, \mathbf{C}, T_G) \to \text{Ni}(G/H, \mathbf{C}/H, T_{G/H})$. This commutes with the action of B_r: $\psi_{G,\mathbf{C},T_G;H}$ is B_r equivariant (*Chap. 3 §*7.1.3).*

Any N between G and $N_{S_n}(G, \mathbf{C})$ that also normalizes H produces an H_r equivariant map $\text{Ni}(G, \mathbf{C}, T_G)/N \to \text{Ni}(G/H, \mathbf{C}, T_{G/H})/(N/H)$.

3.3. Normal fiber products and Galois closure. We inspect the fiber product of two compact Riemann surfaces $\varphi_i : X_i \to \mathbb{P}^1_z$ by comparing two natural choices. According to Prop. 4.9 the naive fiber product $X_1 \times_{\mathbb{P}^1_z} X_2$ will produce an analytic manifold at a point (x'_1, x'_2) lying over $z' \in \mathbb{P}^1_z$ if and only if at least one of the corresponding pairs of ramification orders $e_{x'_i/z'}$ is 1, $i = 1, 2$. It also showed there really should be be $d = (e_{x'_1/z'}, e_{x'_2/z'})$ distinct points (with ramification orders $[e'_1, e'_2]$ over z') in this fiber product corresponding to the pair (x'_1, x'_2). Riemann's Existence Theorem combinatorially gives that by forming a fiber product in the category of compact Riemann surfaces (Prop. 3.4).

3.3.1. *Fiber products of compact Riemann surfaces.* For a given compact Riemann surface Y let $\mathcal{C}^c_X$ be the category of finite covers $\varphi : X \to Y$ of compact Riemann surfaces where a map between two $\varphi_i : X_i \to Y$, $i = 1, 2$, is a map of Riemann surfaces $\psi : X_1 \to X_2$ that commutes with the maps to Y: $\varphi_2 \circ \psi = \varphi_1$. Let $\boldsymbol{y}$ be the union of the branch points for φ_1 and φ_2, and denote $Y \setminus \{\boldsymbol{y}\}$ by $U_{\boldsymbol{y}}$. It is often useful to indicate lengths of disjoint cycles of an element $g \in S_n$ by symbols like $(s_{i,1}) \cdots (s_{i,t_i})$ (Chap. 3 §7.1.4).

Let $\varphi^0_i : X^0_i \to U_{\boldsymbol{y}}$ be the restriction of φ_i over $U_{\boldsymbol{y}}$. Compatible with Def. 1.3, form the unramified fiber product map $\varphi^0_1 \times_{U_{\boldsymbol{y}}} \varphi^0_2 : X^0_1 \times_{U_{\boldsymbol{y}}} X^0_2 \to U_{\boldsymbol{y}}$. This may have several components, even if each of the X^0_i are connected (see and Chap. 3 §8.6.1 and§ 5.1). Thm. 7.16 uses an ordering of points above some base point y_0. With this it corresponds to components of the fiber product a pair of subgroups H_1 and H_2 of $\pi_1(U_{\boldsymbol{y}}, y_0)$. The component of the fiber product corresponds to the subgroup $H_1 \cap H_2$. The maximal pointed cover of $U_{\boldsymbol{y}}$ through which both pr_1 and

pr_2 factor comes from the subgroup $\langle H_1, H_2\rangle = H$ generated by H_1 and H_2. Then, the monodromy group of the fiber product component defined by (H_1, H_2) is the fiber product $G_{H_1} \times_{G_H} G_{H_2}$.

PROPOSITION 3.4. *Let $\varphi_1 \times^c \varphi_2 : X_1 \times^c_Y X_2 \to Y$ be the extension of $\varphi^0_1 \times_{U_{\boldsymbol{y}}} \varphi^0_2$ to the unique manifold completion of $X^0_1 \times_{U_{\boldsymbol{y}}} X^0_2$ given by Cor. 2.9. This satisfies the categorical fiber product in the category $\mathcal{C}^c_Y$.*

Suppose $Y = \mathbb{P}^1_z$ (write $\boldsymbol{z}$ for $\boldsymbol{y}$), and ${}_1\boldsymbol{g}$ and ${}_2\boldsymbol{g}$ are respective branch cycles relative to a classical set of generating homotopy classes for $\pi_(Y_{\boldsymbol{z}}, z_0)$ and orderings of the points X_{i,z_0} of X_i above z_0, $i = 1, 2$. Branch cycles for $\varphi_1 \times^c \varphi_2$ are then

$$(({}_1g_1, {}_2g_1), \dots, ({}_1g_r, {}_2g_r)) \in G_{H_1} \times_H G_{H_2}$$

given by their action on the orbit of points on the component over z_0.

Let $z_i \in \boldsymbol{z}$ and let x'_1 (resp. x'_2) be a point of X_i above z_i. Assume x'_k corresponds to the orbit of ${}_kg_i$ labeled by its disjoint cycle ${}_kg'_i$ (of length ${}_ks'_i$) in the disjoint cycle decomposition of ${}_kg_i$, $k = 1, 2$. Then, points of $\varphi_1 \times^c \varphi_2 : X_1 \times^c_{\mathbb{P}^1_z} X_2$ over both x'_1 and x'_2 correspond one-one with orbits of $({}_1g'_i, {}_2g'_i)$ on pairs of letters in the respective orbits of the cycles ${}_1g'_i$ and ${}_2g'_i$.

PROOF. Since $\varphi_1 \times^c \varphi_2$ is a map of compact Riemann surfaces, it is in the right category. To show it is a fiber product consider what happens if we have maps of compact Riemann surfaces $\varphi : W \to Y$, and $\psi_i : W \to X_i$, $i = 1, 2$, so that $\varphi_i \circ \psi_i = \varphi$, $i = 1, 2$. We only need show there is a unique map $\alpha : W \to X_1 \times^c_Y X_2$ that suits the other maps. Restrict all the existing maps and Riemann surface covers over $U_{\boldsymbol{y}}$, and use ^{0} superscripts to indicate that. Our previous understanding of fiber product produces the corresponding $\alpha^0 : W^0 \to (X_1 \times^c_Y X_2)^0$. Now apply the unique completion property of Cor. 2.9 to get α which then automatically has all desired properties.

Almost everything else is a restatement of previous propositions, though we comment further on the last paragraph of the statement. By relabeling the points in the fibers of X_i over z_0, assume with no loss that ${}_1g'_i$ acts as $(a_1 \dots a_{e_1})$ and ${}_2g'_i$ acts as $(b_1 \dots b_{e_1})$. The final statement says that $({}_1g'_i, {}_2g'_i)$ has $d = (e_1, e_2)$ orbits of length $[e_1, e_2]$ on the pairs $\{(a_u, b_v)\}_{1\le u\le e_1, 1\le v\le e_2}$ [11.12a]. □

DEFINITION 3.5. In Prop. 3.4, $\varphi_1 \times^c \varphi_2 : X_1 \times^c_Y X_2 \to Y$ is the *normal* fiber product of φ_1 and φ_2.

REMARK 3.6 (Use of the word normal). In many problems the fiber product appears as an auxiliary construction. Whether the naive or normal is a better choice depends on circumstances. Usually, however, the normal is best. In our category $\mathcal{C}^c_Y$ it would appear we are stuck with considering only manifolds. For higher dimensional manifolds this result does not work, because it is possible that two manifold (ramified) covers $\varphi_i : X_i \to \mathbb{P}^n$, with $n \ge 2$, $i = 1, 2$, have no manifold fiber product. That is, there is no manifold completion of the fiber product with these properties:

(3.8a) It is the expected fiber product restricted over the unramified locus.
(3.8b) It is a finite cover of $\mathbb{P}^n$.

The correct extension of the Prop. 3.4 uses normal analytic sets (§8.5).

3.3.2. *Geometry of the Galois closure.* Consider a cover $f : Y \to X$ of degree $n = \deg(f)$ with an attached permutation representation $T_f = T : G \to S_n$. When f is an unramified cover, Chap. 3 §8.3.2 constructs the Galois closure of this cover. We want to do the same when the cover ramifies. While the construction goes through using either the naive or normal fiber product (§3.3), we emphasize the latter. So, from this point, when we say fiber product of two covers, we are referring to the normal fiber product.

When f is unramified, we took the fiber product $Y_f^n \overset{\text{def}}{=} Y_X^n$ of φ, n times. Now take the normal fiber product, so the resulting set is a manifold. Then, Y_X^n has components where each point has at least two of the coordinates identical. These form the *fat diagonal*. Remove components of this fat diagonal to give Y^*, which (exactly as in Chap. 3 Thm. 8.9) has as many components as $(S_n : G)$. List one of these components as $\hat{Y}$. Points in $\hat{Y}$ over the branch points no longer have the form of an n-tuple of points in Y. The stabilizer in S_n of $\hat{Y}$ is a conjugate of G. Normalize by choosing $\hat{Y}$ so the stabilizer is actually G.

LEMMA 3.7. *Then, $\hat{\varphi} : \hat{Y} \to X$ is Galois with group G.*

If $X = \mathbb{P}_z^1$ and the cover was in the Nielsen class $\mathrm{Ni}(G, \mathbf{C}, T)$, with $T : G \to S_n$ a faithful permutation representation, the cover $\hat{\varphi}$ has the same conjugacy classes $\mathbf{C}$, but the representation is the regular representation. The Galois cover $\hat{Y} \to Y$ has group $G(1) = G(T, 1)$ where T acts on $G(1)$ cosets. The next lemma (from [**Fri77**, Lem. 2.1]) is just the compactified version of Chap. 3 Lem. 8.8.

LEMMA 3.8. *The centralizer of G in $N_{S_n}(G, \mathbf{C})$ induces the automorphisms of X that commute with φ_T.*

Consider any permutation representation $T' : G \to S_{n'}$. This provides $\varphi_{T'} : X_{T'} \to \mathbb{P}_z^1$; $X_{T'}$ is the quotient $\hat{X}/G(T', 1)$ (with $G(T', 1)$ as in §3.2.1).

From Thm. 2.6 the next observations follow from the analogous statements for unramified covers in Chap. 3 §8.3. A cover (Y, ψ) is *Galois* if the order of $\mathrm{Aut}(Y, \psi)$ is n, as big as it can be. The construction above gives a unique minimal Galois cover $\hat{Y} \xrightarrow{\psi} X$ fitting in a commutative diagram, *the Galois closure diagram*

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\hat{\psi}_Y} & Y \\ & \hat{\psi} \searrow & \downarrow \psi \\ & & X \end{array} \tag{3.9}$$

Suppose $X = \mathbb{P}_z^1$, and $\boldsymbol{g}$ is a branch cycle description of the cover with respect to canonical generators of $\pi_1(U_{\boldsymbol{z}}, z_0)$. The group $\mathrm{Aut}(\hat{Y}, \hat{\psi})$, isomorphic to $G(\boldsymbol{g})$, canonically identifies with elements of $S_{\hat{n}}$ that centralize the image of $G(\boldsymbol{g})$ in its right regular representation where $\hat{n} = \deg(\hat{\psi})$.

For any subgroup H of $\mathrm{Aut}(\hat{Y}, \hat{\psi})$ let $\bar{H}$ be the subgroup of $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ that maps onto H. From $\bar{H}$ we obtain a cover $\psi_H : Y_H \to \mathbb{P}^1$ (Chap. 3 Thm. 8.9) that fits in a commutative diagram

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{\hat{\psi}_H} & Y_H \\ & \hat{\psi} \searrow & \downarrow \psi_H \\ & & \mathbb{P}_z^1 \end{array} \tag{3.10}$$

where $Y \to Y_H$ is Galois with group isomorphic to H. This is a version of the classical *Galois correspondence*.

COROLLARY 3.9. *Let T_H be the coset representation of the group $G(\boldsymbol{g})$ corresponding to a subgroup H. Then $T_H(\boldsymbol{g}) = (T_H(g_1), \dots, T_H(g_r))$ is a description of the branch cycles for the cover $\psi_H : Y_H \to \mathbb{P}^1$.*

PROOF. Let $T_{\bar{H}}$ be the coset representation of $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ corresponding to the subgroup $\bar{H}$, and let $\hat{H}$ be the kernel of the map from $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ given by $[\gamma_i] \to \sigma_i$, $i = 1, \dots, r$, as in Cor. 2.9. Recall that $\hat{H}$ is the maximal normal subgroup of $\pi_1(\mathbb{P}^1 \setminus D(\psi), z_0)$ contained in $\bar{H}$, and the quotient $\bar{H}/\hat{H}$ is isomorphic to H. Then $(T_{\bar{H}}([\gamma_1]), \dots, T_{\bar{H}}([\gamma_r])) = T_H(\boldsymbol{g})$. Since the left side consists of a branch cycle description for (Y_H, ψ_H), this concludes the corollary. □

3.4. Riemann-Hurwitz and the genus of a cover of $\mathbb{P}^1_z$. Let $\boldsymbol{g}$ correspond to $\psi : Y \to {\mathbb{P}_z}^1$ as in Cor. 2.9. Indicate lengths of disjoint cycles of g_i by the symbol $(s_{i,1}) \cdots (s_{i,t_i})$ (Chap. 3 §7.1.4). Points of Y corresponding to cycles of length greater than 1 are *ramified points of* ψ. The *index* of g_i, $\text{ind}(g_i)$, is the integer $\sum_{j=1}^{t_i} (s_{i,j} - 1) = n - t_i$.

3.4.1. *The appearance of $g_{\boldsymbol{g}}$.* Consider the quantity $g_{\boldsymbol{g}}$ defined by the *Riemann-Hurwitz formula*:

$$2(n + g_{\boldsymbol{g}} - 1) = \sum_{z_i \in D(\psi)} \text{ind}(g_i). \tag{3.11}$$

Note!: The following lemma requires Y to be connected. Chap. 3 Ex. 5.12 defines the differential $d\psi$ of the function ψ.

PROPOSITION 3.10. *The expression $t_\psi = \sum_{z_i \in D(\psi)} \text{ind}(g_i) - 2n$ is even. So, $g_{\boldsymbol{g}}$ in* (3.11) *is an integer. Further, t_ψ is the degree of the divisor $(d\psi)$. Finally, $t_\psi = t_{\boldsymbol{g}}$ depends only on Y, and not on ψ or n, and $g_{\boldsymbol{g}} = (t_\psi + 2)/2$ is nonnegative.*

PROOF. The determinant of (the matrix for) g_i is $(-1)^{\text{ind}(g_i)}$ (Chap. 3 §7.1.4); check for each disjoint cycle. The product-one condition implies an even number of g_is have determinant -1. So, for an even number of g_is, $\text{ind}(g_i)$ is odd. In particular, $\sum_{i=1}^{r} \text{ind}(g_i)$ is even, and $g_{\boldsymbol{g}}$ is an integer.

Suppose $\{\varphi_\alpha, U_\alpha\}_{\alpha \in I}$ is the coordinate chart for Y from ψ (Def. 2.8). We may assume the local expression for ψ at $y \in Y$ is $\psi \circ \varphi_\alpha^{-1}(z_\alpha)$ with $\varphi_\alpha(y) = 0$. Then, the leading term is $a_u z_\alpha^u$ ($a_u \neq 0$) and the divisor of $d\psi$ at y is y^{u-1}. For y over z, if $z \in \mathbb{C}$, then $u = e_y$. If, however, $z = \infty$, then $u = -e_y$, and the divisor of $d\psi$ at y is $-e_y - 1$. The expression $-e_y - 1$ summed over $y \in Y_\infty$ is the same as the sum over $e_y - 1 - 2e_y$. Since $\sum_{y \in Y_\infty} e_y = n$ (Lem. 2.1), this gives the formula.

Now use that Y is connected so that $G(\boldsymbol{g})$ is transitive. We show

$$\sum_{i=1}^{r} \text{ind}(g_i) - 2(n-1)$$

is nonnegative. When all the σ's are 2-cycles the result follows if $r \geq 2(n-1)$. That is immediate from the first part of Lem. 3.11. To reduce to that case, write each g_i as $\prod_{u=1}^{\text{ind}(g_i)} h_{u,i}$ with each $h_{u,i}$ a 2-cycle. With $\boldsymbol{h}_i = (h_{1,i}, \dots, h_{\text{ind}(g_i),i})$, replace $\boldsymbol{g}$ by the juxtaposition of these $\boldsymbol{h}_i$s: $\boldsymbol{h} = (\boldsymbol{h}_1, \dots, \boldsymbol{h}_r)$. Then, $\boldsymbol{h}$ satisfies the product-one condition and $\langle \boldsymbol{h} \rangle$ is transitive. (It is S_n: Chap. 3 [9.15e].) Further, $g_{\boldsymbol{h}} = g_{\boldsymbol{g}}$. So, the general formula for the genus of $\boldsymbol{g}$ follows from the case for 2-cycles.

We have only to show $g_{\boldsymbol{g}}$ does not depend on ψ. If ψ^*, however, is another function, then t_ψ and t_{ψ^*} are the respective degrees of the two differentials $d\psi$ and

$d\psi^*$ on the compact Riemann surface Y. The result follows from the statement in §5.3.1 that these degrees are equal. □

3.4.2. *Non-negativity of $g_{\boldsymbol{g}}$.* Let $\mathrm{Ni}(G, \mathbf{C}, T)$ be the Nielsen class for the group G and r of its conjugacy classes $\mathbf{C}$, with $T : G \to S_n$ faithful and transitive.

LEMMA 3.11 (2-cycle Braids). *For $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}, T)$, $t_{\boldsymbol{g}} = \sum_{i=1}^{r} \mathrm{ind}(g_i) - 2n$ is independent of the choice of $\boldsymbol{g}$. When $\boldsymbol{g}$ consists of 2-cycles in S_n generating a transitive subgroup, $(t_{\boldsymbol{g}} + 2)/2 = g_{\boldsymbol{g}} \geq 0$.*

PROOF. The index of an element in S_n is independent of its conjugacy class. Since the conjugacy classes of entries of any $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}, T)$ differs only by permutation from any other, the expression $t_{\boldsymbol{g}}$ is independent of the choise of $\boldsymbol{g}$.

Now apply transitivity of $G(\boldsymbol{g})$ and assume $\boldsymbol{g}$ has entries consisting of 2-cycles. There must be a series of $n-1$ entries of $\boldsymbol{g}$ so that, after the first, each consists of (i_1, i_2) with i_1 in the support of the previous 2-cycles, and i_2 is not. Apply an element $Q \in B_r$ to $\boldsymbol{g}$ to braid these so the $n-1$ entries just chosen come together as the first $n-1$ of the 2-cycles (for help, see [11.8]). Then, the product of the first $n-1$ 2-cycles is an n-cycle.

Now we use the product-one condition: $\prod_{i=1}^{n-1} g_i \prod_{i=n}^{r} g_i = 1$. Since $\prod_{i=1}^{n-1} g_i$ is an n-cycle, that implies $\prod_{i=n}^{r} g_i$ is also. Therefore $\langle g_i, i \geq n-1\rangle$ is also transitive. Now apply the previous argument to $(g_n, \dots, g_r)$ to conclude there are at least $n-1$ of them, giving a total of at least $2(n-1)$. This concludes the proof. □

DEFINITION 3.12 (The genus). Prop. 3.10 defines the *genus* g_ψ of a compact Riemann surface Y presented as a cover $\psi : Y \to \mathbb{P}^1_z$.

Other books on Riemann surfaces give examples of computing g_Y from (3.11). Rarely, however, do they discuss a branch cycle description of ψ and such examples are usually abelian covers from branches of logs (Thm. 8.8 as in Prop. 2.11).

A topologist might say they have an *easier* proof of the Riemann-Hurwitz formula. That suggested proof is likely dependent on having a *triangulation* of Y. The formula then interprets as expressing the *Euler characteristic* of Y (see Rem. 2.19). There are many ways to prove this formula. No matter what the proof, interpreting the integer $g_{\boldsymbol{g}}$, the *genus* of Y, is the key point.

3.5. Hurwitz spaces; inner s-equivalence and conjugacy classes. For each s-equivalence we must consider sets of corresponding covers.

3.5.1. *Notation for Hurwitz spaces.* Suppose $\mathrm{Ni}(G, \mathbf{C})$ is a Nielsen class with r conjugacy classes. Then, any cover in the Nielsen class has an attached set $\boldsymbol{z}$ of r distinct branch points. Label the space of these unordered branch points as U_r. §4.2.1 identifies U_r with $(\mathbb{P}^1_z)^r \setminus \Delta_r / S_r$. For each s-equivalence, the classes of covers with a given $\boldsymbol{z}$ as branch point set is the same as the number of s-equivalence classes in the Nielsen class.

Label the collection of equivalence classes of covers in a given s-equivalence class by using the notation $\mathcal{H}$, denoting a *Hurwitz space*, usually with extra decoration to indicate the type of s-equivalence classes.

There are $|\mathrm{Ni}(G, \mathbf{C}, T)\mathrm{abs}|$ absolute s-equivalence classes of covers with branch points $\boldsymbol{z} \in U_r$ with the data $(G, \mathbf{C}, T)$ attached to them. Prop. 2.18 shows it requires a choice of classical generators (or cuts) canonically corresond these two sets. Denote the set of classes of $\mathrm{Ni}(G, \mathbf{C}, T)\mathrm{abs}$ covers by $\mathcal{H}(G, \mathbf{C}, T)^{\mathrm{abs}}$. A point $\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C}, T)^{\mathrm{abs}}$ has as a representative a cover $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$.

Inner s-equivalence of covers, corresponds exactly to (3.6b). The following pairs correspond to a point $\boldsymbol{p} \in \mathcal{H}_G \stackrel{\text{def}}{=} \mathcal{H}(G,\mathbf{C})^{\text{in}}$:

$$(\hat\varphi : \hat X \to \mathbb{P}^1_z,\ G(\hat X/\mathbb{P}^1_z)\xrightarrow{\alpha} G). \tag{3.12}$$

A given such pair is quivalent to $(\hat\varphi' : \hat X' \to \mathbb{P}^1_z,\ G(\hat X'/\mathbb{P}^1_z)\xrightarrow{\alpha'} G)$ if
(3.13) $\hat\psi : \hat X \to \hat X'$ with $\hat\varphi' \circ \hat\psi = \hat\varphi$ induces α'.

For example: Suppose $g \in G$ maps $\hat X \to \hat X$, changing α by conjugation by g. Then, composing α with g gives a cover inner equivalent to (3.12). On the other hand, composing α with an outer automorphism of G gives a new equivalence class.

PROPOSITION 3.13. *Given a (faithful) permutation representation $T : G \to S_n$, there is a natural map $\Psi_{\text{in,abs}} : \mathcal{H}(G,\mathbf{C})^{\text{in}} \to \mathcal{H}(G,\mathbf{C},T)^{\text{abs}}$ by*

$$(\hat\varphi : \hat X \to \mathbb{P}^1_z,\ G(\hat X/\mathbb{P}^1_z)\xrightarrow{\alpha} G) \mapsto \varphi : \hat X/\alpha^{-1}(G(T,1)) \to \mathbb{P}^1_z.$$

This map is $|N_{S_n}(G,\mathbf{C})/G|$ to 1 over every point of $\mathcal{H}(G,\mathbf{C},T)^{\text{abs}}$.

DEFINITION 3.14. An element $\boldsymbol{g} \in \text{Ni}(G,\mathbf{C})$ is a H-M (Harbater-Mumford) representative if r is even and $\boldsymbol{g} = (g_1, g_1^{-1}, \dots, g_{r/2}, g_{r/2}^{-1})$.

EXAMPLE 3.15 (Comparing H_r inner and absolute orbits). There is a general problem that arises when applying prop. 3.13. Suppose $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ represent two distinct elements of $\mathcal{H}(G,\mathbf{C})^{\text{in}}$ that lie over the same element of $\mathcal{H}(G,\mathbf{C},T)^{\text{abs}}$. When is there a $q \in H_r$ that takes $\boldsymbol{g}_1$ to $\boldsymbol{g}_2$?

With T_n the standard representation of S_n, each element of $\mathcal{H}(A_n,\mathbf{C},T_n)^{\text{abs}}$ has exactly two from $\mathcal{H}(A_n,\mathbf{C})^{\text{in}}$ above it. Suppose $\boldsymbol{g}$ is an H-M rep. from $g_1,\dots,g_{r/2}$ where there is $\alpha \in S_n \setminus A_n$ such that $\alpha g_i \alpha^{-1} = G_i^{-1}$, $i = 1,\dots,r/2$. Then, $(\boldsymbol{g})q = \alpha\boldsymbol{g}\alpha^{-1}$ with $q = q_1 q_3 \cdots q_{r-1} \in H_r$. That solves relating the inner and absolute H_r orbits in this case.

3.5.2. *Conjugacy classes and multiplier groups.* Many times one conjugacy class will appear several times in $\mathbf{C}$. It is easy to label conjugacy classes in S_n. One tricky event is when several entries of $\mathbf{C}$ are distinct conjugacy classes in $G(\boldsymbol{g}) \le S_n$, but generate the same conjugacy class in S_n. We give easy examples here.

Suppose C is a conjugacy class in a group G consisting of elements having order m. Then, for $k \in (\mathbb{Z}/m)^*$ denote the kth powers of elements of C by C^k. For a collection $\mathbf{C}$ of conjugacy classes use the notation $\mathbf{C}^k$, $k \in \hat{\mathbb{Z}}^*$ (integers relatively prime to the order of elements in $\mathbf{C}$).

DEFINITION 3.16. Call $\mathbf{C}$, conjugacy classes in G, a *rational union* if $\mathbf{C}^k = \mathbf{C}$ (both sides counted with multiplicity) for all $k \in \hat{\mathbb{Z}}^*$. There is always a natural *rationalization* $\mathbf{C}'$ of $\mathbf{C}$: The minimal rational collection of conjugacy classes containing $\mathbf{C}$.

Let T_n be the standard representation of S_n, $n \ge 3$. As in Chap. 3 §7.1.4, indicate conjugacy classes in S_n with a simple notation. Give C_i by its cycle type: $(s_{i,1})\cdots(s_{i,t_i})$, $i = 1,\dots,r$. As $\sum_{j+1}^{t_i} s_{i,j} = n$, it is often (not always) convenient to order the $s_{i,j}$ by size: $s_{i,j} \le s_{i,j+1}$. Recall: This class is in A_n if and only if $n - t_i$ (its index, §3.4) is even.

For any homomorphism $\psi : H \to G$ (containment of H in G is the standard case) a conjugacy class C in H generates a conjugacy class C_G in G: For $g \in \text{C}$, C_G is the collection of conjugates of g in G.

DEFINITION 3.17 (Multiplier group). Let C be a conjugacy class C in G whose elements have order m. The *multipier group* of C is $M_{\mathrm{C}} \stackrel{\text{def}}{=} \{k \in (\mathbb{Z}/m)^* \mid \mathrm{C}^k = \mathrm{C}\}$. The *multiplier field* K_{C} is the fixed field in $\mathbb{Q}(e^{2\pi i/k})$ of M_{C}.

3.5.3. *Multiplier groups and fields in A_n.* Each conjugacy class in S_n is rational. It is more complicated for A_n. The following results give valuable examples.

LEMMA 3.18. *For a conjugacy class* C *in A_n, there are two possibilities for* $\mathrm{C}_{S_n} = (s_1)\cdots(s_t)$ *:* $\mathrm{C}_G = \mathrm{C}$, *or* $\mathrm{C}_G = \mathrm{C}\dot{\cup}h\mathrm{C}h$ *with* $h = (1\,2)$. *The former happens if and only if there is an even length cycle or a product of an odd number of disjoint 2-cycles that centralizes any $g \in \mathrm{C}$. The latter happens if and only if*

(3.14) *all the s_j s are odd, $j = 1, \dots, t$, and distinct.*

PROOF. Suppose h is either an m-cycle with m even or it is product of m disjoint 2-cycles with m odd. Then $S_n = A_n\dot{\cup}hA_n$. If h centralizes $g \in \mathrm{C}$, then the orbit of hA_n on g is the same as that of A_n and $\mathrm{C}_{S_n} = \mathrm{C}$.

Conversely, by the class equation if C_{S_n} is larger than C, some nontrivial element of $S_n \setminus A_n$ centralizes g. Suppose m is the length of a disjoint cycle in g and there are t_m of these. Denote by g_m the product of all these disjoint m-cycles in g. Write g as the product of these g_m s running over all distinct integers m. Denote the centralizer of $(1\, m+1\, \dots\, (t_m-1)m+1)\dots(m\, 2m\, \dots\, t_m m)$ by C_m. Then, the centralizer of g is isomorphic to the direct product of the C_m s.

Now we check that the group C_m is the wreath product

$$\mathbb{Z}/m \wr S_{t_m} = (\mathbb{Z}/m)^{t_m} \times^s S_{t_m} \text{ (Chap. 3 §8.4)}$$

regarded as a subgroup of S_{mt_m}. The copy of $(\mathbb{Z}/m)^{t_m}$ identifies with products of powers of the disjoint cycles in g_m. A $\pi \in S_{t_m}$ maps $(i_1, \dots, i_{t_m}) \in (\mathbb{Z}/n)^{t_m}$ to $(i_{(1)\pi}, \dots, i_{(t_m)\pi})$. Example: $\pi = (1\,2)$ acts in S_{mt_m} as $(1\, m+1)(2\, m+2)\cdots(m\, 2m)$, a product of m disjoint 2-cycles. If m is even then C_m contains an m-cycle, that is not in A_{mt_m}. If m is odd, but larger than 1, a 2-cycle $\pi \in S_{t_m}$ acts as a product of m disjoint 2-cycles in A_{mt_m}. So, C_m is in A_{mt_m} if and only t_m is 1 and m is odd. That concludes the proof. □

Assume $g \in \mathrm{C}$ with $\mathrm{C}_{S_n} = (s_1)\cdots(s_t)$ satisfies (3.14), the only possible non-rational conjugacy classes in A_n. The next proposition checks which of those are rational when $\mathrm{C} = (n)$ (n is odd); [11.18b] outlines the general case [**Fri95b**, p. 332].

Recall: p^u *exactly divides* n (written $p^u||n$) if p^u divides n, but p^{u+1} does not. Also, use Euler's Theorem that if p is an odd prime, the invertible integers $(\mathbb{Z}/p^u)^*$ (of $\mathbb{Z}/p^u$) is a cyclic group.

PROPOSITION 3.19 (Irrational Cycles). *Consider the case $n > 4$ is odd and $g \in \mathrm{C}$ with $C_{S_n} = (n)$. Suppose n is not a square. Let J be those primes p that exactly divide n to an odd power $p^{u(p)}$. For any $p \in J$, let $k \in (\mathbb{Z}/n)^*$ have these properties: its image in $(\mathbb{Z}/p^{u(p)})^*$ generates this cyclic group; and its image in $(\mathbb{Z}/p'^{u'})^*$ is 1 for primes $p' \neq p$ that divide n. Then, g^k and g are not conjugate in A_n:* C *is not a rational conjugacy class.*

Denote $\sqrt{\prod_{p\in J}(-1)^{(p-1)/2}p}$ by α_n. For all odd n, $K_{\mathrm{C}} = \mathbb{Q}(\alpha_n)$.

Conversely, if n is an odd square, g^k is conjugate to g in A_n for all $k \in (\mathbb{Z}/n)^*$: C *is a rational conjugacy class.*

PROOF. Suppose n is not a square. With k (and $p \in J$) as in the statement, we show g^k and g aren't conjugate in A_n. With no loss, $g = (1\, \dots\, n)$. So g^k maps

$i \mapsto i+k \mod n$, $i = 1, \ldots, n$. Multiplication by k gives a permutation τ_k of the integers modulo n. Then, $\tau_k^{-1} g \tau_k$ equals g^k:

$$(ki)\tau_k^{-1} g \tau_k = (i) g \tau_k = (i+1)\tau_k = ki + k.$$

We characterize those k with τ_k not in A_n. Apply the Chinese remainder theorem to write $(\mathbb{Z}/n)^* = \prod_{i=1}^t (\mathbb{Z}/p_i^{u_i})^*$ with $p_1, \ldots, p_t$ distinct (odd) primes. So, it suffices to check if $\tau_k \in A_n$ for $k = \boldsymbol{k}_i = (1, \ldots, 1, k_i, 1, \ldots, 1)$; the only non-identity entry is k_i, a generator of the cyclic group $(\mathbb{Z}/p_i^{u_i})^*$, in the i-th position. Consider what happens with k equal $(k_1, 1, \ldots, 1)$.

First, assume $t = 1$, $u_1 = u$ and $k_1 = k$. Consider the cycle structure of τ_k $\mathbb{Z}/p^u$. Multiplication by k on integers of $\mathbb{Z}/p^u$ exactly divisible by p^i, $i < u$, gives one orbit of length $p^{u-i} - p^{u-i-1}$. For each i between 0 and $u-1$, this cycle has even length—not in A_n. (The orbit for $i = u$ has length 1.) Thus, the permutation is a product of u elements not in A_n (and it fixes exactly one integer). The total permutation from multiplication by k is in A_n if and only if u is even.

For the general case, write $\mathbb{Z}/n$ as $\mathbb{Z}/p_1^{u_1} \times \mathbb{Z}/n'$. Multiplication by k is the identity on the second coordinate. Thus, it stabilizes each coset $\mathbb{Z}/p_1^{u_1} \times k'$ with $k' \in \mathbb{Z}/n'$. In particular, τ_k is the product of n' elements coming from the first case above. Thus, $\tau_k \in A_n$ if and only if u_1 is even. The converse comes by noting it suffices to check the elements $\boldsymbol{k}_i$ above.

Finally, we identify the field $\hat{\mathbb{Q}}_n$. Identify the kernel of $\mu : (\mathbb{Z}/n)^* \to \mathbb{Z}/2$ by $k \in (\mathbb{Z}/n)^*$ maps to $\tau_k \mod A_n$. In the above notation, $\boldsymbol{k}_i$ goes to 1 if and only if $i \in J$. The unique quadratic extension of $\mathbb{Q}$ inside $\mathbb{Q}(\zeta_{p_j})$ is $\mathbb{Q}\left(\sqrt{(-1)^{(p_j-1)/2} p_j}\right)$. Conclude by noting the kernel of μ is of index 2 in $(\mathbb{Z}/n)^*$ and it fixes α_n. □

EXAMPLE 3.20. Suppose C_1, C_2 and C_3 are respectively the conjugacy classes of the 5-cycles in A_5 given by $g_1 = (1\,2\,3\,4\,5)$, $g_2 = (1\,3\,5\,2\,4)$ and g_1 again. Then, C_1, C_2, C_3 is not a rational union because the conjugacy class of g_1 appears with multiplicity 2, while its square appears only with multiplicity 1. The collection $\mathbf{C}' = (C_1, C_2, C_1, C_2)$ is its rationalization.

EXAMPLE 3.21 (Rational conjugacy classes in A_9). The conjugacy classes of A_9 that don't remain the same in S_9 are those that become $(1)(3)(5)$ of (9) in S_9. In general, counting the partitions of n into distinct odd integers is a nontrivial combinatorial business (see [11.18d]). [**Al99**] says the number of partitions of n by odd distinct integers equals partitions of n with all parts $\neq 2$, at least 6 apart and at least seven apart if both parts are even. For $n = 25$ this count is

$$12 = |(25), \{(i, 25-i), 1 \le i \le 9, i \ne 2, (1, k, 25-k-1), 7 \le k \le 9\}|.$$

According to Prop. 3.19, there are two rational conjugacy classes A_9 that become (9) in S_9. From [11.18b] the two conjugacy classes C for which $\mathrm{C}_{S_n} = (1)(3)(5)$ are not rational and $M_{\mathrm{C}} = \mathbb{Q}(\sqrt{-3 \cdot 5})$.

4. Applications of the Existence Theorem

This section should surprise the reader at how simple group theory, starting with dihedral groups, reveals serious classical topics. We develop two skills.

- Creating notation for calculating collections of covers.
- Finding algebraic functions to give coordinates on such collections.

For any group G denote by $\mathrm{Aut}(G)$ the full set of automorphisms of G, and by $\mathrm{Inn}(G)$ the autmorphisms induced by conjugation by G. The first nonabelian group that comes up in Galois theory is the dihedral group. Prop. 2.11 shows all abelian covers are algebraic. Covers $\varphi : X \to \mathbb{P}^1_z$ with dihedral monodromy group, even when X has genus 0, are not obviously algebraic. Part of Abel's Theorem is equivalent to displaying functions that show this. There is more to such covers than one would expect from its group theory alone.

We start slowly with dihedral covers, because there is so much history in them, especially about coordinates. §4.1 is a case that function theoretically is almost trivial, though its applications require careful coordinates.

4.1. Dihedral — a ka Tchebychev — polynomials. Suppose a degree n cover $\varphi : X \to \mathbb{P}^1_z$ has genus 0 ($g_X = 0$) and branch cycles $\boldsymbol{g} = (g_1, \dots, g_r)$ (relative to some choice of classical generators) with at least one totally ramified place. That means some g_i, say g_r, is an n-cycle in $G(\boldsymbol{g}) \le S_n$. At first examples use the standard representation T_n of S_n restricted to $G(\boldsymbol{g})$. Apply Riemann-Hurwitz to conclude $\sum_{i=1}^{r-1} \mathrm{ind}(g_i) = n - 1$.

4.1.1. *Cyclic covers and Redei functions.* An element of S_n has index $n-1$ if and only if it is an n-cycle. We draw conclusions from this and the product-one condition, $\Pi(\boldsymbol{g}) = 1$. If there is another n-cycle among the branch cycles, then $r = 2$. By conjugating by an element of S_n we may take $g_1 = (1 \dots n)$ and $g_2 = g_1^{-1}$. There is unique absolute Nielsen class of genus 0 covers with at least two n-cycles: $\mathrm{Ni}(\mathbb{Z}/n, \mathbf{C}_{n,n}, T_n)^{\mathrm{abs}}$. Further, in that class there is exactly one absolute s-equivalence class representing the Nielsen class: $\mathbf{C}$ consists of C and C^{-1}, a conjugacy class in $\mathbb{Z}/n$ and its inverse. The case $n = 2$ is trivial.

For $n \ge 3$, there are $\varphi(n)/2$ inner Nielsen classes of such covers,

$$\mathrm{Ni}(\mathbb{Z}/n, (\mathrm{C}^j, \mathrm{C}^{-j}))^{\mathrm{in}}, \text{ with } (j, n) = 1,\ j \le n/2.$$

As C contains one element, there are two inner s-equivalence class representing each Nielsen class: One with $g \in \mathrm{C}$ (resp. $g^{-1} \in \mathrm{C}^{-1}$) the branch cycle for z_1 (resp. z_2); another with the branch cycles switched.

These abelian covers we can produce by hand. Cases like this where G has a nontrivial center present special problems, as we'll see later. Just consider $\mathbb{P}^1_w \to \mathbb{P}^1_z$ by $w \mapsto w^n$: 0 and ∞ map respectively to 0 and ∞. Put the branch points anywhere using $\alpha \in \mathrm{PGL}_2(\mathbb{C})$ (say $\alpha = \frac{z-z_1}{z-z_2}$) that maps z_1, z_2 to $0, \infty$. Then, $w \mapsto \alpha^{-1}((\alpha(w))^n)$ gives a representing cover $\varphi_{\mathrm{C},\mathrm{C}^{-1},\boldsymbol{z}} : \mathbb{P}^1_w \to \mathbb{P}^1_z$ in the absolute s-equivalence class with branch points $\boldsymbol{z} = \{z_1, z_2\}$. Further, it is z_i that maps to z_i, $i = 1, 2$, by $\varphi_{\mathrm{C},\mathrm{C}^{-1},\boldsymbol{z}}$. We've explicitly written a representative of very s-equivalence class of covers in the Nielsen class.

§4.2.1 discusses r-equivalence classes. In this equivalence, all the covers $\varphi_{\mathrm{C},\mathrm{C}^{-1},\boldsymbol{z}}$ are equivalent. There is just one element in any Nielsen class, for we can put the branch points where we want, and switch the branch points, too. Recall: $\mathbb{P}^1_z(\mathbb{F}_q)$ the values on the Riemann sphere in the finite field $\mathbb{F}_q$ (Chap. 2 [9.19]).

Example 4.1 (Redei functions). The problem solved by Redei functions is to consider the collection of covers $\varphi_{\mathrm{C},\mathrm{C}^{-1},\boldsymbol{z}}$ up to changing φ to $\alpha^{-1}\varphi \circ \alpha$ with $\alpha \in \mathrm{PGL}_2(\mathbb{Q})$. Assume $n \ge 3$ is odd. That is we trying to describe *rational* r-equivalence repesentatives in this Nielsen class. If φ has coefficients in $\mathbb{Q}$, then the set $\{z_1, z_2\}$ is a $\mathbb{Q}$-set (see Lem. 6.4). [**LN83**] discusses Redei functions in detail. They give the easiest examples of *exceptional* functions $f : \mathbb{P}^1_w \to \mathbb{P}^1_z$ that map one-one when restricted to $\mathbb{P}^1_w(\mathbb{F}_q)$, for infinitely many prime powers q. They are perfect for standard cryptography applications, as are Dickson polynomials and other dihedral cover examples.

The branch points $\{0,\infty\}$ and $\{z_1, z_2 \mid z_1 = \sqrt{m}, z_2 = -\sqrt{m}$, m a square-free integer represent the $\mathbb{Q}$ absolute r-equivalence classes [11.15a].

4.1.2. *Twisted Chebychev — a ka Dickson — polynomials.* Here are the conditions for absolute Nielsen classes of Chebychev covers $\varphi : X \to \mathbb{P}^1_z$:

(4.1a) X has genus $g_X = 0$ and $\deg(\varphi)$ is an odd prime p;
(4.1b) $G \le S_p$ is a subgroup of $\mathbb{Z}/p \times^s (\mathbb{Z}/p)^* \stackrel{\text{def}}{=} \mathbb{A}_p$ (acting on $\mathbb{Z}/p$);
(4.1c) $\mathbf{C}$ has an entry, say C_r, that is a p-cycle; and
(4.1d) φ is not a cyclic cover.

Tacitly the permutation representation throughout is the degree p representation T_p on $\mathbb{Z}/p$. Recall, we represent elements of $\mathbb{A}_p$ by 2×2 matrices $\left(\begin{smallmatrix} a & 0 \\ b & 1 \end{smallmatrix}\right)$ with multiplication from matrix multiplication (Chap. 3 Rem. 7.4). Using §4.1.1, we have just one p-cycle of conjugacy classes. Elements of order p are conjugate in $\mathbb{A}_p$ to $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Denote this conjugacy class C_p. The other conjugacy classes in $\mathbb{A}_p$ correspond one-one with non-identity elements of $(\mathbb{Z}/p)^*$. Denote the corresponding conjugacy class to $a \in (\mathbb{Z}/p)^*$ by C_a. For A a subgroup of $(\mathbb{Z}/p)^*$, $\mathbb{Z}/p \times^s A$ is the corresponding subgroup of $\mathbb{A}_p$. Prop. 4.2 and Cor. 4.3 is from [**Fri70**].

Proposition 4.2. *There is only one absolute Nielsen class satisfying* (4.1). *It is* $\mathrm{Ni}(G,\mathbf{C})^{\mathrm{abs}}$ *with* $\mathbf{C} = (\mathrm{C}_{-1}, \mathrm{C}_{-1}, \mathrm{C}_p) \stackrel{\text{def}}{=} \mathbf{C}_{(-1)^2\cdot p}$ *and* $G = \mathbb{Z}/p \times^s \langle -1 \rangle$. *Further, there is one element in this Nielsen class. More generally, for any odd* $n > 0$, *there is a unique absolute representative in the absolute Nielsen class of* $\mathrm{Ni}(D_n, \mathbf{C}_{(-1)^2\cdot n})^{\mathrm{abs}}$.

Proof. With no loss in an absolute Nielsen class take branch cycles so that $\boldsymbol{g}$ has $g_r = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. The other g_is are in C_a, $a \in (\mathbb{Z}/p)^* \setminus \{1\}$, which acts as multiplication by a on $\mathbb{Z}/p$. If m_a is the order of a, then this action has $\frac{p-1}{m_a}$ orbits of length m_a, and one orbit of length 1. The index of such a g_i is thus $\frac{p-1}{m_a}(m_a - 1)$. Now apply Riemann-Hurwitz (3.11) using that $g_X = 0$: $p - 1 = \sum_{i=1}^{r-1} \frac{p-1}{m_{a_i}}(m_{a_i} - 1)$. The expression $\frac{m-1}{m}$ $(m \ge 2)$ is at least $\frac{1}{2}$, with equality if and only if $m = 2$ $(m_a = -1$. Since $r - 1 \ge 2$, the result is $r - 1 = 2$, and $g_i \in \mathrm{C}_{-1}$, $i = 1, 2$.

Now we see there is only one element in this Nielsen class. Fix g_3, and take $g_1 = \left(\begin{smallmatrix} -1 & 0 \\ b & 1 \end{smallmatrix}\right)$, with g_2 determined by the product-one relation: $g_1 g_2 g_3 = 1$. Normalize further by conjugating the 3-tuple by a power of g_3:

$$\left(\begin{smallmatrix} 1 & 0 \\ -k & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 0 \\ b & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ k & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 0 \\ b+2k & 1 \end{smallmatrix}\right).$$

So, by choosing k so $b + 2k = 0 \mod p$ gives a unique representative of $\mathrm{Ni}(\mathbb{Z}/p \times^s \langle -1 \rangle, \mathbf{C}_{(-1)^2\cdot p})^{\mathrm{abs}}$.

□

Up to reduced equivalence, we may place the three branch points $\boldsymbol{z} = \{z_1, z_2, z_3\}$ by whatever three points we want. Given $\varphi : X \to \mathbb{P}^1_z$ (unique up to equivalence) in this equivalence class, we find a polynomial $T_p(w)$ with branch points $-2, +2, \infty$ reduced equivalent to it. Further, from the branch cycle description there is exactly one unramified point of X over each of -2 and $+2$ (use the corresponding between points over branch points and disjoint cycles of the branch cycles). So, by ordinary equivalence, put these at $-1, +1, \infty$ respectively. This is a less trivial case than previously for producing a function on the covers to show they are algebraic.

COROLLARY 4.3.

PROOF. There is one element in the absolute s-equivalence classes of polynomials with dihedral group cover. Suppose f is a monic degree n polynomial over $\bar{F}$ that gives a branched cover $\mathbb{P}^1_T \to \mathbb{P}^1_z$ with two finite branch points $z_1, z_2 \in \bar{F}$, both ramified of order 2. The following observations occur in [**Fri70**]. The geometric Galois group of the Galois closure is a dihedral group. If n is odd, then the Nielsen class of the cover is $\mathrm{Ni}(D_n, \mathbf{C}_{n \cdot 2^{\frac{n-1}{2}} \cdot 2^{\frac{n-1}{2}}})$. Further, since the normalizer of D_n in S_n has no center, any cover with branch points $\{\infty, z_1, z_2\}$ in this Nielsen class is determined up to a unique isomorphism. So, if the unordered branch points are defined over F, then the cover is represented by a unique polynomial over F. As $z_1 + z_2$ are defined over F, changing z to $z - (\frac{z_1+z_2}{2})$ normalizes further to assume the branch points sum to 0. Call these *normalized Chebychev polynomials*. From these observations the following are clear. For any $d \in F^*$, and odd positive integer n define the *Dickson Polynomial* $D_n(a, w)$ to be $a^{n/2}T_n(a^{-1/2}w)$. As a varies we get all the normalized Chebychev polynomials. Clearly two such polynomials are isomorphic over F if and only multiplication by some $b \in F$ maps the branch points of one to the other. □

If, however, n is even, the conjugacy classes defining the Nielsen class are distinct and the branch points all are defined over F. A compensating fact is that $N_{S_n}(D_n)$ has a nontrivial centralizer $Z_{S_n}(D_n) = \langle (1 \ldots n)^{n/2} \rangle$ in S_n (multiplication by -1 leaves $1 + n/2$ invariant modulo n. [**Tu95**] [**Wel69**] [**LN73**] [**Mu80-02**]

4.2. $\mathrm{PGL}_2(\mathbb{C})$ **action, r-equivalence and hyperelliptic covers.** §3.5 explains the sets of covers in $\mathcal{H}(G, \mathbf{C}, T)^{\mathrm{abs}}$ and other s-equivlaence classes. The group $\mathrm{PGL}_2(\mathbb{C})$, as one-one analytic maps of $\mathbb{P}^1_z$ enters immediately to give from each s-equivalence class, a new equivalence (r(educed)-equivalence) from it.

4.2.1. *$\mathbb{P}^r$ as $(\mathbb{P}^1_z)^r/S_r$ and r-equivalence.* Identify the elements of $\mathbb{P}^r$ (Chap. 3 §4.3) as nonzero monic polynomials in a variable z of degree at most r. For example, if $(a_0, a_1, \ldots, a_r)$ represents a point of $\mathbb{P}^r$, and $z_0 \neq 0$, by scaling it by $\frac{1}{z_0}$ assume with no loss $z_0 = 1$. Then, take the polynomial associated to this point as $z^r + \sum_{i=0}^{r-1}(-i)^{r-i}a_{r-i}z^i$. There is a natural permutation action of $\pi \in S_r$ on the entries of $(\mathbb{P}^1_z)^r$: $\pi : (z_1, \ldots, z_r) \mapsto (z_{((1)\pi}, \ldots, z_{(r)\pi})$. Denote the set of distinct r-tuples of elements of $(\mathbb{P}^1_z)^r$ by $U^r = (\mathbb{P}^1_z)^r \setminus \Delta_r$. Call Δ_r the *fat diagonal*: The locus were two or more equal entries.

PROPOSITION 4.4. *Represent the natural quotient map*

$$\Psi_r : (z_1, \ldots, z_r) \in (\mathbb{P}^1_z)^r \mapsto \{\boldsymbol{z}\} \in (\mathbb{P}^1_z)^r/S_r$$

by sending $(z_1, \ldots, z_r)$ to the polynomial $\prod_{i=1}^r (z - z_i)$ in z: If $z_i = \infty$, replace $(z - z_i)$ by 1. This canonically identifies Ψ_r with degree $n!$ analytic map of complex manifolds $(\mathbb{P}^1_z)^r \to \mathbb{P}^r$ [11.14a]. Identify unordered sets of r branch distinct points