Modular Towers and Torsion on Abelian Varieties

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Abstract

The general philosophy of this talk is to provide a link between the regular inverse Galois theory - in particular modular towers theory - and the theory of abelian varieties.

Fix a prime number p, a p-perfect finite group G and a r-tuple $\mathbf{C}$ of p'-conjugacy classes of G. From this data, M. Fried constructs in a canonical way atower of reduced Hurwitz spaces called the *modular tower associated with* $(G, p, \mathbf{C})$. When G is the dihedral group D_{2p} and $\mathbf{C}$ is four copies of the conjugacy class of involutions in G, the resulting modular tower is the usual tower of modular curves $(Y_1(p^{n+1}) \to Y_1(p^n))_{n\geq 1}$. Fried's conjectures generalize the theorems of Manin, Mazur and Merel for the tower of modular curves to any modular towers.

I will begin by constructing a variant of Fried's modular towers I called *abelianized modular towers*. Abelianized modular towers are finite quotients of Fried's modular towers and their arithmetic properties are strongly connected with torsion on abelian varieties *via* class field theory for function fields. In particular, the conjectural generalization of Merel's theorem for abelian varieties of fixed dimension g implies the disappearance of rational points of bounded degree along abelianized modular towers.

Then, I will prove that for any number field k and finite extension $E/k(T)$ regular over k, a profinite group $\tilde{G}$ extension of a finite group by a pro-p group admitting a quotient isomorphic to $\mathbb{Z}_p$ can't be the Galois group of a Galois extension $K/\overline{k}.E$ with field of moduli k. Equivalently, there is no projective system of k-rational points on any tower of Hurwitz spaces associated with $\tilde{G}$ (and, in particular, on any modular towers). This, *via* Faltings' theorem, reduces the conjectural disappearance of rational points along abelianized modular towers when $r = 4$ to a genus computation. We can even improve the above result by showing there is no projective system of k^{cyc}-rational points (where k^{cyc} denotes the cyclotomic closure of k in $\overline{\mathbb{Q}}$) on any abelianized modular towers. In particular, the dihedral groups D_{2p^n}, $n \geq 1$ can't be regularly realized over $\mathbb{Q}^{ab}$ in a compatible way with only order 2 inertia groups. Using an "effective" construction, I will however prove this becomes true when removing the compatibility condition, that is *any dihedral group* D_{2n} *can be regularly realized over* $\mathbb{Q}^{ab}$ *with only order* 2 *inertia groups.*

Conversely, using arithmetic properties of abelianized modular towers which stem from patching methods for G-covers, I will show that several well-known results for abelian varieties over number fields no longer hold for henselian valued field of characteristic 0.

References

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