OBSTRUCTION IN MODULAR TOWERS

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THE MAIN CONJECTURE

Throughout, $G_0$ is a finite group and $C$ is an $r$-tuple of $G_0$-conjugacy classes. The corresponding modular tower is:

$$
\cdots \rightarrow \mathcal{H}(G_1, C) \rightarrow \mathcal{H}(G_0, C) \rightarrow U_r
$$

**Main Conjecture:** For every number field $k$,

$$
\exists N, \forall n > N, \mathcal{H}(G_n, C)(k) = \emptyset.
$$

**Theorem:** $\forall$ number field $k$, $\not\exists$ modular tower that has a projective sequence of $k$-points.

**Strategy:**

1. Characterize projective systems of components defined over a fixed number field $k$, i.e. understand obstruction of components.

2. For $r = 4$, show genera of (reduced) components in projective systems are unbounded.
OUTLINE OF TALK

Part I

1. Reduce obstruction of Hurwitz space components to an obstruction purely in group cohomology.

2. Describe the canonical sequence of finite groups defining modular towers, and enumerate some of their properties that impinge on obstruction.

Part II

1. Review the concepts of cohomological dimension and duality groups.

2. Record consequences of duality for obstruction in modular towers.
PART I
HURWITZ SPACE CONSTRUCTION

If $|z| = r$, $\pi_1(\mathbb{P}^1 \setminus z) \simeq \langle \delta_1, \ldots, \delta_r \mid \delta_1 \cdots \delta_r = 1 \rangle$.

Fix a bouquet $(\delta_1, \ldots, \delta_r)$.
Fix an $r$-tuple $C$ of conjugacy classes of $G_n$.

**Definition:** The Nielsen class $Ni(G_n, C)$ is:

$$\{ \varphi : \pi_1(\mathbb{P}^1 \setminus z) \to G_n \mid \exists \sigma \in S_r, \forall i, \varphi(\delta_i) \in C_{\sigma(i)} \}$$

modulo conjugation by $\pi_1(\mathbb{P}^1 \setminus z)$.

**Definition:** $U_r$ is the configuration space of $z \subset \mathbb{P}_{\mathbb{C}}^1$ such that $|z| = r$.

**Definition:** The Hurwitz space $\mathcal{H}(G_n, C)$ is the covering space corresponding to

$$\pi_1(U_r, z) \to \text{Out}(\pi_1(\mathbb{P}^1 \setminus z)) \to \text{Sym}(Ni(G_n, C)).$$
Let $\psi_n : \pi_1(\mathbb{P}^1_C \setminus z) \to G_n$ represent an element of $\operatorname{Ni}(G_n, C)$ corresponding to $x_n \in \mathcal{H}(G_n, C)$. Given an exact sequence

\[ 1 \to M_n \to G_{n+1} \xrightarrow{\varphi_n} G_n \to 1, \]

let $\phi_n : \mathcal{H}(G_{n+1}, C) \to \mathcal{H}(G_n, C)$ be given by fiber-wise composition by $\varphi_n$. Then, $\phi_n^{-1}(x_n) \neq \emptyset$ iff there is a solution to the embedding problem

$\pi_1(\mathbb{P}^1_C \setminus z)$

$\downarrow \psi_n$

\[ 1 \to M_n \to G_{n+1} \xrightarrow{\varphi_n} G_n \to 1 \]

that lies in $\operatorname{Ni}(G_{n+1}, C)$. 
**p-FRATTINI COVERS**

**Definition:** $F \in \mathcal{C}$ is *Frattini* iff

\[ \forall X \in \mathcal{C}, \forall \phi \in \text{Hom}(X, F), \phi \text{ is an epimorphism}. \]

Let $\varphi : G_{n+1} \to G_n$ be a *p*-Frattini cover (i.e. a Frattini object in the category of epimorphisms onto $G_n$ with $p$-group kernel). Then, every weak solution to the embedding problem is surjective.

**Definition:** A conjugacy class $C$ is a $p'$-c.c. iff

\[ \forall g \in C, p \nmid |\langle g \rangle|. \]

**Lemma:** $\varphi_n$ is a *p*-Frattini cover $\implies$

\[ \forall p'$-c.c. $C_n \subset G_n$, $\exists!$ $p'$-c.c. $C_{n+1} \subseteq \varphi_n^{-1}(C_n). \]

\[ \therefore p'$-c.c. of $G_0$ lift uniquely to $p'$-c.c. of $G_n$ and *elements of the lift have the same order*. \]
A COHOMOLOGICAL OBSTRUCTION

**Definition:** Let \( s = (s_1, \ldots, s_r) \in \mathbb{N}^r \). Then, the *polygonal group* \( \Gamma(s) \) is:

\[
\langle \sigma_1, \ldots, \sigma_r \mid \sigma_1^{s_1} = \ldots = \sigma_r^{s_r} = \sigma_1 \cdots \sigma_r = 1 \rangle.
\]

Assume \( G_{n+1} \to G_n \) is \( p \)-Frattini and every conjugacy class in \( C \) is a \( p' \)-c.c.
For each \( i = 1, \ldots, r \), let \( s_i = |\langle \psi_n(\delta_i) \rangle| \).

1. \( \psi_n \) has a solution lying in \( N_i \left( G_{n+1}, C \right) \) iff \( \overline{\psi_n} \) has a weak solution:

\[
\begin{array}{c}
\pi_1(\mathbb{P}^1 \setminus \mathbb{Z}) \\
\downarrow \\
\Gamma(s) \\
\downarrow \\
\psi_n \\
\end{array}
\]

\[
1 \to M_n \to G_{n+1} \xrightarrow{\varphi_n} G_n \to 1
\]
2. If \( \alpha_n \in H^2(G_n, M_n) \) represents the exact sequence

\[
1 \longrightarrow M_n \longrightarrow G_{n+1} \overset{\varphi_n}{\longrightarrow} G_n \longrightarrow 1,
\]

then \( \overline{\psi}_n : \Gamma(s) \rightarrow G_n \) has a weak solution iff the inflation \( \inf_{\overline{\psi}_n}(\alpha_n) \) of \( \alpha_n \) to \( \Gamma(s) \) is 0.

3. Consider

\[
H \left( G_{n+1}, C \right) \overset{\phi_n}{\longrightarrow} H \left( G_n, C \right)
\]

and let \( \mathcal{O} \) be a connected component of \( H(G_n, C) \) such that \( \phi_n^{-1}(\mathcal{O}) \neq \emptyset \). Then,

\[
H^1(\Gamma(s), M_n) \leftrightarrow \phi_n^{-1}(\mathcal{O}) \rightarrow \mathcal{O}
\]

is a fibration.
THE GRUENBERG-ROGGENKAMP EQUIVALENCE

Definition: The augmentation ideal $\omega_{RG}$ is the kernel of the augmentation map, i.e.

$$0 \to \omega_{RG} \to RG \to R \to 0$$

$g \mapsto 1$

is exact.

Two categories of covers:

$\mathcal{C}_{RG}(\omega_{RG}) = (\{RG\text{-module covers } M \to \omega_{RG}\})$

$\mathcal{C}_{RG}(G) = \left(\{\text{group covers } H \to G \text{ with } RG\text{-module kernel}\}\right)$

Theorem (Gruenberg-Roggenkamp):

$\mathcal{C}_{RG}(\omega_{RG}) \approx \mathcal{C}_{RG}(G')$

where corresponding objects have isomorphic kernels.
FRIED’S MODULAR TOWERS

\[ n+1 \overline{\mathcal{G}} \xrightarrow{\varphi^n} n \overline{\mathcal{G}} \] is the projective Frattini object in 
\[ C_{\mathbb{F}_p[n \overline{\mathcal{G}}]} (n \overline{\mathcal{G}}) \] [unique up to isomorphism].

1. \[ \forall g_n \in n \overline{\mathcal{G}}, \forall g_{n+1} \in \varphi_n^{-1}(g_n), \]
\[ p \text{ divides } |\langle g_n \rangle| \implies |\langle g_{n+1} \rangle| = p \cdot |\langle g_n \rangle|. \]

2. If some conjugacy class in \( C \) is not a \( p’ \)-c.c.,
\[ \forall \text{ number field } k, \exists n, \mathcal{H} \left( n \overline{\mathcal{G}}, C_n \right)(k) = \emptyset. \]
If \( C \) consists of \( p’ \)-c.c. then \( n \overline{\mathcal{G}} \) is \( p \)- perfect.

3. Definition: \( O_{p'}(G) = \max \{ H \triangleleft G \mid p \nmid |H| \} \).
If \( G \) is \( p \)-perfect then \( n \overline{\mathcal{Q}} \) has trivial center:

\[
\begin{array}{ccc}
\text{Pullback:} & n \overline{\mathcal{G}} & \longrightarrow & n \overline{\mathcal{Q}} \\
& \downarrow & \square & \downarrow \\
& G & \longrightarrow & Q = G/O_{p'}(G')
\end{array}
\]

We may assume \( n \overline{\mathcal{G}} \) is \( p \)-perfect and centerless.
A TRICHOTOMY

“Schreier formula:”
\[
\dim_{\mathbb{F}_p}(M_{n+1}) = 1 + p \dim_{\mathbb{F}_p}(M_n) \left( \dim_{\mathbb{F}_p}(M_n) - 1 \right)
\]

1. \( p \nmid |G| \iff \forall n, \dim_{\mathbb{F}_p}(M_n) = 0 \iff \forall n, n_p \tilde{G} = G \)

2. Theorem (Griess-Schmid):
\[
G/O_p(G) \leq (\mathbb{Z}/p^m\mathbb{Z}) \rtimes \mathbb{F}_p^* \text{ if and only if } \forall n, \dim_{\mathbb{F}_p}(M_n) = 1.
\]
Example: \( G = D_{2p} \implies n_p \tilde{G} = D_{2p^{n+1}} \)

3. Otherwise, \( \dim_{\mathbb{F}_p}(M_0) > 1 \).
Example: \( G = (\mathbb{Z}/2\mathbb{Z})^2: \)

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<td>( 1 + 2^{136} )</td>
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LIMITATIONS OF THE INVERSE LIMIT

Lemma: \( \lim_{n \to \infty} p \tilde{G} \simeq p \tilde{G} \), the universal \( p \)-Frattini cover of \( G \).

The construction of \( p \tilde{G} \) uses Zorn’s lemma and so is non-constructive, even when \( G \) is a semi-direct product.

Theorem: Let \( G = N \rtimes H \), where \( p \nmid |H| \) and \( (|N|, |H|) = 1 \). Then, \( p \tilde{G} = p \tilde{N} \ltimes H \).

Lemma: If \( G \) is a \( p \)-group then \( p \tilde{G} \simeq \hat{F}_m(p) \), where \( m \) is the minimal number of generators of \( G \).

Example: Let \( p = 2 \) and \( G = \mathbb{F}_8^+ \ltimes \mathbb{F}_8^* \). Then, \( 2 \tilde{G} \simeq \hat{F}_3(2) \ltimes (\mathbb{Z}/7\mathbb{Z}) \). However, \( \text{Aut}(F_3) \) has no 7-torsion.
CADORET’S ABELIANIZED VARIANT

$p\bar{G} \rightarrow G$ is the projective Frattini object in $\mathcal{C}_{\mathbb{Z}_pG}(G)$ [unique up to isomorphism]. Let $L$ be its kernel.

**Definition:** \( n_p\bar{G} = p\bar{G}/p^nL \)

The argument for Fried’s modular towers works also for Cadoret’s abelianized variants: We may assume \( n_p\bar{G} \) is \( p \)-perfect and centerless, for the purposes of the Main Conjecture.

Note that

\[
0 \rightarrow p^nL/p^{n+1}L \rightarrow n_p^{+1}\bar{G} \rightarrow n_p\bar{G} \rightarrow 1
\]

is exact and that \( p^nL/p^{n+1}L \simeq M_0 \).
SUMMARY OF PART I

1. $\alpha_n : 0 \to M_n \to G_{n+1} \to G_n \to 1$ represents a canonically defined $p$-Frattini cover.

2. $C$ consists of $p'$-c.c.

3. The polygonal group $\Gamma(s)$ is

$$\langle \sigma_1, \ldots, \sigma_r \mid \sigma_1^{s_1} = \ldots = \sigma_r^{s_r} = \sigma_1 \cdots \sigma_r = 1 \rangle.$$ 

4. $x_n \in N_i(G_n, C)$ corresponds to $\Gamma(s) \xrightarrow{\bar{\psi}_n} G_n$.

5. $\exists x_{n+1}$ such that $x_{n+1} \mapsto x_n$

$$0 = \inf_{\bar{\psi}_n} (\alpha_n) \in H^2(\Gamma(s), M_n).$$
PART II

Throughout,

1. $\Gamma$ is a group and $R$ is a commutative ring,

2. $1_{R\Gamma}$ is the $R\Gamma$-module $R$ on which $\Gamma$ acts trivially, and

3. unless otherwise stated, every module is a left module.
FINITENESS CONDITIONS

**Definition:** \( \Gamma \) has cohomological dimension \( n \) over \( R \) (i.e. \( \text{cd}_R(\Gamma) = n \)) iff \( \exists R\Gamma\)-projective resolution of length \( n \) of \( 1_{R\Gamma} \):

\[
0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow 1_{R\Gamma} \longrightarrow 0.
\]

**Lemma:** \( \text{cd}_R(\Gamma) = n \iff \forall k > n, \forall R\Gamma - \text{module } M, H^k(\Gamma, M) = 0 \text{ and } \exists R\Gamma - \text{module } M, H^n(\Gamma, M) \neq 0. \)

**Definition:** \( \Gamma \) is of type \( \text{FP}_\infty \) over \( R \) iff \( \exists \) resolution of \( 1_{R\Gamma} \) by finitely generated, projective \( R\Gamma \)-modules. It is of type \( \text{FP} \) if this resolution may be chosen to have finite length.

**Lemma:** \( \Gamma \) is of type \( \text{FP} \) over \( R \) iff \( \Gamma \) is of type \( \text{FP}_\infty \) over \( R \) and \( \text{cd}_R(\Gamma) < \infty. \)
Example: If $X$ is a finite $n$-dimensional CW-complex with contractible universal cover then $\pi_1(X)$ is of type $\text{FP}$ and $\text{cd}_\mathbb{Z}(\pi_1(X)) \leq n$.

**Theorem:** $\exists K(\Gamma, 1)$ that is the retract of a finite complex $\implies \Gamma$ is of type $\text{FP}$ over $\mathbb{Z}$.

**Definition:** The geometric dimension of $\Gamma$ (geom dim$(\Gamma)$) is the dimension of a minimal-dimensional $K(\Gamma, 1)$.

**Lemma:** $\text{cd}_\mathbb{Z}(\Gamma) \leq \text{geom dim}(\Gamma)$

**Theorem:** Let $n = \begin{cases} 3, & \text{if } \text{cd}_\mathbb{Z}(\Gamma) = 2, \\ \text{cd}_\mathbb{Z}(\Gamma), & \text{otherwise}. \end{cases}$

Then $\exists n$-dimensional $K(\Gamma, 1)$. If $\Gamma$ is finitely presented and of type $\text{FP}$ then this may be chosen to be the retract of a finite complex.
1. Use the completed group algebra $\mathbb{Z}_p[[\Gamma]]$ instead of $\mathbb{Z}_p\Gamma$ (i.e. $R = \mathbb{Z}_p$).

2. Modules are topological and $\Gamma$ acts continuously on them.

3. The *Pontryagin category* $\mathcal{B}_p(\Gamma)$ of modules is the union of

   $\mathcal{D}_p(\Gamma) = \text{discrete torsion modules}$
   $\mathcal{C}_p(\Gamma) = \text{profinite modules}$

   Pontryagin duality $(\text{Hom}_{\mathbb{Z}_p}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p))$ is a contravariant functor on $\mathcal{B}_p(\Gamma)$.

4. A discrete subgroup of a profinite group is “good” iff its cohomology groups are isomorphic to those of its ambient group.
DUALITY GROUPS

Definition: \( \Gamma \) is a duality group of dimension \( n \) over \( R \) iff \( \text{cd}_R(\Gamma) = n \), \( \Gamma \) is of type \( \text{FP} \), and

\[
H^k(\Gamma, R\Gamma) = \begin{cases} 
0, & k \neq n \\
\text{flat } R\text{-module, } & k = n
\end{cases}
\]

Definition: If \( \Gamma \) is a duality group of dimension \( n \) over \( R \) then \( D_R(\Gamma) = H^n(\Gamma, R\Gamma) \) is the dualizing module; this is a right \( R\Gamma \)-module.

Theorem: If \( \Gamma \) is a duality group of dimension \( n \) over \( R \) then, \( \forall k, \exists \) natural isomorphism of functors

\[
H^k(\Gamma, \ast) = H_{n-k}(\Gamma, D_R(\Gamma) \otimes_R \ast),
\]
compatible with the long exact sequences, and where \( \Gamma \) acts diagonally on the tensor product.
POINCARÉ DUALITY GROUPS

**Theorem:** $X$ is a closed $n$-dimensional manifold with contractible universal cover $\implies \pi_1(X)$ is a duality group of dimension $n$ over $\mathbb{Z}$.

**Definition:** A duality group $\Gamma$ over $R$ is a Poincaré duality group iff $D_R(\Gamma) \simeq R$ as an $R$-module.

**Definition:** A Poincaré duality group $\Gamma$ over $R$ is orientable iff $D_R(\Gamma) \simeq 1_{R\Gamma}$ as a right $R\Gamma$-module.

**Corollary:** If $\Gamma$ is an orientable Poincaré duality group of dimension $n$ over $R$ then, $\forall k$, $\exists$ natural isomorphism of functors:

$$H^k(\Gamma, \ast) = H_{n-k}(\Gamma, \ast)$$

**Remark:** It is not known whether existence of an $n$-dimensional $K(\Gamma, 1)$-manifold characterizes Poincaré duality groups over $\mathbb{Z}$. 21
BASIC EXAMPLES

Example: A finitely generated free group is a duality group of dimension 1 over $\mathbb{Z}$. It is Poincaré duality iff it is cyclic (in which case it is orientable).

Example: A finitely generated free pro-$p$ group is a duality group of dimension 1 over $\mathbb{Z}_p$ (with respect to profinite cohomology). It is Poincaré duality iff it is cyclic (in which case it is orientable).

More generally, these are the $p$-projective groups, e.g. $p\tilde{G}$.

Example: The pro-$p$ Poincaré duality groups of dimension 2 over $\mathbb{Z}_p$ (with respect to profinite cohomology) are exactly the Demuškin groups (pro-$p$ one-relator groups).
Finiteness Conditions and Finite-Index Subgroups

Let $1 \neq \gamma \leq \Gamma$ such that $(\Gamma : \gamma) < \infty$.

**Definition:** $\Gamma$ has no $R$-torsion iff $\forall g \in \Gamma$, $|\langle g \rangle|$ is invertible in $R$ if $|\langle g \rangle| < \infty$.

**Theorem (Serre):** If $\Gamma$ has no $R$-torsion then $\text{cd}_R(\Gamma) = \text{cd}_R(\gamma)$.

**Theorem:** $\Gamma$ is of type $\text{FP}_\infty$ iff $\gamma$ is of type $\text{FP}_\infty$.

**Theorem:** If $\Gamma$ has no $R$-torsion then $\Gamma$ is a duality group over $R$ iff $\gamma$ is a duality group over $R$. Furthermore, $\text{Res}_\gamma(D_R(\Gamma)) \cong D_R(\gamma)$.

**Theorem:** Let $\Delta \triangleleft \Gamma$ and let both $\Delta$ and $\Gamma/\Delta$ be duality groups over $R$. Then $\Gamma$ is a duality group over $R$ and

$$\text{cd}_R(\Gamma) = \text{cd}_R(\Delta) + \text{cd}_R(\Gamma/\Delta).$$
THE HURWITZ MONODROMY GROUP

**Definition:** $U_r$ is the configuration space of $z \subset \mathbb{P}^1_C$ such that $|z| = r$. The *Hurwitz monodromy group* $H_r$ is $\pi_1(U_r)$.

For $r \geq 3$,

$$1 \to \mathbb{Z}/2\mathbb{Z} \to H_r \to \text{Mod}_r^+ \to 1$$

is exact, where $\text{Mod}_r^+$ is the orientation-preserving mapping class group of $\mathbb{P}^1_C \setminus z$ (with $|z| = r$).

For $r \geq 4$,

$$1 \to F_{r-2} \rtimes \cdots \rtimes F_2 \to \text{Mod}_r^+ \to S_r \to 1$$

is exact.

**Corollary:** For $p > r \geq 4$, both $\text{Mod}_r^+$ and $H_r$ are (non-Poincaré) duality groups of dimension $r - 3$ over $\mathbb{Z}_p$ and are “good” groups.
THE POLYGONAL GROUP

**Theorem (Weigel):** For prime $p \nmid \prod_{i=1}^{r} s_i$, $\Gamma(s)$ is an orientable Poincaré duality group of dimension 2 over $\mathbb{Z}_p$ and is a “good” group, provided

$$\sum_{i=1}^{r} \left(1 - \frac{1}{s_i}\right) > 2.$$  \hspace{1cm} (1)

**Sketch of proof:** Inequality (1) is equivalent to: for every Galois cover $X \rightarrow \mathbb{P}^1_C$ with ramification prescribed by $s$, the genus of $X$ is greater than 1:

$$\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\Gamma(s)} & X \\
\downarrow & & \downarrow \\
\mathbb{S} & \rightarrow & \mathbb{P}^1_C
\end{array}$$

where $\mathbb{S}$ is an orientable Poincaré duality group of dimension 2 over $\mathbb{Z}$.

$(\Gamma(s) : \mathbb{S}) < \infty$ and the only elements in $\Gamma(s)$ having finite order are the “elliptic” elements (the conjugates of $\sigma_1, \ldots, \sigma_r$). \hfill \square
A CONSEQUENCE FOR MODULAR TOWERS

In the following commutative diagram of exact sequences, \( \bar{\psi}_n : \Gamma(s) \to G_n \) has a weak solution for the top row iff it has a weak solution for the bottom row:

\[
\begin{array}{cccccc}
0 & \to & M_n & \to & G_{n+1} & \to & G_n & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & (M_n)_{G_n} & \to & p\bar{G}_n & \to & G_n & \to & 1 \\
\end{array}
\]

where \( (M_n)_{G_n} = M_n/\omega_{\mathbb{F}_p}G_n M_n \).

In other words, if \( \bar{\alpha}_n \in H^2(G_n,(M_n)_{G_n}) \) corresponds to the bottom row and \( \alpha_n \) to the top, then

\[
\inf \bar{\psi}_n (\alpha_n) = 0 \iff \inf \bar{\psi}_n (\bar{\alpha}_n) = 0.
\]

**Corollary:** If, above a connected component \( \mathcal{O} \) of \( \mathcal{H}(\frac{n}{p}\bar{G}, C) \), \( \exists \) connected component in \( \mathcal{H}(\frac{n+N}{p}\bar{G}, C) \) for a suitably large \( N \), then \( \exists \) projective system of connected components above \( \mathcal{O} \) in the abelianized modular tower.
EXCEPTIONAL POLYGONAL GROUPS

\[ \sum_{i=1}^{r} \left( 1 - \frac{1}{s_i} \right) \leq 2 \]

<table>
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<th>( \Gamma(s) )</th>
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### Spherical triangle groups:

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<th>( \mathbb{Z}/n\mathbb{Z} )</th>
<th>( I_2(n) )</th>
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<tr>
<td>(2, 2, n)</td>
<td>( D_{2n} )</td>
<td>( I_2(n) \sqcup A_1 )</td>
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<td>(2, 3, 3)</td>
<td>( A_4 )</td>
<td>( A_3 )</td>
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<td>(2, 3, 4)</td>
<td>( S_4 )</td>
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<td>(2, 3, 5)</td>
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<td>( H_3 )</td>
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### Euclidean triangle groups:

| (2, 3, 6) | \( \mathbb{Z} \times \mathbb{Z} \rtimes (\mathbb{Z}/6\mathbb{Z}) \) | \( \tilde{H}_2 \) |
| (2, 4, 4) | \( \mathbb{Z} \times \mathbb{Z} \rtimes (\mathbb{Z}/4\mathbb{Z}) \) | \( \tilde{B}_2 \) |
| (3, 3, 3) | \( \mathbb{Z} \times \mathbb{Z} \rtimes (\mathbb{Z}/3\mathbb{Z}) \) | \( \tilde{A}_2 \) |
| (2, 2, 2, 2) | \( \mathbb{Z} \times \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z}) \) | \( \tilde{I}_1 \sqcup \tilde{I}_1 \) |

Each group is the “alternating” subgroup of the corresponding Coxeter group.
HURWITZ SPACES FOR $r = 4$

Let $U^r$ be the configuration space of $\vec{z} \in \left(\mathbb{P}^1_{\mathbb{C}}\right)^r$ with distinct coordinates. Since

$$\text{PSL}_2(\mathbb{C}) \times \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\} \xrightarrow{\sim} U^4 \quad (\alpha, z) \mapsto \alpha(0, 1, \infty, z)$$

the universal cover of $U_4$ is the composition:

$$\text{SL}_2(\mathbb{C}) \times \mathbb{H} \xrightarrow{(\mathbb{Z}/2\mathbb{Z}) \times F_2} U^4 \xrightarrow{S_4} U_4.$$ 

Since the stabilizer in $\text{SL}_2(\mathbb{C})$ of a generic fiber of $U^4 \twoheadrightarrow U_4$ is $(\mathbb{Z}/2\mathbb{Z})^2 \triangleleft S_4$, reduction modulo $\text{SL}_2(\mathbb{C})$ yields

$$\mathbb{H} \xrightarrow{F_2} \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\} \xrightarrow{S_3} \mathbb{C},$$

a $\text{PSL}_2(\mathbb{Z})$-cover ramified over 0 and 1728, since the final map may be taken to be $j$.

Thus, reduced Hurwitz spaces for $r = 4$ are quotients of $\mathbb{H}$ by subgroups of $\text{PSL}_2(\mathbb{Z})$. 

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