PAC Fields, Hilbertian Fields and Fried-Völklein Conjecture

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Abstract

In year 1978, Fried-Jarden [1] showed that a conjecture of Ax, which asserts "there is no proper Galois extension of $\mathbb{Q}$ that is PAC," has counter examples; they proved that every countable Hilbertian field has a Galois extension L which is PAC and Hilbertian. Moreover, $Gal(L/K) = \prod_{n=2}^{\infty} S_n$ where S_n is the symmetric group of order n. Then in 1992 Fried-Völklin [2] in a celebrated result, showed that every finite embedding problem over a PAC and Hilbertian field L is solvable. They combined this result with a result of Iwassawa which says that "if every finite embedding problem over a pro-finite group F is solvable then F is isomorphic to a free pro-finite group of countable rank $\hat{F}_\omega$" to conclude that "**if L is a PAC and Hilbertian field then the absolute Galois group of L is isomorphic to a free pro-finite group of countable rank $\hat{F}_\omega$**". Further, they conjectured that "for a subfield $\mathbb{L}$ of $\tilde{\mathbb{Q}}$, if the absolute Galois group of $\mathbb{L}$, $Gal(\mathbb{L})$, is projective, then $\mathbb{L}$ is Hilbertian if and only if $Gal(\mathbb{L})$ is free pro-finite (of countable rank)". In particular, this conjecture includes Shafarevich's conjecture which asserts "the abelian closure $\mathbb{L}$ of any algebraic number field has free pro-finite absolute Galois group of countable rank."

The purpose of this talk is to provide an overview and some details behind the development of these conjectures.

References

[1] Michael D. Fried, Moshe Jarden, *Diophantine properties of subfields of* $\tilde{\mathbb{Q}}$, Amer. J. Math. **100**, 3, 653–666, 1978.

[2] Michael D. Fried, Helmut Völklein, *The embedding problem over a Hilbertian PAC-field*, Ann. of Math. (2), **135**, 3, 469–481, 1992.