PAC Fields, Hilbertian Fields and Fried-Völklein Conjecture

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PAC Fields

Pseudo Algebrically Closed Fields

Let $K$ be a field. If each nonempty variety defined over $K$ has a $K$-rational point, then $K$ is called pseudo algebraically closed field or PAC field.

1. Let $K$ be a PAC field and $V$ a variety defined over $K$. Then the set $V(K)$ is dense in $V$ in Zariski $K$-topology. In particular $K$ is infinite.

2. Let $L$ be an algebraic extension of an infinite field $K$. Suppose every plane curve defined over $K$ has an $L$-rational point. Then $L$ is PAC.

3. (Ershov): Infinite algebraic extensions of finite fields are PAC fields.

4. (Ax-Roquette): Algebraic extension of a PAC field is a PAC field.
Minimal PAC Fields

The minimal PAC fields are PAC fields whose proper subfields are not PAC fields.

**Example:** Let $K = \mathbb{F}_p$ be a finite prime field. Let $q$ be a prime number and $\mathbb{F}_p^q = \bigcup_{i=1}^{\infty} \mathbb{F}_p^{q^i}$. Therefore $\mathbb{F}_p^q$ is an infinite algebraic extension of $\mathbb{F}_p$ and therefore is a PAC field. It is known that

$$Gal(\mathbb{F}_p^q / \mathbb{F}_p) \cong \mathbb{Z}_q = \varprojlim \mathbb{Z} / q^i \mathbb{Z}.$$

Therefore each proper subfield of $\mathbb{F}_p^q$ is equivalent with a proper closed subgroup of $\mathbb{Z}_q$. If $\mathbb{F}_p^q$ has an infinite proper subfield then $\mathbb{Z}_q$ has a proper closed subgroup of infinite index but $\mathbb{Z}_q$ does not have a proper closed subgroup of infinite index. Thus the only proper subfields of $\mathbb{F}_p^q$ are finite fields which are not PAC. Therefore $\mathbb{F}_p^q$ is a minimal PAC field.
5. Every ultraproduct of PAC fields are PAC field.

6. \textbf{(Ax):} Every nonprincipal ultraproduct of distinct finite fields is a PAC field.

\textbf{Valuation on PAC Fields}

1. Let \((K, v)\) be a valued field, and let \(K'\) be an algebraic extension of \(K\). Then \(v\) has an extension \(v'\) to \(K'\).

2. Let \((K, v)\) be a valued field. Then \((K, v)\) is Henselian if and only if \(v\) has a unique extension to the algebraic closure \(\tilde{K}\) of \(K\).

3. Every algebraic extension of a Henselian valued field is Henselian. Every separably closed field is Henselian with respect to each one of its valuation.
4. Every valued field \((K, v)\) has a Henselian closure \((K^h, v^h)\). \(K^h\) is an algebraic separable extension of \(K\) and \(v^h(K^h) = v(K)\).

5. **(Prestel)**: Let \((K, v)\) be a valued PAC field and let \(\tilde{v}\) be an extension of \(v\) to \(\tilde{K}\). Then \(K\) is \(\tilde{v}\)-dense in \(\tilde{K}\).

6. **(Frey-Prestel)**: Let \((K, v)\) be a valued PAC field with Henselian closure \((K^h, v^h)\). Then \(K^h \cong K^{sep}\), the separable closure of \(K\). Moreover, the residue field \(K^h/v^h\) is separably closed and value group \(v(K^h)\) is a divisible group.

7. Let \((K, v)\) be a Henselian valued field and \(K\) is not separably closed. Then \(K\) is not a PAC field.
The fields $\mathbb{Q}_{ab}$ and $\mathbb{Q}_{nil}$ are not PAC fields. Let $\mathbb{Q}_{p,alg} = \tilde{\mathbb{Q}} \cap \mathbb{Q}_p$. Since $\mathbb{Q}_p$ is complete therefore it is Henselian and every valuation on $\mathbb{Q}_p$ has a unique extension to its closure $\tilde{\mathbb{Q}}_p$. On the other hand

$$Gal(\tilde{\mathbb{Q}}_p/\mathbb{Q}_p) \cong Gal(\tilde{\mathbb{Q}}/\mathbb{Q}_{p,alg})$$

Therefore every valuation on $\mathbb{Q}_{p,alg}$ has also a unique extension to $\tilde{\mathbb{Q}}$ and thus $\mathbb{Q}_{p,alg}$ is Henselian. Let assume $\mathbb{Q}_{nil}$ is PAC field.
\( \mathbb{Q}_{p,alg} \otimes \mathbb{Q}_{nil} \) is an algebraic extension of \( \mathbb{Q}_{p,alg} \). Since \( \mathbb{Q}_{p,alg} \) is Henselian therefore \( \mathbb{Q}_{p,alg} \otimes \mathbb{Q}_{nil} \) is Henselian. \( \mathbb{Q}_{p,alg} \otimes \mathbb{Q}_{nil} \) is an algebraic extension of \( \mathbb{Q}_{nil} \) and since \( \mathbb{Q}_{nil} \) is PAC therefore \( \mathbb{Q}_{p,alg} \otimes \mathbb{Q}_{nil} \) is PAC. \( \mathbb{Q}_{p,alg} \otimes \mathbb{Q}_{nil} \) is PAC and Henselian therefore it is separably closed. It concludes that \( \mathbb{Q}_{p,alg} \otimes \mathbb{Q}_{nil} \cong \tilde{\mathbb{Q}} \). Now we have

\[
\text{Gal}(\mathbb{Q}_{p,alg}) \cong \text{Gal}(\tilde{\mathbb{Q}} / \mathbb{Q}_{p,alg}) \cong \text{Gal}(\mathbb{Q}_{nil} / (\mathbb{Q}_{p,alg} \cap \mathbb{Q}_{nil}))
\]
Since $Gal(\mathbb{Q}_{nil}/(\mathbb{Q}_{p,alg} \cap \mathbb{Q}_{nil}))$ is pronilpotent therefore $Gal(\mathbb{Q}_{p,alg})$ is pronilpotent. Thus Galois group of any extension of $\mathbb{Q}_{p,alg}$ is nilpotent.

Let $p = 5$. Using Eisenstein’s criterion the polynomial $X^3 + 5$ is irreducible over $\mathbb{Q}_5$ and its discriminant is $-27.5^2$. Since $-27 \equiv 3 \pmod{5}$, $-27$ is not a quadratic residue modulo $5$ and $-27.5^2$ is not a square in $\mathbb{Q}_5$. Therefore $X^3 + 5$ is an irreducible polynomial over $\mathbb{Q}_{5,alg}$ and its discriminant is not a square in $\mathbb{Q}_{5,alg}$. Therefore $Gal(X^3 + 5, \mathbb{Q}_{5,alg})$ is $S_3$ which is not nilpotent. This proves that $\mathbb{Q}_{5,alg}\mathbb{Q}_{nil} \neq \tilde{\mathbb{Q}}$. Therefore $\mathbb{Q}_{nil}$ is not a PAC field. Since $\mathbb{Q}_{ab}$ is an algebraic extension of $\mathbb{Q}_{nil}$, thus $\mathbb{Q}_{ab}$ is not also a PAC field.

**Is $\mathbb{Q}_{sol}$ a PAC field?**

It is known that the $Gal(\mathbb{Q}_p) = Gal(\tilde{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a prosolvable group. Therefore any finite dimensional Galois extension of $\mathbb{Q}_p$ is solvable. On the other hand there is a one to one correspondence between finite Galois
field extension of \( \mathbb{Q}_p \) and \( \mathbb{Q}_{p,\text{alg}} \). Therefore any finite Galois extension of \( \mathbb{Q}_{p,\text{alg}} \) is solvable. Since \( \mathbb{Q}_{\text{sol}} \) is compositum of all solvable extensions of \( \mathbb{Q} \) then it is concluded that \( \mathbb{Q}_{p,\text{alg}} \mathbb{Q}_{\text{sol}} \cong \bar{\mathbb{Q}} \).

This shows that \( \mathbb{Q}_{\text{sol}}^h \cong \mathbb{Q}_{p,\text{alg}} \mathbb{Q}_{\text{sol}} \cong \bar{\mathbb{Q}} \) and therefore all valuation on \( \mathbb{Q}_{\text{sol}} \) are non-Henselian. Therefore the statement "Let \( (K, v) \) be a Henselian valued field and \( K \) is not separably closed. Then \( K \) is not a PAC field" is failed to prove \( \mathbb{Q}_{\text{sol}} \) is not a PAC field.
8. \textbf{(Ax)}: Let $K$ be a PAC field. Then $\text{Gal}(K)$ is projective.

9. The following statements hold for every PAC field $K$:

\begin{enumerate}
  \item[(a)] $\text{Gal}(K)$ is projective.
  \item[(b)] $\text{Br}(K)$ is trivial.
  \item[(c)] $\text{cd}(\text{Gal}(K)) \leq 1$.
\end{enumerate}

\textbf{Comments}: The conditions (a) and (c) are equivalent on any field $K$. It is known that $\text{Gal}(\mathbb{Q}_{sol})$ is projective and $\text{Br}(\mathbb{Q}_{sol})$ is trivial. Therefore the conditions in (a)-(c) are failed to make a contradiction to prove $\mathbb{Q}_{sol}$ is not a PAC field.
Hilbertian Fields

Hilbert Sets and Hilbertian Fields

Let $f_1(T, X), \ldots, f_m(T, X)$ be polynomials in $X_1, \ldots, X_n$ with coefficients in $K(T)$, $T = (T_1, \ldots, T_r)$. Let assume that these polynomials are irreducible in the ring $K(T)[X]$. For a non-zero polynomial $g \in K[T]$ the Hilbert subset $H_K(f_1, \ldots, f_m, g)$ of $K^r$ is defined as:

$$H_K(f_1, \ldots, f_m, g) = \{a \in K^r \mid g(a) \neq 0, f_1(a, X), \ldots, f_m(a, X), \text{ are irreducible in} K[X]\}$$

In addition if each $f_i$ is separable in $X$, $H_K(f_1, \ldots, f_m, g)$ is called a separable Hilbert subset of $K^r$. Let $n = 1$, a (separable) Hilbert set of $K$ is defined as a (separable) Hilbert subset of $K^r$ for some positive integer $r$. A field $K$ is called Hilbertian if each separable set of $K$ is non-empty.

1. Each separable Hilbertian set of $K^r$ is dense in $K^r$. Therefore each Hilbertian field is infinite.
2. Let $L$ be a finite separable extension of $K$. If $K$ is Hilbertian then $L$ is Hilbertian.

A **global field** $K$ is either a finite extension of $\mathbb{Q}$ (**number field**) or a function field of one variable over a finite field $\mathbb{F}_p$.

3. Suppose $K$ is a global field or a finitely generated transcendental extension of an arbitrary field $K_0$. Then $K$ is Hilbertian.

4. Let $\aleph$ be a cardinal number and $\{K_\alpha \mid \alpha < \aleph\}$ a transfinite sequence of fields. Suppose that for each $\alpha < \aleph$ the field $K_{\alpha+1}$ is a proper finitely generated regular extension of $K_\alpha$. Then $K = \bigcup_{\alpha < \aleph} K_\alpha$ is a Hilbertian field.
5. **(Fried):** Every field $K$ has a regular extension $F$ which is PAC and Hilbertian. ($F$ is a regular extension of $K$ if $F/K$ is separable and $K$ is algebraically closed in $F$)

6. **Diamond theorem (Haran):** Let $K$ be a Hilbertian field, $L_1$ and $L_2$ Galois extensions of $K$, and $L$ an intermediate field of $L_1L_2/K$. Suppose that $L \not\subseteq L_1$ and $L \not\subseteq L_2$. Then $L$ is Hilbertian.

7. **(Haran-Jarden):** Let $K$ be a Hilbertian field and let $N$ be a Galois extension of $K$ which is not Hilbertian.
Then $N$ is not the compositum of two Galois extensions of $K$ neither of which is contained in the other. In particular, this conclusion holds for separable closure of $K$ i.e., $K_s$.

9. Let $K$ be a Hilbertian field.

- Let $M$ be a separable algebraic extension of $K$ and $M'$ a proper finite separable extension of $M$.

$$
\tilde{M} = \text{Galois closure of } M/K
$$

If $M' \not\subseteq \tilde{M}$ then $M'$ is Hilbertian.
• Let $N$ be a Galois extension of $K$ and $N'$ a finite proper separable extension of $N$.

Then $N'$ is Hilbertian.

• Let $N$ be a Galois extension of $K$ and $L$ a finite separable extension of $K$. Suppose $L \cap N = K$. Then $NL$ is Hilbertian.
\textbullet{} \textbf{\( \mathbb{Q}_{sol} \) is not a Hilbertian field.}

Since \( \mathbb{Q}_{sol} \) is the compositum of all finite solvable extension of \( \mathbb{Q} \) then there is no \( a \in \mathbb{Q}_{sol} \) such that \( X^2 - a \) is irreducible over \( \mathbb{Q}_{sol} \). Therefore if \( f(X, T) = X^2 - T \) then \( H_{\mathbb{Q}_{sol}}(f) \) is empty. This shows that \( \mathbb{Q}_{sol} \) is not a Hilbertian field.

\textbullet{} \textbf{Any finite proper extension of \( \mathbb{Q}_{sol} \) is Hilbertian.}

Using Weissauer’s theorem and since \( \mathbb{Q}_{sol} \) is a Galois extension of \( \mathbb{Q} \) then any finite proper extension of \( \mathbb{Q}_{sol} \) is Hilbertian.

\textbullet{} \textbf{\( \mathbb{Q}_{sol} \) is not a compositum of two separate Galois extensions of \( \mathbb{Q} \).}

Using Haran-Jarden theorem since \( \mathbb{Q}_{sol} \) is a Galois extension of \( \mathbb{Q} \) which is not Hilbertian then \( \mathbb{Q}_{sol} \) is not compositum of two Galois extensions of \( \mathbb{Q} \) neither of which is contained in the other.
10. No Henselian field is Hilbertian.

- \( \mathbb{Q}_p, \mathbb{Q}_{p,alg} \) and formal power series \( K_0[[X]] \) are complete discrete valued fields and therefore Henselian. Thus they are not Hilbertian.

11. (Kuyk): Every abelian extension of a Hilbertian field is Hilbertian.

- \( \mathbb{Q}_{ab} \) is Hilbertian. Abelian closure of any number field is Hilbertian.

A profinite group \( G \) is small if for each positive integer \( n \) the group \( G \) has only finitely many open subgroups of index \( n \). (Example: \( \mathbb{Z}_p \) is small.)

12. Let \( K \) be a Hilbertian field. Then \( \text{Gal}(K) \) is neither prosolvable nor small.

13. Let \( L \) be a Galois extension of a Hilbertian field \( K \). Suppose \( L \neq K_s \). Then \( \text{Gal}(L) \) is neither prosolvable nor it is contained in a closed small subgroup of \( \text{Gal}(K) \).
Let $K$ be a field and $G$ a profinite group. Suppose $K$ has Galois extension $L$ with $Gal(L/K) \cong G$. Then, $G$ occurs over $K$ and $L$ is a $G$-extension of $K$.

14. Let $L$ be a Galois extension of a Hilbertian field $K$. If $Gal(L/K)$ is small then $L$ is Hilbertian.

15. Let $K$ be a Hilbertian field and $p$ a prime number. Then $\mathbb{Z}_p$ occurs over $K$. (Therefore there is a Galois extension $L$ of $K$ such that $Gal(L/K) \cong \mathbb{Z}_p$).

- For each $p$, according to the statement in (15), $\mathbb{Q}$ has a Galois extension $\mathbb{L}_p$ with $Gal(\mathbb{L}_p/\mathbb{Q}) \cong \mathbb{Z}_p$. By (14), $\mathbb{L}_p$ is Hilbertian. Since $\mathbb{Z}_p$ has no nontrivial closed finite subgroups then:

$\mathbb{L}_p$ is a Hilbertian field which is not a proper finite extension of any field.
Fried-Völklein Conjecture

Let $G$ be a profinite group. The **Borel field** of $G$, $\mathcal{B}(G)$ is the $\sigma$-algebra generated by all closed subsets of $G$.

1. Every profinite group has a unique Haar measure on $\mathcal{B}(G)$.

For a field $K$ and for a $\sigma \in Gal(K)^e$ let $K_s(\sigma)$ be the fixed field in $K_s$ of the entries of $\sigma$ by $K_s(\sigma)$.

2. **(Jarden):** Let $K$ be a countable Hilbertian field and $e$ a positive integer. Then $K_s(\sigma)$ is a PAC field for almost all $\sigma \in Gal(K)^e$.

3. **(Fried-Jarden):** Let $K$ be a countable Hilbertian field. Then $K$ has a Galois extension $N$ which is Hilbertian and PAC with $Gal(N/K) \simeq \prod_{k=1}^{\infty} S_k$
Outline of the Proof:

- List all plane curve over $K$ in a sequence $C_1, C_2, C_3, \ldots$.
- Construct a sequence of points $p_1, p_2, p_3, \ldots$ of $\mathbb{A}^2(K)$, and a linearly disjoint sequence $L_1, L_2, L_3, \ldots, \ldots$ of Galois extensions of $K$ satisfying:
  
  1. $\text{Gal}(L_k/K) \cong S_k$, $k = 1, 2, 3, \ldots$
  
  2. $p_i \in C_i(L_k)$

  3. The points $p_1, p_2, \ldots$ are distinct.

- Define $N = \prod_{k=1}^{\infty} L_k$. Then $N$ is a Galois extension and $
\text{Gal}(N/K) \cong \prod_{k=1}^{\infty} S_k$. 

- $N$ is finite proper extension of the Galois extension $\prod_{k=2}^{\infty} L_k$ and thus $N$ is Hilbertian.

- Each plane curve over $K$ has an $N$-rational point and therefore $N$ is a PAC field.
An embedding problem for a profinite group $G$ is a pair

$$(\phi : G \to A, \alpha : B \to A)$$

in which $\phi$ and $\alpha$ are epimorphisms of profinite groups. The $Ker(\alpha)$ is called kernel of the problem. The problem is called finite if $B$ is finite. The problem is called solvable if there exist an epimorphism $\gamma : G \to B$ with $\alpha \circ \gamma = \phi$.

5. **(Iwasawa):** Let $K$ be a countable field. Then if every finite embedding problem over $K$ is solvable then $Gal(K)$ is $\omega$-free. ($Gal(K) \simeq \hat{F}_\omega$)

6. **(Fried-Völklein):** Every finite embedding problem over a Hilbertian PAC field is solvable.

7. **(Fried-Völklein):** Let $K$ be a countable Hilbertian PAC field. Then $Gal(K)$ is $\omega$-free.
**Fried-Völklein Conjecture**
Let $K$ be a countable Hilbertian field. If the absolute Galois group of $K$, $\text{Gal}(K)$, is projective then $\text{Gal}(K)$ is $\omega$-free.

(Class Field Theory: Absolute Galois group of abelian closure of any number field is projective.)

**Shafarevich Conjecture**
Let $K$ be abelian closure of a number field. Then the absolute Galois group of $K$, $\text{Gal}(K)$, is $\omega$-free.