Siegel's theorem and the Chabauty-Kim method

Let X be a smooth affine curve, defined over a number field K. Then $X = \bar{X} - D$, where $\bar{X}$ is a smooth and projective curve, say of genus g, and $D = \{x_1, \ldots, x_n\}$ is a non-empty set of distinct, not necessarily K-rational points. The points x_i are called the *points at infinity.*

Let $M \subset X(K)$ be a set of K-rational points. We call M *quasi-integral* if there exists a subring $R \subset K$ which is of finite type over $\mathbb{Z}$ and an embedding $X \subset \mathbb{A}^m_K$ into an affine space over K such that every point $x \in M$ has coordinates in R^m.

Theorem: (Siegel) If $2g - 2 + n > 0$ then very quasi-integral set $M \subset X(K)$ is finite.

As far as I know, all available proofs of this theorem use, in some way, methods of diophantine approximation and heights. A recent and notable exception can be found in [1]. In this paper, Kim proves Siegel's theorem in the rather special case $K = \mathbb{Q}$ and $X = \mathbb{P}^1 - \{0, 1, \infty\}$, i.e. for $g = 0$, $n = 3$, using a method which is radically different from the classical approach. It is reasonable to hope that in the near future this new method can be used to prove diophantine theorems which are difficult to attack using more classical methods.

Kim's method is a nonabelian variant of the classical Chabauty method. It is a way to produce nonconstant p-adic analytic functions on X which vanish on a given quasi-integral set. Since quasi-integral sets are contained in a compact subset of X (with respect to the p-adic topology), finiteness follows from standard properties of analytic functions. In the classical case considered by Chabauty, X is projective and the desired analytic functions are obtained by pullback from functions on the Jacobian of X. The main idea of Kim is to replace the Jacobian by the *motivic fundamental group.*

In my talk I will try to explain the main ideas used in Kim's paper. If time permits it, I will briefly mention work in progress by Tamas Szamuely and myself which aims at using a relative version of Kim's method to prove finiteness of integral points on the complement in $\mathbb{P}^2$ of a divisor of degree ≥ 4.

References

[1] M. Kim. The motivic fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ and the theorem of Siegel. *Inventiones Math.*, 161(3):629–656, 2005.

[2] Serre. *Lectures on the Mordell-Weil theorem.* Aspects of Mathematics. Viehweg, 1989.

[3] T. Szamuely and S. Wewers. The Chabauty-Kim method in dimension two. In preparation.