Generalized Fuchsian groups and
the p-reduction theory of
elements in Hurwitz spaces

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A. Hurwitz, 1859-1919
§1 Hurwitz groups

Definition. • A Hurwitz group \( H(m) \) (of signature \( m \)) is a group with presentation

\[
H(m) = \langle z_1, \ldots, z_r \mid z_1^{m_1} = \cdots = z_r^{m_r} = \ldots \rangle
\]

where \( m = (m_1, \ldots, m_r) \) is a finite sequence of integers satisfying \( m_i \geq 2 \).

• Let

\[
\mu(m) := \sum_{i=1}^{r} (1 - \frac{1}{m_i}).
\]

Then

\[
H(m) \text{ is finite} \quad \iff \quad \mu(m) < 2,
\]

\[
H(m) \text{ is Bieberbach} \quad \iff \quad \mu(m) = 2,
\]

\[
H(m) \text{ is Fuchsian} \quad \iff \quad \mu(m) > 2.
\]


§2 Hurwitz spaces

Definition. • Let $H(m)$ be a Hurwitz group, let $G$ be a finite group and let $C_1, \ldots, C_r$ be $G$-conjugacy classes with $g \in C_i$ of order $m_i$. Let $\mathcal{C} := (C_1, \ldots, C_r)$ denote the ordered sequence of $G$-conjugacy classes. The set of group homomorphisms

$$\mathcal{H}(G, \mathcal{C}) := \{ \phi : H(m) \to G \mid \phi \text{ surjective and } \phi(z_i) \in C_i \}$$

is called the Hurwitz space of $(G, \mathcal{C})$. Put

$$[\phi] := (\phi(z_1), \ldots, \phi(z_r)).$$

• Composition with inner automorphisms of $G$ yields a left action of $\text{Inn}(G)$ on $\mathcal{H}(G, \mathcal{C})$. The orbit space

$$\mathcal{H}^{\text{inn}}(G, \mathcal{C}) := \text{Inn}(G) \setminus \mathcal{H}(G, \mathcal{C})$$

is called the inner Hurwitz space of $(G, \mathcal{C})$. 
§3.1 Circular braid groups

Definition. • The (abstract) circular braid group \( \Omega_r \) is the group generated by elements \( Q_1, \ldots, Q_r \) with subject to the relations

\[
Q_i Q_j = Q_j Q_i \quad \text{for } i - j \not\equiv \pm 1 \pmod{r}, \\
Q_i Q_j Q_i = Q_j Q_i Q_j \quad \text{for } i - j \equiv \pm 1 \pmod{r}.
\]

• \( s: Q_i \rightarrow Q_{i+1} \) is an automorphism of \( \Omega_r \). The group

\[
\tilde{\Omega}_r = \langle s \rangle \rtimes \Omega_r
\]

will be called the extended circular braid group.

• It is still an open problem whether the abstract circular braid group coincides with the geometric circular braid group.
§3.2 The action of the circular braid group

- Let $G$ be a group and

$$S_r(G) := \{ (g_1, \ldots, g_r) \in G^r \mid g_1 \cdots g_r = 1 \}. $$

The assignment

$$(g_1, \ldots, g_r).Q_i :=
(g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+1}^{-1}g_{i}g_{i+1}, g_{i+2}, \ldots, g_r)$$

$$(g_1, \ldots, g_r).Q_r := (g_{1}^{-1}g_1g_1^{-1}, g_2, \ldots, g_{r-1}, g_1),$$

$$(g_1, \ldots, g_r).s := (g_2, \ldots, g_r, g_1).$$

defines a right action of $\tilde{\Omega}_r$ on $S_r(G)$ which commutes with the action of $\text{Inn}(G)$. 
§3.3 Reduced Hurwitz spaces

- Put \( \mathcal{H}(G, \mathcal{C}^*) := \bigsqcup_{\sigma \in S_r} \mathcal{H}(G, \mathcal{C}^\sigma) \). Then
  \[
  \mathcal{H}^{\text{red}}(G, \mathcal{C}^*) := \mathcal{H}(G, \mathcal{C}^*) / \tilde{\Omega}_r,
  \]
  is called the reduced Hurwitz space of \((G, \mathcal{C})\).

- \( \mathcal{H}^{\text{red}}(G, \mathcal{C}^*) = \mathcal{H}^{\text{red,inn}}(G, \mathcal{C}^*) \) in M.D.Fried’s notation.

- Let \( \phi \in \mathcal{H}(G, \mathcal{C}), \omega \in \tilde{\Omega}_r \). Then

\[
\begin{array}{c}
H(m) \\
\downarrow \omega \\
H(m^{\omega^{-1}}) \quad \xrightarrow{\phi \circ \omega} \quad G
\end{array}
\]
§4 Profinite Hurwitz groups

Let $p$ be a prime number which is coprime to $N(m) := m_1 \cdots m_r$. Let

$$\iota : H(m) \longrightarrow \hat{H}(m)$$

denote the profinite completion of the Hurwitz group $H(m)$.

- Every homomorphism $\phi : H(m) \rightarrow G$ onto a finite group $G$ extends in a unique way to a homomorphism $\hat{\phi} : \hat{H}(m) \rightarrow G$.

- $\hat{H}(m)$ is a $p$-perfect group, i.e.,

$$\text{Hom}(\hat{H}(m), \mathbb{F}_p) = 0.$$ 

- [W] The profinite group $\hat{H}(m)$ is an orientable $p$-Poincaré duality group of dimension 2.
§5 P-projective profinite groups

**Theorem.** (W.Gaschütz; Cossey, O.H.Kegel & L.Kovacs; M.D.Fried & Ershov; M.D.Fried & M.Jarden; et al.) Every (pro)finite group $G$ has a universal $p$-Frattini cover $\pi : \hat{pG} \to G$. It coincides with the minimal $p$-projective cover.

**Theorem.** (K.W.Gruenberg) Let $\hat{G}$ be a profinite group. Then

$$\text{cd}_p(\hat{G}) \leq 1 \iff \hat{G} \text{ is } p\text{-projective}.$$
§6 Cusp branches

Definition. One says that the Hurwitz element $\phi \in \mathcal{H}(G, \mathbb{C})$ has a cusp branch, if there exists a mapping $\beta : \hat{\mathcal{H}}(m) \rightarrow p\tilde{G}$ making the diagram

$\hat{\mathcal{H}}(m)$

\[ \downarrow \beta \]

\[ \phi \]

\[ \downarrow \]

$p\tilde{G}$

\[ \pi \]

\[ \rightarrow \]

$G$

commute. The mapping $\beta$ is necessarily surjective.

• Question: Is it possible to characterize Hurwitz elements which have a cusp branch?
§7.1 Harbater-Mumford elements

Definition. The element $\phi \in \mathcal{H}(G, \mathbb{C})$ is called a Harbater-Mumford element, if there exists $\omega \in \tilde{\Omega}_r$ such that $[\phi \circ \omega] = (g_1, \ldots, g_r)$ has the following property: There exists $i_1, \ldots, i_k$, $k \geq 2$, such that for all $j = 1, \ldots, k$

- $G_j := \langle g_{i_j-1+1}, \ldots, g_{i_j} \rangle \leq G$ is a $p'$-group,

- $g_{i_j-1+1} \cdots g_{i_j} = 1$. 
§7.1 Harbater-Mumford elements (cont.)

One has a commutative diagram

\[
\begin{array}{c}
\hat{H}(m) \xrightarrow{\hat{\alpha}} \bigotimes_{j=1}^{k} \hat{G}_j \\
\downarrow \beta \quad \downarrow \phi \quad \downarrow \xi \\
\tilde{p} \tilde{G} \xrightarrow{\pi} G \\
\end{array}
\]

• by hypothesis \( \hat{\phi} = \hat{\phi}_o \circ \hat{\alpha} \),

• \( \tilde{p} \tilde{G} \) \( p \)-projective \( \Rightarrow \) \( \xi \) exists,

• \( \beta = \xi \circ \hat{\alpha} \).

Thus, \( \phi \) has a cusp branch.
§8.1 Good reduction (g-p’-cusps)

Definition. \( \phi \in \mathcal{H}(G, \mathcal{C}) \) is called strongly reducible, if exists an element \( \omega \in \tilde{\Omega}_r \) such that for \( [\phi \circ \omega] = (g_1, \ldots, g_r) \) there exists \( i_1, \ldots, i_k, k \geq 2 \), such that

- \( G_j := \langle g_{i_j-1+1}, \ldots, g_{i_j} \rangle \) are finite \( p' \)-groups.

- Let \( y_j := g_{i_j-1+1} \cdots g_{i_j} \in G_j \), and put

\[
K := \langle G_j \mid y_1 \cdots y_k = 1 \rangle.
\]

- Let \( Y := \langle y_1, \ldots, y_k \rangle \leq G \), and let \( \mathcal{C}_j' \) denote the \( Y \)-conjugacy class of \( Y \) containing \( y_j \).

- Let \( n_j := \text{ord}(y_j) \), and \( \underline{n} := (n_1, \ldots, n_k) \). Then

\[
\psi : H(\underline{n}) \longrightarrow Y \in \mathcal{H}(Y, \mathcal{C}')
\]
§8.1 Good reduction (cont.)

Theorem. (M.D.Fried)

\[ \phi \in \mathcal{H}(G, C) \text{ has a cusp branch} \iff \psi \in \mathcal{H}(Y, C') \text{ has a cusp branch.} \]

Consider

\[
\begin{array}{c}
\hat{H}(m) \\
\downarrow \beta \\
p\tilde{G}
\end{array} \xrightarrow{\hat{\alpha}} \begin{array}{c}
\hat{K} \\
\downarrow \xi \\
G
\end{array} \xrightarrow{\pi} \begin{array}{c}
p\tilde{G} \\
\downarrow \beta
\end{array}
\]

Then:

\[ \exists \beta \iff \exists \xi. \]
§8.2 Bad reduction (o-p’-cusps)

**Definition.** \( \phi \in \mathcal{H}(G, \mathcal{C}) \) is called weakly reducible, if there exists an element \( \omega \in \tilde{\Omega}_r \) such that for \( [\phi \circ \omega] = (g_1, \ldots, g_r) \) there exists \( i_1, \ldots, i_k, k \geq 2, \) such that

- \( y_j : = g_{i_{j-1}+1} \cdots g_{i_j} \) are elements of \( p' \)-order.

- Let \( s_i : = \text{ord}(g_i), t_j : = \text{ord}(y_j) \) and put \( \underline{s}^* : = (s_{i_{j-1}+1}, \ldots, s_{i_j}, t_j), \underline{t} : = (t_1, \ldots, t_k). \)

- Let
  \[
  Y_j : = \langle g_{i_{j-1}+1}, \ldots, g_{i_j} \rangle,
  \]
  \[
  Y^* : = \langle y_1, \ldots, y_k \rangle.
  \]

- Hence for such an element one has \( k + 1 \) Hurwitz elements
  \[
  \psi_j : H(\underline{s}^*) \longrightarrow Y_j, \ 1 \leq j \leq k,
  \]
  \[
  \psi^* : H(\underline{t}) \longrightarrow Y^*.
  \]
§8.2 Bad reduction (cont.)

Theorem. (M.D.Fried) Assume that $\psi^*$ and $\psi_j$ have cusp branches for all $j \in \{1, \ldots, k\}$. Then $\phi$ has a branch cusp.

• $\Leftarrow$ is wrong: Let

$$C : = \langle g_1, \ldots, g_r, y_1, \ldots, y_k \mid g_i^{m_i} = y_j^{t_j} = g_{i, j-1+1} \cdots g_{i, j} y_j^{-1} = y_1 \cdots y_k = 1 \rangle,$$

The existence of $\xi$ implies the existence of $\beta$ but not vice versa.
§9 M.D.Fried’s conjecture

Conjecture 1. (M.D.Fried\(^1\)) \(\phi \in \mathcal{H}(G, \mathcal{C})\) weakly reducible \(\Rightarrow \phi\) strongly reducible \((r = 4)\).

Conjecture 2. \(\phi\) has a cusp branch \(\Rightarrow \phi\) is strongly reducible.

Conjecture 3. Let \(\phi: \hat{\mathcal{H}}(m) \rightarrow p\tilde{G}\) be a continuous homomorphism of profinite groups such that

- \(m = (m_1, m_2, m_3), (\hat{\mathcal{H}}(m)\) is a profinite triangle group),
- \((p, m_1m_2m_3) = 1\).

Then \(\text{im}(\phi)\) is finite.

\(^1\)During the meeting I learnt through conversations with Mike Fried and Darren Semmen, that Conjecture 3 cannot hold in this form. Indeed, on the last day of the meeting Darren and me, we constructed morphism onto profinite Frobenius groups which violate Conjecture 3, and therefore Conjecture 1 and 2 for \(r = 3\). However, it might be possible, that some version of these conjectures (with additional hypothesis) is true.)
§10 A test case

**Theorem.** (Stallings; Swan) Let $\tilde{F}$ be a (discrete) group. Then

$$\text{vcd}(\tilde{F}) \leq 1 \iff \tilde{F} \text{ is virtually free}.$$ 

**Theorem.** (J-P.Serre) Let $a, b, c \in \tilde{F}$ be elements of finite order in the finitely generated virtually free group $\tilde{F}$ satisfying $a \cdot b \cdot c = 1$. Then $\langle a, b, c \rangle$ is finite.

**Theorem.** (W.) Let $g = (g_1, \ldots, g_r)$ be a sequence of elements of finite order in the finitely generated virtually free group $\tilde{F}$ satisfying

$$g_1 \cdots g_r = 1.$$ 

Then there exists an element in the extended circular braid group $\omega \in \tilde{\Omega}_r$ such that

$$\langle (g_\omega)_1, (g_\omega)_2 \rangle$$

is a finite group.
§11.1 P-projective profinite groups and $\mathbb{Z}_p$-trees

Theorem. (W.) Let $\hat{G}$ be a finitely generated $p$-projective virtual pro-$p$ group. Then $\hat{G}$ has an action on a $\mathbb{Z}_p$-tree $\hat{T}$ with the following properties:

(i) $\hat{G}$ is acting without inversion of edges,

(ii) every vertex and every edge stabilizer is a (finite) $p'$-group,

(iii) every (finite) subgroup of $\hat{G}$ of $p'$-order fixes a vertex,

(iv) the vertex group $V(\hat{T})$ and the edge group $E(\hat{T})$ are $p$-projective $\hat{G}$-modules.

• If $\hat{G}$ is a finitely generated virtual pro-$p$ group acting on a $\mathbb{Z}_p$-tree $\hat{T}$ such that (i), (ii) and (iv) are satisfied, then $\hat{G}$ is $p$-projective.
§11.2 The difference to Bass-Serre

- There are finitely generated $p$-projective virtual pro-$p$ groups $\hat{G}$ that can act on a $\mathbb{Z}_p$-tree $\hat{T}$, such that (i), (ii), (iv) is satisfied, but (iii) fails.

- There are finitely generated $p$-projective virtual pro-$p$ groups $\hat{G}$ which cannot act with finitely many orbits on vertices and edges on a $\mathbb{Z}_p$-tree $\hat{T}$ satisfying (i) and (ii).

- **Question:** Assume that the finitely generated $p$-projective virtual pro-$p$ group $\hat{G}$ is acting on a $\mathbb{Z}_p$-tree such that (i), (ii), (iii) and (iv) are satisfied. Let $g \in \hat{G}$ be an element of $p'$-order. Is it true that $\hat{T}^g$ is connected?
§11.3 Boolean sets

- A boolean (or profinite) set is a compact totally disconnected Hausdorff space. Let bool denote the category of boolean sets.

- Let $\text{ab}_p$ denote the category of abelian pro-$p$ groups. The forgetful functor $\text{for}: \text{ab}_p \to \text{bool}$ has a left adjoint

$$\mathbb{Z}_p[-]: \text{bool} \longrightarrow \text{ab}_p.$$
§11.4 What is a $\mathbb{Z}_p$-tree?

- A profinite graph $\hat{\Gamma}$ is the collection of a boolean set $\mathcal{V}(\hat{\Gamma})$, the vertices, a boolean set $\mathcal{E}(\hat{\Gamma})$, the edges, an origin mapping $o : \mathcal{E}(\hat{\Gamma}) \to \mathcal{V}(\hat{\Gamma})$, a terminus mapping $t : \mathcal{E}(\hat{\Gamma}) \to \mathcal{V}(\hat{\Gamma})$ and an inversion mapping $\bar{\cdot} : \mathcal{E}(\hat{\Gamma}) \to \mathcal{E}(\hat{\Gamma})$ satisfying the usual identities. (All mappings are mappings in bool).

- One puts

$$V(\hat{\Gamma}) := \mathbb{Z}_p[\mathcal{V}(\hat{\Gamma})],$$

$$E(\hat{\Gamma}) := \mathbb{Z}_p[\mathcal{E}(\hat{\Gamma})]/\langle e + \bar{e} | e \in \mathcal{E}(\hat{\Gamma}) \rangle.$$

Then $\partial : E(\hat{\Gamma}) \to V(\hat{\Gamma}), \partial(e) := t(e) - o(e)$, is a morphism of abelian pro-$p$ groups.

- The profinite graph $\hat{\Gamma}$ is called $\mathbb{Z}_p$-connected, if $\ker(\partial) = 0$, and a $\mathbb{Z}_p$-tree, if it is connected and $\ker(\partial) = 0$. 
§12 Generalized Fuchsian groups

Definition. A geodesic space \((X, d)\) with a non-empty subset \(\mathcal{P}\) of closed subspaces satisfying

- \((P, d) \cong \mathbb{H}^2\) or \(\mathbb{R}^2\) for \(P \in \mathcal{P}\),
- for \(P, Q \in \mathcal{P}, P \neq Q\), one has \(|P \cap Q| \leq 1\),
- for all \(x \in X\), \(|\{P \in \mathcal{P} \mid x \in P\}| < \infty\),

will be called a geodesic plane arrangement.

Definition. A (discrete) group \(G\) is called a generalized Fuchsian group, if it has a faithful, discontinuous and co-compact action on a contractible geodesic plane arrangement \((X, d, \mathcal{P})\), such that \(G_P\) acts co-compactly on \(P\) for all \(P \in \mathcal{P}\).
§12 Generalized Fuchsian groups (cont.)

Proposition. The group $K$ of §8.1 is (in general) a generalized Fuchsian group, $C$ of §8.2 is not.

Proposition. Let $G$ be a generalized Fuchsian group acting on the geodesic plane arrangement $(X, d, \mathcal{P})$. Then $X$ is an $E_\delta G$ of $G$. In particular, $\text{cd}_\delta(G) = 2$. 