FAMILIES OF SUPERELLIPTIC JACOBIANS AND ISOGENY CLASSES

YURI G. ZARHIN

Let K be a field of algebraic functions in one variable over the field $\mathbf{C}$ of complex numbers, i.e., K is finitely generated and of degree of transcendency 1 over $\mathbf{C}$. We write $\bar{K}$ for an algebraic closure of K. Let X be an abelian variety of positive dimension over K.

Definition 0.1. We say that X is *completely non-isotrivial* if for every abelian variety W over $\mathbf{C}$ there are no non-zero homomorphisms between X and W over $\bar{K}$.

Definition 0.2. Let ℓ be a prime number. We write $\mathrm{Isog}(X, K, \ell)$ for the set of K-isomorphism classes of abelian varieties Y over K such that there exists an ℓ-isogeny $Y \to X$ that is defined over K. We write $\mathrm{Isog}_1(X, K, \ell)$ for the subset of $\mathrm{Isog}(X, K, \ell)$ that consists of all (isomorphism) classes of Y with a principal polarization defined over K.

Theorem 0.3. *Suppose that $K = \mathbf{C}(t)$ is the field of rational functions in one variable. Let $f(x) \in \mathbf{C}(t)[x]$ be an irreducible polynomial of degree $n \geq 3$, whose Galois group acts doubly transitively on the set of roots of $f(x)$. Let p be a prime that does not divide n, let r be a positive integer and $q = p^r$. If $(n, p) = (3, 2)$ then we assume additionally that $r = 1, q = p = 2$. let C_f be the smooth projective model of the affine curve $y^q = f(x)$ over $\mathbf{C}(t)$ and let $J(C_f)$ be the jacobian of C_f.*

Then $J(C_f)$ is an abelian variety over $\mathbf{C}(t)$ that is completely non-isotrivial.

Example 0.4. It is known that the polynomial $f(x) = x^n - x - t$ is irreducible over $\mathbf{C}(t)$ and has Galois group $\mathbf{S}_n$, which is doubly transitive.

Theorem 0.5. *In notations and assumptions of Theorem 0.3 assume that*

$$n = 3, \; p > 2, \; r = 1, \; q = p.$$

Then there exists a positive integer d and a degree d cyclic extension $L/\mathbf{C}(t)$ that enjoy the following properties:

- (i) *d divides $2p$.*
- (ii) *The field extension $L/\mathbf{C}(t)$ is unramified outside the places of bad reduction of $J(C_f)$.*
- (iii) *The set $\mathrm{Isog}(J(C_f), L, \ell)$ is infinite for all but finitely many primes ℓ.*
- (iv) *The set $\mathrm{Isog}_1(J(C_f) \times J(C_f), L, \ell)$ is infinite for all but finitely many primes ℓ with $4 \mid (\ell - 1)$.*

References

[1] Yu. G. Zarhin, A. N. Parshin, *Finiteness problems in Diophantine geometry.* Amer. Math. Soc. Transl. (2) **143** (1989), 35–102.

[2] J. de Jong and R. Noot, Jacobians with complex multiplications. In: *Arithmetic algebraic geometry* (eds. G. van der Geer, F. Oort and J. Steenbrink). Progress in Math., vol. **89** (Birkhäuser, 1991), pp. 177–192.

[3] Yu. G. Zarhin, *Isogeny classes of abelian varieties over function fields.* arXiv:math.AG/0504523 .

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

Mathematical Sciences Research Institute, Berkeley, CA 94720, USA
E-mail address: `zarhin@math.psu.edu`