Connectedness of families of sphere covers of Atomic-Orbital type

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Abstract. Our basic question: Restricting to covers of the sphere by a compact Riemann surface of a *given type*, do all such compose one connected family? Or failing that, do they fall into easily discerned components? The answer has often been "Yes!," figuring in such topics as the connectedness of the moduli space of curves of genus g (geometry), Davenport's problem (arithmetic) and the genus 0 problem (group theory). One consequence: We then know the definition field of the family components.

Our connectedness story considers the existence of unramified p-group extensions attached to a compact Riemann surface cover of the sphere. This translates to existence of a sequence of spaces – a **M(odular) T(ower)** whose levels correspond to an integer $k \geq 0$. Connectedness results ensure certain cusp types lie on the tower level boundaries. One cusp type – conjoining papers of Harbater and Mumford – guarantees the full sequence of these spaces (and so the group extensions) are nonempty. Another, called a p cusp, contributes to the Main **MT** Conjecture: When all tower levels are defined over some fixed number field K, high tower levels have general type and no K points.

Modular curve towers have both cusp types, and no others. General **MT**s can have another cusp type, though these often disappear at high levels, preserving our expectations. This happens in examples of Liu-Osserman, allowing us to prove the Main Conjecture holds in an infinity of cases. A combinatorial description of cusps enables a different type of group theory–modular representations–than used by representation and automorphic function people. A graphical device — the *sh-incidence matrix*, coming from a natural *pairing* on cusps — simplifies the display of results. A *lifting invariant* — used by the author and Serre — appears often to explain both components and cusps.

The **S(trong) T(orsion) C(onjecture)** — bounding the $\mathbb{Q}$ torsion on abelian varieties of a fixed dimension — implies the Main Conjecture, giving it a R(egular) I(nverse) G(alois) P(roblem) interpretation. Any success (as given here) or failure of the Main Conjecture, can use this cusp description as an explicit test of the **STC**.

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2000 *Mathematics Subject Classification.* Primary 11F32, 11G18, 11R58; Secondary 20B05, 20C25, 20D25, 20E18, 20F34.

http://www.math.uci.edu/ mfried has been revamped for adding/updating concise definitions/aids in understanding **MT**s. The end of §1.3.3 has the paths to two definition helpfiles.

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1. Framework for the problem

§1.1 describes the main results as a by-product of understanding cusps. We use cusps (and their types) — introduced in [**Fr06a**, §3.2] — to show how they quickly produce connectedness results, and then lead the way to many applications. They do so by giving group/geometric character to components.

1.1. A brief overview. Let G be p-perfect group (§1.3.3) and let $\mathbf{C}$ be p' conjugacy classes of G (multiplicity of the appearance of conjugacy classes matters). Each such $(G, \mathbf{C}, p)$ produces a projective system of algebraic varieties $\{\mathcal{H}_k\}_{k=0}^{\infty}$ that is still mysterious to many, though written on plenty. A **MT** is a projective system of absolutely irreducible components on $\{\mathcal{H}_k\}_{k=0}^{\infty}$.

1.1.1. *Comparison with modular curves.* Rational points on each $\mathcal{H}_k$ correspond to regular realizations of a Frattini covering group $G_k \to G$ with p-group kernel (of exponent p^k). One well-known case is where the **MT** corresponds to dihedral groups $\{D_{p^{k+1}}\}_{k=0}^{\infty}$ (p odd) and the conjugacy classes are four repetitions of the involution (order 2) class: the tower of modular curves (minus their cusps) $\{Y_0(p^{k+1}\}_{k=0}^{\infty}$. Components at level k correspond to (Artin or Hurwitz) braid orbits on a combinatorial object —*Nielsen class* attached to $(G_k, \mathbf{C}, p)$ (§1.3). To understand, however, the general level k, it is necessary to nail level 0 ($G_0 = G$).

Sometimes a **MT** level will have nothing over it at the next level (an *obstructed* level; some tower levels may even be empty) and sometimes a **MT** level has several components (even at level 0). Ex. 2.18 and Ex. 6.2 give, respectively, infinitely many examples of each, including distinctions between *absolute* and *inner* spaces.

Being obstructed is not modular curve-like. Yet, effective homological results show precisely when there is obstruction (Prop. 2.15). So the problem of obstruction —both interesting and controllable —is transparent to our *lifting invariant* technique. These are especially effective when $p = 2$ using Invariance Prop. 2.12. §2.1.3 reformulates the description of **MT**s, as a problem in classifying p group extensions of any finite p-perfect group. The p-perfect condition (resp. p' conjugacy classes) is necessary for any level (resp. any level past the 0th) to be nonempty.

From now on, assume a **MT** refers to a projective system with all levels nonempty. We say a **MT** is over a field K when all levels (and maps between them) are defined over K. The Main Conjecture for these expresses a modular curve-like property. Now take K to be a fixed number field. The weakest version of the Main Conjecture says that high tower levels have no K points. This is trivial unless the **MT** has definition over a number field.

1.1.2. *The role of cusps.* The cusp types from §2.3.2 that naturally generalize those on modular curves are the g-p' cusps and the p cusps. Example: The two cusps on $X_0(p)$ have width p and width 1. In our identifications, the former is a H(arbater)-M(umford) cusp (that happens to be a p cusp) and the latter, the *shift* of the former, is a very special g-p' cusp.

As Thm. 2.17 shows, that there might be no p cusps at any level of a **MT** is what makes the Main Conjecture hard. Mysterious components are the culprit in detecting p cusps. Appropriate connectedness results allow *recognizing* one of **MT** levels by *distinguishing cusps* (on the boundaries of their compactifications). To show how this works, we establish the Main Conjecture for infinitely many cases where G is an alternating group and $p = 2$.

§1.2 lists results on connected spaces leading to proving the Main Conjecture. §1.3 reviews the combinatorial framework. By proving this for abelianized **MT** s (§2.1) we get a stronger result. Also, the abelianized towers have properties more akin to modular curve towers and a more direct connection to the Strong Torsion Conjecture. For example, existence of a nonempty abelianized tower is a simpler test than for general towers (Prop. 2.15).

The proofs come clear from a list of sh-incidence matrix Tables; based on the *cusp pairing* on reduced Hurwitz spaces introduced in [**BF02**, §2.10]. Tables 2–5 display our main theorem (Props. 4.9 and 4.10). These make all components, cusp-types and elliptic ramification contributions transparent. The remaining sh-incidence tables show the difference between assuming spaces of genus 0 covers (say, in the Liu-Osserman examples) and the case of higher genus covers.

1.2. Spaces whose components appear here. Liu and Osserman [**LOs06**, Cor. 4.11] consider all connected covers, $\varphi : X \to \mathbb{P}^1_z$ of the Riemann sphere $\mathbb{P}^1_z$ (uniformized by z) with the genus $\mathbf{g}_X$ of X equal 0, and the degree $\deg(\varphi) = n$, having r specific *pure-cycles* as branch cycles. Stipulate one of their examples with the pure-cycle lengths (with no loss) as $\boldsymbol{d} = (d_1, \dots, d_r)$. For many purposes, assume $d_1 \le d_2 \le \cdots \le d_r$. Their result: The space of such covers form one connected family $\mathcal{H}^{\mathrm{abs}}_{\boldsymbol{d}}$. Here, abs denotes *absolute equivalence* (the usual notion, see §2.1.1) of covers in the family.

1.2.1. *Liu-Osserman examples.* Compare the Liu-Osserman genus 0 result, with [**Fr06b**, Thm. A and B] where the pure-cycles are all 3-cycles, but $\mathbf{g}_X$ is any fixed non-negative integer. Here, if the genus exceeds 0, the spaces have exactly two components, distinguishable using our main tool, the *spin invariant* (related in this case to Riemann's half-canonical classes).

The spin invariant has many uses. Two used here: deciding when a none $2(=p)$ cusp has above it only 2 cusps; and formulating a natural umbrella result containing both [**LOs06**] and [**Fr06b**] (§6.3.3). Our main results apply when all the pure-cycles have odd order and $r = 4$. Then, $G = A_n$ and, with no loss, the pure-cycle lengths are $d_1 \le d_2 \le d_3 \le d_4$ with $\sum_{i=1}^4 d_i - 1 = 2(n-1)$. We redo, while generalizing, part of their results for two reasons.

(1.1a) [**LOs06**, Cor. 4.11] is on *absolute* equivalence, but the Inverse-Galois and modular-curve-comparison applications are on *inner* equivalence of Galois covers, and results for this case are related but sometimes different (compare $n \equiv 1 \mod 8$ with $n \equiv 5 \mod 8$ in Prop. 4.1).

(1.1b) Redoing their hardest case, $r = 4$, using our combinatorial description of cusps shows quickly its advantage (Table 1 of Lem. 4.2).

When $p = 2$, Ex. 3.11 applies Invariance Prop. 2.12 to describe exactly which of the Liu-Osserman examples are the bottom level of at least one abelianized **MT**. When $p \ne 2$ is a prime dividing $n!/2$ (but none of the d_i s), then each Liu-Osserman example is the bottom level of at least one **MT**.

Again, §6.5 examples show if you drop the condition that these be spaces of genus 0 covers, the story is much richer. Yet, the lifting invariant tells much — though, not all — of the tale.

1.2.2. *Connecting to the RIGP.* Each space in §1.2.1 plays a role in the R(egular) I(nverse) G(alois) P(roblem), another modular curve-like property. I explain.

Spaces $\mathcal{H} = \mathcal{H}^{\mathrm{in}}$ attached to inner equivalence and a centerless group G come with a uniquely defined Galois cover $\Psi : \mathcal{Y} \to \mathcal{H}^{\mathrm{in}} \times \mathbb{P}^1_z$, with group G. Attached to a K point $\boldsymbol{p} \in \mathcal{H}$ is the fiber $\Psi_{\boldsymbol{p}} : \mathcal{Y}_{\boldsymbol{p}} \to \boldsymbol{p} \times \mathbb{P}^1_z$. This is a geometric cover attached to a K regular realization of G.

Assume p is an odd prime. For modular curves, an old story gives an RIGP way to look at the K rational points of $\{Y_1(p^{k+1})\}_{k=0}^{\infty}$, the space parametrizing elliptic curves with a p^{k+1} division point up to isomorphism. Any $\boldsymbol{p} \in Y_1(p^{k+1})(K)$ corresponds to a regular realization of the dihedral group $D_{p^{k+1}}$ of order $2p^{k+1}$ with four *involution* (order 2) branch cycles ([**Fr78**, §4], for notation §1.3). In fact, the whole Strong Torsion Conjecture for hyperelliptic Jacobians is tantamount to considering: Where are involution realizations of dihedral groups?

Mazur's Theorem describes explicitly the possible orders of $\mathbb{Q}$ torsion points on elliptic curves over $\mathbb{Q}$. One version is that you need at least six branch points to find a $\mathbb{Q}$ regular realization of D_m if m is odd and exceeds 7 [**DFr94**, Thm. 5.1] (§7.2 gives the context more dramatically and generally).

An action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathcal{H}'_k$ produces a *reduced* space $\mathcal{H}'_k/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\mathrm{def}}{=} \mathcal{H}'^{\mathrm{rd}}_k$. A K point on $\mathcal{H}'_k$ produces a K point on $\mathcal{H}'^{\mathrm{rd}}_k$, a complex analytic space of dimension $r-3$. Let $\bar{\mathcal{H}}^{\mathrm{rd}}_k$ be the unique (projective) normalization of $\mathbb{P}^r/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\mathrm{def}}{=} \bar{J}_r$ in the function field of $\mathcal{H}^{\mathrm{rd}}_k$.

Assume $G_0 = G$ is centerless. As it is p-perfect, then so is G_k centerless for all $k \geq 0$ [**BF02**, Prop. 3.21]. So, even one $\mathbb{Q}$ point $\boldsymbol{p}$ (not a cusp) on $\mathcal{H}'_k$ (a manifold) gives a geometric cover $\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ over $\mathbb{Q}$ with group G_k (§1.2.2). Running over $\beta \in \mathrm{PGL}_2(\mathbb{Q})$ the covers $\beta \circ \varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z$ give ∞-ly many inner inequivalent covers, with group G_k, over $\mathbb{Q}$. These, however, are all reduced equivalent.

1.2.3. *Main Conjecture(s).* For any **MT**, $\{\mathcal{H}_k\}_{k=0}^{\infty}$, over a number field K, we expect properties like those of the standard modular curve tower. For example, high levels should have no K points; one, of two, aspects of the Main **MT** Conjecture.

(1.2a) Arithmetic MC: For any number field K, $\mathcal{H}'^{\mathrm{rd}}_k(K) = \emptyset$ for $k >> 0$.

(1.2b) Geometric MC: For $k >> 0$, $\bar{\mathcal{H}}'^{\mathrm{rd}}_k$ has general type.

When $r = 4$ the reduced levels are curves, and (1.2a) is equivalent to (1.2b). In this case, the failure of the Main Conjecture has an explicit statement using p cusps (Prop. 2.8). For many Nielsen classes we cannot be certain that $\boldsymbol{p} \in \mathcal{H}'^{\mathrm{rd}}_0(K)$ is the image of some $\boldsymbol{p}^{\mathrm{rd}} \in \mathcal{H}'_0(K)$. Still, (1.2b) holds if (and only if) the statement replacing $\mathcal{H}'^{\mathrm{rd}}_k$ by $\mathcal{H}'_k$ holds. Further, it holds if $\mathcal{H}'^{\mathrm{rd}}_{k^*}(K)$ is finite for some k^*.

[**Cad05b**] has shown the *S(trong) T(orsion) C(onjecture)* (§7.3 lists various versions) from abelian varieties implies a stronger version of (1.2a). For that, replace G_k by $G^{\mathrm{ab}}_k \stackrel{\mathrm{def}}{=} G_k/U_k$ with U_k the commutator subgroup of $\ker(G_k \to G)$: This we call the Arithmetic $\mathrm{MC}^{\mathrm{ab}}$, with a similar decoration for the Geometric $\mathrm{MC}^{\mathrm{ab}}$. It clearly implies (1.2a).

For the prime $p = 2$, when even one of the d_i is even, the monodromy group of a cover in the Nielsen class is S_n, which is not 2-perfect. So, to to satisfy the 2-perfect condition, a Liu-Osserman example must have all the d_i s odd.

Congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ appear in the standard definition of modular curves. This make a simple arithmetic matter for finding their genuses. When $r = 4$, the (reduced) **MT** levels are defined by finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$, though they are rarely congruence subgroups and certainly not in the Liu-Osserman cases. Still, the (compactified) levels have a genus (usually different from that of the curves their points parametrize). The $\mathrm{MC}^{\mathrm{ab}}$ holds —Arithmetic (for all number fields K) or Geometric —if the and only if some tower level genus exceeds 1.

When all the d_i s are odd, then $G = A_n$ for some n. For the most refined results, we take the subcase the d_i are equal: $\mathbf{C} = \mathbf{C}_{(\frac{n+1}{2})^4}$ is four repetitions of an $\frac{n+1}{2}$-cycle (odd only if $n \equiv 1 \mod 4$). Props. 4.9 ($n \equiv 5 \mod 8$) and 4.10 ($n \equiv 1 \mod 8$) prove the Arithmetic $\mathrm{MC}^{\mathrm{ab}}$ as a corollary of computing the genuses of the reduced level 0 Hurwitz spaces. A graphic understanding of these very different cases —the first proven cases of the Main Conjecture for infinitely many distinct, non-modular curve, examples —comes from explicit sh-incidence matrices.

The case $n \equiv 1 \mod 8$ has special interest because there are two braid orbits. That is $\mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ is not connected. The explanation does not come from the lifting invariant. Rather it is a case of the outer automorphism of A_n taking one component to the other. So, here is an infinite set of cases with the covering group

$G = A_n$ simple, where it is not immediately obvious what are the definition fields of these two components.§6.4 shows, in fact, the two components are conjugate over a natural quadratic extension of $\mathbb{Q}$.

The case $n \equiv 5 \mod 8$, where $\mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ has just one component, generalizes level 0 of the major example ($n = 5$) that guided so much of [**BF02**]. The difference: [**BF02**, §9] went deeply into level 1 —even figuring the sh-incidence matrix for it. While here we manage just enough about level 1 in an infinity of cases to draw modular curve parallels and get the Main Conjecture.

1.2.4. *Results phrased as locating p cusps. None* of the $\mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in,rd}}$ spaces has a 2 cusp, though each **MT** over them does have a 2 cusp by level 1 (Cor. 5.1). We expand on the §1.1.2 discussion on modular curve cusps. Cusps on a **MT** form a projective tree, and one way to tackle the nature of the tree is to compare it with the cusp tree of modular curves. For that purpose we call the type of subtree that arises over the long cusp of $\bar{X}_0(p)$ a *p-Spire*.

For the cases $n \equiv 5 \mod 8$, the H-M cusps at level 1 are the base of a p-Spire (Cor. 5.2), a property that is stronger than any of the Main Conjectures, at least when $r = 4$. Considering the existence of a p-Spire is meaningful for any $r \ge 4$.

§6.1 gives an approach to proving the Main Conjectures for Liu-Osserman examples for primes different from 2. While we are certain this technique will work, it misses a piece of modular representation theory at this time. §6.2, however, shows it working for $(A_5, \mathbf{C}_{3^4})$ and the prime 5. Finally, §6.3 remarks on other phenomena appearing in proving Main Conjecture for the rest of the Liu-Osserman examples not completed here. We felt it time to regroup before taking what we we'd learned here to a new stage in understanding the Main **MT** conjectures.

1.3. Using Classical Generators of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}, z_0)$. Let $\varphi : X \to \mathbb{P}^1_z$ be a (nonconstant) function on a compact Riemann surface X. Then, φ defines a number of quantities:

(1.3a) A group G for a minimal Galois closure cover $\hat{\varphi} : \hat{X} \to \mathbb{P}^1_z$: Automorphisms $(\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z))$ of $\hat{X}$ commuting with $\hat{\varphi}$;
(1.3b) Unordered branch points $\boldsymbol{z} = \{z_1, \dots, z_r\} \in U_r$;
(1.3c) Conjugacy classes $\mathbf{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$ in G; and
(1.3d) A *Poincaré extension* of groups:

$$\psi_{\hat{\varphi}} : M_{\hat{\varphi}} \to G \text{ with } \ker_\psi \stackrel{\text{def}}{=} \ker(M_{\hat{\varphi}} \to G) = \pi_1(\hat{X}).$$

Further, (1.3a) produces a permutation representation of G, by its action on the cosets of $\mathrm{Aut}(\hat{X}/X)$ in $\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z)$. Here the coset of the identity is given canonically, but there is no natural labeling of the other cosets.

1.3.1. *Homomorphisms and Nielsen class elements.* If we put an ordering on $\boldsymbol{z}$, then following App. A, we may consider a set of classical generators of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}, z_0)$. This ordering doesn't assume C_i is the conjugacy class attached to z_i. Denote their isotopy class as r generators of the fundamental group by $\bar{\boldsymbol{g}} = (\bar{g}_1, \dots, \bar{g}_r)$. Refer to their images in $M_{\hat{\varphi}}$ also as $\bar{\boldsymbol{g}}$, and their images in G by $(g_1, \dots, g_r) = \boldsymbol{g}$.

Then, $\boldsymbol{g}$ is in the *Nielsen class* of $(G, \mathbf{C})$:

$$\mathrm{Ni}(G, \mathbf{C}) \stackrel{\text{def}}{=} \{\boldsymbol{g} \in \mathbf{C} \mid \langle \boldsymbol{g} \rangle = G, \Pi(\boldsymbol{g}) \stackrel{\text{def}}{=} g_1 \cdots g_r = 1\}.$$

In English: The set consisting of ordered r-tuple generators of G having product one, and falling (in some order, multiplicity counted) in the conjugacy classes $\mathbf{C}$.

Given classical generators $\bar{\boldsymbol{g}} = (\bar{g}_1, \dots, \bar{g}_r)$ of M_φ, an element of $\mathrm{Ni}(G, \mathbf{C})$ is exactly what we need to form a canonical map $M_{\hat{\varphi}} \to G$, by mapping $\bar{g}_i \mapsto g_i$, $i = 1, \dots, r$. The notation $M_{\bar{\boldsymbol{g}}}$ applied to $M_{\hat{\varphi}}$ is useful, and then it is convenient to rename $\psi_{\hat{\varphi}}$ to $\psi_{\bar{\boldsymbol{g}}, \boldsymbol{g}}$.

The classical generators of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}, z_0)$ form a homogeneous space for the action of the combinatorial *Hurwitz monodromy* group. We use the phrase *braid* equivalence of these homomorphisms by this action (§2.1.5). When there is just one such equivalence class, we call the space of such homomorphisms *connected*. This corresponds to an actual *Hurwitz* space $\mathcal{H}(G, \mathbf{C})$ being connected.

Actually, there are four types of Hurwitz space that appear in this paper attached to any Nielsen class: $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$, $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}$ (attached to a permutation representation of G and their reduced versions $\mathcal{H}(G, \mathbf{C})^{\mathrm{in,rd}}$ and $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs,rd}}$. Each corresponds to a further equivalence on $\mathrm{Ni}(G, \mathbf{C})$ (§2.1.5). Connectedness of $\mathcal{H}(G, \mathbf{C})^{\mathrm{in}}$ and $\mathcal{H}(G, \mathbf{C})^{\mathrm{in,rd}}$ (resp. $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs}}$ and $\mathcal{H}(G, \mathbf{C})^{\mathrm{abs,rd}}$) are equivalent, though we emphasize cusps, and their combinatorial counterparts, that belong to the reduced spaces. Significant cases, however, are that connectedness absolute spaces does not imply connectedness of their associatedinner space.

App. A summarizes the literature on this correspondence and these spaces. Most of this paper is about the braid orbits; the translation to applications is by quoting results already in print. Applications depend on our figuring from this useful properties of the Hurwitz space, or their reduction by an action of $\mathrm{PSL}_2(\mathbb{C})$ so it has dimension $r - 3$. Also, applications don't start from one group or one Nielsen class. Rather, initial data about some problem produces a collection of groups that could be the monodromy group of a cover solving the problem. The following is the meaning of the phrase:

The R(egular)I(nverse)G(alois)P(roblem) holds for G:

There is a geometric Galois cover of the sphere with group G, with all its automorphisms defined over $\mathbb{Q}$. Such a regular realization of G corresponds to a rational point on an inner Hurwitz space associated to some Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ for some rational union of conjugacy classes in G (§7.2). Part of the point of this theory is that if a Hurwitz space has no rational points, then there will be no regular realizations corresponding to those conjugacy classes.

§7.3 reminds that **MT**s result from a ramification restriction on the RIGP, akin to, but far less restrictive than that used in number fields for the Fontaine-Mazur Conjecture. The Main Conjecture thus says, for each p-perfect finite group, there are p-perfect covers of it for which require increasingly unbounded numbers of conjugacy classes to produce any regular realization of them.

1.3.2. *Three expositional sections.* There is an html definition file for the RIGP: http://www.math.uci.edu/˜ mfried → Sect. I.b. → Definitions: Arithmetic of covers and Hurwitz spaces → * The R(egular) I(nverse) G(alois) P(roblem): RIGP.html.

Similarly for an overview of **MT**s:

Outline of how Modular Towers generalizes modular curve towers:
http://www.math.uci.edu/˜ mfried → Sect. I.a. → Articles: Generalizing modular curve properties to Modular Towers → Item #1 mt-overview.html.

Hurwitz spaces are abstract. They have algebraic descriptions, though it is inefficient to rely on their equations. For G a p-perfect group, you can't get their Hurwitz spaces from Kummer theory; they come from nonabelian covers of $\mathbb{P}^1$. The essential data about Hurwitz spaces we use comes through connectedness results.

We start with $r = 4$ (supported by $r = 3$), where the reduced Hurwitz spaces are curves (with natural compactifications) that are upper-half plane quotients and j-line covers.

Properties come from knowing about cusps through the braid class of our homomorphisms $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$. When the spaces are connected, or their components separate by discrete invariants, we know their definition fields.

Example conclusion: In Davenport's problem (§7.1), there are no nontrivial pairs of indecomposable polynomials over $\mathbb{Q}$ with the same value sets modulo all but finitely many primes. There are, however, class calls. For several Nielsen classes, representing a finite number of possible degrees, there are families of polynomial pairs (in particular, genus 0 covers) that do give the same values over a finite extension of $\mathbb{Q}$. These families have each more than one connected component, with none defined over $\mathbb{Q}$. A cover gives a bundle (in this case over $\mathbb{P}^1_z$. Then, each Hurwitz space component attached to a given Nielsen class in Davenport's problem, defines the same family of bundles over $\mathbb{Q}$.

Second example: §7.2 reminds of the Conway-Fried-Parker-Völklein framework for connectedness results. A rough statement for that: If you repeat conjugacy classes *sufficiently*, you know explicitly the connected components (and their definition fields) of the Hurwitz spaces (absolute or inner) of covers of the sphere in a given Nielsen class $\text{Ni}(G, \mathbf{C})$. Our examples engage expectations from the word *sufficiently*, by considering imformation embodied in cusps.

This applied to show how to find Nielsen classes for which the corresponding inner Hurwitz space has a connected component with definition field $\mathbb{Q}$. They must exist, and some of them must have $\mathbb{Q}$ points, for each centerless group G if the RIGP is correct. Still, the version of the Conway-Fried-Parker-Voelklein result in [**FV92**] required unknown large values of r. It applied to create presentations of $G_{\mathbb{Q}}$, the absolute Galois group of $\mathbb{Q}$, the first, and still, only such proven presentations. The version of CFPV in §7.2 allows us to state connnectedness problems very close to the Liu-Osserman examples that reflect on all aspects of this paper, especially how explicitly lifting invariants tie to connectedness results.

Finally, §7.3 comments on Cadoret's observation that the S(trong) T(orsion) C(onjecture) implies the abelianized version of the Main Conjecture, more evidence that connected results help solve practical problems.

1.3.3. *Notation.* Group notation: Denote the cyclic group of order N by $\mathbb{Z}/N$. For any finite group G, with conjugacy classes $\mathbf{C} = \{\mathrm{C}_1, \dots, \mathrm{C}_r\}$, denote the least common multiple of orders of elements in $\mathbf{C}$ by $N_{\mathbf{C}}$. We say G is p-perfect if $p||G|$, but there is no surjective homomorphism $G \to \mathbb{Z}/p$.

Equation notation: If V is a (quasi-projective) algebraic variety (open subspace of a projective variety), and K is a field, then $V(K)$ denotes the points on V with coordinates in K.

Our main results are on pure-cycle Nielsen classes. A *pure-cycle* conjugacy class in $G \le S_n$ is one in which each element in the class has exactly one nontrivial (length greater than 1) disjoint cycle. Some displays simplify by using the notation $x_{i,j}$ for $(i\, i{+}1 \cdots j)$. This is assuming $1 \le i < j \le n$. The inverse of this element is $x_{i,j}^{-1} = x_{j,i}$. So, long as we don't cycle around $\mod n$, the notation for inverse should cause no problems. We use the following easy lemma often.

LEMMA 1.1. *Consider $x_{a,b}$ with $a < b$ with $b-a \equiv 0 \mod 4$ (resp. $\equiv 2 \mod 4$). Then, $(b\,a)(b{-}1\,a{+}1)\cdots(b'\,a')$ with $(b'\,a') = (b{-}\frac{b-a-2}{2}\; a{+}\frac{b-a-2}{2})$, an even (resp. odd) permutation, conjugates $x_{a,b}$ to its inverse. It has parity $(-1)^{\frac{b-a}{2}}$.*

Also, if $b-a+1 \equiv 0 \mod 4$ (resp. $\equiv 2 \mod 4$) then $(b\,a)(b{-}1\,a{+}1)\cdots(b'\,a')$ with $(b'\,a') = (b{-}\frac{b-a-1}{2}\; a{+}\frac{b-a-1}{2})$ is even (resp. odd) permutation, conjugates $x_{a,b}$ to its inverse. It has parity $(-1)^{\frac{b-a+1}{2}}$.

We often use the acronym *R-H* (App. A) for the Riemann-Hurwitz formula given by the genus of a cover of a sphere from a branch cycle description for it.

2. Tools and MT definitions

§2.1 introduces the braid group and certain of its quotients and subgroups, in the service of a natural equivalence on group extensions. The definitions of **MT**s and their abelianization appear here. §2.2 has the combinatorial definition of cusps that appears in the statement of the paper's precise results. Their relation to the Main Conjecture is in §2.3.

§2.4 has the main homological tool, the *spin lifting invariant* and how it applies to deciding braid orbits and existence of **MT**s. Finally, §2.5 introduces the precise Nielsen classes for our main result. Here there are examples of how to apply the lifting invariant for information on cusps at the next level.

2.1. Braid actions and MTs.

We start with braid actions on sphere covers.

2.1.1. *Deformation equivalence of extensions.* If $\varphi : X \to \mathbb{P}^1$ is Galois with group G with G abelian, we could write equations for it by hand. From, however, G being p-perfect, it isn't. Further, why deal one cover at-a-time? Consider all covers with $(G, \mathbf{C})$ as their data: In the Nielsen class.

We have a topological need: to decide the nature of connected components of all covers in a given Nielsen class. For that we have a deformation conclusion: Each component has a cover with any a priori fixed (collection of r distinct) branch points $\boldsymbol{z}^0$. That is, any cover (with branch points $\boldsymbol{z}$) deforms through covers with r branch points to a cover with branch points $\boldsymbol{z}^0$. Further, if $(\boldsymbol{g}, \mathbf{C})$ is associated to the, consider a homomorphism from §1.3.1: $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$. Then, $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$ and any of its extensions deform with it. This, the identification of *Hurwitz Monodromy group* H_r with $\pi_1(U_r, \boldsymbol{z}^0)$, and the explicit action (with representing paths) on $\bar{\boldsymbol{g}}$ in (2.1) is in [**Fr77**, §4].

For further help, H_r — related to classical braid group discussions — and their consequences are reviewed in [**BF02**, §2.2] (full proofs compatible with our use are in [**Fr07**, Chap. 4, 5]; exposition in html definition files as in §1.3.2). We especially use H_4 though, generally: H_r is the group of automorphisms of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}^0, z_0)$ that preserves an (transitive) action on classical generators. Given classical generators, it identifies with $\pi_1(U_r, \boldsymbol{z}^0)$.

We give the generators of H_r by their actions on $\bar{\boldsymbol{g}}$:

(2.1a) Shift: $\mathbf{sh} : \bar{\boldsymbol{g}} \mapsto (\bar g_2, \dots, \bar g_r, \bar g_1)$; and
(2.1b) 2nd Twist: $q_2 : \bar{\boldsymbol{g}} \mapsto (\bar g_1, \bar g_2 \bar g_3 \bar g_2^{-1}, \bar g_2, \bar g_4, \dots)$.

For each $i = 1, \dots, r{-}1$ there is an i-twist $q_i \stackrel{\text{def}}{=} \mathbf{sh}^{i-2} q_2 \mathbf{sh}^{-i+2}$ ($i \mod r{-}1$). Our formulas are best seen using $i = 2$ when $r = 4$.

2.1.2. **MT** *definitions.* Denote the maximal p-Frattini cover of G with elementary p group kernel by $G_1 \to G = G_0$. Let $G_{k+1} = G_1(G_k)$. Note: We drop most p notation. Still, if you change p, the new G_k for $k > 0$ is a different group.

DEFINITION 2.1 (**MT**). A projective system of H_r orbits on $\{\mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$ is a M(odular) T(ower). Let $\ker_{k,0} = \ker(G_k \to G_0 = G)$. An *abelianized* **MT** is similarly a projective system, except the braid orbits are in the Nielsen classes from replacing G_k by $G_k/(\ker_{k,0}, \ker_{k,0}) = G_{k,\mathrm{ab}}$ (as in [**BF02**, Prop. 4.16]).

Denote the projective limit of all $G_{k,\mathrm{ab}}$ s by ${}_p\tilde{G}/(\ker_0, \ker_0) = {}_p\tilde{G}_{\mathrm{ab}}$. Though $G_{1,\mathrm{ab}} = G_1$, for $k > 1$, the natural map $G_k \to G_{k,\mathrm{ab}}$ has (known) degree exceeding 1 if and only if $\dim_{\mathbb{Z}/p} \ker(G_1 \to G) > 1 \Leftrightarrow G_0$ is not p super-solvable [**BF02**, §5.7].

Let $M_{\bar{\boldsymbol{g}},\mathrm{ab}}$ be the natural quotient of $M_{\bar{\boldsymbol{g}}}$ with $\ker(M_{\bar{\boldsymbol{g}},\mathrm{ab}} \to G)$ the homology of the Riemann surface for which $\ker(M_{\bar{\boldsymbol{g}}} \to G)$ is its fundamental group. Finding extensions of $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$ to ${}_p\tilde{G}_{\mathrm{ab}}$ is equivalent to its extension to $M_{\bar{\boldsymbol{g}},\mathrm{ab}} \to G$.

Any p-perfect group G has a universal p-Central extension $\psi^* : R^*_{G,p} \to G$. Universal here means that if $\mu_{H,G} : H \to G \to 1$ is a central p extension, than a unique map $\psi : R^*_{G,p} \to H$ commutes between ψ^* and $\mu_{H,G}$. Let $\mu_k : R_k \to G_k$ be the universal exponent p central extension of G_k:

- $G_{k+1} \to G_k$ factors through μ_k.
- $\ker(R_k \to G_k) =$ max. elementary p-quotient of the Schur multiplier of G_k.

2.1.3. *Group form of* **MT***s.* For a prime p dividing $|G|$, we ask the following.

(2.2a) When does $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$ extend to all covers $H \to G$ with p-group kernel?
(2.2b) How does this depend on $\boldsymbol{g}$?
(2.2c) What equivalence reasonably describes all extensions of $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$?

2.1.4. *Basic Reductions.* The following reductions apply to considering when there is an affirmative answer to (2.2a).

(2.3a) Complete $M_{\bar{\boldsymbol{g}}}$ so $\ker_{\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}}$ is the pro-p completion of $\pi_1(X)$.
(2.3b) Restrict to p-Frattini covers $H \to G$ (no H proper in G maps onto).
(2.3c) Any $g \in \mathbf{C}$ must have order prime to p.
(2.3d) G must be p-perfect (it has no $\mathbb{Z}/p$ quotient).

Equivalent: When are all p-Frattini covers $H \to G$ achieved by unramified extensions $Y_H \to X$ extending $X \to \mathbb{P}^1$?

Here is the source of these reductions. For (2.3b), consider a p extension

$$\mu : H \to G \to 1.$$

Take any subgroup $H^* \leq H$ for which μ_{H^*} is still surjective. A minimal such is a p-Frattini cover of G. Assuming you can extend to that, you can extend through μ. Any element of order p in $G = G_0$ has all its lifts to G_1 of order p^2 [**FK97**, Lifting Lem. 4.1]. That explains (2.3c). The most used geometric implication is for cusps (first paragraph of Prop. 2.7): Suppose a cusp at level k_0 has ramification index divisible by p. Then, over it are only cusps whose ramification indices are divisible by an additional power of p.

Finally, for (2.3d), we now know all elements of $\mathbf{C}$ are p'. So, entries of a Nielsen class element cannot generate if G has $\mathbb{Z}/p$ as a quotient (as in [**Fr06a**, Lem. 2.1]).

Since the G_k s (of §2.1.2) are co-final in all p-Frattini covers of G, goal (2.2a) needs only for H to run over the G_k s.

2.1.5. *Braid Comments.* Through the action (2.1), the H_r action on $\bar{\boldsymbol{g}}$ extends to an action on the image of $\bar{\boldsymbol{g}}$ in any quotient group G of $M_{\bar{\boldsymbol{g}}}$. Then, it acts compatibly on the following sets:

(2.4a) Inner Nielsen Classes: $\mathrm{Ni}(G,\mathbf{C})/G \stackrel{\mathrm{def}}{=} \mathrm{Ni}^{\mathrm{in}}$.
(2.4b) Absolute Nielsen classes: $\mathrm{Ni}(G,\mathbf{C})/N_{S_n}(G) \stackrel{\mathrm{def}}{=} \mathrm{Ni}^{\mathrm{abs}}$ (given $G \le S_n$ a permutation representation).
(2.4c) Poincaré extensions: $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}} : M_{\bar{\boldsymbol{g}}} \to G$.

Since we want extensions of homomorphism $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$, the action (starting from $q \in H_r$ acting on $\bar{\boldsymbol{g}}$, from the left) is given by $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}} \mapsto \psi_{\bar{\boldsymbol{g}},(\boldsymbol{g})q^{-1}}$, an action on the right. Any extension properties of $\psi_{\bar{\boldsymbol{g}},\boldsymbol{g}}$ are preserved by a braid orbit.

Given $(G,\mathbf{C},p)$, here is our first (albeit, naive) goal.

Conjecture 2.2 (Goal 1). Understand projective systems of H_r orbits acting on $\{\mathrm{Ni}(H,\mathbf{C})^{\mathrm{in}}\}_{H\to G}$: Running over p-Frattini covers $H \to G$.

Remark 2.3. The actions of (2.4) come through uniquely deforming covers by deforming branch points. This won't work for covers in positive characteristic.

2.2. Cusp types. When reduced Hurwitz spaces are not connected, the critical problem is deciding on which components the cusps belong. For each M(odular) T(ower) (Def. 2.1), there is a prime p (dividing $|G|$) attached to the problem, and a notion of p cusp.

2.2.1. *Hurwitz space Cusps.* We understand H_r orbits through their *cusps*. A combinatorial definition gives them as an H_r suborbit of a *cusp group* $\mathrm{Cu}_r < H_r$.

(2.5a) For $r \ge 5$: $\mathrm{Cu}_r = \langle q_2 \rangle$.
(2.5b) For $r = 4$: $\mathrm{Cu}_r = \langle q_2, \mathbf{sh}^2, q_1 q_3^{-1} \rangle$.

Much data is from the conjugacy class of Cu_r. So —except for normalizations related to identifications with upper half-plane objects —if done consistently, we could substitute q_i for the appearance of q_2 in Cu_r.

The following definition appears often in our results.

Definition 2.4. A *p cusp* is the Cu_r orbit of $\boldsymbol{g} \in O$ for which $p^{\mu_p(\boldsymbol{g})}||\mathrm{ord}(g_2g_3)$, $\mu_p(\boldsymbol{g}) > 0$ (p-multiplicity of $\boldsymbol{g}$).

The definition doesn't depend on the representive of the p cusp, as changing the representative changes (g_2, g_3) to (hg_2h^{-1}, hg_3h^{-1}) with h a power of g_2g_3. For $r = 4$, to see that being a p cusp is independent of the representative, you would substitute (g_4, g_1) (resp. (g_1, g_4)) for (g_2, g_3) to see the condition for a p cusp is unchanged by applying $\mathbf{sh}^2$ (resp. $q_1q_3^{-1}$) to $\boldsymbol{g}$. When $r = 4$, we call $\mathrm{ord}(g_2g_3)$ the *middle product* of $\boldsymbol{g}$, denoted $(\boldsymbol{g})\mathbf{mp}$.

2.2.2. *Other cusp types for* $r = 4$. App. B gives these for $r > 4$.

(2.6a) g(roup)-p': $U_{1,4}(\boldsymbol{g}) = \langle g_1, g_4\rangle$ and $U_{2,3}(\boldsymbol{g}) = \langle g_2, g_3\rangle$ are p' groups
(2.6b) o(nly)-p': p cusp, but $U_{1,4}(\boldsymbol{g})$ or $U_{2,3}(\boldsymbol{g})$ not p'.

H(arbater)-M(umford) cusps for all even $r = 2s$ have a cusp orbit representative of form $(g_1, g_1^{-1}, \dots, g_s, g_s^{-1})$. When $r = 4$, its shift $(g_1^{-1}, g_2, g_2^{-1}, g_1)$ is a representative for a g-p' cusp, no matter what is p since the middle product is 1.

Existence of an H-M rep. requires special conjugacy classes. They must be pairable, in the form $\mathrm{C}_1, \mathrm{C}_1^{-1}, \dots, \mathrm{C}_s, \mathrm{C}_s^{-1}$ where the -1 exponent denotes the conjugacy class of the inverse of an element in a given conjugacy class. Consider a Liu-Osserman pure-cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{\boldsymbol{d}})^{\mathrm{abs}}$ (§1.2). So, with $d_1 < d_2 < \cdots < d_u$, it is necessary that that the conjugacy classes defining the Nielsen class have form $\mathbf{C}_{\boldsymbol{d}}$, with $\boldsymbol{d} = d_1^{e_1} \cdots d_u^{e_u}$ and each e_i even.

The only time there are two distinct conjugacy classes of length d_i is when $G = A_n$, and $d_u = n$ (if n is odd) or $d_u = n{-}1$ (when n is even). In these cases denote the conjugacy class pairs by $\mathrm{C}(n)', \mathrm{C}(n)''$ (resp. $\mathrm{C}(n{-}1)', \mathrm{C}(n{-}1)''$).

PROPOSITION 2.5 (g-p' **MT**). *If a braid orbit O_0 has a g-p' cusp, then a* **MT**, *$\mathcal{O} = \{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$, lies over it.*

Consider a pure-cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C}_{\boldsymbol{d}})^{\mathrm{abs}}$ (§1.2), with all the d_i s odd, and G a transitive, but not cyclic, subgroup of A_n. Then, Prop. 3.10 says $G = A_n$. This Nielsen class contains an H-M *rep. if and only if one of the following:*

(2.7a) *$n \equiv 1 \mod 4$ (resp. $n{-}1 \equiv 1 \mod 4$), $d_u = n$ (resp. $d_u = n{-}1$) and exactly half the e_u conjugacy classes of length d_u are equal $\mathrm{C}(n)'$ (resp. $\mathrm{C}(n{-}1)'$); or*

(2.7b) *$n \equiv 3 \mod$ (resp. $n{-}1 \equiv 3 \mod 4$), $d_u = n$ (resp. $d_u = n{-}1$) and each of the conjugacy classes $\mathrm{C}(n)'$ and $\mathrm{C}(n)''$ (resp. $\mathrm{C}(n{-}1)'$ and $\mathrm{C}(n{-}1)''$) appear with even multiplicity.*

PROOF. The first result applies to the general definition of g-p' cusp as in App. B [**Fr06a**, Fratt. Princ. 3.6]. The necessity of (2.7) is a consequence of the definition of H-M rep., and the congruence condition for declaring when $\mathrm{C}(n)', \mathrm{C}(n)''$, etc. each contain the inverse of any element in them (say, Lem. 1.1).

Finally, to fulfill an H-M rep. under these conditions requires only producing transitive pure-cycles whose lengths in order are given by the symbol $d_1^{e'_1} \cdots d_u^{e'_u}$ with e' denoting $\frac{e}{2}$. Since, $\sum_{i=1}^{u} \frac{e'}{2}(d_i{-}1) \geq n-1$, this is quite easy. Start with $g_1 = (1 \ldots d_1)$, and continue inductively, starting the next pure-cycle —and its increasing integer sequence —with the last integer occuring in the previous pure-cycle. When you get to n, cycle around to 1. Example: For $n = 7$, $r = 6$ and $\boldsymbol{d} = 3^4 \cdot 5^2$, so $e'_1 = 2, e'_2 = 1$, take $g_1 = (1\,2\,3), g_2 = (3\,4\,5)$, $g_3 = (5\ 6\,7\,1\,2)$. □

REMARK 2.6 ($G = S_n$ and genus not 0 in Prop. 2.5). If in Prop. 2.5 one of the d_i s is even then $G = S_n$, and the conjugacy classes considerations for top-length cycles is much simpler. On the other hand, there are a few more exceptions [**LOs06**, Thm. 5.3]: Cyclic groups for one. The Liu-Osserman case is when the Nielsen class has genus 0. So, if $r \geq 3$, a cyclic (transitive) subgroup of S_n generated by pure-cycles is impossible.

Also, if $r = 3$, there is a non-trivial possibility of having, $n = 5$, and $\boldsymbol{d} = 4^2 \cdot 5$ where the group G is the non-standard representation of S_5 in S_6. The action of S_5 on the normalizer N of a 5-Sylow. If we expand beyond the genus 0 case, we can have $A_5 = G$ in this degree six representation (all conjugacy classes among the two 5-cycle conjugacy classes), too. So, for higher genus, these sporadic representations of pure-cycle cases require careful accounting.

2.3. MT Geometric correspondence. A **MT** is a projective system of braid orbits $\mathcal{O} = \{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}\}_{k=0}^{\infty}$. This corresponds to a projective system (a tower) of —inner Hurwitz spaces —$\{\mathcal{H}_k\}_{k=0}^{\infty}$ where the $\mathcal{H}_k$ s are (normal) absolutely irreducible affine algebraic varieties (dim=r) each covering $U_r \subset \mathbb{P}^r$.

Here is the Main Conjecture, about rational points on the spaces $\mathcal{H}_k$ (not on their compactifications).

CONJECTURE 2.7 (Main Conj.). For K a number field, $\mathcal{H}_k(K) = \emptyset$ for $k >> 0$.

The Main Conjecture is trivial unless G is p-perfect, for otherwise $\mathrm{Ni}(G, \mathbf{C})$ is empty when $\mathbf{C}$ are p' conjugacy classes [**Fr06a**, Lem. 2.1]. It is trivial, too, unless the **MT** has some number field K as definition field: all levels (simultaneously, with the maps between them) have K as definition field. [**Fr06a**, Prop. 3.3] reduces the Main Conjecture to this case:

(2.8) The prime p does not divide the order of the center of G.

From here on, assume these things hold, including G is p-perfect and has p' center.

2.3.1. *What tower levels look like if the Main Conjecture is wrong.* Prop. 2.8, shows we may replace Hurwitz spaces by their reduced versions, and also the reliance of the Main Conjecture on knowing about cusps. Our main results concentrate on the case $r = 4$, where reduced tower levels are curves.

PROPOSITION 2.8. *The conclusion of Conj. 2.7 holds if and only if it holds with each $\mathcal{H}_k$ replaced by its corresponding reduced space $\mathcal{H}_k/\mathrm{PGL}_2(\mathbb{C}) = \mathcal{H}_k^{\mathrm{rd}}$. Let $\bar{\mathcal{H}}_k^{\mathrm{rd}}$ be the unique (projective) normalization of $\mathbb{P}^r/\mathrm{PGL}_2(\mathbb{C}) \stackrel{\mathrm{def}}{=} \bar{J}_r$ in the function field of $\mathcal{H}_k^{\mathrm{rd}}$. If there is at least one p cusp at level k_0, the relative degree $\deg(\bar{\mathcal{H}}_{k+1}^{\mathrm{rd}}/\bar{\mathcal{H}}_k^{\mathrm{rd}}) \stackrel{\mathrm{def}}{=} d_{k+1,k}$, is some integer multiple of p for $k \ge k_0$. It is always true that $\limsup_{\leftarrow k} d_{k+1,k} > 1$.*

For $r = 4$, Conj. 2.7 holds unless for $k >> 0$, either the cover $\bar{\mathcal{H}}_{k+1}^{\mathrm{rd}} \to \bar{\mathcal{H}}_k^{\mathrm{rd}}$

- *doesn't ramify and each $\bar{\mathcal{H}}_k^{\mathrm{rd}}$ has genus 1,*
- *or it is equivalent to a degree p polynomial cover of $\mathbb{P}^1_w \to \mathbb{P}^1_z$,*
- *or it is equivalent to a degree p rational (Redyi) function ramified (of order p) at two points.*

PROOF. Everything in this proposition is already in [**Fr06a**, §5] except the observation showing the impossibility of $d_{k+1,k} = 1$ for all large k. Assume it is 1 for all $k \ge k_0$. Then, let $\boldsymbol{p}^{\mathrm{rd}} \in \mathcal{H}_{k_0}^{\mathrm{rd}}$ be any point defined over some number field K that is the image of $\boldsymbol{p} \in \mathcal{H}_{k_0}(K)$ (a K point of the non-reduced space).

Since G is p-perfect and its center is p', [**BF02**, §2.2.2] shows that all the G_k s also have p' center. Start with a stronger condition —satisfied by all examples in this paper: it has no center at all. Then, the non-reduced spaces, given as fiber products [**BF02**, §2.2.2], $\{\mathcal{H}_k = \mathcal{H}_k^{\mathrm{rd}} \times_{J_r} U_r\}_{k=0}^{\infty}$, all have fine moduli [**FV91**, Cor. 1]. As the maps between the reduced spaces have degree one, they induce degree 1 maps $\mathcal{H}_{k+1} \to \mathcal{H}_k$, identifying them as covers of U_r. So, the points on each space identified to $\boldsymbol{p}$ correspond to a projective sequence of covers $X_k \to \mathbb{P}^1_z$ realizing the G_k s as a Galois group over K. This contradicts [**BF02**, Prop. 6.8]: No such projective sequence can exist over a number field K. This implies $\limsup_{\leftarrow k} d_{k+1,k} > 1$.

Suppose, G_0 does have a nontrivial p' center Z. Then, we cannot immediately assume the conclusion above immediately holds. Here is why. While the point $\boldsymbol{p}$ has coordinates in K, since the spaces $\mathcal{H}_k$ don't have fine moduli, there may be no Galois cover of $\mathbb{P}^1_z$ associated to $\boldsymbol{p}$ defined (with its automorphisms) over K. App. §C completes the argument in this case. □

The following comments elaborate on the geometric correspondence between p cusps and geometric cusps.

(2.9a) Cusps at level k correspond to divisors on the normal compactification $\bar{\mathcal{H}}_k^{\mathrm{rd}}$ over J_r.

(2.9b) F(rattini) Princ. 1 [**Fr06a**, Princ. 3.5]: If ${}_0\boldsymbol{g} \in O_0$ is a p cusp —with μ_p as in (2.4), then $(\boldsymbol{g})\mathbf{mp}$ —then $\mu_p({}_k\boldsymbol{g}) = k + \mu_p({}_0\boldsymbol{g})$.

(2.9c) If $r = 4$, then, $\mathcal{H}_k^{\rm rd}$ is an upper-half plane quotient, j-line cover, with ramification order dividing 3 (resp. 2) over 0 (resp. 1).

The combinatorics theoretically allows us to compute the genus of the reduced **MT** levels when $r = 4$. Consider a $\bar{M}_4 \stackrel{\rm def}{=} H_4/\langle \mathbf{sh}^2, q_1 q_3^{-1}\rangle$ orbit $O \le \mathrm{Ni}(G, \mathbf{C})^{*,\rm rd}$. Take $*$ to be any appropriate equivalence; here either inner or absolute. [**Fr06a**, §3.1.2] gives the formula for computing the genus of a reduced Hurwitz space component $\mathcal{H}_O^*$ corresponding to O. Take the induced permutation actions of q_1q_2, or $\mathbf{sh}$, or q_2) on O and call these respectively $\gamma_0', \gamma_1', \gamma_\infty'$. Then (R-H), the genus $\boldsymbol{g}_O$ satisfies

$$2(|O| + \mathbf{g}_O - 1) = \mathrm{ind}(\gamma_0') + \mathrm{ind}(\gamma_1') + \mathrm{ind}(\gamma_\infty'). \tag{2.10}$$

The following problem is very close to the Main Conjecture when $r = 4$. We suspect a more refined statement is close to the Geometric MC$^{\rm ab}$ for all $r \ge 4$.

PROBLEM 2.9 (Goal 2). Given a **MT**, $\{O_k \subset \mathrm{Ni}(G_k, \mathbf{C})^{\rm in}\}_{k=0}^\infty$, classify when there is a p cusp on O_k for $k >> 0$.

CONJECTURE 2.10. p Cusp Generation: If $\mathcal{O}$ has no p cusp branches, then high tower levels are relatively unramified, and they have no number field as a uniform definition field for all levels.

Otherwise, we expect the relative monodromy groups $G(O_{k+1}/O_k)$ to be p groups, generated by the inertial groups of their relative p cusps.

p Cusp Existence: In particular, this implies a **MT** $\mathcal{O}$ defined over a number field has a p cusp at all high levels.

2.3.2. *Finding p cusps and using g-p' cusps.* Finding p cusps, and the nature of the components on which they lie is the most sophisticated ingredient in this paper. Combining works of the author, J.-P. Serre [**Ser90**], and Thomas Weigel [**Wei05**] gives an effective tool for locating p cusps in the cases here of the **MT** Main Conjecture. The p-adic completions of the groups $M_{\boldsymbol{g}}$ have a property called p-Poincaré duality, crucial to our analysis of the p cusp problem. This tool first appeared through interpreting the Main **MT** conjecture as a problem of computing braid equivalence classes of extensions of the groups $M_{\boldsymbol{g}}$. Its computational gist is Prop. 2.15. The production of full (not abelianized) **MT**s has only come about so far through Prop. 2.5 using g-p' cusps (see (2.6a)). On the other hand, we know many examples where a braid orbit has no g-p' cusp, but there is an abelianized **MT** over that orbit. The $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$ of § 6.5 is sterling for that and for examples of general geometric properties of **MT**s.

The most occurring g-p' cusps in this paper are *shifts of H(arbater)-M(umford)* cusps because Prop. 3.10 applies to our cases. We see the role of H-M cusps classically by translating the **MT** view to modular curves. There are two cusps on the curve called $X_0(p)$. The *long* cusp is a p cusp, and the *short* cusp is a shift of an H-M cusp. Complications in making precise calculations to prove the Main Conjecture arise often because o-p' cusps (as in (2.6b)) can occur, too. These have no analog on modular curves.

These definitions become explicit when we restrict to groups generated by odd pure-cycle Nielsen classes, to which results of Liu-Osserman [**LOs06**] apply. We prove the **MT** Main Conjecture in many of their cases. By comparing their result with [**Fr06b**, Thms. A and B] we develop techniques toward a proof of the Main Conjecture in general for $r = 4$. §6.3.3 uses the Spin-lifting invariant of (2.11) for an explicit conjectured umbrella to the combined Liu-Osserman and Fried results on connected pure-cycle Nielsen classes.

2.4. The Spin lifting invariant. I'll denote the universal central extension of A_n by Spin_n. It happens that $\ker(\mathrm{Spin}_n \to A_n)$ is $\mathbb{Z}/2$ ($n \ge 4$). To present it, embed A_n in the determinant 1 elements $\mathrm{SO}_n(\mathbb{R})$ of the orthogonal group. The fundamental group of $\mathrm{SO}_n(\mathbb{R})$ ($n \ge 4$) is $\mathbb{Z}/2$, so $\mathrm{SO}_n(\mathbb{R})$ has a 2-sheeted cover, $\mathrm{Spin}_n(\mathbb{R})$. Then, Spin_n is the pullback of A_n in $\mathrm{Spin}_n(\mathbb{R})$. It arises in practice often. For example, $A_5 = \mathrm{PSL}_2(\mathbb{Z}/5)$ — 2×2 matrices of determinant 1, mod $\pm I_2$ — and Spin_5 is just $\mathrm{SL}_2(\mathbb{Z}/5)$.

2.4.1. *Universal central extensions.* Examples of Schur multipliers in this paper occur when G is a subgroup of A_n (often it will be A_n). Then, consider the pullback $\hat{G}$ to Spin_n. (It depends on the embedding in A_n, but that will be clear from the context.) If $\hat{G} \to G$ is a nonsplit extension, then $\ker(\hat{G} \to G)$ is a $\mathbb{Z}/2$ quotient of the Schur multiplier of G.

In this situation, assume $\mathbf{C}$ consisting of odd ($2'$) conjugacy classes in G. Then, we attach to $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ a *lifting invariant* $s_{\hat{G}/G}(\boldsymbol{g})$ defined as follows. Take the unique $2'$ lift $\hat{g}_i \in \hat{G}$ lying over g_i, $i = 1, \dots, r$. Then,

$$s_{\hat{G}/G}(\boldsymbol{g}) \stackrel{\text{def}}{=} \hat{g}_1, \dots, \hat{g}_r \in \{\pm 1\}. \tag{2.11}$$

These basic definitions easily generalize to arbitrary groups and arbitrary primes. This paper concentrates, in our special case, on the meaning of the lifting invariant and our ability (often) to compute it explicitly thanks to [**Ser90**] and [**Fr06b**]. With this we establish the Main Conjecture in our cases.

REMARK 2.11 (Well-definedness of lifting invariant). Though the covers in an absolute Nielsen class (§2.1.5) such as $\mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{abs}}$ are not Galois, the lifting invariant still makes sense. When we pass to the Galois closure of the cover, it checks whether Spin_n is realized as an unramified cover of it. The Galois closure cover is only canonical up to an inner isomorphism of the Galois group with G. So, that requires knowing the lifting invariant doesn't depend on changing that isomorphism. That follows because any inner isomorphism of G lifts to a canonical inner isomorphism of $\hat{G} \to G$.

2.4.2. *Using the lifting invariant.* It makes sense to replace $\mathrm{Ni}(G, \mathbf{C})$ under the hypotheses of §2.4 with $\mathrm{Ni}(\hat{G}, \mathbf{C})$: replace G by the nonsplit degree two to extension $\hat{G}$. This gives a natural one-one (often not onto) map $\mathrm{Ni}(\hat{G}, \mathbf{C}) \to \mathrm{Ni}(G, \mathbf{C})$. Associate to this map one of three possible symbols: $\oplus$ if it is onto; $\ominus$ if $\mathrm{Ni}(\hat{G}, \mathbf{C})$ is empty; and $\oplus\ominus$ if neither of the first two happen.

When the symbol is $\oplus\ominus$ there must be at least two braid orbits on $\mathrm{Ni}(G, \mathbf{C})$ (two Hurwitz space components). That is because the lifting invariant is a braid invariant. Conway-Fried-Parker-Völklein (C-F-P-V, §7.2) in this case says that if each class in $\mathbf{C}$ appears "suitably often," then there are *exactly* two braid orbits on $\mathrm{Ni}(G, \mathbf{C})$ with these two components represented by the symbol $\oplus\ominus$.

Prop. 2.12 (proofs in [**Ser90**] or [**Fr06b**, Cor. 2.3]) gets repeated use here. Many of our applications are really partial generalizations of it. For odd order $g \in A_n$, let $w(g)$ count length l disjoint cycles in g with $(l-1)/2 \equiv 1$ or $2 \mod 4$.

PROPOSITION 2.12 (Invariance). *Let $n \ge 3$. If $\varphi : X \to \mathbb{P}^1$ is in the Nielsen class $\mathrm{Ni}(A_n, \mathbf{C}_{3^{n-1}})^{\mathrm{abs}}$, then $\deg(\varphi) = n$, X has genus 0, and $s(\varphi) = (-1)^{n-1}$.*

Generally, for any genus 0 Nielsen class of odd order elements, and representing $\boldsymbol{g} = (g_1, \dots, g_r)$, $s(\boldsymbol{g})$ is constant, equal to $(-1)^{\sum_{i=1}^r w(g_i)}$.

Examples 2.13 and 2.14 are a warmup to recognizing the significance of the phrase "suitably often," mean. In Ex. 2.13, the covers in the Nielsen class have genus 0, and Prop. 2.12 shows they only achieve one value of the lifting invariant. In fact, [**LOs06**] there is just one braid orbit. In Ex. 2.14 the covering group is $G_1(A_4)$, the 1st 2-Frattini extension of A_4. It has a Schur multiplier of order 4, and correspondingly, there are four braid orbits, each corresponding to a different value of the lifting invariant.

2.4.3. *Pure cycles and the invariance Corollary.* Our chief source of examples is from *pure-cycle* Nielsen classes: $\mathbf{C}$ consists of pure-cycles (§1.3.3).

Let $d_1, \dots, d_r$ be the disjoint cycle lengths. We often use $d_1 \cdots d_r$, sometimes with exponents to indicate repetitions. R-H: A cover in this Nielsen class has genus

(2.12) $\mathbf{g} = \mathbf{g}_{d_1 \cdots d_r} \stackrel{\text{def}}{=} \frac{\sum_{i=1}^r d_i - 1}{2} - (n-1)$, a non-negative integer.

Suppose $G \le S_n$ and $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, a pure cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C})$, with the image of $\mathbf{C}$ in S_n equal to $\mathbf{C}^{S_n} \stackrel{\text{def}}{=} \mathbf{C}_{d_1 \cdots d_r}$. Assume you have chosen branch points, and a set of classical generators (§1.3), so $\varphi : X \to \mathbb{P}^1_z$ corresponds to $\boldsymbol{g}$ in this Nielsen class. Here is a case of computing the lifting invariant that shows what we mean by "explicit."

Example 2.13 (Genus 0 pure-cycles). When the Nielsen class is odd pure-cycle, and the genus is 0, the lifting invariant (in additive notation) is $\sum_{i=1}^r \frac{d_i^2-1}{8}$ mod 2. For example, if $r = 3$, n is odd, and $d_1 = d_2 = \frac{n+1}{2}$ and $d_3 = n$, then $G = \langle g_1, g_2 \rangle = A_n$, and

$$s_{\mathrm{Spin}_n/A_n}(\boldsymbol{g}) = \frac{n^2-1}{8} \mod 2 = \begin{cases} 0 & \text{if } n \equiv \pm 1 \mod 8 \\ 1 & \text{if } n \equiv \pm 3 \mod 8 \end{cases}. \tag{2.13}$$

The following example reappears in §6.5. It illustrates many points about braid orbits in this section, and also some expectations in going beyond the genus 0 condition of Liu-Osserman.

Example 2.14. When G is $G_1(A_4)$, and extension of A_4 with kernel of order 2^5, the Schur multiplier (for $p = 2$) is $(\mathbb{Z}/2)^2$. Then, there are exactly four braid orbits on $\mathrm{Ni}(G_1(A_4), \mathbf{C}_{\pm 3^2})$.

2.4.4. *Inductive criterion for existence of a non-empty* **MT**. We can check that a braid orbit at level k has above it (a nonempty) braid orbit at level $k+1$. §1.3.1 says this is equivalent to extending a given $M_{\boldsymbol{g}} \to G_k$ to $M_{\boldsymbol{g}} \to G_{k+1}$.

In the abelianized case, the inductive procedure simplifies to just one test to see if $M_{\boldsymbol{g}} \to G$ extends to $M_{\boldsymbol{g}} \to G_{k,\mathrm{ab}}$ for all k. The first statement is from [**BF02**, Prop. 3.21]. The last two are from [**Fr06a**, Cor. 4.19] (using results of [**Wei05**]).

Recall (§2.1): $R^*_{G,p} \to G_k$ is the representation cover whose kernel is the maximal exponent p quotient of the) Schur multiplier of G_k. The Schur multiplier is always a finite group, and so $\ker(R^*_{G,p} \to G_k)$ has a finite exponent p^{u_0}. Below, denote the subgroup of $\ker(R{*}_{G,p} \to G)$ generated by all elements of exponent larger than p^k ($k \le u_0$) by U_k.

Proposition 2.15. *If G has p' center (as in (2.8)), then so does G_k, $k \ge 1$.*

The Hurwitz orbit $H_r(\boldsymbol{g}_k) \subset \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in}}$ is in the image of $\mathrm{Ni}(G_{k+1}, \mathbf{C})^{\mathrm{in}}$ if and only if $\boldsymbol{g}_k$ is in the image of $\mathrm{Ni}(R_k, \mathbf{C})$.

Also, $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ is in the image of $\mathrm{Ni}(G_{k,\mathrm{ab}}, \mathbf{C})$ if and only if it is in the image of $\mathrm{Ni}(R{*}_{G,p}/U_k, \mathbf{C})$. If $k \ge u_0$, the conclusion holds exactly when $\boldsymbol{g}$ is in the image of $\mathrm{Ni}(R{*}_{G,p}, \mathbf{C})$.

The following example works because we know precisely the Schur multiplier of alternating groups. Indeed, it works as well for all simple groups, though there is as yet no Invariance Cor. 2.12 to make the calculation of the lifting invariant for other simple groups.

EXAMPLE 2.16. Suppose $G = A_n$, $n \geq 4$, with $p = 2$ and $\mathbf{C}$ classes of odd order elements. Then, for $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$, there is an extension of $\psi_{\bar{\boldsymbol{g}}} : M_{\boldsymbol{g}} \to G$ to ${}_p\tilde{G}_{\mathrm{ab}}$ if and only if $s_{\mathrm{Spin}/A_n}(\boldsymbol{g})$ is trivial. If $p \neq 2$ (where $\mathbf{C}$ are p') then there is always an extension to ${}_p\tilde{G}_{\mathrm{ab}}$.

2.5. Organizing our choice of MTs. Our examples show how connectedness allows computing when **MT**s have p cusps. We will do this on general principles, thereby getting a sense of what we can compute as we go up the tower.

2.5.1. *General approach.* Level 0 requires some knowledge of conjugacy classes of a group G_0 with no normal p subgroup. Often that might be a simple group. Going up to level 1 requires knowing certain precise facts about Schur multipliers of p-perfect subgroups of G_0.

In proving the Main Conjecture for ∞-ly many not modular curve cases, we have chosen examples where G_0 is an alternating group, and the prime p is 2. This is a serious challenge, with phenomena revealing of what will occur in general, though without tremendous group theory complications. Here are guesses (justified by examples) for what happens in a general **MT** $\mathcal{O} \stackrel{\text{def}}{=} \{O_k\}_{k=0}^{\infty} \leq \{\mathrm{Ni}(G_k, \mathbf{C})\}_{k=0}^{\infty}$.

Now consider the case $r = 4$ and the Main Conjecture. Use the correspondence $O_k \Leftrightarrow \bar{\mathcal{H}}'_k$ with the space on the right an absolutely irreducible component of $\bar{\mathcal{H}}(G_k, \mathbf{C})^{\mathrm{in,rd}}$. Proving p Cusp Existence is close to proving the Main Conjecture as a consequence of [**Fr06a**, Thm. 5.1]. Denote the genus of $\bar{\mathcal{H}}'_k$ by $\mathbf{g}'_k$.

THEOREM 2.17. *If $\mathbf{g}'_0 > 0$, then a single p cusp branch, starting at some level k_0, implies $\mathbf{g}_k > 1$, $k > k_0$, and the Main Conjecture holds for $\mathcal{O}$.*

The Main Conjecture holds if $\mathbf{g}'_0 = 0$, and some level has three p cusps.

2.5.2. *Pure-cycle Nielsen classes.* Now return to the case a Nielsen class is pure-cycle (all conjugacy classes have one disjoint cycle; §2.4.2). We will do the case $p = 2$, the pure-cycle lengths are $d_1 \leq d_2 \leq \cdots \leq d_r$, and the genus of covers in the Nielsen class is 0.

All the Liu-Osserman examples stand out because there are no 2 cusps at level 0, but they do appear at level 1. Reminder: When $r = 4$, then there are two places were genus 0 may come up: the inner reduced Hurwitz spaces may have genus 0; for all the spaces, their points represent covers of genus 0.

Recall: Since S_n is not 2-perfect, it does not enter into the Main Conjecture for the prime $p = 2$. So, for the start, here are our general assumptions.

(2.14a) We have a pure-cycle Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ of odd order elements, $G \leq A_n$ a transitive subgroup and $p = 2$; and

(2.14b) covers in the absolute Nielsen class all have genus 0.

So, even though what we prove here will only apply only when (2.14) holds, it gives us infinitely many cases where the Main Conjecture holds. Further, even here we can often conclude the Main Conjecture for other primes, not just $p = 2$. §6.3.2 considers the extension of this result on existence of 2 cusps in genus 0 odd pure-cycles from the case $r = 4$ to general $r \geq 3$.

2.5.3. *Comparing Liu-Osserman with other cases.* Connectedness of the Liu-Osserman spaces is not without precedent. There are a small number of other Nielsen class collections of genus 0 covers long known to give connected spaces. The first such is the space of simple-branched (2-cycle) covers that Clebsch used 140 years ago to show the connectedness of the moduli of curves of genus g. Less obvious, yet more relevant for general applications, are modular curves.

We assume p is odd. Recall: The dihedral group is $D_{p^{k+1}} = \mathbb{Z}/p^{k+1} \times^s (\mathbb{Z}/p^{k+1})^*$ and $\mathbf{C}_{2^4}$ is four repetitions of the involution conjugacy, represented by multiplication by -1 on $\mathbb{Z}/p$. So, the Nielsen class is $\text{Ni}(G = D_{p^{k+1}}, \mathbf{C}_{2^4}, p)$. Note: G here is $G_k(D_p)$ in the notation of §2.1.2. We use the absolute Nielsen class, where the representation of G has degree p^{k+1}.

Notice that $(p^{k+1}-1)/2$ gives the number of length two orbits of multiplication by -1 on $\mathbb{Z}/p^{k+1}$. Check easily from this that (2.14b) holds: R-H gives the genus as $\boldsymbol{g} = 0$ satisfying $2(p^{k+1} + \boldsymbol{g} - 1) = 4(p^{k+1} - 1)/2$. For $p \geq 5$, the conjugacy classes are not pure-cycle. This is the elementary modular curve case, where $\bar{\mathcal{H}}_0^{\text{in,rd}}$ is $X_1(p^{k+1})$ and $\bar{\mathcal{H}}_0^{\text{abs,rd}}$ is $X_0(p^{k+1})$. Of course, though [**Fr78**, §4] has been around a long time, this is not the traditional way to look at this space.

Since it will come up later, we now consider cases that are like the Clebsch case. You fix a group and you vary the conjugacy classes. In Ex. 2.18 you fix a group G and one conjugacy class C within it. Then, you vary the multiplicity of that conjugacy class to consider different Nielsen classes. In both cases denote the collection of absolutely irreducible components by I, and consider the natural map from $i \in I$ to the conjugacy class collection for that component. The listing of components for absolute classes and inner classes is the same (that does not hold up in all the Liu-Osserman examples as we shall see).

Example 2.18 (Dihedral and Alternating cases). If $G = D_{p^{k+1}}$ with p odd, and $\mathbf{C}^* = \{\text{C}_2\}$ (conjugacy class of an involution), then $i \mapsto \mathbf{C}_{2^{r_i}}$ is one-one and onto, with the r_i s running over all even integers ≥ 4. Also, H_i^{rd} identifies with the space of cyclic p^{k+1} covers of hyperelliptic jacobians of genus $\frac{r_i-2}{2}$ [**DFr94**, §5].

If $G = A_n$ with $\mathbf{C}^* = \{\text{C}_3\}$, class of a 3-cycle, then $i \mapsto \mathbf{C}_{3^{r_i}}$ with $r_i \geq n$ is two-one (Main Result of [**Fr06b**]). Denote indices mapping to r by $i_r^{\pm}$. Those covers in $\mathcal{H}_{i_r^{\pm}}$ are Galois closures of degree n covers $\varphi : X \to \mathbb{P}_z^1$ with 3-cycles for local monodromy. Also, write the divisor $(d\varphi)$ of the differential of φ as $2D_\varphi$. Then, $\varphi \in \mathcal{H}_{i_r^+}$ (resp. $\mathcal{H}_{i_r^-}$) if the linear system of D_φ has even (resp. odd) dimension; it is an even (resp. odd) θ characteristic. For $r_i = n-1$ the map $i \mapsto \mathbf{C}_{3^{r_i}}$ is one-one.

3. Cusp Principles

We assume the Liu-Osserman conditions (2.14) hold from this point for the rest of the paper unless otherwise said. We use several principles to compute properties of the absolute and inner reduced spaces defined by some of their Nielsen classes. §3.1 nails the description of pure-cycle Nielsen classes when $r = 3$.

For $r = 4$, §3.2 is a basic tool kit for this the contribution of the cusps to the genus of the reduced spaces. Its three principles detect when we have p cusps; especially (Princ. 3.3) on how their existence gives the Main Conjecture. One conclusion: once you know an H-M cusp is a p cusp, a subtree of the cusp tree on a **MT** resembles that for the whole cusp tree on modular curves. In this case, we clearly see the number of p cusps grow with the levels.

We contrast this with Princ. 3.6. This says we get *only* pure-cycle cusps — $(\boldsymbol{g})\mathbf{mp}$ is pure-cycle (or trivial) for all $\boldsymbol{g}$ in the Nielsen class — precisely when the Nielsen class is $\mathrm{Ni}_{(\frac{n+1}{2})^4}$. One conclusion from is that case is there are no 2 cusps at level 0. Yet, by applying [**Fr06a**, Fratt. Princ. 3], §3.3 shows in all cases some 2 cusps appear at level 1, proving the Main Conjecture.

3.1. Detecting p cusps in the cusp tree. For $r \geq 4$, all reduced spaces have well-defined cusps. Their combinatorial definition (§2.2.1) applied to an element $\boldsymbol{g}$ in the Nielsen class starts by imposing a grouping on the entries of $\boldsymbol{g}$. When $r = 4$, we inspect the ordered pairs (g_2, g_3) and (g_4, g_1) both for their products and the groups they generate. This inductive definition appears clearly in App. B. So, we can't dismiss the case $r = 3$, the case most Riemann surface people find comforting because its attached reduced space consists just of points with no need for cusps.

3.1.1. *The case* $r = 3$. Assume $r = 3$ and the genus of the covers in the Nielsen class is 0. With no loss, by applying a braid from H_3, assume $d_1 \geq d_2 \geq d_3$. Let $g_3 = (1 \dots d_3{-}u \dots d_3)$ for some integer $1 \leq u \leq d_3{-}1$. Now consider the following two elements based on another integer t with $n{-}t = d_3 + 1$:

$$\begin{aligned} g_2 &= (d_3\, d_3{-}1 \dots d_3{-}u\, n \dots n{-}t), \text{ and} \\ g_1 &= (1 \dots d_3{-}u{-}1\, n \dots n{-}t\, d_3)^{-1}. \end{aligned} \tag{3.1}$$

Note these properties:

(3.2a) (g_1, g_2, g_3) has product-one.
(3.2b) The genus is 0.
(3.2c) This represents the unique element in the statement of the theorem.

Principle 3.1. *For* $r = 3$ *and* $\mathbf{g}_{d_1 \cdot d_2 \cdot d_3} = 0$*, there is a unique*

$$\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C}_{d_1 \cdot d_2 \cdot d_3})^{\mathrm{abs}} \text{ with } \mathrm{ord}(g_i) = d_i, i = 1, 2, 3.$$

3.1.2. *p Cusps and Main* **MT** *Conjecture.* The projective system of cusp orbits forms a natural directed tree. A *cusp branch* is a projective system of representatives

$$\tilde{\boldsymbol{g}} = \{{}_k\boldsymbol{g} \stackrel{\mathrm{def}}{=} [{}_kg_1, {}_kg_2] \in \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{in,rd}}\}_{k=0}^{\infty}.$$

If all representatives are H-M reps., call it an *H-M branch*. Its projective system of braid orbits defines an H-M **MT**, or an H-M *component branch*. Assume, with no loss (see the start of [**Fr06a**, §5]), that the p-part of the center of all the G_k s is trivial. That is, the center of G_k is a (the same) p' group.

It will simplify some expressions to use a short-hand, $[g_1, g_2]$, for the H-M rep. $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$.

Principle 3.2. *We have the following formulas:*

$$([g_1, g_2])q_1 = [g_1^{-1}, g_2], ([g_1, g_2])q_3 = [g_1, g_2^{-1}] \text{ and } ([g_1, g_2])q_1q_2 = [g_1^{-1}, g_2^{-1}].$$

Among them is a p cusp if and only if one of $p|\mathrm{ord}(g_1^{\pm 1}g_2)$.

Proof. The display just repeats the definition of the actions of q_1 and q_2. By definition, these are p cusps if and only if the respect products $g_1^{\pm 1}g_2^{\pm 1}$ have order divisible by p. Notice, however, that the inner equivalent H-M rep. $g_1[g_1, g_2]g_1^{-1}$ has middle product

$$g_1^{-1}g_1g_2g_1^{-1} = g_2g_1^{-1} = (g_1g_2^{-1})^{-1}.$$

This shows the middle product order of $g_1g_2^{-1}$ is already among the middle product orders $\mathrm{ord}(g_1^{\pm 1}g_2)$, etc. □

If the following held (we state this just for H-M reps., though p is arbitrary), then the Main Conjecture would follow for an H-M component branch from Princ. 3.3.

(3.3) [Non-Weigel cusp branch] For $k >> 0$, ${}_k\boldsymbol{g}$ does not define an o-p' cusp. So, from FP1 (2.9b), there is a k_0 so that for $k \ge k_0$, $p^{k-k_0+1}||({}_k\boldsymbol{g})\mathbf{mp}$.

It is significant to figure out when condition (3.3) is automatic.

Assuming (3.3), we can spin off low ramification p cusps for general p. With no loss by changing k_0 to 0 (replace G_0 by G_{k_0}), assume the cusp branch starts with an H-M rep. $(g_1, g_1^{-1}, g_2, g_2^{-1}) = [g_1, g_2] = \boldsymbol{g}_0$ with middle product $d \cdot p^u$, $u \ge 1$. Take the case $u = 1$ to simplify notation. Now let ${}_{k+1}\boldsymbol{g}$ be the level $k+1$ representative in our projective sequence, with

$$\bar{c}_k = ({}_{k+1}g_1^{-1}\,{}_{k+1}g_2)^{d_k} \in \ker(G_{k+1} \to G_k),$$

and d_k the middle product order of ${}_k\boldsymbol{g}$. So

$${}_{k+1}\boldsymbol{g}' = ({}_{k+1}\boldsymbol{g})q_2^{2d_k} = ({}_{k+1}g_1, \bar{c}_k({}_{k+1}g_1^{-1})\bar{c}_k^{-1}, \bar{c}_k({}_{k+1}g_2)\bar{c}_k^{-1}, {}_{k+1}g_2^{-1}).$$

Further, $\bar{c}_k$ that will satisfy product-one in this expression must centralize $g_1^{-1}g_2$. Now use that $\langle {}_{k+1}g_1, {}_{k+1}g_1\,{}_{k+1}g_2\rangle = G_{k+1}$ and the center of this group has no p-part. Since $\bar{c}_k$ commutes with the second of these generators, it can't commute with the first. Conclude:

$$(3.4) \qquad {}_{k+1}\boldsymbol{g}' \ne {}_{k+1}\boldsymbol{g}.$$

Now form $({}_{k+1}\boldsymbol{g}')\mathbf{sh}$, whose 2nd and 3rd entries are $(\bar{c}_k({}_{k+1}g_2)\bar{c}_k^{-1}, {}_{k+1}g_2^{-1})$, with exactly one power of p dividing their product.

Principle 3.3. *As above,* $(\boldsymbol{g}'_{k+1})\mathbf{sh}$ *is a new* p *cusp. So,* (3.3) *for the* H-M *cusp branch of* $\tilde{\boldsymbol{g}}$ *implies the main conjecture for its* H-M *component branch.*

Proof. The first paragraph of the proof of [**Fr06a**, Prop. 5.5] shows how spinning of new p cusps implies the number of p cusps grows with k: The cusps at level k produced by this have above them only cusps with middle product divisible by one more power of p. So, this new cusp cannot equal any of them.

The Main Conjecture counterexample towers have at most two p cusps at each level [**Fr06a**, Thm. 5.1] (or Prop. 2.8). This concludes the proposition. □

Example 3.4. In A_5, consider $g_1 = (1\,2\,3\,4\,5)$ and $g_2 = (1\,2\,3)$. They generate A_5, and the middle product $([g_1, g_2])\mathbf{mp}$ is $(1\,2\,3)$ while $([g_1, g_2^{-1}])\mathbf{mp}$ is $(5\,4\,2\,3\,1)$. This shows middle products of H-M reps. are not a braid orbit invariant.

3.2. Two cusp-type Principles. Princ. 3.5, a version of [**BF02**, Prop. 2.17], makes transparent the width of most cusps. Princ. 3.6 smooths the way between the case $r = 3$ and $r = 4$ when dealing with pure-cycle Nielsen classes. It is a version of [**LOs06**, §4], the hardest combinatorial part of their paper, where $r = 4$. Our simplification results from using cusps to improve the combinatorial efficiency in computing braid orbits.

3.2.1. *The Twisting Principal.* For $(g, g') \in G \times G$, denote $(g, g') \mapsto (gg'g^{-1}, g)$ by tw. It is just the q_2 operator restricted to 2-tuples, instead of 4-tuples. As such the iterated action of tw starting from (g, g') has an orbit in $G \times G$.

Principle 3.5. *Given* (g, g')*, denote* gg' *by* g''*. Assume* $g^{-1} \ne g'$ *and for simplicity that* $Z(\langle g, g'\rangle) \cap \langle g''\rangle$ *is trivial (see Rem. 3.8). Then, the* tw *orbit length is* $2 \cdot \mathrm{ord}(\boldsymbol{g})$ *unless*

(3.5) $\mathrm{ord}(\boldsymbol{g}) = o$ *is odd, and* $\mathrm{ord}((gg')^{\frac{o-1}{2}}g) = 2$.

In turn, (3.5) *is equivalent to*

$$(g, g')\mathrm{tw} = (g'')^{\frac{o+1}{2}}(g, g')(g'')^{-\frac{o+1}{2}}. \tag{3.6}$$

PROOF. The first paragraph is [**BF02**, Prop. 2.17]. Now assume o is odd and let $j = \frac{o+1}{2}$. If (3.6) holds, then $gg'g^{-1} = (g'')^j g(g'')^{-j}$, or $g' = (g'')^{\frac{o-1}{2}} g(g'')^{-\frac{o-1}{2}}$

$$\implies g'(g'')^{\frac{o-1}{2}} = (g'')^{\frac{o-1}{2}} g.$$

This is reversible; the last restates (3.5) that $(g'')^{\frac{o-1}{2}} g$ has order 2. □

3.2.2. *The Pure-Cycle Cusp Principle.* The next principal shows exactly why, among the Liu-Osserman cases, the Nielsen classes $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ (§4.1) stand out. It is these for which all cusps are pure-cycle (or have trivial middle product; Def. 3.7). Ex. 3.9 does a split-cycle case in detail to assure the reader of the notation.

PRINCIPLE 3.6. *Apply R-H to see, with no loss, each* $j \in \{1, \dots, n\}$ *is in the support of exactly two of the* g_i *s modulo one of two possibilities:*

(3.7a) *Either there is an* i_0 *in the support of all four* g_i *s; or*

(3.7b) *There are two integers* i_0 *and* k_0 *in the support of exactly three* g_i *s.*

Consider the common support of (g_2, g_3) *for* $\boldsymbol{g}$ *in the Nielsen class. With no loss, unless it is empty, take it to be* $\{1, \dots, k\}$*. Then, consider also the overlap* $U(\boldsymbol{g})$ *of that with* $(\boldsymbol{g})\mathbf{mp}$*. This consists of at most two integers.*

If $|U(\boldsymbol{g})| = 1$ *(with no loss take it to be* k*) then* (g_2, g_3) *has the form*

$$((k \dots 1\,\boldsymbol{v}), (1 \dots k\,\boldsymbol{w})) \text{ with } \boldsymbol{v}, \boldsymbol{w} \text{ and } \{1, \dots, k\} \text{ mutually disjoint.}$$

Here, $(\boldsymbol{g})\mathbf{mp} = (k\,\boldsymbol{w}\,\boldsymbol{v})$*, an odd pure-cycle.*

If $|U(\boldsymbol{g})| = 2$*, then there is no overlap in the support of* (g_4, g_1, g_4g_1)*. Further,* (g_2, g_3) *has the form*

$$((k \dots i_0{+}1\,\boldsymbol{v}_1\,i_0 \dots 1\,\boldsymbol{v}_2), (1 \dots i_0\,\boldsymbol{w}_1\,i_0{+}1 \dots k\,\boldsymbol{w}_2)),$$

with $\boldsymbol{v}_1$*,*$\boldsymbol{v}_2$*,* $\boldsymbol{w}_1$*,* $\boldsymbol{w}_2$ *and* $\{1, \dots, k\}$ *mutually disjoint and* $(\boldsymbol{g})\mathbf{mp} = (k\,\boldsymbol{w}_2\,\boldsymbol{v}_2)(i_0\,\boldsymbol{w}_1\,\boldsymbol{v}_1)$*. Two disjoint cycles are* g_1 *and* g_4*, giving conditions (see* (3.8)*) on the lengths (orders) of* $\boldsymbol{v}_i$ *and* $\boldsymbol{w}_i$*,* $i = 1, 2$ *so* $\boldsymbol{g}$ *entries have the right orders.*

The condition $|U(\boldsymbol{g})| = 2$ *happens for some rep. in each allowed Nielsen class if and only if it is not* $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ *for some* $n \geq 4$*.*

PROOF. For the first part of this argument start with general r. The product one condition $\prod_{i=1}^r g_i$ implies each integer must appear in at least two of the g_i s. For each $j \in \{1, \dots, n\}$ let k_j be the number of g_i s containing j in their support, and let $k_i - 2 = k'_i \geq 0$. Apply R-H, using $\mathbf{g} = 0$ to conclude

$$2(n-1) = \sum_{i=1}^r d_i - 1 = 2n + \sum k'_j - r.$$

So, $\sum k'_j = r - 2$. If $r = 4$, then either some $k'_j = 2$, or two of them equal 1. In the former case (3.7a) holds, and in the latter it is (3.7b). Now we characterize each. For a segment labeled $\boldsymbol{v}$ (or $\boldsymbol{w}$) in the calculations, compatible with previous notation, denote its length $o(\boldsymbol{v})$.

If $U(\boldsymbol{g})$ is empty, then g_2 and g_3 are disjoint. Otherwise, assume $k \in U(\boldsymbol{g})$. If no other letter is in $U(\boldsymbol{g})$, then consider the effect of g_2g_3 to see that by reordering $1, \dots, k$, we may assume g_3 maps $i \mapsto i{+}1$, and g_2 reverses this, for $i = 1, \dots, k{-}1$.

So, these integers disappear in the support of the product, and (g_2, g_3) has the shape given in the proposition statement. The length of $(\boldsymbol{g})\mathbf{mp}$ is $1 + o(\boldsymbol{v}) + o(\boldsymbol{w})$, and $2k + o(\boldsymbol{v}) + o(\boldsymbol{w}) = d_2 + d_3$. Since d_2 and d_3 are both odd, conclude $o(\boldsymbol{v}) + o(\boldsymbol{w})$ is even, and the length of $(\boldsymbol{g})\mathbf{mp}$ is odd.

It is similar for $|U(\boldsymbol{g})| = 2$. Now consider, by cases, what happens with the complementary pair (g_4, g_1).

Suppose $|U(\boldsymbol{g})| = 2$. Then, two integers having three supports among the entries of $\boldsymbol{g}$ appear in $(\boldsymbol{g})\mathbf{mp}$. Apply the argument to $(g_4, g_1, g_4g_1 = (g_2g_3)^{-1})$ that we used on (g_2, g_3, g_2g_3). If there were further integers in the common support of (g_4, g_1, g_4g_1) and (g_4, g_1), that would give at least three integers appearing in the common support of three entries of $\boldsymbol{g}$. So, that can't happen. Similarly, if $|U(\boldsymbol{g})| = 1$, then the common support for $(g_4, g_1, (g_4g_1)^{-1})$ has also cardinality 1, different from the integer in $U(\boldsymbol{g})$.

Finally, consider the production of split-cycle cusps. When the Nielsen class is $\mathrm{Ni}_{(\frac{n+1}{2})^4}$, all pairs of entries of $\boldsymbol{g}$ must have overlapping support, so there can be no split-cycle cases. Given $\boldsymbol{d} \neq (\frac{n+1}{2})^4$, if we have a split-cycle, then may apply braids to assume, with $d_1 \leq d_2 \leq d_3 \leq d_4$, that $o(g_i) = d_{i+1}$, $i = 1, \dots, 4 \mod 4$. This assures the two smallest lengths are at positions 1 and 4. The genus 0 condition shows $d_1 + d_2 \leq n$. Here are equations expressing segment lengths using $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$\begin{array}{ll} 1 + o(\boldsymbol{v}_1) + o(\boldsymbol{w}_1) = d_1, & 1 + o(\boldsymbol{v}_2) + o(\boldsymbol{w}_2) = d_2, \\ k + o(\boldsymbol{v}_1) + o(\boldsymbol{v}_2) = d_3, & k + o(\boldsymbol{w}_1) + o(\boldsymbol{w}_2) = d_4. \end{array} \tag{3.8}$$

Solve the equations, as in Ex. 3.9, to canonically, up to absolute equivalence, produce a split-cycle cusp. For example, $d_1 - 1 + d_2 - 1 = d_3 - k + d_4 - k$, determining k. This concludes the proposition. □

Suppose $\boldsymbol{g} \in \mathrm{Ni}_{d_1 \cdot d_2 \cdot d_3 \cdot d_4}$. Then, consider the cusp $\mathrm{Cu}_4(\boldsymbol{g})$ it generates.

DEFINITION 3.7. Assume $\boldsymbol{g}$ is not the shift of an H-M rep. (so $(\boldsymbol{g})\mathbf{mp}$ is not trivial). Corresponding to the cases in Princ. 3.6, $\mathrm{Cu}_4(\boldsymbol{g})$ is a *pure-cycle* (resp. *split-cycle*) cusp if $|U(\boldsymbol{g})| = 1$ (resp. $|U(\boldsymbol{g})| = 2$ or 0).

REMARK 3.8. In our applications here the triviality of $H = Z(\langle g, g' \rangle) \cap \langle g'' \rangle$ assumption in Princ. 3.5 holds. As in [**BF02**, Prop. 2.17], modding out by H gives a completely general result.

EXAMPLE 3.9 (Split-cycle cusp). Let $n = 9$ and $(d_1, d_2, d_3, d_4) = (3, 5, 5, 7)$, so $\mathrm{Ni}(A_9, \mathbf{C}_{3 \cdot 5 \cdot 5 \cdot 7})$ satisfies the genus 0 assumption. Make a split-cycle cusp $\mathrm{Cu}_4(\boldsymbol{g})$ where $(o_1, o_2, o_3, o_4) = (3, 5, 7, 5)$ by assigning values to $i_0, k, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1\boldsymbol{w}_2$ in the formula in Princ. 3.6. As in (3.8), $1 + o(\boldsymbol{v}_1) + o(\boldsymbol{w}_1) = 5$, $1 + o(\boldsymbol{v}_2) + o(\boldsymbol{w}_2) = 3$, $k + o(\boldsymbol{v}_1) + o(\boldsymbol{v}_2) = 5$ and $k + o(\boldsymbol{w}_1) + o(\boldsymbol{w}_2) = 7$.

So, $4 + 2 = 5 - k + 7 - k$, or $k = 3$ and $o(\boldsymbol{v}_1) = o(\boldsymbol{v}_2) = o(\boldsymbol{w}_2) = 1, o(\boldsymbol{w}_1) = 3$. With no loss:

$$\boldsymbol{v}_1 = |4|, \boldsymbol{v}_2 = |5|, \boldsymbol{w}_1 = |6\,7\,8|, \boldsymbol{w}_2 = |9|. \tag{3.9}$$

For $i_0 = 1$, $g_2 = (3\,2\,4\,1\,5)$ and $g_3 = (1\,6\,7\,8\,2\,3\,9)$ $((1\,6\,7\,8\,4)(3\,9\,5) = (\boldsymbol{g})\mathbf{mp})$; and for $i_0 = 2$ $g_2 = (3\,4\,2\,1\,5)$ and $g_3 = (1\,2\,6\,7\,8\,3\,9)$ (middle product $(2\,6\,7\,8\,4)(3\,9\,5)$).

3.3. 2-cusps and the Liu-Osserman examples. Prop. 3.10 assumes the genus 0, pure-cycle hypotheses of (2.14). For general r it gives a precise criterion for an abelianized **MT** over any braid orbit on $\mathrm{Ni}(A_n, \mathbf{C}_{d_1 \cdots d_r}) \stackrel{\text{def}}{=} \mathrm{Ni}_{d_1 \cdots d_r}$. It says there are no 2 cusps at (**MT**) level 0 for $r = 4$, but indicates how to find 2 cusps at level 1. Denote the order of $g \in G$ by $o(g)$. Cusp notation is from §2.2.2. Keep in mind — though Prop. 3.10 doesn't use it — that [**LOs06**, Cor. 4.11] says there is only one braid orbit on $\mathrm{Ni}^{\mathrm{abs}}_{d_1\cdots d_r}$. Further, if there are two braid orbits on $\mathrm{Ni}^{\mathrm{in}}_{d_1\cdots d_r}$, then conjugating by an element of $S_n \setminus A_n$ (as in Prop. 4.1) joins them.

PROPOSITION 3.10. *Assume* $\mathrm{Ni}^{\mathrm{abs}}_{d_1\cdots d_r}$ *is a Nielsen class of odd pure-cycle genus 0 covers,* $r \geq 3$*. Then,* $G = A_n$*,* $n \geq 4$*. For* $p = 2$*, there is a (nonempty) abelianized* **MT** *above any component of* $\mathcal{H}(A_n, \mathbf{C}_{d_1\cdots d_r})^{\mathrm{in}}$ *if and only if*

$$\sum_{i=1}^{r} \frac{o(g_i)^2 - 1}{8} \equiv 0 \mod 2. \tag{3.10}$$

For $p \neq 2$*, there is always a (nonempty) abelianized* **MT** *above any component of* $\mathcal{H}(A_n, \mathbf{C}_{d_1\cdots d_r})^{\mathrm{in}}$*. Further, if the* d_i *s are equal in pairs, there is always (irrespective of* p*) a (full — not abelianized)* **MT** *over any component of* $\mathcal{H}(A_n, \mathbf{C}_{d_1\cdots d_r})^{\mathrm{in}}$*. From here on assume* (3.10) *holds (Ex. 3.11 has Liu-Osserman examples when it doesn't).*

Now assume $r = 4$*. Then, all cusps (Def. 3.7) in* $\mathrm{Ni}_{d_1\cdot d_2\cdot d_3\cdot d_4}$ *are either g-2′ or o-2′. All g-2′ cusps are shifts of* H-M *reps, and all* H-M *reps give o-2′ cusps.*

Let $\mathrm{Cu}_4(\boldsymbol{g})$ *be an o-2′ pure-cycle cusp. Then,* $U_{2,3}(\boldsymbol{g}) = A_u$ *and* $U_{1,4}(\boldsymbol{g}) = A_v$ *for some* $u, v \geq 4$*. All level 1 cusps above it are 2 cusps if and only if*

$$\frac{o(g_2)^2 - 1}{8} + \frac{o(g_3)^2 - 1}{8} + \frac{o(g_2 g_3)^2 - 1}{8} \equiv 1 \mod 2. \tag{3.11}$$

Let $\mathrm{Cu}_4(\boldsymbol{g})$ *be a split-cycle cusp with* $|U(\boldsymbol{g})| = 2$ *and* $(d_1, d_4) = d'$*. Then,* $U_{2,3}(\boldsymbol{g}) = A_n$ *and* $U_{1,4}(\boldsymbol{g}) = \mathbb{Z}/d_1 \times_{\mathbb{Z}/d} \mathbb{Z}/d_4$ *(natural fiber product over* $\mathbb{Z}/d$*). Assume* (3.10) *holds. Then, both 2 cusps and o-2′ cusps at level 1 lie over* $\mathrm{Cu}_4(\boldsymbol{g})$*.*

PROOF. All appearances of alternating groups come directly from [**Wm73**], whose hypothesis are a noncyclic, transitive subgroup of A_n, generated by odd pure-cycles. Then, the group must be A_n, $n \geq 4$. [**LOs06**, Thm. 5.3] gives more detail (see Rem. 2.6). If we can exclude that G is cyclic, then $G = A_n$, $n \geq 4$ in any such Nielsen class. If, however, $G = \langle h \rangle$, then transitivity implies h is an n-cycle. Apply the pure-cycle and genus 0 conditions. Conclude: all the g_i s are invertible powers of h. So, by R-H: $2(n-1) = r(n-1)$ and $r = 2$, contrary to hypothesis.

Consider, then, (3.10). Inv. Prop. 2.12 says this is the exact condition that $\mathrm{Ni}(\mathrm{Spin}_n, \mathbf{C}_{d_1\cdots d_r})$ has a nonempty H_r orbit over any H_r orbit of $\mathrm{Ni}(A_n, \mathbf{C}_{d_1\cdots d_r})$. Since the representation cover of A_n is Spin_n, Prop. 2.15 says this is also the exact condition there be an abelianized **MT** for $p = 2$ over any braid orbit of $\mathrm{Ni}(A_n, \mathbf{C}_{d_1\cdots d_r})$. It also says there is no condition for this when $p \neq 2$.

Now take $r = 4$ and apply Princ. 3.6 to consider cases for $\mathrm{Cu}_4(\boldsymbol{g})$. First suppose it is a g-2′ cusp. That means both $U_{2,3}(\boldsymbol{g})$ and $U_{1,4}(\boldsymbol{g})$, each generated by odd pure-cycles, have orders prime to 2. From the previous paragraph this implies they are cyclic groups. Therefore, $(\boldsymbol{g})\mathbf{mp}$ generates a common normal subgroup of these two groups. This would be a normal subgroup of an alternating group G. The only possibility G is not simple, is if it is A_n, $n = 3$ or 4. In both cases, however, the entries of $\boldsymbol{g}$ would have to be 3-cycles. Apply R-H to see this gives $2(n-1) = 4 \cdot 2$, which works for neither $n = 3$ or 4. So, the middle product is trivial. Apply the shift to conclude $(\boldsymbol{g})\mathbf{sh}$ is an H-M rep.

Now suppose $\mathrm{Cu}_4(\boldsymbol{g})$ is a pure-cycle cusp. (This includes the H-M rep. case.). First exclude that $U_{2,3}(\boldsymbol{g})$ is cyclic. As above that would require all the odd pure-cycles g_2, g_3, g_2g_3 (as given by Princ. 3.6) to be (invertible) powers of some pure-cycle h. Since the support of h would also have to contain the supports of g_1 and g_4, that would require h to be an $n-$cycle, contradicting again the genus 0 condition. Thus, $U_{2,3}(\boldsymbol{g})$ and $U_{1,4}(\boldsymbol{g})$ are both alternating groups of degree exceeding 3.

Now consider $\boldsymbol{g}' \in \mathrm{Ni}(G_1, \mathbf{C})$ ($G_1 = G_1(A_n)$) lying over $\boldsymbol{g}$, an o-$2'$ cusp, where G_1 is the 1st p-Frattini of A_n. The exact condition that only p cusps ($p = 2$) in $\mathrm{Ni}(G_1, \mathbf{C}_{d_1\cdot d_2\cdot d_3\cdot d_4})$ lie over $\mathrm{Cu}_4(\boldsymbol{g})$ is that there are no o-$2'$ cusps $\mathrm{Ni}(G_1, \mathbf{C}_{d_1\cdot d_2\cdot d_3\cdot d_4})$ over $\boldsymbol{g}$. Let $h = (\boldsymbol{g})\mathbf{mp}$, denoting its ($2'$) order by d. We simplify notation by not indicating in which conjugacy class $h \in A_u$ lies, and establish the contrapositive.

In one direction, if $\boldsymbol{g}' \in \mathrm{Ni}(G_1, \mathbf{C})$ gives an o-$2'$ cusp over $\boldsymbol{g}$, then its image in $\boldsymbol{g}^* \in \mathrm{Ni}(\mathrm{Spin}_n, \mathbf{C}_{\boldsymbol{d}})$ (uniquely determined by $\boldsymbol{g}$ given (3.10) is also o-$2'$. Since (the cusp of) $\boldsymbol{g}$ is o-$2'$, with $\langle g_2, g_3\rangle = U_{2,3}(\boldsymbol{g}) = A_u$, Invariance Prop. 2.12 applies to (g_2, g_3, h^{-1}). Then, the negation of (3.11) —that its left side is $\equiv 0 \mod 2$ —is the exact condition there is $(g_2^*, g_3^*, (h^*)^{-1}) \in \mathrm{Ni}(\mathrm{Spin}_u, \mathbf{C}_{d_2,d_3,d})$ over (g_2, g_3, h^{-1}). That is a consequence of $\boldsymbol{g}'$ existing. The converse —constructing such an o-$2'$ $\boldsymbol{g}'$ given the negation of (3.11) —is harder.

Consider $\langle g_1, g_4\rangle = U_{1,4}(\boldsymbol{g}) = A_v$ and the analogous condition for (g_4, g_1, h) being lifted to an element of $\mathrm{Ni}(\mathrm{Spin}_v, \mathbf{C}_{d_2,d_3,d})$. This is now automatic from combining (3.10) and the negation of (3.11). As a special case of [**Fr06a**, Princ. 4.24] (called F(rattini) Princ. 3), these respective Nielsen class elements give an o-$2'$ cusp of $\mathrm{Ni}(G_1(A_n), \mathbf{C}_{\boldsymbol{d}})$ over $\mathrm{Cu}_4(\boldsymbol{g})$, showing the desired outcome of (3.11) not holding.

All that is left is to establish the analog for a split-cycle cusp. In this case, since g_1 and g_4 have disjoint support, the analogous expression to the left side of (3.10) is just the left side of (3.11). Given that the former holds then, the latter reads as its negation. So, the previous argument —the lifting invariant does not need pure-cycle elements to apply —gives that there are both 2 cusps and o-$2'$ cusps above $\boldsymbol{g}$. □

Example 3.11 (Empty **MT**s over Liu-Osserman Nielsen classes). For $r = 4$, check how to get $\boldsymbol{d}$, satisfying $2(n-1) \equiv \sum_{i=1}^4 d_i - 1$ (genus zero), for which there is no abelianized **MT** over $\mathrm{Ni}_{\boldsymbol{d}}$. Equivalent to this is failing (3.10): There is 1 or 3 of the d_i s equivalent to $\pm 3 \mod 8$. If $G = A_n$ with $n \equiv 1 \mod 4$, then a case check shows the genus 0 condition automatically forces (3.10).

For, however, $n \equiv 3 \mod 4$, $\boldsymbol{d}$ with these unordered mod 8 entries will work:

$$1, 1, 1, -3; -1, -1, -1, 3; 1, -3, -3, -3.$$

For example, $\boldsymbol{d} = (5, 9, 9, 9)$ with $n = 15$. The case n is even can also happen, though we leave that to the reader.

4. The Liu-Osserman case $\mathrm{Ni}_{(\frac{n+1}{2})^4}$

This section shows the structure of the reduced spaces in the subcase of Liu-Osserman from §2.5.2: Their cusps and their genuses, in particular. We denote this case $\mathrm{Ni}_{(\frac{n+1}{2})^4}$ with $G = A_n$: the conjugacy classes are four reps. of $\frac{n+1}{2}$-cycles. This is where level 0 of the **MT** contains an H-M rep., though it has a further specialness. Our graphical presentation with **sh**-incidence matrices is the most memorable way to consider the vast information coming from cusps.

4.1. Cusps for $\mathrm{Ni}_{(\frac{n+1}{2})^4}$. Use the pure-cycle notation of §1.3.3. Prop. 3.10 says all g-2′ cusps in this case are **sh** applied to H-M cusps $(g_1, g_1^{-1}, g_2, g_2^{-1})$. Further, all remaining cusps are pure-cycle. With $x_{i,j} = (i\, i{+}1 \cdots j)$, each inner H-M class has one of two representatives:

$$\begin{aligned} \text{H-M}_1 &\stackrel{\text{def}}{=} (x_{\frac{n+1}{2},1}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}) \\ \text{H-M}_2 = (\text{H-M}_1)q_1 &\stackrel{\text{def}}{=} (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}) \end{aligned}$$

The proof of the next result takes up the rest of this section, for which subsections do the respective cases of absolute and inner cusps.

PROPOSITION 4.1. *For $n \equiv 5 \mod 8$, H-M$_1$ and H-M$_2$ are not inner equivalent. So, there is one braid orbit on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.*

For $n \equiv 1 \mod 8$, if $h \in S_n \setminus A_n$, there is no braid between $\boldsymbol{g}$ and $h\boldsymbol{g}h^{-1}$. So, there are two braid orbits on $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{in}}$.

4.1.1. *Cusps of $\mathrm{Ni}^{\mathrm{abs,rd}}_{(\frac{n+1}{2})^4}$.* We first show how **sh** applied to the cusp of H-M$_1$ gives representatives of all the absolute cusps. For $U \le \{1, \dots, n\}$, S_U is the collection of permutations of elements of U.

LEMMA 4.2. *A complete list of cusp representatives for $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs,rd}}$ comes from applying **sh** to elements (read subscripts $\mod n$) in*

$$\mathrm{Cu}_4(\text{H-M}_1) = \{\text{H-M}_{1,t} = (x_{\frac{n+1}{2},1}, x_{1+t,\frac{n+1}{2}+t}, x_{\frac{n+1}{2}+t,n+t}, x_{n,\frac{n+1}{2}})\}_{t=0}^{n-1}.$$

*Also, the order of the **mp** of the cusp representative equals the cusp width.*

PROOF. Table 1 consists of **sh** applied to elements of Cu$_4$(H-M$_1$). I've put the expression "ord(g_2g_3):" (order of $(\boldsymbol{g})$**mp**) at the head of each table row.

TABLE 1. **sh** applied to elements of Cu$_4$(H-M$_1$)

$$\begin{array}{ll}
[1:]_1 & (\text{H-M}_{1,0})\mathbf{sh} = (x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}) \\
[3:]_1 & (\text{H-M}_{1,1})\mathbf{sh} = (x_{2,\frac{n+3}{2}}, (\frac{n+3}{2} \dots n\,1), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}) \\
[5:]_1 & (\text{H-M}_{1,2})\mathbf{sh} = (x_{3,\frac{n+5}{2}}, (\frac{n+5}{2} \dots n\,1\,2), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}) \\
 & \dots \\
[n:]_1 & (\text{H-M}_{1,\frac{n-1}{2}})\mathbf{sh} = (x_{\frac{n+1}{2}\,n}, (n\,1 \dots \frac{n-1}{2}), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}) \\
[n:]_2 & (\text{H-M}_{1,\frac{n+1}{2}})\mathbf{sh} = ((\frac{n+3}{2} \dots n\,1), x_{1,\frac{n+1}{2}}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}) \\
 & \dots \\
[5:]_2 & (\text{H-M}_{1,n-2})\mathbf{sh} = ((n{-}1\,n\,1 \dots \frac{n-3}{2}), x_{\frac{n-3}{2},n-2}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}) \\
[3:]_2 & (\text{H-M}_{1,n-1})\mathbf{sh} = ((n\,1 \dots \frac{n-1}{2}), x_{\frac{n-1}{2},n-1}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})
\end{array}$$

We want to know when the list above gives a complete set of representatives of all cusps in the absolute case. Given $\boldsymbol{g} = (g_1, g_2, g_3, g_4)$ in the Nielsen class, denote the centralizer (in S_n) of the pure cycle g_2g_3 by $\mathrm{Cen}_{(\boldsymbol{g})\mathbf{mp}}$. Assume $\mathrm{ord}(g_2g_3) = k$. Princ. 3.1 shows any $\boldsymbol{g}'$ with $\mathrm{ord}(g_2'g_3') = k$ has an absolute Nielsen class rep. in

$$T_{\boldsymbol{g}} \stackrel{\text{def}}{=} \{\boldsymbol{g}_\alpha \stackrel{\text{def}}{=} (g_1, \alpha g_2 \alpha^{-1}, \alpha g_3 \alpha^{-1}, g_4)\}_{\alpha \in \mathrm{Cen}_{(\boldsymbol{g})\mathbf{mp}},\ \text{with } \langle \boldsymbol{g}_\alpha \rangle \text{ transitive}}.$$

So, we need to know if this collection of $\boldsymbol{g}_\alpha$ (for a given k) appears in the collection of cusps listed above. For each odd $k > 1$, there are two rows headed by $[k:]$. Denote them respectively $[k:]_1$ and $[k:]_2$.

To fix the ideas, look at $[3:]_1$, denoting its 4-tuple by $\boldsymbol{g}$. The respective $\mathrm{Cen}_{(\boldsymbol{g})\mathbf{mp}}$ of the middle product is $\langle(\frac{n+1}{2}\,1\,\frac{n+3}{2}), S_{2,\dots,\frac{n-1}{2},\frac{n+5}{2},\dots,n}\rangle$, but if α moves one of $\frac{n+5}{2},\dots,n$ to one of $2,\dots,\frac{n-1}{2}$, then $\boldsymbol{g}_\alpha$ won't be transitive. Further, $S_{2,\dots,\frac{n-1}{2}}$ is $\mathrm{Cen}_{\langle g_2,g_3\rangle}$ and $S_{\frac{n+5}{2},\dots,n}$ is $\mathrm{Cen}_{\langle g_1,g_4\rangle}$. Finally:

(4.1a) For $\alpha \in T_{\boldsymbol{g}}$, $\boldsymbol{g}_\alpha$ and $\boldsymbol{g}_{u_{1,4}\alpha u_{2,3}}$, $u_{2,3} \in \mathrm{Cen}_{\langle g_2,g_3\rangle}$ and $u_{1,4} \in \mathrm{Cen}_{\langle g_1,g_4\rangle}$, represent the same Nielsen class.

(4.1b) The cusp generated by $\boldsymbol{g}'_\alpha \in T_{\boldsymbol{g}}$ consists of $\{\boldsymbol{g}_\alpha\}_{\alpha\in\langle(\frac{n+1}{2}\,1\,\frac{n+3}{2})\rangle\alpha'}$.

Adjusting for the other rows marked $[5:]_1, [7:]_1 \dots, [n:]_1$ shows each absolute cusp with middle product k is in the row marked $[k:]_1$. In particular, (3.5) from Princ. 3.5 implies the cusp width is also k. □

4.1.2. *Cusps of* $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$. For all but the width 2 cusp there are two inner cusps for each absolute cusp. We inspect Table 1 more closely to find which of the inner cusps have representatives in it.

Lemma 4.3. *For* $n \equiv 5 \mod 8$*, there is one braid orbit on* $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$*, with these cusps:* (H-M$_1$)**sh** *represents the unique cusp of middle product 1 (width 2); and for each odd* $3 \le k \le n$*, there are exactly two inner cusps of width* k*. For* $\frac{k-1}{2}$ *odd* ($3 \le k < n$) $[k:]_1$ *and* $[k:]_2$ *represent these in Table 1, while for* $\frac{k-1}{2}$ *even,* $[k:]_1$ *and* $[k:]_2$ *represent the same cusp.*

For $n \equiv 1 \mod 8$*, there are two braid orbits* O_1 *and* O_2 *on* $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$*, respectively containing the following cusp representatives:* H-M$_1$ *and* βH-M$_1\beta^{-1}$*,* $\beta \in S_n \setminus A_n$ *(cusp width 1). For each* $j = 1, 2$*, and odd* $1 \le k \le n$*, there is exactly one inner cusp of width* k *in* O_j*.*

Proof. Now we adjust Lem. 4.2 for inner cusps. Use [**BiF82**, Lem. 3.8]: For each $h \in G$ there is a braid that goes from $\boldsymbol{g} \in \mathrm{Ni}(G, \mathbf{C})$ to $h\boldsymbol{g}h^{-1}$. When $G = A_n$, for any $h \in S_n$, there is a braid from $\boldsymbol{g}$ to $h\boldsymbol{g}h^{-1}$ if and only if there is such a braid for one case of $h \in S_n \setminus A_n$.

Now replace H-M$_1$ with H-M$_2$. The analogous table of absolute cusps for H-M$_2$ must, by the argument above, be exactly the same as that for H-M$_1$. Apply q_1 to H-M$_2$ to get H-M$_1$. This braid is equivalent to conjugation by

$$\beta' = (2\ \tfrac{n+1}{2})(3\ \tfrac{n-1}{2})\cdots(\tfrac{n-1}{4}\ \tfrac{n+1}{4}).$$

We have listed every absolute class that comes from applying $\bar{M}_4$ to H-M$_1$ in Table 1. So, we get one (resp. 2) $\bar{M}_4$ orbits on $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})^4}$, if and only if β is (resp. is not) in $S_n \setminus A_n$. These cases occur, respectively, for $n \equiv 5 \mod 8$ (resp. $1 \mod 8$), and in these cases H-M$_1$ represents an inner cusp of width 2 (resp. 1).

Now consider the case $n \equiv 5 \mod 8$. Then, ((H-M$_1$)**sh**)q_2 = (H-M$_2$)**sh**. By the above, this means there is one inner, reduced cusp of width 2, consisting of the shifts of the H-M reps. Now we show the rows in Table 1 corresponding to $[3:]_1$ and $[3:]_2$ are resp. reps. of the *two* inner cusps.

From Lem. 4.2, we know some element of S_n conjugates $[3:]_2$ into the cusp of $[3:]_1$. The statement in the lemma requires showing this is an element of $S_n \setminus A_n$. Princ. 3.5 tells precisely the cusp orbit of $[3:]_1$ (or $[3:]_2$): run through the orbit of conjugation of the middle pair of $[3:]_1 = (g_1, g_2, g_3, g_4)$ by (powers of)

$$(([3:]_1)\mathbf{mp} = g_2g_3 = (1\ \tfrac{n+3}{2}\ \tfrac{n+1}{2}) \stackrel{\text{def}}{=} h.$$

This adds $(g_1, hg_2h^{-1}, hg_3h^{-1}, g_4)$ and $(g_1, h^{-1}g_2h, h^{-1}g_3h, g_4)$ to get the full orbit.

Conjugate $[3:]_2$ by the inverse of $x_{\frac{n+1}{2},1}$, and then by $(n\ \frac{n+3}{2})$ —total conjugation by an element of $S_n \setminus A_n$ —to get its outer (1st and 4th) entries to be the same as those of $[3:]_1$. So, by our cusp characterization, the result must be in cusp of $[3:]_1$. We are done with this case.

To see the continuing pattern we'll do a similar calculation comparing $[5:]_1$ and $[5:]_2$ (the result holds for $n = 5$, but note the next sentence). In this case, the cusp orbit of $[5:]_1$ has Nielsen class reps. from the orbit from conjugating the middle pair in $[5:]_1$ by the $\frac{5+1}{2}$-th (3rd) power of $([5:]_1)\mathbf{mp}$. Conjugate $[5:]_2$ by $(x_{\frac{n+1}{2},1})^{-2}$ times $(n{-}1\ \frac{n+3}{2})(n\ \frac{n+5}{2})$ (just one disjoint cycle if $n = 5$) to get

$$[5:]_2' \stackrel{\text{def}}{=} (x_{3,\frac{n+5}{2}}, \bullet, \bullet, x_{\frac{n+1}{2},1}),$$

which again is in the cusp of $[5:]_1$. For $n = 5$, only the first disjoint cycle appears. The pattern appears from this.

Substituting H-M$_2$ for H-M$_1$ gives the same pattern, but in each line of Table 1, you get the complementary cusp. The case $n \equiv 1 \mod 8$ is much simpler. □

When $n \equiv 1 \mod 8$, Lem. 4.3 says there are two separate braid orbits, so two projective space components $\bar{\mathcal{H}}^i(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{in,rd}}$, $i = 1, 2$. Each maps one-one to $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs,rd}}$. This identifies geometric cusps on each $\bar{\mathcal{H}}^i(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{in,rd}}$ with those of $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs,rd}}$. So, we can denote the cusps simply as Cusp_k, running over $1 \le k \le n$ odd.

Prop. 6.3 finds (and explains) that as moduli spaces, while $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs,rd}}$ has definition field $\mathbb{Q}$, the spaces $\bar{\mathcal{H}}^i(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{in,rd}}$ are conjugate over a quadratic extension of $\mathbb{Q}$. For $n \equiv 5 \mod 8$, $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{in,rd}} \to \bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs,rd}}$ is a degree 2 cover of absolutely irreducible $\mathbb{Q}$ varieties, and we need more intricate notation for the cusps.

Definition 4.4. For $n \equiv 5 \mod 8$ denote the unique width 2 cusp by Cusp_1. For $k \equiv 3 \mod 4$, denote the distinct cusps represented by $[k:]_i$ by $\text{Cusp}_{k,i}$, $i = 1, 2$. For $k \equiv 1 \mod 4$, denote the cusp represented by $[k:]_1$ by $\text{Cusp}_{k,1} = \text{Cusp}_{\text{H-M}_{1,\frac{k-1}{2}}}$, ($[k:]_2$ represents the same cusp) and the cusp $\text{Cusp}_{\text{H-M}_{2,\frac{k-1}{2}}}$ by $\text{Cusp}_{k,2}$. You may assume representatives $\boldsymbol{g}^i \in \text{Cusp}_{k,i}$, $i = 1, 2$ both have $x_{\frac{n+1}{2},1}$ in their 4th positions. Then, their first entries differ by a conjugate of $S_n \setminus A_n$.

4.2. sh-incidence for $\text{Ni}^{\text{in,rd}}_{(\frac{n+1}{2})^4}$**.** §4.2.1 introduces the **sh**-incidence matrix. Then §4.2.2 uses it to put order in the naming of the inner cusps, and computes it for $\text{Ni}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$. §4.3 computes the matrix for all $\text{Ni}^{\text{in,rd}}_{(\frac{n+1}{2})^4}$. Prop. 4.9 uses this to compute the genus of the spaces $\bar{\mathcal{H}}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{in,rd}}$, $n \equiv 5 \mod 8$.

4.2.1. *Using the* **sh***-incidence matrix.* The **sh**-incidence matrix is a pairing on cusps (as Cu_r orbits of reduced Nielsen classes) [**BF02**, §2.10]:

$$(O, O') \mapsto |O \cap (O')\mathbf{sh}|.$$

It makes sense for all $r \ge 4$, and applies to all *reduced* Nielsen class types. For $r = 4$ it is symmetric. It summarizes much data on inner and absolute classes to simplify computing the genus of reduced Hurwitz spaces. For example, we have the following. Take γ_0', γ_1' and γ_∞' to be the actions of $\gamma_0, \gamma_1, \gamma_\infty$ on $\mathrm{Ni}(G,\mathbf{C})^{*,\mathrm{rd}}$ with $*$ the inner or absolute classes as in §2.3.

A tautological appearance of entries comes from this identity:

$$(((\boldsymbol{g})q_2^j)\mathbf{sh})^{\mathrm{in,rd}} = (((\boldsymbol{g})q_2^j))^{\mathrm{in,rd}})\mathbf{sh}. \tag{4.2}$$

In words: The pairing of the cusp of $\boldsymbol{g}$ with the cusp of each shift in the cusp of $\boldsymbol{g}$ contributes 1 to a position in the matrix.

The next lemma shows fixed points of either γ_0' or γ_1' contribute to the main diagonal of the **sh**-incidence matrix. Still, the converse doesn't hold. For example, Table 6, for the **sh**-incidence matrix labeled Ni_0^+, has a nonzero diagonal entry, though neither γ_0' nor γ_1' has a fixed point.

LEMMA 4.5. *The* **sh** *incidence pairing applied to all cusps in a reduced Nielsen class has its irreducible blocks corresponding one-one to the braid orbits (components). Further, we can replace the shift (represented by $q_1q_2q_1$) by q_1q_2 (representing γ_0) to form the matrix.*

In particular, if either γ_0' or γ_1' fix a representative in the Nielsen class then that contributes an element in the diagonal of the matrix.

PROOF. The first sentence is from [**BF02**, Lem. 2.26]. So, we show the second.

Consider the effect of $q_1q_2 \mod \mathrm{Cu}_4$ on $\boldsymbol{g} \in {}_cO_{\boldsymbol{g}}$. It gets mapped to exactly the same place as $q_2q_1q_2$ maps $\boldsymbol{g}' = (\boldsymbol{g})q_2^{-1} \in {}_cO_{\boldsymbol{g}}$. One of the braid relations is $q_2q_1q_2 = q_1q_2q_1$ and $\mod \mathrm{Cu}_4$, this is the shift. By the definition of the **sh**-incidence matrix, something fixed by the shift will be in the intersection of O and $(O)\mathbf{sh}$. This concludes the proof. □

4.2.2. **sh**-*incidence matrices and naming cusps.* Continue with $n \equiv 5 \mod 8$. To start naming cusps, denote the unique cusp of middle product (**mp**) 1 and width 2 by $O_{1,2}$. Therefore,

$$O_{1,2} = \mathrm{Cusp}_{(\text{H-M}_1)\mathbf{sh}} = \mathrm{Cusp}_{(\text{H-M}_2)\mathbf{sh}} = ((\text{H-M}_1)\mathbf{sh})^{\mathrm{in,rd}} \cup ((\text{H-M}_2)\mathbf{sh})^{\mathrm{in,rd}}.$$

Already this fills in the column pairing $O_{1,2}$ with the other cusps — see Table 2 — as the sum of the entries in the column will be the cusp width (2 in this case) and pairing of $O_{1,2}$ with $\mathrm{Cusp}_{(\text{H-M}_i)}$ is 1, fulfilled by $((\text{H-M}_i)\mathbf{sh})^{\mathrm{in,rd}}$.

Denote the cusp represented by (both) $[n:]_i$, $i = 1,2$, by $O_{n,n;1}$ (the first of two cusps of **mp** n and width n). The following identifies this with $\mathrm{Cusp}_{\text{H-M}_2}$:

$$(\text{H-M}_2)\mathbf{sh}^2 = (x_{\frac{n+1}{2},n}, x_{n,\frac{n+1}{2}}, x_{1,\frac{n+1}{2}}, x_{\frac{n+1}{2},1}) = ([n:]_1)q_2^{-1}. \tag{4.3}$$

That is, $[n:]_i$s are in the cusp of H-M$_2$. So, $O_{n,n;1}$ is the cusp of H-M$_2$. The n fulfilling elements for all nontrivial intersections of $(O_{n,n;2})\mathbf{sh}$ comprise the list of Table 1. To whit: The column for $O_{n,n;1}$ in the pairing of the **sh**-incidence matrix is fulfilled by elements listed in Table 1. These add up to n elements. All that is missing is a naming of the cusps, and this gives us one.

Denote the cusp of $[k:]_i$ by $O_{k,k;i}$ if $k \equiv 3 \mod 4$. Otherwise, for everything else in Table 1 of odd width, denote the cusp of $[k:]_i$ by $O_{k,k;1}$, and the other cusp of width k by $O_{k,k;2}$. Table 2 does the case $n = 5$ so we can see what we've already determined of the sh-incidence matrix.

Between the symmetry of Table 2, and the comments above (including that column entries sum to the cusp width), the only unfinished point is the value of $O_{3,3,i}$ paired with itself, leaving that the pairing of $O_{3,3,1}$ with $O_{3,3,2}$ is 1.

TABLE 2. **sh**-incidence Matrix: $r = 4$ and $\mathrm{Ni}_{3^4}^{\mathrm{in,rd}}$

Cusp orbit	$O_{5,5;1}$	$O_{5,5;2}$	$O_{3,3;1}$	$O_{3,3;2}$	$O_{1,2}$
$O_{5,5;1}$	0	2	1	1	1
$O_{5,5;2}$	2	0	1	1	1
$O_{3,3;1}$	1	1	0	1	0
$O_{3,3;2}$	1	1	1	0	0
$O_{1,2}$	1	1	0	0	0

PROPOSITION 4.6. *The genus* $\mathbf{g}_{3^4}$ *of* $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ *is 0.*

There are no 2 cusps at level 0. Yet, every cusp at level 1 over each of the non H-M *cusps, is a 2-cusp. So, there are at least four 2 cusps on every level 1 component of the* **MT** *for* $p = 2$. *In particular, the Main Conjecture holds for* $p = 2$ *and all component branches.*

The only other prime of consideration is $p = 5$, *and the Main Conjecture holds for this prime, too, for all* H-M *component branches.*

PROOF. Apply (2.10) to get $\gamma_0', \gamma_1', \gamma_\infty'$ acting on the unique braid orbit on $(A_5, \mathbf{C}_{3^4})$. The diagonal entries of the **sh**-incidence matrix has only 0's. According to Lem. 4.5, that means neither γ_0' nor γ_1' has a fixed point. So, we have a degree 18 cover of the j-line. R-H gives its genus $\mathbf{g}_{3^4}$ through

$$
\begin{aligned}
(4.4) \qquad 2(18 + \mathbf{g}_{3^4} - 1) = \quad & \mathrm{ind}(\gamma_0') + \mathrm{ind}(\gamma_1') + \mathrm{ind}(\gamma_\infty') = \\
& 2(18/3) + 18/2 + (1 + 2(2+4)) : \mathbf{g}_{3^4} = 0.
\end{aligned}
$$

Apply Prop. 3.10 to the cusps of width 3 and 5. Respectively, since $\frac{3^2-1}{8}$ and $\frac{5^2-1}{8}$ are $\equiv 1 \mod 2$, all cusps about these are 2 cusps. In particular, any level 1 component has at least four 2 cusps. Therefore, the Main Conjecture is automatic for any component branch for $p = 2$ (Prop. 2.8).

The two H-M rep. cusps are already 5 cusps at level 0. Princ. 3.3 therefore shows for $p = 5$, there are at least three 5 cusps on every H-M level 1 component. In particular the Main Conjecture holds for any component branch going through a level 1 H-M component. If, however, a component is not H-M at level 1, we only can guarantee two 5 cusps. □

REMARK 4.7 (Distinction between non-H-M and H-M level 1 components). An H-M component — a Hurwitz space component containing an H-M cusp (§3.1.2) — figures in many applications of the moduli approach to properties of $G_{\mathbb{Q}}$ (as in §7.2). Level 0 of a **MT** over $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ being an H-M component doesn't guarantee the same for level 1, even an abelianized **MT**. For example, $\bar{\mathcal{H}}(G_1(A_4), \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ (that's A_4, not A_5; §6.5.2) has six components. Three of those are level 1 of a $p = 2$ **MT** over $\mathcal{H}(A_4, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ and one of those is not H-M.

Further, to show the MC for all allowable primes for just the infinite collection of Nielsen classes of Prop. 4.6 will require general principles, to avoid detailed calculations about non-H-M components. §6.2 completes Prop. 4.6 by showing all level 1 components for a $p = 5$ **MT** over $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\mathrm{in,rd}}$ have at least three 5-cusps, so the MC holds. A guess at a general braid orbit principle applied here, though we computed using [**GAP00**] to assure it held.

4.3. Rest of $\mathrm{Ni}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$ **sh-inc. matrix.** The $[3:]_1$ row of Table 1 is $(x_{2,\frac{n+3}{2}}, (\frac{n+3}{2}\ \dots\ n\,1), x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$, with middle product $(1\,\frac{n+3}{2}\,\frac{n+1}{2})$. Here are (2nd, 3rd) positions of reps. of $((([3:]_1)q_2^j)\mathbf{sh})^{\mathrm{in}}$, $j = 1, 2$:

$$\begin{aligned} \boldsymbol{h}^{3,1} &= \ ((n\ \dots\ \tfrac{n+5}{2}\ \tfrac{n+1}{2}\ 1), x_{\frac{n+1}{2},1}) \\ \boldsymbol{h}^{3,2} &= \ ((n\ \dots\ \tfrac{n+5}{2}\ 1\ \tfrac{n+3}{2}), x_{\frac{n+1}{2},1}). \end{aligned} \tag{4.5}$$

Prop. 4.8 reduces the computation of the inner, reduced sh-incidence matrix to deciding if a collection of conjugations between pairs of length $\frac{n+1}{2}$ pure-cycles is or is not in $A_{\frac{n+1}{2}}$. The actual sh-incidence display is completed in §4.3.2.

4.3.1. *Reduction to finding conjugating elements.* To aid in following notation, we make explicit asides on the sh-incidence column for the cusp of $[3:]_1$. The respective products of entries of $\boldsymbol{h}^{3,1}$ and $\boldsymbol{h}^{3,2}$ are

$$(n\ \dots\ \tfrac{n+5}{2}\ \tfrac{n+1}{2}\ \dots\ 2) \text{ and } (n\ \dots\ \tfrac{n+5}{2}\ 1\ \tfrac{n+1}{2}\ \dots\ 2\ \tfrac{n+3}{2}).$$

Denote the alternating (resp. symmetric) group acting on $\{\frac{n+3}{2}, \dots, n\}$ by $A_{\frac{n+3}{2},n}$ (resp. $S_{\frac{n+3}{2},n}$) and the group generated by $x_{\frac{n+1}{2},1}$ and $A_{\frac{n+3}{2},n}$ (resp. $S_{\frac{n+3}{2},n}$) by U_n^a (resp. U_n^s). Also, denote $k-1-|2u-(k-1)|$ by $m_{k,u}$. A key to Prop. 4.8 is applying $q_1^{-1}q_3$, leaving the reduced Nielsen class unchanged.

Proposition 4.8. *The cusps of*

$$([n:]_1)^{\mathrm{in,rd}}, ([n:]_2)^{\mathrm{in,rd}} \text{ and } ([n-2:]_2)^{\mathrm{in,rd}}$$

each have one intersection with $(\mathrm{Cusp}_{[3]_1})\mathbf{sh}$. *This accounts for all nonzero entries in the column of the cusp of* $[3:]_j$, $j = 1, 2$.

Denote $(1\ \dots\ \frac{k-1}{2}\ \frac{n+k}{2}\ \frac{n+k-2}{2}\ \dots\ \frac{n+1}{2})$ *by* α_k. *Use notation from Def. 4.4. Then,* $\mathrm{Cusp}_{\ell,1}$ *and* $\mathrm{Cusp}_{\ell,2}$ *consist of inner reduced Nielsen classes represented by*

$${}_{k,u}\boldsymbol{g} \stackrel{\mathrm{def}}{=} (\alpha_k^u x_{n,\frac{n+1}{2}} \alpha_k^{-u}, \bullet, \bullet, x_{\frac{n+1}{2},1})$$

with $\ell = n - m_{k,u}$. *Conjugation of* ${}_{k,u}\boldsymbol{g}$ *by an element of* U_n^s *leaves the 4th entry fixed. Further,* ${}_{k,u}\boldsymbol{g}$ *is then in* $\mathrm{Cusp}_{\ell,1}$ *if and only if* $\alpha_k^u x_{n,\frac{n+1}{2}} \alpha_k^{-u} \mod U_n^a$ *is* $x_{\frac{n+1}{2}-\frac{m_{k,u}}{2},n-\frac{m_{k,u}}{2}}$ *(the first entry of* $[\ell:]_1$ *in Table 1).*
.

Proof. Three distinct sets of form $\boldsymbol{g}^{\mathrm{in,rd}}$ comprise $(\mathrm{Cusp}_{[3]_1})\mathbf{sh}$. §4.2.2 shows $([n:]_1)^{\mathrm{in,rd}}$ and $([n:]_2)^{\mathrm{in,rd}}$ give two. Then, $\boldsymbol{g}^* = ((\frac{n+1}{2}\ \frac{n+5}{2}\ \dots\ n\ \frac{n+3}{2}), \boldsymbol{h}^{3,1}, x_{2,\frac{n+3}{2}})$ is the 3rd, and it is in the cusp of one of $([n-2:]_1)^{\mathrm{in,rd}}$ or $([n-2:]_2)^{\mathrm{in,rd}}$.

Apply $q_1^{-1}q_3$ to $\boldsymbol{g}^*$ to get the 4th entry $x_{\frac{n+1}{2},1}$:

$$\boldsymbol{g}^{n-2,2} \stackrel{\mathrm{def}}{=} ((n\ \dots\ \tfrac{n+5}{2}\ \tfrac{n+1}{2}\ 1), \bullet, \bullet\, x_{\frac{n+1}{2},1}).$$

Conjugate $[n-2:]_2 = ((\frac{n+5}{2}\ \dots\ n\,1\,2), x_{2,\frac{n+3}{2}}, x_{n,\frac{n+1}{2}}, x_{\frac{n+1}{2},1})$ by $x_{\frac{n+1}{2},1}$ to get

$$\boldsymbol{g}^{3,1} \stackrel{\mathrm{def}}{=} ((\tfrac{n+5}{2}\ \dots\ n\ \tfrac{n+1}{2}\ 1), \bullet, \bullet, x_{\frac{n+1}{2},1}).$$

Apply the strategy of Def. 4.4: With $\beta = (n\ \frac{n+5}{2})(n-1\ \frac{n+7}{2})\cdots$, $\beta^{-1}\boldsymbol{g}^{n-2,2}\beta$ is in the cusp of $\boldsymbol{g}^{3,1}$. Further, β is a product of $(n-(\frac{n+5}{2}))/2$ disjoint cycles. This is an even number since $n \equiv 5 \mod 8$. That accounts for the column in the sh-incidence matrix for the cusp of $[3:]_1$. There is a similar accounting for $[3:]_2$, one intersection with each of the cusps for H-M$_1$ and H-M$_2$, and one intersection with the cusp of $[n{-}2{:}]_1$.

Now we extend the pattern above for computing into which cusp does the shift of inner, reduced elements fall in the cusp columns of the other $[k{:}]_j$, $j=1,2$. With α_k, as in the statement, apply Princ. 3.5 and $q_1^{-1}q_3$ as above. This gives

$$_{k,u}\boldsymbol{g} = (\alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}, \bullet, \bullet, x_{\frac{n+1}{2},1}).$$

The following steps give the sh-incidence matrix.

(4.6a) As (k,u) varies, $3 \le k \le n$ odd and $0 \le u \le k{-}1$, compute

$$\ell = \mathrm{ord}((_{k,u}\boldsymbol{g})\mathbf{mp}) = \mathrm{ord}(x_{\frac{n+1}{2},1} \cdot \alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}).$$

(4.6b) Then, (k,u) contributes 1 to $\mathrm{Cusp}_{\ell,1}$ (resp. $\mathrm{Cusp}_{\ell,2}$) if $\alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}$ is (resp. isn't) conjugate by U_n^a to the first entry of $[\ell{:}]_1$.

For $k=5$ and u from 0 to the value of $\alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}$ attached to (k,u) appears on the left in (4.7); and for $k=7$ and u from 0 to 6 on the right. To the left of each entry is $\ell = \mathrm{ord}((_{k,u}\boldsymbol{g})\mathbf{mp})$ followed by a ":". Here ℓ is n minus two numbers: the cardinality of the subset of $\{\frac{n+k}{2},\dots,\frac{n+3}{2}\}$ (resp. of $\{1,2,\dots,\frac{n+1}{2}\}$) missing from $\alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}$ (resp. moved by $\alpha_k^u x_{n,\frac{n+1}{2}}\alpha_k^{-u}$ as in $x_{1,\frac{n+1}{2}}$).

$$\begin{array}{rl}
n: & x_{n,\frac{n+1}{2}}\\
n{-}2: & (n\ \dots\ \frac{n+7}{2}\ \frac{n+3}{2}\ \frac{n+1}{2}\ 1)\\
n{-}4: & (n\ \dots\ \frac{n+7}{2}\ \frac{n+1}{2}\ 1\,2)\\
n{-}2: & (n\ \dots\ \frac{n+7}{2}\ 1\,2\ \frac{n+5}{2})\\
n: & (n\ \dots\ \frac{n+7}{2}\ 2\ \frac{n+5}{2}\ \frac{n+3}{2})
\end{array}
\qquad
\begin{array}{rl}
n: & x_{n,\frac{n+1}{2}}\\
n{-}2: & (n\ \dots\ \frac{n+9}{2}\ \frac{n+5}{2}\ \frac{n+3}{2}\ \frac{n+1}{2}\ 1)\\
n{-}4: & (n\ \dots\ \frac{n+9}{2}\ \frac{n+3}{2}\ \frac{n+1}{2}\ 1\,2)\\
n{-}6: & (n\ \dots\ \frac{n+9}{2}\ \frac{n+1}{2}\ 1\,2\,3)\\
n{-}4: & (n\ \dots\ \frac{n+9}{2}\ 1\,2\,3\ \frac{n+7}{2})\\
n{-}2: & (n\ \dots\ \frac{n+9}{2}\ 2\,3\ \frac{n+7}{2}\ \frac{n+5}{2})\\
n: & (n\ \dots\ \frac{n+9}{2}\ 3\ \frac{n+7}{2}\ \frac{n+5}{2}\ \frac{n+3}{2}).
\end{array} \tag{4.7}$$

The pattern is clear, even if cumbersome in those fractionally represented integers. For u from 0 to $\frac{k-1}{2}$ (resp. $\frac{k+1}{2}$ to $k{-}1$), $m_{k,u}$ is u (resp. $k{-}1-u$). First consider $0 \le u \le \frac{k-1}{2}$, so the entry in (4.7) for a general (n,k) has this shape:

$$(n\ \dots\ \tfrac{n+k+2}{2}|x_{\frac{n+k-2u}{2},\frac{n+3}{2}}|\tfrac{n+1}{2}\ x_{1,u}). \tag{4.8}$$

The dividers | are just to visually separate sections of the permutation.

For such a u, $_{k,u}\boldsymbol{g}$ is in $\mathrm{Cusp}_{\ell,1}$ if and only if (4.8) is in $x_{\frac{n+1}{2}-u,n-u} \mod U_n^a$ (the first entry of $[\ell{:}]_1$). Conjugate by a power of $x_{1,\frac{n+1}{2}} \in U_n^a$ to change the segment $|\frac{n+1}{2}\ x_{1,u}|$ to $|x_{\frac{n+1-2u}{2},\frac{n+1}{2}}|$. By cycling the last two segments to the front,

$$(x_{\frac{n+k-2u}{2},\frac{n+3}{2}}|x_{\frac{n+1-2u}{2},\frac{n+1}{2}}|x_{n,n-u+1}|x_{n-u,\frac{n+k+2}{2}}) \tag{4.9}$$

is the same permutation.

For $\frac{k+1}{2} \le u \le k{-}1$, denote $k{-}1-u$ by u'. Then, here is the analog of (4.9):

$$(x_{\frac{n+1-2u'}{2},\frac{n+1}{2}}|x_{\frac{n+k}{2},\frac{n+3+2u'}{2}}|x_{n,n-u'+1}|x_{n-u',\frac{n+k+2}{2}}). \tag{4.10}$$

This proof is done; §4.3.2 uses this to produce the precise sh-incidence matrix. □

4.3.2. *Completing sh-incidence entries.* The conclusion of this section is the desired sh-incidence matrix. It behooves breaking the parameter u into two ranges. We start with $0 \le u \le \frac{k-1}{2}$. As at the conclusion of the Prop. 4.8 proof, for each such (k,u) check if an element of $A_{\frac{n+3}{2},n}$ conjugates (4.9) to $x_{\frac{n+1}{2}-u,n-u}$.

Consider this example: In (4.7), compare the $k=7$ column, the first position labeled $n{-}4$ $(u=2)$ —using the element from the inner, reduced representative

$$\boldsymbol{g}' = (\tfrac{n+3}{2}\ \tfrac{n-3}{2}\ \tfrac{n-1}{2}\ \tfrac{n+1}{2}|n\, n{-}1|n{-}2\ \dots\ \tfrac{n+9}{2})$$

in (4.9), except I've removed the first division —with the first position

$$g'' = (\tfrac{n-3}{2}\ \tfrac{n-1}{2}\ \tfrac{n+1}{2}\ \tfrac{n+3}{2}|\tfrac{n+5}{2}\ \tfrac{n+7}{2}|\tfrac{n+9}{2}\ \dots\ n{-}2)$$

of $[n{-}2:]_1$. We want the parity of the permutation from $S_{\frac{n+3}{2},n}$ that conjugates between these two. The divisions in the two permutations correspond, and the desired permutation is $\beta_{7,2} = \beta_{7,2,1}\beta_{7,2,2}\beta_{7,2,3}$ with these comprising βs as follows:

(4.11) $\beta_{7,2,1}$, a 4-cycle, cycles $\frac{n-3}{2}$, $\frac{n-1}{2}$, $\frac{n+1}{2}$, $\frac{n+3}{2}$; $\beta_{7,2,2} = (n\ \frac{n+5}{2})(n{-}1\ \frac{n+7}{2})$; and $\beta_{7,2,3}$ is the inverting permutation from Lem. 1.1 whose parity in this case is $(-1)^{(n-2-\frac{n+9}{2})/2)} = 1$.

Conclude: $\beta_{7,2}$ has parity -1, and the $(7,2)$ term in (4.7) is in $\mathrm{Cusp}_{13,7,2}$.

We fill notation for the general case using the (7,2) example plan (above). Remove the first division in (4.9). We form $\beta_{k,u,j}$, $j=1,2,3$, to conjugate each of three divisions in (4.9) to its correspondant in the first position of $[n{-}2u:]_1$ for k:

$$(x_{\frac{n+1-2u}{2},\frac{n+k-2u}{2}}|x_{\frac{n+k-2u+2}{2},\frac{n+k}{2}}|x_{\frac{n+k+2}{2},n-u}). \tag{4.12}$$

Conjugate $|x_{\frac{n+1-2u}{2},\frac{n+k-2u}{2}}|$ to $|x_{\frac{n+3}{2},\frac{n+k-2u}{2}}\ x_{\frac{n+1-2u}{2},\frac{n+1}{2}}|$ using $x^{t'}_{\frac{n+1-2u}{2},\frac{n+k-2u}{2}}$ with $t' = \frac{n+k-2u}{2} - \frac{n+3}{2} + 1 = \frac{k-2u-1}{2}$. To get $\beta_{k,u,1}$ combine this with the Lem. 1.1 conjugation inverting $x_{\frac{n+3}{2},\frac{n+k-2u}{2}}$. The former has parity $(-1)^{t'\cdot(\frac{k-1}{2})} = (-1)^{(\frac{k-1}{2}-u)\cdot(\frac{k-1}{2})}$. The latter has parity $(-1)^{(\frac{n+k-2u}{2}-\frac{n+3}{2})/2} = (-1)^{(\frac{k-3}{2}-u)/2}$ if $\frac{k-3}{2} \equiv u \mod 2$, and $(-1)^{(\frac{k-1}{2}-u)/2}$ otherwise.

The parity of $\beta_{k,u,1}$ is the product of these. Check the cases: For $\frac{k-1}{2}-u$ even (resp. odd), this parity is $(-1)^{(\frac{k-1}{2}-u)/2}$ (resp. $(-1)^{(\frac{k-1}{2}-u+1)/2}$).

As with our example $(k=7, u=2)$, $\beta_{k,u,2}$ is clearly a product of u disjoint transpositions, so its parity is $(-1)^u$.

Similarly, if $\frac{n-k-2}{2}-u$ is even (resp. odd), apply Lem. 1.1 to get the parity of $\beta_{k,u,3}$ to be $(-1)^{(\frac{n-k-2}{2}-u)/2}$ (resp. $(-1)^{(\frac{n-k}{2}-u)/2}$). Since $n \equiv 5 \mod 8$, an n-free form of the parity replaces it by 5 to get $(-1)^{(\frac{k-3}{2}+u)/2}$ (resp. $(-1)^{(\frac{k-5}{2}+u)/2}$).

There's no need to start all over with the case $\frac{k+1}{2} \le u \le k{-}1$. Just check if an element of $A_{\frac{n+3}{2},n}$ conjugates (4.9) to (4.10), where in the latter u' is set equal to u. If so they are both in the same inner, reduced cusp, and otherwise in the complementary cusps for the given value of the middle product. In both (4.9) and (4.10) join the 3rd and 4th divisions, and in the former replace by the equivalent permutation where the first division has been placed at the end. We are left to compare the two rows in this expression:

$$\begin{array}{l}(x_{\frac{n+1-2u}{2},\frac{n+1}{2}}|x_{n,\frac{n+k+2}{2}}|x_{\frac{n+k-2u}{2},\frac{n+3}{2}})\\ (x_{\frac{n+1-2u}{2},\frac{n+1}{2}}|x_{\frac{n+k}{2},\frac{n+3+2u}{2}}|x_{n,\frac{n+k+2}{2}}).\end{array} \tag{4.13}$$

To continue, conjugate the corresponding integers in the 2nd division of the bottom row to the integers in the 3rd division of the top row:

$$\tfrac{n+k}{2} \mapsto \tfrac{n+k-2u}{2}, \dots, \tfrac{n+3+2u}{2} \mapsto \tfrac{n+3}{2}.$$

A product of $\frac{k-2u-1}{2}$ disjoint transpositions, $\gamma_{k,u,1}$, of parity $(-1)^{\frac{k-2u-1}{2}}$ gives this.

To finish, conjugate by $\gamma_{k,u,2} = (x_{\frac{n+k-2u}{2},\frac{n+3}{2}} x_{n,\frac{n+k+2}{2}})^{t''}$ to cycle the 2nd division of $|x_{\frac{n+k-2u}{2},\frac{n+3}{2}}|x_{n,\frac{n+k+2}{2}}|$ past the 1st: t'' is the length, $\frac{n+k-2u}{2} - \frac{n+1}{2}$, of $x_{\frac{n+k-2u}{2},\frac{n+3}{2}}$. So, $\gamma_{k,u,2}$ has parity $(-1)^{\frac{k-2u-1}{2}\cdot(\frac{n-2u-3}{2})}$. Again, since $n \equiv 5 \mod 8$ this has the n-free form $(-1)^{\frac{k-2u-1}{2}(u-1)}$. Then, let $\gamma_{k,u} = \gamma_{k,u,1}\gamma_{k,u,2}$.

The next expression is for convenient use of the previous computations in §4.3.3. Since $\frac{k+1}{2} - u + \frac{3-k}{2} - u = 1 - 2u$, an odd number, if $\frac{k+1}{2} - u$ is odd (resp. even) then $\frac{3-k}{2} - u$ is even (resp. odd). This gives us simple expressions for $\beta_{k,u}$ differentiating between just two cases.

(4.14a) If $\frac{k+1}{2} - u$ is odd, then $\beta_{k,u} = (-1)^{(\frac{k-1}{2}-u+1)/2}(-1)^u(-1)^{(\frac{3-k}{2}-u)/2}$.

(4.14b) If $\frac{k+1}{2} - u$ is even, then $\beta_{k,u} = (-1)^{(\frac{k-1}{2}-u)/2}(-1)^u(-1)^{(\frac{3-k}{2}-u+1)/2}$.

4.3.3. *Actual display of the sh-incidence matrix.* Following Table 2, label the general sh-incidence matrix for $n \equiv 5 \mod 8$ so its columns (or rows) in order are

$$O_{n,n;1}, O_{n,n;2}, O_{n-2;n-2;1}, O_{n-2,n-2;2}, \dots, O_{3,3;1}, O_{3,3;2}, O_{1,2}.$$

Notation for the $u, v; w$ subscript: u is the middle product of a cusp rep.; v indicates the cusp width (as a ramified point in the j-line cover); and w distinguishes the inner reduced cusps with given u, v (if there is more than one).

For a given n, our previous parameters (k, u) where a convenient way to list inner reduced Nielsen class elements, with the understanding the pattern of what cusp into which its shift fell, as (k, u) varied $\mod 8$. We now use these to consider notation for the row denoted $O_{\ell,\ell;1}$; it will be immediate then how to get $O_{\ell,\ell;2}$. Since that notation is redundant —meant to consider more general cases where subscripts won't be repeated —replace $O_{\ell,\ell;j}$ with $O'_{\ell;j}$ to shorten the matrix display. Leave the notation $O_{1,2}$ for the one exceptional cusp.

Denote the sh-incidence matrix by $I(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in}$, with its $((\ell;j),(\ell';j'))$-entry labeled $I_{(\ell;j),(\ell';j')}$. To make less of a distinction with the (1,2) subscript, we sometimes use $\bar\ell$ to denote $(\ell;j)$, and then include $(1,2)$ as another $\bar\ell$.

Proposition 4.9. *For $n \equiv 5 \mod 8$, the $I_{(\ell;j),(\ell';j')}$ are all 0, 1 or 2, and they satisfy these additional rules.*

(4.15a) *Symmetry: $I_{\bar\ell,\bar\ell'} = I_{\bar\ell,\bar\ell'}$.*

(4.15b) *Width sum: Entries in the row for $O_{\ell;j}$ (resp. $O_{1,2}$) sum to ℓ (resp. 2).*

(4.15c) *$I_{(\ell;1),(\ell';1)} + I_{(\ell;1),(\ell';2)} = 2$ (resp. 1) if $\ell' = n-2u$ and $0 \le u < \frac{k-1}{2}$ (resp. $u = \frac{k-1}{2}$) and 0 otherwise.*

Diagonal entries occur only when ?? The genus of $\mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\rm in,rd}$ is $\boldsymbol{g}_{(\frac{n+1}{2})^4}$ in the following expression.

Proof. As a warmup, here are the last three rows with a listing of the columns in a style used below. The first two are from the opening paragraph of Prop. 4.8, with the last from the first line of Prop. 4.1.

Table 3. Rows for $O'_{3,1}$, $O'_{3,2}$ and $O_{1,2}$

Cusp orbit	$O'_{n;1}$	$O'_{n;2}$	$O'_{n-2;1}$	$O'_{n-2;2}$	…	$O'_{3;1}$	$O'_{3;2}$	$O_{1,2}$
$O'_{3;1}$	1	1	0	1	…	0	0	0
$O'_{3;2}$	1	1	1	0	…	0	0	0
$O_{1,2}$	1	1	0	0	…	0	0	0

TABLE 4. Intersections of $O'_{5,1}$, $O'_{7,1}$, $O'_{9,1}$, $O'_{11,1}$ with $O_{\ell,1}$

Cusp orbit	$O'_{n;1}$	$O'_{n-2;1}$	$O'_{n-4;1}$	$O'_{n-6;1}$	$O'_{n-8;1}$	$O'_{n-10;1}$	$O'_{n-12;1}$	...
$O'_{11;1}$	1	1	0	1	0	0	0	0
$O'_{9;1}$	0	1	0	0	0	0	0	0
$O'_{7;1}$	1	1	0	0	0	0	0	0
$O'_{5;1}$	0	1	0	0	0	0	0	0

TABLE 5. Intersections of $O'_{5,1}$, $O'_{7,1}$, $O'_{9,1}$, $O'_{11,1}$ with $O_{\ell,2}$

Cusp orbit	$O'_{n;2}$	$O'_{n-2;2}$	$O'_{n-4;2}$	$O'_{n-6;2}$	$O'_{n-8;2}$	$O'_{n-10;2}$	$O'_{n-12;2}$	...
$O'_{11;1}$	1	1	0	1	0	0	0	0
$O'_{9;1}$	2	1	0	0	0	0	0	0
$O'_{7;1}$	1	1	0	0	0	0	0	0
$O'_{5;1}$	2	1	0	0	0	0	0	0

Expression (4.6) gives values $I_{(\ell,1),(\ell',1)}$ of the row of $O'_{\ell,1}$, $\ell = 5, 6, 9$ and 11: $\ell = n - m_{k,u}$ with $m_{k,u} = k-1 - |2u - (k-1)|$, as in Prop. 4.8. This is compatible with reading columns of (4.7) as rows, thereby fixing ℓ (and k), and changing u.

The column for $O'_{n,1}$ in Table 4 comes from Lem. 4.3 and choice of representatives for $O'_{n,j}$ given in §4.2.2. The rest is from computing the values of $\beta_{k,u}$, $0 \le u \le \frac{k-1}{2}$, and $\beta_{k,u-\frac{k-1}{2}}\gamma_{k,u-\frac{k-1}{2}}$ for $\frac{k+1}{2} \le u \le k$, with k running over 5,7,9,11 from (4.14). The formulas (4.15) show why it suffices to know just Tables 3 and 4 to completely fill the whole sh-incidence matrix. For example, the respective entries in the columns for $O'_{n,j}$ in Tables 4 and 5 sum to 2 (as expected from (4.15c)), as does the rest of Table 5 follow from Table 4.

□

PROPOSITION 4.10.

REMARK 4.11. There is a sh-incidence matrix $I(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs}}$ as well.

5. 2 cusps on Liu-Osserman MTs

5.1. Spin invariant assures 2 cusps at level 1. 2cusps-level1 2spire-level1

COROLLARY 5.1. *Describe exactly why all* $(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$ *have 2 cusps at level 1 on any* $\mathbf{MT}$ *over* $(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})$.

COROLLARY 5.2. *That for the case* $n \equiv 5 \mod 8$ *that they have a 2-Spire at level one.*

6. How to approach primes different from 2

§6.3 considers the rest of the odd order pure-cycles cases of Liu-Osserman. Finally, §6.5 gives one example from the list of Ex. 2.18. This shows issues involved in dropping the condition that the absolute spaces represent genus 0 covers in (2.14b). Considering it may seem slight, since $G = A_4$ is such an "easy" group. Yet, it is our most important example for using this paper to head toward a general proof of the Main Conjecture.

Much of the idea of this section is general. The missing general ingredient is a purely modular representation step. We consider if there are non-H-M braid orbits on $\text{Ni}(G_1(A_5), \mathbf{C}_{3^4}, p=5)$. The Main Conjecture holds for any component branch through them if there are at least three 5 cusps at level 1.

To prove the following result we look carefully at how to write out the level 1 sh-incidence matrix for $\text{Ni}(\text{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4}, p=5)^{\text{in,rd}}$, recognizing the Hurwitz components for this cover the unique component for $\text{Ni}(A_5, \mathbf{C}_{3^4})$, and the level 1 components all factor through some one of these components.

Proposition 6.1. *Each braid orbit on* $\text{Ni}(\text{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\text{in,rd}}$ *has two representives over* H-M_1. *Therefore, the Main Conjecture holds for all* **MT***s for* $(A_5, \mathbf{C}_{3^4}, p=5)$ *and therefore for all* **MT***s with level 0 equal to* $\text{Ni}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$.

§6.1 shows why it suffices to consider $\text{Ni}(\text{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\text{in,rd}}$ to conclude Prop. 6.1. §6.2 considers how to compute the cusps of $\text{Ni}(\text{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\text{in,rd}}$, and the corresponding sh-incidence matrix.

6.1. Relation of $\text{Ni}(G_1(A_5), \mathbf{C}_{3^4}, p=5)$ **to** $\text{Ni}(\text{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})$**.** Make use of $\text{PSL}_2(\mathbb{Z}/5) = A_5$ using the notation 0_2 (resp. I_2) for the 2×2 zero (resp. identity) matrix. Then, $A_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has order 2 $\mod \{\pm I_2\}$. We can see A_4 in A_5 as a Klein 4-group with a $\mathbb{Z}/3$ action. Nonzero representatives of the Klein 4-group are order 2 matrices commuting with $A_1 \mod \pm I_2$: $A_2 = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ and $A_3 = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ are representatives of the two non-identity classes. Note: Traces of the involution conjugacy class are 0.

A generator $\alpha \in \mathbb{Z}/3$ in A_4 conjugates A_1 to A_2: $\alpha A_1 = A_2\alpha$: $\alpha = \left(\begin{smallmatrix} 2 & -3 \\ 1 & -1 \end{smallmatrix}\right)$ is a trace 1 representative. So, ± 1 is the trace of all elements in the order 3 conjugacy $\text{PSL}_2(\mathbb{Z}/5)$ class. To get A_5, throw into this copy of A_4 an element of order 5 by finding a representative $\beta \in \text{SL}_2(\mathbb{Z}/5)$ of trace 2 or 3: $\beta = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ will do. Note: These representatives canonically lift to have determinant 1 in $\text{SL}_2(\mathbb{Z}/5^2)$.

From [**Fr95**, Rem. 2.10]:

(6.1a) $\ker(G_1(A_5) \to A_5)$ is a module with Loewy display $U_5 \to U_5$ with U_5 the trace 0 matrices in $\mathbb{M}_2(\mathbb{Z}/5)$; and

(6.1b) $G_1(A_5) \to A_5$ factors through $\text{PSL}_2(\mathbb{Z}/5^2) \to \text{PSL}_2(\mathbb{Z}/5)$.

Finally, we find in $\text{PSL}_2(\mathbb{Z}/5)$ two H-M reps. with middle product order 5. As $\alpha\beta = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \gamma$ has trace 1, take $\text{H-M}_1 = (\gamma^{-1}, \gamma, \alpha, \alpha^{-1})$ as one H-M rep. and $\text{H-M}_2 = (\gamma, \gamma^{-1}, \alpha, \alpha^{-1})$ as the other. Use the same integer entries of α and γ to give representatives of all lifts of H-M_1 to $\text{Ni}(\text{PSL}_2(\mathbb{Z}/5^2), \mathbf{C}_{3^4})^{\text{in,rd}}$:

$$\boldsymbol{g}_{A_{\gamma^{-1}}, A_\gamma, A_\alpha, A_{\alpha^{-1}}} \stackrel{\text{def}}{=} (\gamma^{-1}(I_2 + 5A_{\gamma^{-1}}), \gamma(I_2 + 5A_\gamma), \alpha(I_2 + 5A_\alpha), \alpha^{-1}(I_2 + 5A_{\alpha^{-1}})),$$

modulo conjugation by $\ker(\text{PSL}_2(\mathbb{Z}/5^2) \to \text{PSL}_2(\mathbb{Z}/5))$ subject to these conditions.

(6.2a) Entries in $\text{PSL}_2(\mathbb{Z}/5^2)$: Entries of $(A_{\gamma^{-1}}, A_\gamma, A_\alpha, A_{\alpha^{-1}})$ have trace 0.

(6.2b) Product-one: $\gamma^{-1}A_{\gamma^{-1}}\gamma + A_\gamma + \alpha A_\alpha \alpha^{-1} + A_{\alpha^{-1}} = 0_2$.

The effect of conjugation of U by $I_2 + 5B$ sends the former to

$$(I_2 + 5B)(U)(I_2 - 5B) = U + 5([B, U]),$$

with $[B, U] = BU - UB$.

With no loss, assume $A_\gamma = 0_2$, and consider the case A_α also is 0_2. Write $A_{\gamma^{-1}} = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, so $\gamma^{-1}A_{\gamma^{-1}}\gamma = -A_{\alpha^{-1}}$. With $\gamma^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$,

$$A_{\alpha^{-1}} = -\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right) = -\left(\begin{smallmatrix} a-c & c \\ b-c+2a & c-a \end{smallmatrix}\right).$$

It is meaningful to have $q \in \bar{M}_4$ act on $\boldsymbol{g}_{A_{\gamma^{-1}},A_\gamma,A_\alpha,A_{\alpha^{-1}}}$, by acting on its 4-tuple. The Main Conjecture follows if for each q, so that its induced action $\mod 5$ leaves $(\gamma^{-1},\gamma,\alpha,\alpha^{-1})$ invariant, while not leaving $\boldsymbol{g}_{A_{\gamma^{-1}},A_\gamma,A_\alpha,A_{\alpha^{-1}}}$ invariant.

6.2. sh-incidence matrix for $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2),\mathbf{C}_{3^4})$. §4.2.2 has the sh-incidence matrix at level 0. For $\boldsymbol{g}$ in some Nielsen class $\mathrm{Ni}(G,\mathbf{C})$, denote the full collection of elements in its reduced Nielsen class (its orbit under $\langle G, \langle \mathbf{sh}, q_1q_3^{-1}\rangle\rangle$) by $\boldsymbol{g}^{\mathrm{in,rd}}$. The cusp containing $\boldsymbol{g}$ (as a subset of $\mathrm{Ni}(G,\mathbf{C})$) is the union of $\{((\boldsymbol{g})q_2^j)^{\mathrm{in,rd}}\}$ running over all integers j. Of course you only need at most the first $2\cdot(\boldsymbol{g})\mathbf{mp}$ values of j. We denote this set by $\mathrm{Cu}_{\boldsymbol{g}}$, the cusp of $\boldsymbol{g}$.

Using this notation, Lem. 4.2 gives the 5 cusps of $\mathrm{Ni}(A_5,\mathbf{C}_{3^4})$ as $\mathrm{Cu}_{(\text{H-M}_1)q_2^j\mathbf{sh}}$, $j = 0,1,2,3,4$, with $j = 0$ the unique cusp of width 2, $j = 1,4$ the cusps of width 3, and $j = 2,3$ the cusps of width 5. Now suppose $\text{H-M}_1'$ lies over H-M_1 in $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2),\mathbf{C}_{3^4})$. Then, we get the complete set of representatives of cusps for the spaces corresponding to braid orbits on $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2),\mathbf{C}_{3^4})$ by considering the collection $\mathrm{Cu}_{(\text{H-M}_1')q_2^j)\mathbf{sh}}$, $j = 0,1,2,3,4$.

Let $\text{H-M}_1''$ denote another representative over H-M_1. A contribution to the sh-incidence matrix of $\mathrm{Ni}(\mathrm{PSL}_2(\mathbb{Z}/5^2),\mathbf{C}_{3^4})^{\mathrm{in,rd}}$ over the level 0 position of (i,j) comes from $(\text{H-M}_1')q_2^j\mathbf{sh})^{\mathrm{in,rd}} = (\text{H-M}_1')q_2^i)^{\mathrm{in,rd}}$ for some $\text{H-M}_1'$ and $\text{H-M}_1''$. So, to find such contributions requires only looking at the cases where RETURN

6.2.1. **sh**-*incidence Matrix: $r = 4$ and* $\mathrm{Ni}^{\mathrm{in,rd}}_{(\frac{n+1}{2})}$. Here is how to look at the level 0 sh-incidence matrix.

6.3. The rest of the Liu-Osserman Examples. Throughout this subsection assume we are given an odd-cycle Liu-Osserman Nielsen class $\mathrm{Ni}(A_n,\mathbf{C}$.

6.3.1. *Remaining odd-cycle Liu-Osserman examples for $r = 4$.* Give What changes if we don't have the $(\frac{n+1}{2})^4$ case?

6.3.2. *What about general $r \ge 3$?*

Example 6.2. Give the alternating group obstructed components here.

6.3.3. *Umbrella result.*

6.4. Rational functions representing elements of $\mathrm{Ni}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4})^{\mathrm{abs}}$. For $n \equiv 1 \mod 4$ (and especially for $n \equiv 1 \mod 8$) we consider rational functions representing $\mathrm{Ni}(A_n,\mathbf{C}_{(\frac{n+1}{2})^4}$. Here's the rubric.

Rational functions in $\mathbb{Q}$ with branch points in $\mathbb{Q}$, of which (with no loss) we take three to be $\{0,1,\infty\}$ and the other as z'. So, we can write such an $f \stackrel{\mathrm{def}}{=} f_{x'}(x) : \mathbb{P}^1_x \to \mathbb{P}^1_z$ as $h_1(x)x^{\frac{n+1}{2}}/h_2(x)$ with h_1,h_2 of degree $\frac{n-1}{2}$. This automatically puts 0 (resp. ∞) as the ramified point over 0 (resp. ∞). The following equations encode the rest of the conditions at the branch points. These make $1 \in \mathbb{P}^1_x$ the ramified point over $z=1$, to determine $f \stackrel{\text{def}}{=} f_{x'}(x)$ with x' the ramified point over z':

$$\begin{aligned} h_1(x)x^{\frac{n+1}{2}} - h_2(x) &= (x-1)^{\frac{n+1}{2}} m_1(x) \\ h_1(x)x^{\frac{n+1}{2}} - z'h_2(x) &= (x-x')^{\frac{n+1}{2}} m_2(x). \end{aligned} \tag{6.3}$$

We can solve for h_1, h_2 as a function of m_1 and m_2:

$$\begin{aligned} &(a) & (z'-1)h_2 &= (x-1)^{\frac{n+1}{2}} m_1 - (x-x')^{\frac{n+1}{2}} m_2 \\ &(b) & (z'-1)x^{\frac{n+1}{2}} h_1 &= z'(x-1)^{\frac{n+1}{2}} m_1 - (x-x')^{\frac{n+1}{2}} m_2. \end{aligned} \tag{6.4}$$

So, we want coefficients (total of $n+1$ coefficients) on the degree $\frac{n-1}{2}$ polynomials m_1, m_2 polynomials so that h_1 and h_2 both have degree $\frac{n-1}{2}$, simultaneously figuring x' as a function of z'.

Proposition 6.3. *As x' varies in $\mathbb{P}^1_x \setminus \{0,1,\infty\}$, we run over the connected set of $f_{x'}(x)$ in the Nielsen class of covers with ordered branch points by solving the equations* (6.4) *according to the stipulations above. For x' lying in a field K, the solution for $f_{x'}(x)$ has coefficients also lying in K.*

Consider the cover $\Psi_n^{\text{in,abs}} : \mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{in}} \to \mathcal{H}(A_n, \mathbf{C}_{(\frac{n+1}{2})^4})^{\text{abs}}$. The top space has one (resp. 2) components, defined over $\mathbb{Q}$ (resp. the unique quadratic extension K_n of $\mathbb{Q}$ in $\mathbb{Q}(e^{\frac{n+1}{2}})$) when $n \equiv 5 \mod 8$ (resp. $n \equiv 1 \mod 8$).

Proof. What we actually need to know, as x' runs over $\mathbb{Q}$ is is that the discriminant of the cover $f_{x'} : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is not locally a square in $\mathbb{Q}\{\{x'\}$. We will show for $n \equiv 5 \mod 8$, it has expression in the square root of x', while for $n \equiv 1 \mod 8$ you must extend the constants by

Expand the zeros of $f_{x'}(x) = z$ about z'. □

6.5. Pure-cycle cases of non-genus zero covers. When $r=4$, the reduced Hurwitz space of a pure-cycle Nielsen class has a birational embedding in $\mathbb{P}^1_j \times \mathbb{P}^1_j$. It doesn't matter if the covers in the family have genus 0 or not. To see that consider such a cover $\varphi : X \to \mathbb{P}^1_z$. Then, map the four branch points $\varphi_{\boldsymbol{z}}$ to their j invariant $j_{\varphi_{\boldsymbol{z}}}$. Above each branch point z_i is a unique ramified point x_i. So, that gives the j invariant of $\boldsymbol{x}$, which we denote $j_{\varphi_{\boldsymbol{x}}}$. The birational embedding is $\varphi \mapsto (j_{\varphi_{\boldsymbol{z}}}, j_{\varphi_{\boldsymbol{x}}})$. Notice this also holds for modular curves. There is a common reason for both cases, though they do differ.

Lemma 6.4. *Suppose $r=4$, and $\mathbf{C}$ has the property that each conjugacy class is represented by elements with a disjoint cycle of distinguished length, and also the gcd of all cycle lengths in the conjugacy class is 1. Then, the reduced space embeds in $\mathbb{P}^1_j \times \mathbb{P}^1_j$. This applies to the modular curves $X_0(p)$ because they are the Nielsen class of $(D_p, \mathbf{C}_{2^4})$, and the conjugacy class of multiplication on $\mathbb{Z}/p$ fixes just 0. Why doesn't this work for $(D_{p^{k+1}}, \mathbf{C}_{2^4})$?*

6.5.1. *Start of the* **MT** *for* $(A_4, \mathbf{C}_{\pm 3^2}, p=2)$. Here there is only the prime 2 to consider. This is the "easiest" case of pure-cycle covers of genus exceeding 0. [**Fr06a**, Prop. 6.12] considers this case to show that both level 0 components of the reduced absolute spaces are nonmodular curves, despite —like modular curves —that they embed in $\mathbb{P}^1_j \times \mathbb{P}^1_j$ just as do modular curves.

Cusp representatives —1st 3 for Ni^+, 2nd 3 for Ni^- —of the various cusp orbits are in this list using the corresponding subscripts.

- $\boldsymbol{g}_{1,1} = ((1\,2\,3), (1\,3\,2), (1\,3\,4), (1\,4\,3))$
- $\boldsymbol{g}_{1,3} = ((1\,2\,3), (1\,2\,4), (1\,4\,2), (1\,3\,2))$

TABLE 6. **sh**-Incidence Matrix for Ni_0^+

Orbit	$O_{1,1}$	$O_{1,3}$	$O_{3,1}$
$O_{1,1}$	1	1	2
$O_{1,3}$	1	0	1
$O_{3,1}$	2	1	0

TABLE 7. **sh**-Incidence Matrix for Ni_0^-

Orbit	$O_{1,4}$	$O_{3,4}$	$O_{3,5}$
$O_{1,4}$	2	1	1
$O_{3,4}$	1	0	0
$O_{3,5}$	1	0	0

- $\boldsymbol{g}_{3,1} = ((1\,2\,3), (1\,3\,2), (1\,4\,3), (1\,3\,4))$
- $\boldsymbol{g}_{1,4} = ((1\,2\,3), (1\,2\,4), (1\,2\,3), (1\,2\,4))$
- $\boldsymbol{g}_{3,4} = ((1\,2\,3), (1\,2\,4), (1\,2\,4), (4\,3\,2))$
- $\boldsymbol{g}_{3,5} = ((1\,2\,3), (1\,2\,4), (1\,4\,3), (2\,3\,1))$

Some comments: $\boldsymbol{g}_{1,1}$ is an H-M rep, and a 2 cusp, while $\boldsymbol{g}_{1,3}$ is the shift of an H-M rep. On the other hand, the cusp orbit of $\boldsymbol{g}_{3,5}$ has length three by Princ. **??**. From Princ. 3.3 we know immediately that the Main Conjecture holds for any H-M cusp branch. Here, however, is a harder question.

QUESTION 6.5. FP 3 says there is at least one H-M component branch defining a **MT** for $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$. Does the Main Conjecture hold for every component branch?

Not much of a question if there is only one component branch, or slightly worse there are several component branches, all H-M. Neither of these, however, holds.

6.5.2. *Level 1 of* **MT***s for* $(A_4, \mathbf{C}_{\pm 3^2}, p = 2)$.

7. Connectedness Applications

7.1. Two problems: Davenport's and Genus 0.

7.2. Fried-Voelklein. Whatever is $N_{G,p} = |\ker(R^*_{G,p} \to G)|$, then the braid orbits on $\mathrm{Ni}(G, \mathbf{C})$ with $\mathbf{C}$ a collection of p' conjugacy classes realizing giving lifting invariants can be as large as $N_{G,p}$. That certainly happens if the conjugacy classes in $\mathbf{C}$ are repeated sufficiently often. The following example appears again in §6.5.1. It shows the .

Note the many uses of H-Mreps as in [**FV91**] or [**?**].

7.3. STC. Divide responding to the following two — from (**??**) — into two parts.

(7.1a) Given $\mathrm{Ni}(G, \mathbf{C}, p)$, when does it support a **MT** defined over a number field K?

(7.1b) When $r \geq 5$, what relations can we expect among the two **MT** conjectures and the STC.

(2.3c) is a grand use of the B(ranch) C(ycle) L(emma) for which the following is a summary: Gist of [**FK97**, Thm. 4.4] and [**D06**].

Start with any number r_0, and consider finding $\mathbb{Q}$ regular realizations of all p-Frattini extensions of your favorite p-perfect G using no more than r_0 branch points, without any declaration as to the kind of conjugacy classes.

Example: With $p = 5$, find such regular realizations of all the dihedral groups $\{D_{5^{k+1}}\}_{k=0}^{\infty}$ with no more than three trillion branch points. The general conclusion is there are ($\le r_0$) p' conjugacy classes of G for which there is a (nonempty) **MT** whose levels have points corresponding to those realizations. (For the dihedral case that means that you can take the conjugacy classes in the regular realizations to be no more than r_0 repetitions of the involution class.) That restates (2.3d). Further, all of whose levels have a $\mathbb{Q}$ point. The Main Conjecture is that it is not possible that all levels of a **MT** could have points over a given number field K.

Appendix A. Classical Generators of $\pi_1(\mathbb{P}^1_z \setminus \boldsymbol{z}^0, z_0)$

Start with R(iemann)-H(urwitz) to compute the genus of an elements in a Nielsen class. Put here also the minimal points on what is a Hurwitz space, and the distinction between inner and absolute classes, and the reduced versions of each.

Appendix B. Classification of cusps

Appendix C. A p' moduli argument

Finish the argument of Prop. 2.8 about non-degree 1 when there is a non-trivial p' center. [**Fr06a**, Rem. 3.4] notes the universal p-Frattini cover ${}_p\tilde{G}$ of G identifies with the fiber product over G/Z of G and the universal p-Frattini cover of G/Z. Thus, Z is the center of ${}_p\tilde{G}$. Conclude: The (one-one) image of Z by the map ${}_p\tilde{G} \to G_k$ then identifies with the center of G_k.

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