SIMULTANEOUS SURFACE RESOLUTION IN CYCLIC GALOIS EXTENSIONS

SHREERAM S. ABHYANKAR AND NAN GU

Abstract. We show that simultaneous surface resolution is not always possible in a cyclic extension whose degree is greater than three and is not divisible by the characteristic. This answers a recent question of Ted Chinburg.

Section 1: Introduction

Let K be a two dimensional algebraic function field over an algebraically closed ground field k. Recall that K/k has a minimal model means that amongst all the nonsingular projective models of K/k there is one which is dominated by all others (basic reference [?] or [?]). Also recall that K/k has a minimal model if and only if it is not a ruled function field, i.e., K is not a simple transcendental field extension of a one dimensional algebraic function field over k (see [?]). A finite algebraic field extension L/K is said to have a simultaneous resolution if there exist nonsingular projective models V and W of K/k and L/k, respectively, such that W is the normalization of V in L. Given any positive integer q which is not divisible by the characteristic char(K) of K and letting Z_q denote a cyclic group of order q, in [?] it was shown that if $q \leq 3$ and L/K is a Z_q extension, i.e., a Galois extension whose Galois group is a cyclic group of order q, then it has a simultaneous resolution, whereas if K/k has a minimal model and $q > 3$ with q being a prime number, then there exists a Z_q extension L/K which has no simultaneous resolution. Here we shall extend this second result to those nonprimes q which are divisible by the square of some prime p. By taking $q = 4$, this answers a question raised by Ted Chinburg at the March 2006 AMS Meeting in New Hampshire to the effect whether every Z_2 by Z_2 extension L/K, i.e., a Z_2 extension L/J of a Z_2 extension J/K, has a simultaneous resolution. By using a Theorem of David Harbater and Florian Pop, we generalize our extended result by replacing Z_q by its direct sum $H \oplus Z_q$ with any finite group H. For related matter see [?].

In Lemma (2.2) of Section 2 we shall give a consequence of the Harbater-Pop Theorem to be used in proving our generalized extended result. In Lemma (2.1) of Section 2 we shall summarize some technical results form our previous papers [?] and [?]. These technical results deal with the structure of the integral closure of a normal noetherian domain in a cyclic extension. They are used in the proof of Theorem (3.1) of Section 3 which gives a sufficient condition for a two dimensional local domain to be nonregular. Theorem (3.1) is used in proving the special case of Theorem (3.2) of Section 3 which corresponds to our extended result, i.e., the $H = 1$ case of our generalized extended result. The general case of Theorem (3.2), which corresponds to our generalized extended result, then follows by using Lemma (2.2).

1991 *Mathematics Subject Classification.* Primary 14A05.

Section 2: Two Lemmas

Let $M(R)$ denote maximal ideal of a local ring R. In Lemma (2.1) we summarize some properties of the integral closure of a normal noetherian domain in a cyclic extension. In Lemma (2.2) we give a consequence of the Harbater-Pop Theorem.

LEMMA (2.1). Let R be a normal noetherian domain with quotient field K, let S be the integral closure of R in a finite algebraic field extension L of K, and let $[L : K] = q$. Assume that q is a unit in R, and L contains a nonzero element z such that $L = K(z)$ and

$$z^q = u \prod_{j=1}^{d} x_j^{a(j)}$$

where u is a unit in R, d is a nonnegative integer, $a(j)$ is an integer such that $\mathrm{GCD}(a(j), q) = 1$ for $1 \le j \le d$, and $x_1, \ldots, x_d$ are elements in R such that $x_1 R, \ldots, x_d R$ are pairwise distinct minimal (= height one) prime ideals in R. Let $b(i,j)$ and $c(i,j)$ be the unique integers such that

$$b(i,j) = a(j)i + c(i,j)q \quad \text{and} \quad 0 \le b_(i,j) < q.$$

Let

$$z_i = z^i \prod_{j=1}^{d} x_j^{c(i,j)}.$$

Then we have the following:

1. $(z_0, \cdots, z_{q-1})$ is a free R-basis of S.
2. If R is a local domain and $d \ge 1$, then S is a local domain and for its maximal ideal $M(S)$ we have $M(S) = M(R)S + (z_1, \cdots, z_{q-1})S$ with $S/M(S) = R/M(R)$.
3. If R is a regular local domain and $d \ge 2$ then S is a nonregular local domain.

PROOF. For (1) and (2) see Theorem 7 [**?**]. For (3) see Theorem 6 [**?**] with the observation that, although in the context of this theorem q is a prime number, the primeness of q was never used in its proof. A different version of (1) and (2) can also be found in Theorems 4 and 5 [**?**]; see Remark 2 on page 28 of [**?**].

LEMMA (2.2). Let K/k be a two dimensional algebraic function field over an algebraically closed ground field k. For any finite group H, there exists a Galois extension $\widetilde{L}/K$ with Galois group H.

PROOF. It follows from Theorem 4.4 [**?**] or the Corollary to Theorem A [**?**] that given any finite group H and any one dimensional algebraic function field E over an algebraically closed ground field k, there exists a Galois extension F/E whose Galois group is H. The following argument, provided by Harbater and Pop, shows how the desired two-variable existence follows from this.

Given a two dimensional algebraic function field K over k, choose a separating transcendence basis x, y for K over k. So K is a finite separable field extension of $k(x, y)$. Let E be the algebraic closure of $k(x)$ in K. E is finite over $k(x)$, since K is finite over $k(x, y)$ and since $k(x)$ is algebraically closed in $k(x, y)$. Thus E is a one dimensional algebraic function field over k and so, by the one-variable existence theorem, H is the Galois group of a finite extension F of E. Since E is algebraically closed in K and since F is algebraic over E, it follows that F and K are linearly disjoint over E. So the compositum $\widetilde{L} = KF$ (in an algebraic closure of K) is a Galois extension of K with Galois group H, completing the proof.

Section 3: Two Theorems

In Theorem (3.1) we give a sufficient condition for a local domain to be nonregular. In Theorem (3.2) we construct our examples of simultaneous nonresolvability.

THEOREM (3.1). Let R be a two dimensional regular local domain, let (X, Y) be generators of its maximal ideal $M(R)$, and let K be its quotient field. Let $R_0 = R$. For all $n > 0$, let $Y_n = Y/X^n$ and let R_n be the localization of the ring $R_{n-1}[Y_n]$ at the maximal ideal in it generated by (X, Y_n). Note that then R_n is a two dimension regular local domain with quotient field K such that R_n dominates R_{n-1} and (X, Y_n) are generators of $M(R_n)$.

Let q be a positive integer which is a unit in R. Assume that $q = pm$ where p is a prime number and m is a positive integer divisible by p. Assume that K contains q distinct q-th roots of 1. Let L be a splitting field over K of the polynomial of $Z^q - XY^m$. Let S_n be the integral closure of R_n in L.

Then L/K is a Z_q extension and for every nonnegative integer n, the ring S_n is a two dimensional nonregular local domain.

PROOF. Let w be the discrete valuation whose valuation ring is the one dimensional regular local domain obtained by localizing the ring R at the prime ideal in it generated by X. Then $w(XY^m) = 1$ and hence the polynomial $Z^q - XY^m$ is irreducible in $K[Z]$ and L/K is a Z_q extension. Let $z \in L$ be a root of the said polynomial. Then $z^q = XY^m$ and $L = K(z)$. Let $\overline{X} = z^p/Y$ and $J = K(\overline{X})$. Then $\overline{X}^m = X$ and hence J/K is a Z_m extension. By (2.1)(2) the integral closure T_n of R_n in J is a two dimensional regular local domain whose maximal ideal $M(T_n)$ is generated by $(\overline{X}, Y_n)$. Also $z^p = \overline{X}Y = \overline{X}^{1+nm}Y_n$ and, since m is assumed divisible by p, upon letting $\zeta = z/\overline{X}^{nm/p}$ we get $L = J(\zeta)$ with $\zeta^p = \overline{X}Y_n$. Now L/J is a Z_p extension with $L = J(\zeta)$, and S_n is the integral closure of T_n in L. Therefore by (2.1)(3) we see that S_n is a two dimensional nonregular local domain.

THEOREM (3.2). Let K/k be a two dimensional algebraic function field over an algebraically closed ground field k. Assume that K/k has a minimal model V^*. Let q be a positive integer which is not divisible by $\mathrm{char}(K)$. Assume that $q = pm$ where p is a prime number and m is a positive integer divisible by p. Then, given any finite group H, there exists a Galois extension L'/K with Galois group $H \oplus Z_q$ such that L'/K has no simultaneous resolution.

PROOF. By (2.2) there exists a Galois extension $\widetilde{L}/K$ with Galois group H. Take R in (3.1) to be the local ring of a point of V^* which is not ramified in $\widetilde{L}$. Let L' be a compositum of $\widetilde{L}$ and L. It is easy to see that L'/K is a Galois extension whose Galois group is $H \oplus Z_q$.

By [**?**] Lemma 12, there exists a unique valuation v of K dominating R_n for all $n \geq 0$. By construction each R_{n+1} is the immediate quadratic transform of R_n along v. Let $\widetilde{v}$ be an extension of v to $\widetilde{L}$. Let, if possible, V and W be nonsingular projective models of K/k and L'/k respectively, such that W is the normalization of V in L'. Then by the minimality of V^*, V must dominate V^*. Consequently by [**?**] Theorem 3 the local ring of the center P of v on V must equal R_n for some nonnegative integer n. Since R_n dominates R and R is not ramified in $\widetilde{L}$, R_n is not ramified in $\widetilde{L}$. Let $\widetilde{V}$ be the normalization of V in $\widetilde{L}$, and $\widetilde{R}_n$ be the local ring of the center $\widetilde{P}$ of $\widetilde{v}$ on $\widetilde{V}$, then $\widetilde{P}$ lies above P in $\widetilde{V}$ and $\widetilde{R}_n$ is two dimension regular local ring whose maximal ideal $M(\widetilde{R}_n)$ is generated by (X, Y_n). Now L' is a Z_q extension of $\widetilde{L}$ constructed from $\widetilde{L}$ in the same way as L is constructed from K in (3.1), and W is the normalization of $\widetilde{V}$ in L'. By (3.1), the point of W lying above $\widetilde{P}$ is not a simple point, which is a contradiction.

REMARK (3.3). The construction of a Z_q extension L/K having no simultaneous resolution does not use the results of Harbater and Pop. Their results plus the fact that a regular system of parameters lifts to a regular system of parameters through a unramified local ring extension allow us to mimic such construction to get a $H \oplus Z_q$ extension. Similar arguments will show that the statement of (3.2) remains true if $q > 3$ is a prime number; see [**?**] Theorem 11 for details.

References

[Ab1] S. S. Abhyankar, *On the valuations centered in a local domain,* American Journal of Mathematics, 78 (1956), 321-348.

[Ab2] S. S. Abhyankar, *Simultaneous resolution for algebraic surfaces,* American Journal of Mathematics, 78 (1956), 761-790.

[Ab3] S. S. Abhyankar, *Uniformization of Jungian local domains,* Mathematische Annalen, 159 (1965), 1-43.

[Ab4] S. S. Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces,* Springer Verlag (1998).

[Ab5] S. S. Abhyankar, *Lectures on Algebra I,* World Scientific, 2006.

[AbK] S. S. Abhyankar and M. Kumar, *Simultaneous surface resolution in quadratic and biquadratic Galois extensions,* Contemporary Mathematics, 390 (2005), 1-8.

[Har] D. Harbater, *Fundamental groups and embedding problems in characteristic p,* Contemporary Mathematics, 186 (1995), 353-369.

[Pop] F. Pop, *Étale Galois covers of affine smooth curves,* Invent. Math., 120 (1995), 555-578.

[Zar] O. Zariski, *The problem of minimal models in the theory of algebraic surfaces,* American Journal of Mathematics, 80 (1958), 146-184.

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907
E-mail address: ram@cs.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907
E-mail address: ngu@math.purdue.edu