CLASS FIELD THEORY AND MODULAR TOWERS

TH.WEIGEL

Let $\phi\colon \mathcal{H}(\underline{m}) \to G$ be a surjective homomorphism of the Fuchsian group $\mathcal{H}(\underline{m})$ of signature $\underline{m}\colon = (m_1, \ldots, m_r)$ onto the finite group G, let p be prime number coprime to m_i, $i = 1, \ldots, r$, and let $\pi\colon \tilde{G} \to G$ denote the universal p-abelian Frattini cover of G, i.e., $\tilde{G}$ is the maximal Frattini extension of G which kernel is an abelian pro-p group.

A very delicate problem in the theory of modular towers is to decide whether there exists a mapping $\psi\colon \mathcal{H}(\underline{m}) \to \tilde{G}$ making the diagram

(1)
$$\begin{tikzcd} & \tilde{G} \arrow[d, "\pi"] \\ \mathcal{H}(\underline{m}) \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"] & G \end{tikzcd}$$

commute or not. For easier version of this type of embedding problem, M.Fried's small lifting invariant ([1, Part II]) together with [3, Prop.3.2, (3.6)] gives a very satisfactory answer. Using abstract class field theory (cf. [2]) we will show the following theorem generalizing the just mentioned results:

Theorem A. *The mapping ϕ defines a canonical class $c_\phi \in H_2(G, \mathbb{Z}_p)$. Moreover, the embedding problem* (1) *has a solution, if and only if $c_\phi = 0$.*

Corollary B. *Assume that the p-Sylow subgroups of G are either cyclic or quaternion ($p = 2$). Then the embedding problem* (1) *has a solution.*

Although class field theory is a very powerful tool in number theory, it has not been used very often in other contexts. For our purpose we will establish the following theorem supporting the idea that one can expect more application of this theory:

Theorem C. *Every profinite group $\hat{G}$ has a canonical class field theory $(\mathbf{C}, \eta)$.*

The profinite completion $\hat{\mathcal{H}}(\underline{m})$ of $\mathcal{H}(\underline{m})$ is an orientable p-Poincaré duality group of dimension 2. The canonical p-class field theory for this type of groups has some particular features. In particular, it will turn out that the class c_ϕ of Theorem A coincides with the Heller translate of the embedding of the universal norms into the class field module.

References

[1] M.D. Fried, *Introduction to modular towers: generalizing dihedral group–modular curve connections*, Recent developments in the inverse Galois problem (Seattle, WA, 1993), Contemp. Math., vol. 186, Amer. Math. Soc., Providence, RI, 1995, pp. 111–171.

[2] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften, vol. 323, Springer-Verlag, Berlin, 2000.

[3] Th. Weigel, *Maximal l-Frattini quotients of l-Poincaré duality groups of dimension 2*, Arch. Math. (Basel) **85** (2005), no. 1, 55–69.