Absolute anabelian cuspidalizations of configuration spaces over finite fields

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Abstract

In the present paper, we study the cuspidalization problem for fundamental groups of configuration spaces of proper hyperbolic curves over finite fields. The goal of this paper is to show that any Frobenius-preserving isomorphism of the geometrically pro-l fundamental groups of hyperbolic curves induces an isomorphism of the geometrically pro-l fundamental groups of the associated configuration spaces.

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0 Introduction

In this paper, we study the following problem, which is called the "cuspidalization problem" (cf. [7], Problem 0.2):

Problem 0.1. *Let r be a positive integer. Then can one reconstruct the* (*arithmetic*) *fundamental group*

$$\pi_1(U_{X_{(r)}})$$

of the r-th configuration space $U_{X_{(r)}}$ of a hyperbolic curve X over a field K (*i.e., the open subscheme of the r-th product of X [over K] whose complement consists of the diagonals "$\{(x_1, \cdots, x_r) \mid x_i = x_j\}$" $[i \neq j]$ from the* (*arithmetic*) *fundamental group $\pi_1(X)$ of X?*

Let r be a positive integer, X a proper hyperbolic curve over a finite field K, and l a prime number that is invertible in K. We shall denote by $\Pi_{X_{(r)}}$ (respectively, $\Pi_{X_{\underline{(r)}}}$) the geometrically pro-$l$ fundamental group of the r-th configuration space $U_{X_{(r)}}$ of X (respectively, the fiber product $\overbrace{X \times_K \cdots \times_K X}^{r}$ of r copies of X over K), i.e., the quotient of $\pi_1(U_{X_{(r)}})$ (respectively, $\pi_1(\overbrace{X \times_K \cdots \times_K X}^{r})$) by the closed normal subgroup obtained as the kernel of the natural projection from $\pi_1(U_{X_{(r)}} \otimes_K \overline{K})$ (respectively, $\pi_1((\overbrace{X \times_K \cdots \times_K X}^{r}) \otimes_K \overline{K})$) to its maximal pro-$l$ quotient, and by $p_{U_{X_{(r-1)}}:i} : U_{X_{(r)}} \to U_{X_{(r-1)}}$ the projection obtained by forgetting the i-th factor ($i = 1, \cdots, r$). Let Y be a proper hyperbolic curve over a finite field L in which l is invertible; moreover, we shall use similar notations for Y. Then the main result of this paper is as follows (cf. Theorem 4.32):

Theorem 0.2. *Let*

$$\alpha_{(1)} : \Pi_X \stackrel{\text{def}}{=} \Pi_{X_{(1)}} \xrightarrow{\sim} \Pi_Y \stackrel{\text{def}}{=} \Pi_{Y_{(1)}}$$

be a Frobenius-preserving isomorphism (cf. Definition 2.11). Then, for any positive integer r, there exists a unique isomorphism

$$\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}} ,$$

well-defined up to composition with a cuspidally inner automorphism (i.e., a $\mathrm{Ker}\,(\Pi_{Y_{(r)}} \twoheadrightarrow \Pi_{Y_{\underline{(r)}}})$*-inner automorphism), which is compatible with the natural respective actions of the symmetric group on r letters such that, for $i = 1, \cdots, r+1$, the following diagram commutes:*

$$\begin{array}{ccc}
\Pi_{X_{(r+1)}} & \xrightarrow{\alpha_{(r+1)}} & \Pi_{Y_{(r+1)}} \\
{\scriptstyle \text{via } p_{U_{X_{(r)}}:i}} \downarrow & & \downarrow {\scriptstyle \text{via } p_{U_{Y_{(r)}}:i}} \\
\Pi_{X_{(r)}} & \xrightarrow[\alpha_{(r)}]{} & \Pi_{Y_{(r)}} .
\end{array}$$

Note that Theorem 0.2 is a generalization of [14], Theorem 3.10. (In [14], Theorem 3.10, the case where $r = 2$ is proven.)

An essential part of the proof of this main theorem is to show that the profinite group $\Pi_{X_{(r+1)}}$ can be reconstructed from $\Pi_{X_{(r)}}$ "group-theoretically". This "group-theoretic" reconstruction of the profinite group $\Pi_{X_{(r+1)}}$ from the given profinite group $\Pi_{X_{(r)}}$ is performed as follows: Let $X^{\log}_{(r)}$ be the r-th log configuration space of X (cf. [7], Definition 1.1). Then the interior of $X^{\log}_{(r)}$ is naturally isomorphic to the (usual) r-th configuration space $U_{X_{(r)}}$ of X; moreover, it follows from the log purity theorem that the natural open immersion $U_{X_{(r)}} \hookrightarrow X^{\log}_{(r)}$ induces an isomorphism of the geometrically pro-l fundamental group $\Pi_{X_{(r)}}$ of $U_{X_{(r)}}$ with the geometrically pro-l log fundamental group of $X^{\log}_{(r)}$. Therefore, to reconstruct $\Pi_{X_{(r+1)}}$, it is enough to reconstruct the geometrically pro-l log fundamental group of $X^{\log}_{(r+1)}$. Now it follows from a similar argument to the argument used in the proof of [7], Theorem 2.5, that the images of the geometrically pro-l log fundamental groups of certain irreducible components (equipped with the log structures induced by the log structure of $X^{\log}_{(r+1)}$) of the divisor at infinity of the log scheme $X^{\log}_{(r+1)}$ topologically generate the desired profinite group $\Pi_{X_{(r+1)}}$. On the other hand, there exists a topological group $\Pi^{\mathrm{Lie}}_{X_{(r+1)}}$ *which arises from the pro-graded Lie algebra obtained by considering the weight filtration of the pro-l fundamental group* $\Delta_{X_{(r+1)}}$ *of* $U_{X_{(r+1)}} \otimes_K \overline{K}$ such that the desired profinite group $\Pi_{X_{(r+1)}}$ is naturally embedded in $\Pi^{\mathrm{Lie}}_{X_{(r+1)}}$; moreover, this topological group $\Pi^{\mathrm{Lie}}_{X_{(r+1)}}$ can be reconstructed "group-theoretically" from the given profinite group $\Pi_{X_{(r)}}$ by considering the *Galois invariant splitting* of the subquotients of $\Delta_{X_{(r+1)}}$ with respect to the weight filtration. Therefore, if one can *reconstruct* "group-theoretically" the natural images in $\Pi^{\mathrm{Lie}}_{X_{(r+1)}}$ of the geometrically pro-l log fundamental groups of certain irreducible components (equipped with the log structures) of the divisor at infinity of the log scheme $X^{\log}_{(r+1)}$, then one can construct a subgroup *which is isomorphic to the desired profinite group* $\Pi_{X_{(r+1)}}$ as the subgroup which is topologically generated by the images *reconstructed*.

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Notations and Terminologies:

Numbers:

We shall denote by $\mathfrak{Prime}$ the set of all prime numbers, by $\mathbb{N}$ the monoid of rational integers $n \geq 0$, by $\mathbb{Z}$ the ring of rational integers, by $\mathbb{Q}$ the field of rational numbers, by $\widehat{\mathbb{Z}}$ (respectively, $\mathbb{Z}_l$) the profinite completion of $\mathbb{Z}$ (respectively, pro-l completion of $\mathbb{Z}$ for a prime number l), and by $\mathbb{Q}_l$ the field of fractions of $\mathbb{Z}_l$.

Let Σ be a set of prime numbers, and n an integer. Then we shall say that n is a Σ-*integer* if the prime divisors of n are in Σ.

Groups:

Let G be a profinite group and Σ a (non-empty) set of prime numbers. We shall refer to the quotient

$$\varprojlim G/H$$

of G (where the projective limit is over all open normal subgroups $H \subseteq G$ such that the order $[G:H]$ of H is a Σ-integer) as the *maximal pro-*Σ *quotient of* G. We shall denote by $G^{(\Sigma)}$ the maximal pro-Σ quotient of G.

For a topological group G, we shall denote by G^{ab} the abelianization of G, i.e., the quotient of G by the closed normal subgroup $[G,G]$ generated by the commutators of G.

For a Hausdorff topological group G, we shall denote by $\mathrm{Aut}(G)$ the group of continuous automorphisms, and by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the subgroup $\mathrm{Inn}(G)$ of inner automorphisms of G.

Let G be a center-free Hausdorff topological group and H a topological group. Then there exists a natural exact sequence:

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1$$

(where $G \to \mathrm{Aut}(G)$ is defined by letting G act on G by conjugation). For a continuous homomorphism $H \to \mathrm{Out}(G)$, we shall denote by

$$G \overset{\mathrm{out}}{\rtimes} H$$

the group obtained by pulling-back the above exact sequence via the continuous homomorphism $H \to \mathrm{Out}(G)$, i.e.,

$$G \overset{\mathrm{out}}{\rtimes} H \overset{\mathrm{def}}{=} \mathrm{Aut}(G) \times_{\mathrm{Out}(G)} H\,.$$

Note that it is immediate that $G \overset{\mathrm{out}}{\rtimes} H$ fits into the following natural exact sequence:

$$1 \longrightarrow G \longrightarrow G \overset{\mathrm{out}}{\rtimes} H \longrightarrow H \longrightarrow 1\,.$$

Note that if G is *topologically finitely generated*, then by considering a basis of the topology of G consisting of characteristic open subgroups of G, we may regard $\mathrm{Aut}(G)$ as being equipped with a topology. This topology on $\mathrm{Aut}(G)$ induces a topology on $\mathrm{Out}(G)$, hence also a topology on $G \overset{\mathrm{out}}{\rtimes} H$.

Log schemes:

Let $\mathcal{P}$ be a property of schemes [for example, "quasi-compact", "connected", "normal", "regular"] (respectively, morphisms of schemes [for example, "proper", "finite", "étale", "smooth"]). Then we shall say that a log scheme (respectively, a morphism of log schemes) satisfies $\mathcal{P}$ if the underlying scheme (respectively, the underlying morphism of schemes) satisfies $\mathcal{P}$.

For a log scheme $X^{\log}$ (respectively, a morphism $f^{\log}$ of log schemes), we shall denote by X the underlying scheme (respectively, by f the underlying morphism of schemes). For fs log schemes $X^{\log}$, $Y^{\log}$, and $Z^{\log}$, we shall denote by $X^{\log} \times_{Y^{\log}} Z^{\log}$ the fiber product of $X^{\log}$ and $Z^{\log}$ over $Y^{\log}$ in the category of *fs log schemes*. In general, the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is *not naturally isomorphic* to $X \times_Y Z$. However, since strictness (a morphism $f^{\log} : X^{\log} \to Y^{\log}$ of log schemes is called *strict* if the induced morphism on the sheaves of monoids determining the log schemes is an isomorphism) is stable under base-change in the category of *arbitrary* log schemes, if $X^{\log} \to Y^{\log}$ is strict, then the underlying scheme of $X^{\log} \times_{Y^{\log}} Z^{\log}$ is naturally isomorphic to $X \times_Y Z$.

If there exist both schemes and log schemes in a commutative diagram, then we regard each scheme in the diagram as the log scheme obtained by equipping the scheme with the trivial log structure.

We shall refer to the largest open subset (possibly empty) of the underlying scheme of a log scheme on which the log structure is trivial as the *interior* of the log scheme.

Let $X^{\log}$ be a log scheme, and $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ the log structure of $X^{\log}$. Then we shall refer to the quotient $\mathcal{M}_X/\alpha^{-1}(\mathcal{O}_X^*)$ of $\mathcal{M}_X$ as the *characteristic sheaf* of $X^{\log}$.

Curves:

Let $f : X \to S$ be a morphism of schemes. Then we shall say that f is a *curve* if f is a smooth, geometrically connected morphism whose geometric fibers are one-dimensional. Moreover, we shall say that f is a *hyperbolic curve* (respectively, *tripod*) if there exist a proper curve $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ whose geometric fibers are of genus g and a relative divisor $D \subseteq X^{\mathrm{cpt}}$ which is finite étale over S of relative degree r such that X and $X^{\mathrm{cpt}} \setminus D$ are isomorphic over S, and (g, r) satisfies $2g - 2 + r > 0$ (respectively, $(g, r) = (0, 3)$).

We shall denote by $\overline{\mathcal{M}}_{g,r}$ the moduli stack of r-pointed stable curves of genus g whose r sections are equipped with an ordering (cf. [9]), and by $\overline{\mathcal{M}}_{g,r}^{\log}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings which parametrizes singular curves. Moreover, we shall write $\overline{\mathcal{M}}_g \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{g,0}$ and $\overline{\mathcal{M}}_g^{\log} \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{g,0}^{\log}$.

Fundamental groups:

For a locally noetherian, connected scheme X (respectively, log scheme $X^{\log}$) equipped with a geometric point $\overline{x} \to X$ (respectively, log geometric point $\widetilde{x}^{\log} \to X^{\log}$), we shall denote by $\pi_1(X, \overline{x})$ (respectively, $\pi_1(X^{\log}, \widetilde{x}^{\log})$) the fundamental group of X (respectively, log fundamental group of $X^{\log}$). Since one knows that the fundamental group is determined up to inner automorphisms independently of the choice of base-point, we shall often omit the base-point, i.e., we shall often denote by $\pi_1(X)$ (respectively, $\pi_1(X^{\log})$) the fundamental group of X (respectively, log fundamental group of $X^{\log}$).

For a set Σ of prime numbers and a locally noetherian, connected scheme X (respectively, log scheme $X^{\log}$), we shall refer to the maximal pro-Σ quotient of $\pi_1(X)$ (respectively, $\pi_1(X^{\log})$) as the *pro-Σ fundamental group of* X (respectively, *pro-Σ log fundamental group* of $X^{\log}$). Moreover, for a geometrically connected scheme X (respectively, log scheme $X^{\log}$) which is locally of finite type over a field K, we shall refer to the quotient of $\pi_1(X)$ (respectively, $\pi_1(X^{\log})$) by the closed normal subgroup obtained as the kernel of the natural projection from $\pi_1(X \otimes_K K^{\mathrm{sep}})$ (respectively, $\pi_1(X^{\log} \otimes_K K^{\mathrm{sep}})$) (where K^{sep} is a separable closure of K) to its maximal pro-Σ quotient $\pi_1(X \otimes_K K^{\mathrm{sep}})^{(\Sigma)}$ (respectively, $\pi_1(X^{\log} \otimes_K K^{\mathrm{sep}})^{(\Sigma)}$) as the *geometrically pro-Σ fundamental group* of X (respectively, *geometrically pro-Σ log fundamental group* of $X^{\log}$). Thus, the geometrically pro-Σ fundamental group $\pi_1(X)^{\underline{(\Sigma)}}$ of X (respectively, geometrically pro-Σ log fundamental group $\pi_1(X^{\log})^{\underline{(\Sigma)}}$ of $X^{\log}$) fits into the following exact sequence:

$$1 \longrightarrow \pi_1(X \otimes_K K^{\mathrm{sep}})^{(\Sigma)} \longrightarrow \pi_1(X)^{\underline{(\Sigma)}} \longrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow 1$$

(respectively,

$$1 \longrightarrow \pi_1(X^{\log} \otimes_K K^{\mathrm{sep}})^{(\Sigma)} \longrightarrow \pi_1(X^{\log})^{\underline{(\Sigma)}} \longrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow 1).$$

1 Exactness properties of the graded Lie algebras arising from a family of curves

In this section, we consider some exactness properties of graded Lie algebras arising from a family of curves.

Definition 1.1. Let l be a prime number, G, H, and A topologically finitely generated pro-l groups, and $\phi : H \twoheadrightarrow A$ a (continuous) surjective homomorphism. Suppose further that A is abelian, and that G is an l-adic Lie group. Then (cf. [14], Definition 3.1):

(i) We shall refer to the central filtration

$$\{H(n)\} \ \ (n \geq 1)$$

of H defined as

$$H(1) \stackrel{\text{def}}{=} H\,;$$

$$H(2) \stackrel{\text{def}}{=} \operatorname{Ker}\phi\,;$$

$$H(m) \stackrel{\text{def}}{=} \langle [H(m_1), H(m_2)] \mid m_1 + m_2 = m \rangle \ \text{ for } \ m \geq 3$$

(where $\langle N_i \mid i \in I \rangle$ is the group topologically generated by the N_i $[i \in I]$) as the *central filtration with respect to the surjection* ϕ.

Let a, b, $n \in \mathbb{Z}$ such that $1 \leq a \leq b$, $n \geq 1$; we shall write

$$H(a/b) \stackrel{\text{def}}{=} H(a)/H(b)\,;$$

$$\operatorname{Gr}(H)(n) \stackrel{\text{def}}{=} \bigoplus_{m \geq n} H(m/m+1)\,;$$

$$\operatorname{Gr}(H) \stackrel{\text{def}}{=} \operatorname{Gr}(H)(1)\,;$$

$$\operatorname{Gr}(H)(a/b) \stackrel{\text{def}}{=} \operatorname{Gr}(H)(a)/\operatorname{Gr}(H)(b)\,;$$

$$H(a/\infty) \stackrel{\text{def}}{=} \varprojlim H(a/b)$$

(where the projective limit is over all integers $b \geq a+1$).

(ii) We shall denote by $\operatorname{Lie}(G)$ the Lie algebra over $\mathbb{Q}_l$ determined by the l-adic Lie group G. We shall say that G is *nilpotent* if there exists a positive integer m such that if we denote by $\{G(n)\}$ the central filtration with respect to the natural surjection $G \twoheadrightarrow G^{\text{ab}}$ (cf. (i)), then $G(m) = \{1\}$. If G is nilpotent, then $\operatorname{Lie}(G)$ is a nilpotent Lie algebra over $\mathbb{Q}_l$, hence determines a connected, unipotent linear algebraic group $\operatorname{Lin}(G)$, which we shall refer to as the *linear algebraic group associated to* G. In this situation, there is a natural (continuous) homomorphism (with open image)

$$G \longrightarrow \operatorname{Lin}(G)(\mathbb{Q}_l)$$

which is determined by the condition that it induces the identity morphism on the associated Lie algebras (cf. [14], Remark 3.3.2). In the situation of (i), if $1 \leq a \in \mathbb{Z}$, then we shall write

$$\operatorname{Lie}(H(a/\infty)) \stackrel{\text{def}}{=} \varprojlim \operatorname{Lie}(H(a/b)) \ \ ; \ \ \operatorname{Lin}(H(a/\infty)) \stackrel{\text{def}}{=} \varprojlim \operatorname{Lin}(H(a/b))$$

(where the projective limit is over all integers $b \geq a+1$). (Note that each $H(a/b)$ is an l-adic Lie group.)

Let K be a separably closed field, and l a prime number that is invertible in K. Let S be a connected locally noetherian *normal* scheme over K. Let $g \geq 2$ and r be natural numbers. Let $f : X \to S$ be a *hyperbolic curve of type* (g, r) (i.e., there exists a proper, smooth, geometrically connected morphism $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ whose geometric fibers are curves of genera g such that f factors as the composite $X \hookrightarrow X^{\mathrm{cpt}} \xrightarrow{f^{\mathrm{cpt}}} S$ of an open immersion $X \hookrightarrow X^{\mathrm{cpt}}$ onto the complement $X^{\mathrm{cpt}} \setminus D$ of a relative divisor D which is finite étale over S of relative degree r, and (g, r) satisfies $2g - 2 + r > 0$). We shall denote by

$$\pi_1(X)^{\underline{(l)}}$$

the geometrically pro-l fundamental group of X.

Lemma 1.2. *Let $\overline{s} \to S$ be a geometric point of S. Then the homomorphism $\pi_1(X)^{\underline{(l)}} \to \pi_1(S)$ induced by f fits into an exact sequence:*

$$1 \longrightarrow \pi_1(X \times_S \overline{s})^{(l)} \overset{\text{via } \pi_1(\mathrm{pr}_1)}{\longrightarrow} \pi_1(X)^{\underline{(l)}} \overset{\text{via } \pi_1(f)}{\longrightarrow} \pi_1(S) \longrightarrow 1\,.$$

Proof. If the finite étale covering $D = X^{\mathrm{cpt}} \setminus X \to S$ is empty or trivial (i.e., D is a disjoint union of copies of S, and the covering $D \to S$ is induced by the identity morphism of S), then this follows from [20], Proposition 2.3. In general, let $S' \to S$ be a connected finite étale covering of S such that $D \times_S S' \to S'$ is trivial, then we obtain a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X \times_S \overline{s})^{(l)} \arrow[r] \arrow[d, equal] & \pi_1(X \times_S S')^{\underline{(l)}} \arrow[r, "\text{via } \pi_1(\mathrm{pr}_2)"] \arrow[d] & \pi_1(S') \arrow[r] \arrow[d] & 1 \\
 & \pi_1(X \times_S \overline{s})^{(l)} \arrow[r, "\text{via } \pi_1(\mathrm{pr}_1)"'] & \pi_1(X)^{\underline{(l)}} \arrow[r, "\text{via } \pi_1(f)"'] & \pi_1(S) \arrow[r] & 1\,,
\end{tikzcd}$$

where the horizontal sequences are exact, and the vertical arrows are injective. Thus, $\pi_1(X \times_S \overline{s})^{(l)} \overset{\text{via } \pi_1(\mathrm{pr}_1)}{\to} \pi_1(X)^{\underline{(l)}}$ is injective. □

We shall denote by

$$\Delta_{X/S}$$

the kernel of the homomorphism $\pi_1(X)^{\underline{(l)}} \to \pi_1(S)$ induced by f. Then by Lemma 1.2, this pro-l group $\Delta_{X/S}$ is isomorphic to the pro-l fundamental group of a connected smooth hyperbolic curve $X \times_S \overline{s}$ of type (g, r) (over a separably closed field). We shall write

$$\Delta^{\mathrm{cpt}}_{X/S} \overset{\mathrm{def}}{=} \Delta_{X^{\mathrm{cpt}}/S}\,,$$

i.e., the pro-l fundamental group of a geometric fiber of a (unique, up to canonical isomorphism [cf. the discussion entitled "*Curves*" in [12], Section 0]) compactification $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ of $f : X \to S$. Then we have a natural surjection:

$$\Delta_{X/S} \twoheadrightarrow \Delta^{\mathrm{cpt}}_{X/S}\,.$$

We shall denote by

$$\{\Delta_{X/S}(n)\}$$

the central filtration of $\Delta_{X/S}$ with respect to the composite of the natural surjections (cf. Definition 1.1, (i)):

$$\Delta_{X/S} \twoheadrightarrow \Delta_{X/S}^{\mathrm{cpt}} \twoheadrightarrow (\Delta_{X/S}^{\mathrm{cpt}})^{\mathrm{ab}}\,.$$

Remark 1.3. As is well-known, the graded Lie algebra $\mathrm{Gr}(\Delta_{X/S})$ (where "Gr" is taken with respect to the central filtration defined above) is *center-free* (cf. e.g., [2], Theorem 1, (ii), together with [2], Proposition 5).

Now by Lemma 1.2, we obtain an outer representation:

$$\rho_{X/S} : \pi_1(S) \longrightarrow \mathrm{Out}(\Delta_{X/S})\,.$$

We shall denote by

$$\mathrm{Out}^*(\Delta_{X/S}) \subseteq \mathrm{Out}(\Delta_{X/S})$$

the subgroup of $\mathrm{Out}(\Delta_{X/S})$ whose elements preserve the central filtration $\{\Delta_{X/S}(n)\}$ of $\Delta_{X/S}$.

Remark 1.4. If $r \geq 2$, then by the definition of $\mathrm{Out}^*(\Delta_{X/S})$, we obtain

$$\mathrm{Out}^*(\Delta_{X/S}) \neq \mathrm{Out}(\Delta_{X/S})\,.$$

Indeed, this follows immediately from the definition of $\{\Delta_{X/S}(n)\}$, together with the fact that the assumption that $r \neq 0$ implies that the profinite group $\Delta_{X/S}$ is a $free$ pro-l group.

Proposition 1.5. *The outer representation $\rho_{X/S}$ factors through* $\mathrm{Out}^*(\Delta_{X/S})$.

Proof. This follows from the fact that the exact sequence obtained in Lemma 1.2 fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \pi_1(X)^{\underline{(l)}} & \xrightarrow{\text{via } f} & \pi_1(S) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_{X/S}^{\mathrm{cpt}} & \longrightarrow & \pi_1(X^{\mathrm{cpt}})^{\underline{(l)}} & \xrightarrow[\text{via } f]{} & \pi_1(S) & \longrightarrow & 1\,,
\end{array}$$

where the horizontal sequences are exact (cf. Lemma 1.2). □

Definition 1.6. We shall say that f is *of pro-l-exact type* if the sequence

$$1 \longrightarrow \Delta_{X/S} \longrightarrow \Delta_X \xrightarrow{\text{via } f} \Delta_S \longrightarrow 1$$

naturally induced by the exact sequence obtained in Lemma 1.2 is *exact*, where Δ_X (respectively, Δ_S) is the pro-l fundamental group of X (respectively, S).

Proposition 1.7. *The image of the composite*

$$\pi_1(S) \xrightarrow{\rho_{X/S}} \mathrm{Out}^*(\Delta_{X/S}) \longrightarrow \mathrm{Aut}((\Delta_{X/S}^{\mathrm{cpt}})^{\mathrm{ab}})$$

is a pro-l group (e.g., the action of $\pi_1(S)$ on $(\Delta_{X/S}^{\mathrm{cpt}})^{\mathrm{ab}}$ is trivial) if and only if f is of pro-l-exact type.

Proof. It is immediate that if f is of pro-l-exact type, then $\rho_{X/S}$ factors through Δ_S. Thus, we prove that if the composite in the statement of Proposition 1.7 factors through Δ_S, then f is of pro-l-exact type. It follows from [11], Lemma 3.1, (i), that the kernel of the natural morphism

$$\mathrm{Out}^*(\Delta_{X/S}) \longrightarrow \mathrm{Aut}((\Delta_{X/S}^{\mathrm{cpt}})^{\mathrm{ab}})$$

is a pro-l group. Therefore, the assumption implies that the homomorphism $\rho_{X/S}$ factors through Δ_S. Now let us write

$$\Gamma \stackrel{\mathrm{def}}{=} \Delta_{X/S} \stackrel{\mathrm{out}}{\rtimes} \Delta_S$$

(cf. the discussion entitled "*Groups*" in Introduction). Then we have a natural morphism $\pi_1(X)^{\underline{(l)}} \to \Gamma$ that fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \pi_1(X)^{\underline{(l)}} & \xrightarrow{\text{via } f} & \pi_1(S) & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \Gamma & \xrightarrow[\mathrm{pr}_2]{} & \Delta_S & \longrightarrow & 1\,,
\end{array}$$

where the horizontal sequences are exact. Note that since $\pi_1(S) \to \Delta_S$ is surjective, $\pi_1(X)^{\underline{(l)}} \to \Gamma$ is also surjective, and that since $\Delta_{X/S}$ and Δ_S are pro-l, Γ is also pro-l. Now we shall denote by N_1 (respectively, N_2) the kernel of the natural surjection $\pi_1(X)^{\underline{(l)}} \to \Delta_X$ (respectively, $\pi_1(X)^{\underline{(l)}} \to \Gamma$). Then the following hold:

(i) $N_1 \subseteq N_2$. (This follows from the fact that Γ is pro-l.)

(ii) $\Delta_{X/S} \cap N_2 = \{1\}$. (This follows from the above diagram.)

(iii) $\Delta_{X/S} \cap N_1 = \{1\}$. (This follows from (i) and (ii).)

By (ii) and (iii), the following natural sequence is exact

$$1 \longrightarrow \Delta_{X/S} \longrightarrow \Delta_X \longrightarrow \pi_1(S)/N_3 \longrightarrow 1\,,$$

where N_3 is the image of N_1 via the surjection $\pi_1(X)^{\underline{(l)}} \twoheadrightarrow \pi_1(S)$. Moreover, by (i), this exact sequence fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \pi_1(X)^{\underline{(l)}} & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \Delta_X & \longrightarrow & \pi_1(S)/N_3 & \longrightarrow & 1 \\
& & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \Gamma & \longrightarrow & \Delta_S & \longrightarrow & 1\,,
\end{array}$$

where the horizontal sequences are exact, and all vertical arrows are surjective. Since Δ_X is pro-l, the group $\pi_1(S)/N_3$ is also pro-l. Thus, the right-hand lower vertical arrow $\pi_1(S)/N_3 \to \Delta_S$, hence also, $\Delta_X \to \Gamma$ is an isomorphism. This completes the proof of Proposition 1.7. $\square$

Let A_X and A_S be profinite *abelian* groups, and $\Delta_X \twoheadrightarrow A_X$ and $\Delta_S \twoheadrightarrow A_S$ (continuous) surjections. Then we shall denote by

$$\{\Delta_X(n)\} \ \text{(respectively, } \{\Delta_S(n)\})$$

the central filtration with respect to the surjection $\Delta_X \twoheadrightarrow A_X$ (respectively, $\Delta_S \twoheadrightarrow A_S$) (thus, $A_X \simeq \Delta_X(1/2)$ and $A_S \simeq \Delta_S(1/2)$).

Now we assume that f is *of pro-l-exact type.* Moreover, we also assume that the surjections $\Delta_X \twoheadrightarrow A_X$ and $\Delta_S \twoheadrightarrow A_S$ fit into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X/S} & \longrightarrow & \Delta_X & \xrightarrow{\text{via } f} & \Delta_S & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & (\Delta_{X/S}^{\text{cpt}})^{\text{ab}} & \longrightarrow & A_X & \longrightarrow & A_S & \longrightarrow & 1\,, \end{array}$$

where the bottom sequence is also *exact.* By the commutativity of the above diagram, the morphisms $\Delta_{X/S} \to \Delta_X$ and $\Delta_X \to \Delta_S$ preserve the central filtrations on these groups associated to the abelian quotients in the bottom sequence.

Definition 1.8. We assume that f is of pro-l-exact type. Then we shall say that $(f,\, \Delta_X \to A_X,\, \Delta_S \to A_S)$ is *of Lie-exact type* if the sequence of graded Lie algebras

$$1 \longrightarrow \mathrm{Gr}(\Delta_{X/S}) \longrightarrow \mathrm{Gr}(\Delta_X) \xrightarrow{\text{via } f} \mathrm{Gr}(\Delta_S) \longrightarrow 1$$

(where "Gr" is taken with respect to the central filtrations defined above) naturally induced by the exact sequence in Definition 1.6 is *exact.*

Proposition 1.9. *We assume that f is of pro-l-exact type. Then the following conditions are equivalent:*

(*i*) $(f,\, \Delta_X \to A_X,\, \Delta_S \to A_S)$ *is of Lie-exact type.*

(*ii*) *The action of Δ_X on $\Delta_{X/S}(n/n{+}1)$ and the action of $\Delta_X(2)$ on $\Delta_{X/S}(n/n{+}2)$ (induced via conjugation) are trivial for any $n \geq 1$.*

(*ii′*) *The action of Δ_S on $\Delta_{X/S}(n/n{+}1)$ and the action of $\Delta_S(2)$ on $\Delta_{X/S}(n/n{+}2)$ (induced via $\rho_{X/S}$) are trivial for any $n \geq 1$.*

(*iii*) *The action of $\Delta_X(m)$ on $\Delta_{X/S}(n/n+m)$ (induced via conjugation) is trivial for any $n,\, m \geq 1$.*

Proof. First, we prove that (i) implies (ii). If (ii) does not hold, then there exists $x \in \Delta_{X/S}(n)$ and $\sigma \in \Delta_X(m)$ (where $m = 1$ or 2) such that $\sigma \cdot x \cdot \sigma^{-1} \cdot x^{-1} \notin \Delta_{X/S}(n+m)$. On the other hand, by the definition of the filtration $\{\Delta_X(n)\}$, we have that $\sigma \cdot x \cdot \sigma^{-1} \cdot x^{-1} \in \Delta_X(n+m) \cap \Delta_{X/S}$. Thus, $\Delta_{X/S}(n+m) \neq \Delta_X(n+m) \cap \Delta_{X/S}$. This implies that the natural morphism $\mathrm{Gr}(\Delta_{X/S}) \to \mathrm{Gr}(\Delta_X)$ is not injective. Thus, (i) does not hold.

Next, we prove that (ii) implies (iii). This proof will be by induction on m. The assertion for $m = 1$ and 2 follows from (ii). Assume that $m \geq 3$. Then it follows from the induction hypothesis and an well-known identity due to P. Hall (i.e.,

$$[A, [B, C]] \subseteq [B, [C, A]] \cdot [C, [A, B]]$$

for closed normal subgroups A, B, and C of an ambient group [cf. e.g., [10], Theorem 5.2]) that

$$[\Delta_{X/S}(n), [\Delta_X(m_1), \Delta_X(m_2)]] \subseteq \Delta_{X/S}(n+m)$$

for positive integers m_1 and m_2 such that $m_1+m_2 = m$. Thus, since, in general, for a finite set I,

$$\langle [G, H_i] \mid i \in I \rangle = [G, \langle H_i \mid i \in I \rangle]$$

for closed normal subgroups H_i $(i \in I)$ of an ambient group G, we thus obtain an inclusion

$$[\Delta_{X/S}(n), \Delta_X(m)] \subseteq \Delta_{X/S}(n+m)$$

by the definition of the filtration $\{\Delta_X(n)\}$. Therefore, we conclude that (iii) holds.

The assertion that (iii) implies (i) follows from a similar argument to the argument used in the proof of [11], Proposition 3.2 (cf. also Remark 1.3 and [11], Lemma 3.2).

The equivalence of (ii) and (ii′) follows immediately from the exactness of the following sequences:

$$1 \longrightarrow \Delta_{X/S} \longrightarrow \Delta_X \longrightarrow \Delta_S \longrightarrow 1\,;$$

$$1 \longrightarrow \Delta_{X/S}(2) \longrightarrow \Delta_X(2) \longrightarrow \Delta_S(2) \longrightarrow 1\,.$$

□

Lemma 1.10. *Let I^{cpt} be the kernel of the surjection*

$$\Delta_{X/S} \twoheadrightarrow \Delta^{\mathrm{cpt}}_{X/S}\,.$$

Let $\overline{s} \to S$ be a geometric point of S. We shall write

$$D_{\overline{s}} \stackrel{\mathrm{def}}{=} D \times_S \overline{s}\,,$$

where $D \subseteq X^{\mathrm{cpt}}$ is the reduced relative divisor over S obtained as the complement of X in X^{cpt}. Then the following hold:

(*i*) *The submodule*

$$(\Delta^{\mathrm{cpt}}_{X/S})^{\mathrm{ab}} = \Delta_{X/S}(1/2) \subseteq \mathrm{Gr}(\Delta_{X/S})$$

and the submodule

$$I^{\mathrm{cpt}}/(\Delta_{X/S}(3) \cap I^{\mathrm{cpt}}) \subseteq \Delta_{X/S}(2/3) \subseteq \mathrm{Gr}(\Delta_{X/S})$$

generate the graded Lie algebra $\mathrm{Gr}(\Delta_{X/S})$ (as a Lie algebra). In particular, if f is of pro-l-exact type, then the following conditions are equivalent:

(*1*) *The action of Δ_X on $\Delta_{X/S}(n/n+1)$ (induced via conjugation) is trivial for any $n \geq 1$.*

(*1′*) *The action of Δ_S on $\Delta_{X/S}(n/n+1)$ (induced via $\rho_{X/S}$) is trivial for any $n \geq 1$.*

(2) *The action of* Δ_X *on* $(\Delta^{\mathrm{cpt}}_{X/S})^{\mathrm{ab}}$ *and* $I^{\mathrm{cpt}}/(\Delta_{X/S}(3) \cap I^{\mathrm{cpt}})$ *(induced via conjugation) is trivial.*

(2′) *The action of* Δ_S *on* $(\Delta^{\mathrm{cpt}}_{X/S})^{\mathrm{ab}}$ *and* $I^{\mathrm{cpt}}/(\Delta_{X/S}(3) \cap I^{\mathrm{cpt}})$ *(induced via* $\rho_{X/S}$*) is trivial.*

(*ii*) *The submodule*

$$I^{\mathrm{cpt}}/(\Delta_{X/S}(3) \cap I^{\mathrm{cpt}}) \subseteq \Delta_{X/S}(2/3)$$

is a free $\mathbb{Z}_l$*-module in the formal generators* ζ*, where* ζ *ranges over the elements of the underlying set of* $D_{\overline{s}}$*. Moreover, the action of* Δ_S *on* $I^{\mathrm{cpt}}/(\Delta_{X/S}(3) \cap I^{\mathrm{cpt}})$ *(induced via* $\rho_{X/S}$*) is compatible with the natural action of* Δ_S *on* $D_{\overline{s}}$*.*

Proof. This follows immediately from [8], Proposition 1. □

Corollary 1.11. *If the quotient* $\Delta_S \to A_S$ *of* Δ_S *coincides with the abelianization of* Δ_S*, and the action of* $\pi_1(S)$ *on* $(\Delta^{\mathrm{cpt}}_{X/S})^{\mathrm{ab}}$ *and on* $I^{\mathrm{cpt}}/\Delta_{X/S}(3) \cap I^{\mathrm{cpt}}$ *(induced via* $\rho_{X/S}$*) are trivial, then* f *is of pro-*l*-exact type, and* $(f, \Delta_X \to A_X, \Delta_S \to \Delta_S^{\mathrm{ab}}\ [= A_S])$ *is of Lie-exact type.*

Proof. This follows immediately from Propositions 1.7; 1.9; Lemma 1.10, together with the well-known identity due to P. Hall applied in the proof of Proposition 1.9. □

Definition 1.12. Let m be a natural number.

(i) We shall say that

$$X_m \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_{m-2}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 = \operatorname{Spec} K\,,$$

is a *successive extension of hyperbolic curves of product type* if there exist proper hyperbolic curves C_i $(i = 0, \cdots, m-1)$ over K which satisfy the following condition: The morphism $f_i : X_{i+1} \to X_i$ factors as the composite

$$X_{i+1} \hookrightarrow C_i \times_K X_i \xrightarrow{\mathrm{pr}_2} X_i$$

of an open immersion $X_{i+1} \hookrightarrow C_i \times_K X_i$ onto the complement $(C_i \times_K X_i) \setminus D_i$ of a relative divisor D_i which is finite étale over X_i.

Note that it is immediate that X_i is a *regular scheme of dimension* i, that f_i is a smooth family of connected hyperbolic curves, and that the f_i's induce an open immersion $X_i \hookrightarrow C_0 \times_K \cdots \times_K C_{i-1}$.

(ii) Let

$$X_m \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_{m-2}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 = \operatorname{Spec} K$$

be a successive extension of hyperbolic curves of product type. Then we shall denote by

$$\{\Delta_{X_i}(n)\}$$

the central filtration with respect to the composite of the natural surjections

$$\Delta_{X_i} \twoheadrightarrow \Delta_{C_0 \times_K \cdots \times_K C_{i-1}} \twoheadrightarrow \Delta^{\mathrm{ab}}_{C_0 \times_K \cdots \times_K C_{i-1}} (\simeq \Delta^{\mathrm{ab}}_{C_0} \times \cdots \times \Delta^{\mathrm{ab}}_{C_{i-1}})\,,$$

where the first arrow is the morphism induced by the open immersion $X_i \hookrightarrow C_0 \times_K \cdots \times_K C_{i-1}$ (cf. (i)).

Note that it is immediate that the following sequence is exact:

$$1 \longrightarrow \Delta_{X_{i+1}/X_i}(1/2) \longrightarrow \Delta_{X_{i+1}}(1/2) \xrightarrow{\text{via } f_i} \Delta_{X_i}(1/2) \longrightarrow 1\,.$$

Corollary 1.13. *Let*

$$X_m \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_{m-2}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 = \operatorname{Spec} K$$

be a successive extension of hyperbolic curves of product type, and $0 \leq i \leq m-1$ *an integer. Then the following hold:*

(*i*) *The morphism* f_i *is of pro-l-exact type.*

(*ii*) *The following conditions are equivalent:*

(*1*) *The relative divisor* D_i (*which appears in Definition 1.12,* (*i*)) *is empty or the finite étale covering* $D_i \to X_i$ *is trivial* (*i.e.,* D_i *is a disjoint union of copies of* X_i*, and the covering* $D_i \to X_i$ *is induced by the identity morphism of* X_i).

(*2*) $(f_i, \Delta_{X_{i+1}} \to \Delta_{X_{i+1}}(1/2), \Delta_{X_i} \to \Delta_{X_i}(1/2))$ *is of Lie-exact type.*

Proof. First, we prove assertion (i). Since the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X_{i+1}/X_i} & \longrightarrow & \pi_1(X_{i+1})^{\underline{(l)}} & \xrightarrow{\text{via } f_i} & \pi_1(X_i) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^{\text{cpt}}_{X_{i+1}/X_i} & \longrightarrow & \pi_1(C_i \times_K X_i)^{\underline{(l)}} & \xrightarrow{\text{via } \text{pr}_2} & \pi_1(X_i) & \longrightarrow & 1 \\
 & & \| & & \| & & \| & & \\
1 & \longrightarrow & \Delta_{C_i} & \longrightarrow & \Delta_{C_i} \times \pi_1(X_i) & \xrightarrow{\text{pr}_2} & \pi_1(X_i) & \longrightarrow & 1
\end{array}$$

commutes, the action of $\pi_1(X_i)$ on $\Delta^{\text{cpt}}_{X_{i+1}/X_i}$ is trivial; thus, assertion (i) follows from Proposition 1.7.

Next, we prove assertion (ii). Assume that condition (1) holds. Then, by Lemma 1.10, (ii), the action of Δ_{X_i} on $I^{\text{cpt}}/(\Delta_{X_{i+1}/X_i}(3) \cap I^{\text{cpt}})$ is trivial. Thus, in light of the triviality of the action of $\pi_1(X_i)$ on $\Delta^{\text{cpt}}_{X_{i+1}/X_i}$ (observed in the proof of assertion (i)), we conclude that the action of Δ_{X_i} on $\Delta_{X_{i+1}/X_i}(n/n+1)$ is trivial for any $n \geq 1$ (cf. Lemma 1.10, (i)). Thus, it follows from the equivalence of (i) and (ii′) in Proposition 1.9 that it is enough to show that the action of $\Delta_{X_i}(2)$ on $\Delta_{X_{i+1}/X_i}(n/n+2)$ is trivial for any $n \geq 1$. Moreover, by the triviality of the action of $\pi_1(X_i)$ on $\Delta^{\text{cpt}}_{X_{i+1}/X_i}$ (observed in the proof of (i)), together with the well-known identity due to P. Hall applied in the proof of Proposition 1.9, the action of $[\Delta_{X_i}, \Delta_{X_i}]$ on $\Delta_{X_{i+1}/X_i}(n/n+2)$ is trivial for any $n \geq 1$. Since $\Delta_{X_i}(2)$ is generated by $[\Delta_{X_i}, \Delta_{X_i}]$ and the kernel I of the natural surjection $\Delta_{X_i} \twoheadrightarrow \Delta_{C_0 \times_K \cdots \times_K C_{i-1}} (\simeq \Delta_{C_0} \times \cdots \times \Delta_{C_{i-1}})$, it is enough to show that the action of I on $\Delta_{X_{i+1}/X_i}(n/n+2)$ is trivial for any $n \geq 1$. On the other hand, I is topologically normally generated by the inertia subgroups (well-defined, up to conjugation) of Δ_{X_i} determined by the irreducible components of the divisor with normal crossings $(C_0 \times_K \cdots \times_K C_{i-1}) \setminus X_i \subseteq C_0 \times_K \cdots \times_K C_{i-1}$ (by the purity theorem [cf. [4], Exposé X, Theorem 3.4], together with the regularity of $C_0 \times_K \cdots \times_K C_{i-1}$), it is enough to show that the action of these inertia subgroups on $\Delta_{X_{i+1}/X_i}(n/n+2)$ is trivial for any $n \geq 1$.

For any positive integer N, we shall denote by $C_{i\,\underline{(N)}}$ (respectively, $U_{C_{i\,(N)}}$) the fiber product of N copies of C_i over $\operatorname{Spec} K$ (respectively, the N-th configuration space of C_i, i.e., the scheme which represents the open subfunctor

$$S \mapsto \{(s_1, \cdots, s_N) \in C_{i\,\underline{(N)}}(S) = C_i(S)^{\times N} \mid s_n \neq s_m \ \text{if} \ n \neq m\}$$

of the functor represented by $C_{i\,\underline{(N)}}$). By (1), if we denote by r the degree of the (trivial) covering $D_i \to X_i$, then there exist "classifying morphisms" $X_i \overset{g_i}{\to} U_{C_{i\,(r)}}$ and $X_{i+1} \overset{g_{i+1}}{\to} U_{C_{i\,(r+1)}}$ that fit into the following *cartesian* diagram

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{g_{i+1}} & U_{C_{i\,(r+1)}} \\ {\scriptstyle f_i}\big\downarrow & & \big\downarrow \\ X_i & \xrightarrow[g_i]{} & U_{C_{i\,(r)}} \,, \end{array}$$

where the right-hand vertical arrow is the morphism induced by the morphism $C_{i\,\underline{(r+1)}} \to C_{i\,\underline{(r)}}$ obtained by forgetting the $(r+1)$-st factor. Thus, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X_{i+1}/X_i} & \longrightarrow & \Delta_{X_{i+1}} & \xrightarrow{\text{via } f_i} & \Delta_{X_i} & \longrightarrow & 1 \\ & & \big\downarrow & & {\scriptstyle \text{via } g_{i+1}}\big\downarrow & & \big\downarrow{\scriptstyle \text{via } g_i} & & \qquad (*) \\ 1 & \longrightarrow & \Delta_{U_{C_{i\,(r+1)}}/U_{C_{i\,(r)}}} & \longrightarrow & \Delta_{U_{C_{i\,(r+1)}}} & \longrightarrow & \Delta_{U_{C_{i\,(r)}}} & \longrightarrow & 1\,, \end{array}$$

where the horizontal sequences are exact, and the left-hand vertical arrow is an isomorphism. Note that the sequence

$$U_{C_{i\,(r)}} \longrightarrow U_{C_{i\,(r-1)}} \longrightarrow \cdots \longrightarrow U_{C_{i\,(2)}} \longrightarrow C_i \longrightarrow \operatorname{Spec} K$$

(where the morphism $U_{C_{i\,(N+1)}} \longrightarrow U_{C_{i\,(N)}}$ [where $1 \leq N \leq r-1$] is the morphism induced by the morphism $C_{i\,\underline{(N+1)}} \to C_{i\,\underline{(N)}}$ obtained by forgetting the $(N+1)$-st factor) is a successive extension of hyperbolic curves of product type; thus, the filtration $\{\Delta_{U_{C_{i\,(r)}}}(n)\}$ is defined (cf. Definition 1.12, (ii)); moreover, since the sequence

$$1 \longrightarrow \mathrm{Gr}(\Delta_{U_{C_{i\,(r+1)}}/U_{C_{i\,(r)}}}) \longrightarrow \mathrm{Gr}(\Delta_{U_{C_{i\,(r+1)}}}) \longrightarrow \mathrm{Gr}(\Delta_{U_{C_{i\,(r)}}}) \longrightarrow 1$$

(naturally induced by the bottom sequence in the commutative diagram $(*)$) is *exact* (cf. [11], Proposition 3.2, (i)), by the equivalence in Proposition 1.9, (i) and (ii$'$), the action of $\Delta_{U_{C_{i\,(r)}}}(2)$ on $\Delta_{U_{C_{i\,(r+1)}}/U_{C_{i\,(r)}}}(n/n+2)$ is trivial for any $n \geq 1$. Thus, by the commutativity of the above diagram $(*)$ and the fact that the left-hand vertical arrow in the above diagram $(*)$ is an *isomorphism*, to prove assertion that condition (1) implies condition (2), it is enough to show that the composite $X_i \overset{g_i}{\to} U_{C_{i\,(r)}} \hookrightarrow C_{i\,\underline{(r)}}$ extends to the generic points of the irreducible components of the divisor with normal crossings $(C_0 \times_K \cdots \times_K C_{i-1}) \setminus X_i \subseteq C_0 \times_K \cdots \times_K C_{i-1}$. However, this follows from the properness of $C_{i\,\underline{(r)}}$. This completes the proof that condition (1) implies condition (2).

Next, we assume that $(f_i, \Delta_{X_{i+1}} \to \Delta_{X_{i+1}}(1/2), \Delta_{X_i} \to \Delta_{X_i}(1/2))$ is of Lie-exact type. Then the equivalence of (i) and (ii′) in Proposition 1.9 and the equivalence of (1′) and (2′) in Lemma 1.10, (i) imply that the action of Δ_{X_i} on $I^{\mathrm{cpt}}/(\Delta_{X_{i+1}/X_i}(3) \cap I^{\mathrm{cpt}})$, where I^{cpt} is the kernel of the natural surjection $\Delta_{X_{i+1}/X_i} \twoheadrightarrow \Delta^{\mathrm{cpt}}_{X_{i+1}/X_i}$, is trivial. Therefore, by Lemma 1.10, (ii), we conclude that either the relative divisor D_i is empty, or the finite étale covering $D_i \to X_i$ is trivial. $\square$

Remark 1.14. Note that the fact that

> *the action of the inertia subgroups of* $\Delta_{X_i}(2)$ *on* $\Delta_{X_{i+1}/X_i}(n/n+2)$ *is trivial for any* $n \geq 1$.

can also be proven as follows. Note that we showed the above claim in the proof of Corollary 1.13 by means of [11], Proposition 3.2, (i), which is proven via *transcendental techniques*; however, the following proof is *purely algebraic*:

To prove the assertion, it is immediate that we may assume that there exists a *finite field* k such that $X_{i+1} \xrightarrow{f_i} X_i$ descends to k. (We denote by G_k the absolute Galois group of k, by $\mathrm{Fr}_k \in G_k$ the Frobenius element, and by q_k the cardinality of k.) Then by the "Riemann hypothesis for abelian varieties over finite fields" (cf. e.g., [16], p. 206) (respectively, as is well-known), the eigenvalues of the action of Fr_k on the G_k-module $\Delta_{X_{i+1}/X_i}(n/n+1)$ (respectively, the inertia subgroup) are algebraic numbers all of whose complex absolute values are equal to $q_k^{n/2}$ (respectively, q_k), i.e., the G_k-module $\Delta_{X_{i+1}/X_i}(n/n+1)$ (respectively, the inertia subgroup) is "of weight n" (respectively, "of weight 2"). In particular, the G_k-module

$$\mathrm{Hom}_{G_k}(\Delta_{X_{i+1}/X_i}(n/n+1), \Delta_{X_{i+1}/X_i}(n+1/n+2))$$

is "of weight 1". On the other hand, since the action of the inertia subgroup on $\Delta_{X_{i+1}/X_i}(n/n+1)$ and $\Delta_{X_{i+1}/X_i}(n+1/n+2)$ is trivial, by the exactness of the sequence

$$1 \longrightarrow \Delta_{X_{i+1}/X_i}(n+1/n+2) \longrightarrow \Delta_{X_{i+1}/X_i}(n/n+2) \longrightarrow \Delta_{X_{i+1}/X_i}(n/n+1) \longrightarrow 1\,,$$

the action of the inertia subgroup on $\Delta_{X_{i+1}/X_i}(n/n+2)$ determines (and is determined by!) a G_k-*equivariant* homomorphism from the inertia subgroup to

$$\mathrm{Hom}_{G_k}(\Delta_{X_{i+1}/X_i}(n/n+1), \Delta_{X_{i+1}/X_i}(n+1/n+2))\,.$$

Thus, by considering the "weights" of the domain and codomain of this G_k-equivariant homomorphism, we conclude that the G_k-equivariant homomorphism is *trivial*; in particular, the action of the inertia subgroup on $\Delta_{X_{i+1}/X_i}(n/n+2)$ is trivial.

2 Fundamental groups of configuration spaces over finite fields

In this section, we consider the group-theoretic properties of the fundamental groups of configuration spaces.

Let K be a field, and l a prime number that is invertible in K. We shall fix a separable closure K^{sep} of K. We shall denote by G_K the Galois group of K^{sep} over K. Moreover, in the following, let X be a proper hyperbolic curve of genus $g_X \geq 2$ over K.

Definition 2.1. Let r be a natural number.

(i) We shall denote by $X_{\underline{(r)}}$ the fiber product of r copies of X over $\operatorname{Spec} K$, i.e.,

$$X_{\underline{(r)}} \stackrel{\mathrm{def}}{=} \overbrace{X \times_K \cdots \times_K X}^{r} .$$

For an integer $1 \leq i \leq r$, we shall denote by $p_{X_{\underline{(r-1)}}:i} : X_{\underline{(r)}} \to X_{\underline{(r-1)}}$ the morphism obtained by forgetting the i-th factor.

(ii) We shall denote by $U_{X_{(r)}} \subseteq X_{\underline{(r)}}$ the r-th configuration space of X, i.e., the scheme which represents the open subfunctor

$$S \mapsto \{(f_1, \cdots, f_r) \in X_{\underline{(r)}}(S) = X(S)^{\times r} \mid f_i \neq f_j \text{ if } i \neq j\}$$

of the functor represented by $X_{\underline{(r)}}$. For an integer $1 \leq i \leq r$, we shall denote by $p_{U_{X_{(r-1)}}:i} : U_{X_{(r)}} \to U_{X_{(r-1)}}$ the morphism induced by $p_{X_{\underline{(r-1)}}:i}$. Let $1 \leq i < j \leq r$ be an integers. Then we shall denote by $D_{X_{\underline{(r)}}\{i,j\}} \subseteq X_{\underline{(r)}}$ the closed subscheme of $X_{\underline{(r)}}$ which represents the closed subfunctor

$$S \mapsto \{(f_1, \cdots, f_r) \in X_{\underline{(r)}}(S) = X(S)^{\times r} \mid f_i = f_j\}$$

of the functor represented by $X_{\underline{(r)}}$. Then it is immediate that

$$U_{X_{(r)}} = X_{\underline{(r)}} \setminus \bigcup_{1 \leq i < j \leq r} D_{X_{\underline{(r)}}\{i,j\}} .$$

(iii) We shall denote by $\Pi_{X_{\underline{(r)}}}$ the geometrically pro-l fundamental group of $X_{\underline{(r)}}$, and by $\Delta_{X_{\underline{(r)}}}$ the kernel of the natural surjection

$$\Pi_{X_{\underline{(r)}}} \twoheadrightarrow G_K .$$

Thus, we have an exact sequence

$$1 \longrightarrow \Delta_{X_{\underline{(r)}}} \longrightarrow \Pi_{X_{\underline{(r)}}} \longrightarrow G_K \longrightarrow 1 .$$

Moreover, we shall write

$$\Pi_X \stackrel{\mathrm{def}}{=} \Pi_{X_{(1)}} \ ; \ \ \Delta_X \stackrel{\mathrm{def}}{=} \Delta_{X_{(1)}} .$$

(iv) We shall denote by $\Pi_{X_{(r)}}$ the geometrically pro-l fundamental group of $U_{X_{(r)}}$, by $\Delta_{X_{(r)}}$ the kernel of the natural surjection

$$\Pi_{X_{(r)}} \twoheadrightarrow G_K \,,$$

and by $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$ the kernel of the surjection

$$\Delta_{X_{(r)}} \overset{\text{via } p_{U_{X_{(r-1)}}:i}}{\twoheadrightarrow} \Delta_{X_{(r-1)}} \quad (i = 1, \cdots, r)\,.$$

Thus, we have exact sequences

$$1 \longrightarrow \Delta_{X_{(r)}} \longrightarrow \Pi_{X_{(r)}} \longrightarrow G_K \longrightarrow 1\,;$$

$$1 \longrightarrow \Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \longrightarrow \Delta_{X_{(r)}} \overset{\text{via } p_{U_{X_{(r-1)}}:i}}{\longrightarrow} \Delta_{X_{(r-1)}} \longrightarrow 1\,;$$

$$1 \longrightarrow \Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \longrightarrow \Pi_{X_{(r)}} \overset{\text{via } p_{U_{X_{(r-1)}}:i}}{\longrightarrow} \Pi_{X_{(r-1)}} \longrightarrow 1\,.$$

Note that since the sequence obtained as the base-change of

$$U_{X_{(r)}} \overset{p_{U_{X_{(r-1)}}:r}}{\longrightarrow} U_{X_{(r-1)}} \overset{p_{U_{X_{(r-2)}}:r-1}}{\longrightarrow} \cdots \overset{p_{U_{X_{(1)}}:2}}{\longrightarrow} X \longrightarrow \operatorname{Spec} K$$

from K to K^{sep} is a successive extension of hyperbolic curves of product type (cf. Definition 1.12, (i)), the family of smooth curve $U_{X_{(r)}} \otimes_K K^{\mathrm{sep}} \overset{\text{via } p_{U_{X_{(r-1)}}:i}}{\to} U_{X_{(r-1)}} \otimes_K K^{\mathrm{sep}}$ is of pro-l exact type (cf. Corollary 1.13, (i)); thus the pro-l group $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$ is isomorphic to the pro-l fundamental group of the geometric fiber of the family of smooth curve $U_{X_{(r)}} \otimes_K K^{\mathrm{sep}} \overset{\text{via } p_{U_{X_{(r-1)}}:i}}{\to} U_{X_{(r-1)}} \otimes_K K^{\mathrm{sep}}$ at a geometric point of $U_{X_{(r-1)}} \otimes_K K^{\mathrm{sep}}$.

Proposition 2.2. *Let r be a positive integer. Then the profinite groups $\Delta_{X_{\underline{(r)}}}$, $\Delta_{X_{(r)}}$, and $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$ are slim.*

Proof. The slimness of $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$ (in particular, the slimness of Δ_X) follows from [1], Propositions 8; 18. The slimness of $\Delta_{X_{\underline{(r)}}}$ follows from the slimness of Δ_X, together with the fact that $\Delta_{X_{\underline{(r)}}}$ is the product of r copies of Δ_X. The slimness of $\Delta_{X_{(r)}}$ follows from induction on r, the slimness of $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$, and the exactness of the sequence

$$1 \longrightarrow \Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \longrightarrow \Delta_{X_{(r)}} \overset{\text{via } p_{U_{X_{(r-1)}}:i}}{\longrightarrow} \Delta_{X_{(r-1)}} \longrightarrow 1$$

in Definition 2.1, (iv). $\square$

Next, let us recall the theory of log configuration schemes (cf. [7], Section 1). Let us denote by $X^{\log}_{(r)}$ the r-th log configuration scheme of X, i.e.,

$$X^{\log}_{(r)} \overset{\text{def}}{=} \operatorname{Spec} K \times_{\overline{\mathcal{M}}^{\log}_g} \overline{\mathcal{M}}^{\log}_{g,r}\,,$$

where the (1-)morphism $\mathrm{Spec}\,K \to \overline{\mathcal{M}}_g^{\log}$ is the classifying morphism of the curve $X \to \mathrm{Spec}\,K$, and the (1-)morphism $\overline{\mathcal{M}}_{g,r}^{\log} \to \overline{\mathcal{M}}_g^{\log}$ is the (1-)morphism obtained by forgetting the sections; and by $p^{\log}_{X_{(r-1)}:i} : X^{\log}_{(r)} \to X^{\log}_{(r-1)}$ the morphism induced by the (1-)morphism $\overline{\mathcal{M}}_{g,r}^{\log} \to \overline{\mathcal{M}}_{g,r-1}^{\log}$ obtained by forgetting the i-th section (cf. [7], Definition 1.1). Then, by definition, the interior of the log scheme $X^{\log}_{(r)}$ is naturally isomorphic to the usual r-th configuration space $U_{X_{(r)}}$ of X, and we have a natural commutative diagram:

$$\begin{tikzcd}
U_{X_{(r)}} \arrow[r] \arrow[d, "p_{U_{X_{(r-1)}}:i}"'] & X^{\log}_{(r)} \arrow[r] \arrow[d, "p^{\log}_{X_{(r-1)}:i}"] & X_{\underline{(r)}} \arrow[d, "p_{X_{\underline{(r-1)}}:i}"] \\
U_{X_{(r-1)}} \arrow[r] & X^{\log}_{(r-1)} \arrow[r] & X_{\underline{(r-1)}} .
\end{tikzcd}$$

This diagram induces a sequence

$$\Pi_{X_{(r)}} \longrightarrow \pi_1(X^{\log}_{(r)})^{\underline{(l)}} \longrightarrow \Pi_{X_{\underline{(r)}}} ,$$

where $\pi_1(X^{\log}_{(r)})^{\underline{(l)}}$ is the geometrically pro-l log fundamental group of $X^{\log}_{(r)}$. Now by [7], Lemma 2.7, the first morphism $\Pi_{X_{(r)}} \to \pi_1(X^{\log}_{(r)})^{\underline{(l)}}$ (in the above sequence) is an *isomorphism.*

Let I be a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^{\#} \geq 2$. We denote by $D^{\log}_{X_{(r)}I}$ the log scheme defined in [7], Definition 1.10, and by $\delta^{\log}_{X_{(r)}I} : D^{\log}_{X_{(r)}I} \hookrightarrow X^{\log}_{(r)}$ the strict closed immersion defined in [7], Definition 1.10. Now if $1 \leq i < j \leq r$ are integers, then $p^{\log}_{X_{(r)}:i} \circ \delta^{\log}_{X_{(r+1)}\{i,j\}} = p^{\log}_{X_{(r)}:j} \circ \delta^{\log}_{X_{(r+1)}\{i,j\}}$ (cf. the proof of [7], Lemma 1.14), and these composite are morphisms of type $\mathbb{N}$ (cf. [6], Definition 4.1; [7], Lemma 1.14). Let $\overline{x} \to X^{\log}_{(r)}$ be a geometric point whose image lies on the interior $U_{X_{(r)}}$ of $X^{\log}_{(r)}$. Then we obtain the following commutative diagram:

$$\begin{tikzcd}
D^{\log}_{X_{(r+1)}\{i,j\}} \times_{X^{\log}_{(r)}} \overline{x} \arrow[r, "\mathrm{pr}_1"] \arrow[d] & D^{\log}_{X_{(r+1)}\{i,j\}} \arrow[r, "p^{\log}_{X_{(r)}:i} \circ \delta^{\log}_{X_{(r+1)}\{i,j\}}"] \arrow[d, "\delta^{\log}_{X_{(r+1)}\{i,j\}}"] & X^{\log}_{(r)} \arrow[d, equal] \\
X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x} \arrow[r, "\mathrm{pr}_1"'] & X^{\log}_{(r+1)} \arrow[r, "p^{\log}_{X_{(r)}:i}"'] & X^{\log}_{(r)} .
\end{tikzcd}$$

This diagram induces a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(D^{\log}_{X_{(r+1)}\{i,j\}} \times_{X^{\log}_{(r)}} \overline{x})^{(l)} \arrow[r, "\text{via } \mathrm{pr}_1"] \arrow[d] & \pi_1(D^{\log}_{X_{(r+1)}\{i,j\}})^{\underline{(l)}} \arrow[r, "\text{via } p^{\log}_{X_{(r)}:i} \circ \delta^{\log}_{X_{(r+1)}\{i,j\}}"] \arrow[d, "\text{via } \delta^{\log}_{X_{(r+1)}\{i,j\}}"] & \Pi_{X_{(r)}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta^{(i)}_{X_{(r+1)}/X_{(r)}} \arrow[r] & \Pi_{X_{(r+1)}} \arrow[r, "\text{via } p^{\log}_{X_{(r)}:i}"'] & \Pi_{X_{(r)}} \arrow[r] & 1 ,
\end{tikzcd}$$

where the horizontal sequences are exact (cf. [6], Proposition 4.22; [7], Remark 2.8, (i)). By [6], Proposition 4.22, we have $\pi_1(D^{\log}_{X_{(r+1)}\{i,j\}} \times_{X^{\log}_{(r)}} \overline{x})^{(l)} \xrightarrow{\sim} \mathbb{Z}_l(1)$; moreover, by the definition of $D^{\log}_{X_{(r+1)}\{i,j\}}$, it follows that the left-hand vertical arrow $\pi_1(D^{\log}_{X_{(r+1)}\{i,j\}} \times_{X^{\log}_{(r)}} \overline{x})^{(l)} \to \Delta^{(i)}_{X_{(r+1)}/X_{(r)}}$ is *injective*, and this image is the inertia subgroup (well-defined, up to conjugation) associated to the cusp (of the geometric fiber of $p_{U_{X_{(r)}}:i} : U_{X_{(r+1)}} \to U_{X_{(r)}}$ at a geometric point of $U_{X_{(r)}}$) determined by the divisor $D_{\underline{X_{(r+1)}}\{i,j\}} \subseteq \underline{X_{(r+1)}}$. In particular, the vertical arrow $\pi_1(D^{\log}_{X_{(r+1)}\{i,j\}})^{\underline{(l)}} \to \Pi_{X_{(r+1)}}$ in the above diagram is also *injective*.

Definition 2.3. Let $r \geq 2$ be an integer, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^{\#} \geq 2$. Then we shall denote by $\mathfrak{D}_{X_{(r)}I}$ the image of the morphism $\pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \to \Pi_{X_{(r)}}$ induced by $\delta^{\log}_{X_{(r)}I}$, where $\pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}}$ is the geometrically pro-l log fundamental group of $D^{\log}_{X_{(r)}I}$. We shall denote by $\mathfrak{D}^{\Delta}_{X_{(r)}I}$ the intersection of $\mathfrak{D}_{X_{(r)}I}$ and $\Delta_{X_{(r)}}$. Note that these subgroups are well-defined, up to conjugation in $\Pi_{X_{(r)}}$.

Moreover, if $I^{\#} \geq 3$, then by [7], Proposition 1.12, (iii), the composite

$$D^{\log}_{X_{(r)}I} \overset{\delta^{\log}_{X_{(r)}I}}{\hookrightarrow} X^{\log}_{(r)} \xrightarrow{p^{\log}_{X_{(r-1)}:i}} X^{\log}_{(r-1)}$$

factors through $\delta^{\log}_{X_{(r-1)}I^{[i]}} : D^{\log}_{X_{(r-1)}I^{[i]}} \hookrightarrow X^{\log}_{(r-1)}$, where $I^{[i]}$ is a *unique* subset of $\{1, 2, \cdots, r-1\}$ such that for $1 \leq j \leq r-1$, $j \in I^{[i]}$ if and only if

$$\begin{cases} j \in I & \text{if } j < i \\ j+1 \in I & \text{if } j \geq i\,. \end{cases}$$

On the other hand, by a similar argument to the argument in the proof of [7], Lemmas 1.14; 1.19, there exists a morphism

$$D^{\log}_{X_{(r)}I} \longrightarrow X^{\log}_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^{\#}+1}$$

which is of type $\mathbb{N}$; moreover, these morphisms fit into a commutative diagram

$$\begin{array}{ccc} D^{\log}_{X_{(r)}I} & \longrightarrow & X^{\log}_{(r-I^{\#}+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,I^{\#}+1} \\ \big\downarrow & & \big\downarrow \\ D^{\log}_{X_{(r)}I^{[i]}} & \longrightarrow & X^{\log}_{(r-(I^{[i]})^{\#}+1)} \times_K \overline{\mathcal{M}}^{\log}_{0,(I^{[i]})^{\#}+1}\,, \end{array}$$

where the left-hand vertical arrow is the morphism induced by the composite $p^{\log}_{X_{(r-1)}:i} \circ \delta^{\log}_{X_{(r)}I}$, and if $i \notin I$ (respectively, $i \in I$), then the right-hand vertical arrow is the morphism obtained as the base-change of the morphism $p^{\log}_{X_{(r-I^{\#})}:i'} : X^{\log}_{(r-I^{\#}+1)} \to X^{\log}_{(r-(I^{[i]})^{\#}+1)} = X^{\log}_{(r-I^{\#})}$ (respectively, $\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1} \to \overline{\mathcal{M}}^{\log}_{0,(I^{[i]})^{\#}+1} = \overline{\mathcal{M}}^{\log}_{0,I^{\#}}$ obtained by forgetting the i'-th section), where i' is the integer such that $\{1, 2, \cdots, r\} \setminus I = \{i_1, i_2, \cdots, i_{r-I^{\#}}\}$; $i_1 \leq i_2 \leq \cdots \leq i_{r-I^{\#}}$; $i = i_{i'}$ (respectively, $I = \{i_1, i_2, \cdots, i_{I^{\#}}\}$; $i_1 \leq i_2 \leq \cdots \leq i_{I^{\#}}$; $i = i_{i'}$). Now it follows from [6], Proposition 4.22; Remark 4.24, together with a similar argument to the argument in the proof of [7], Lemma 2.7, (iv); (v), that the above diagram induces a commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z}_l(1) & \longrightarrow & \pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} & & \\
& & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}_l(1) & \longrightarrow & \pi_1(D^{\log}_{X_{(r)}I^{[i]}})^{\underline{(l)}} & &
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \Pi_{X_{(r-I^{\#}+1)}} \times_{G_K} \pi_1(\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1})^{\underline{(l)}} & \longrightarrow & 1 \\
& \downarrow & & \\
\longrightarrow & \Pi_{X_{(r-(I^{[i]})^{\#}+1)}} \times_{G_K} \pi_1(\overline{\mathcal{M}}^{\log}_{0,(I^{[i]})^{\#}+1})^{\underline{(l)}} & \longrightarrow & 1\,,
\end{array}$$

where $\pi_1(\overline{\mathcal{M}}^{\log}_{0,-})^{\underline{(l)}}$ is the geometrically pro-l log fundamental group of $\overline{\mathcal{M}}^{\log}_{0,-}$, and the horizontal sequences are exact; moreover, by considering the restriction of $D^{\log}_{X_{(r)}I} \to D^{\log}_{X_{(r)}I^{[i]}}$ to the generic point of $D^{\log}_{X_{(r)}I}$, the left-hand vertical arrow is an *isomorphism*. Thus, the kernel of the morphism $\pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \to \pi_1(D^{\log}_{X_{(r)}I^{[i]}})^{\underline{(l)}}$ (in the above diagram) is isomorphic to the kernel of the morphism $\Pi_{X_{(r-I^{\#}+1)}} \overset{\text{via } p^{\log}_{X_{(r-I^{\#})}:i'}}{\to} \Pi_{X_{(r-I^{\#})}}$ (respectively, $\pi_1(\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1})^{\underline{(l)}} \to \pi_1(\overline{\mathcal{M}}^{\log}_{0,I^{\#}})^{\underline{(l)}}$ induced by the morphism $\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1} \to \overline{\mathcal{M}}^{\log}_{0,I^{\#}}$ obtained by forgetting the i'-th section). Now the fiber of the morphism $\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1} \to \overline{\mathcal{M}}^{\log}_{0,I^{\#}}$ (obtained by forgetting the i'-th section) at a geometric point of $\operatorname{Spec} K^{\mathrm{sep}} \to \overline{\mathcal{M}}^{\log}_{0,I^{\#}}$ whose image lies on the interior of $\overline{\mathcal{M}}^{\log}_{0,I^{\#}}$ is isomorphic to the log scheme obtained by equipping $\mathbb{P}^1_{K^{\mathrm{sep}}}$ with the log structure associated to the reduced divisor consisting of $I^{\#}$ elements of $\mathbb{P}^1_{K^{\mathrm{sep}}}(K^{\mathrm{sep}})$; thus, if $i \in I$, then the kernel of the morphism $\pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \to \pi_1(D^{\log}_{X_{(r)}I^{[i]}})^{\underline{(l)}}$ (induced by the composite $p^{\log}_{X_{(r-1)}:i} \circ \delta^{\log}_{X_{(r)}I}$) is the free profinite group of rank $I^{\#}-1$. More precisely, if we denote by $\Delta_{\mathbb{P}\setminus I^{\#}}$ the pro-l fundamental group of the log scheme obtained by equipping $\mathbb{P}^1_{K^{\mathrm{sep}}}$ with the log structure associated to the reduced divisor consisting of $I^{\#}$ elements of $\mathbb{P}^1_{K^{\mathrm{sep}}}(K^{\mathrm{sep}})$, then the kernel of $\pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \to \pi_1(D^{\log}_{X_{(r)}I^{[i]}})^{\underline{(l)}}$ is naturally isomorphic to $\Delta_{\mathbb{P}\setminus I^{\#}}$; moreover, by base-changing the exact sequence

$$1 \longrightarrow \mathbb{Z}_l(1) \longrightarrow \pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \longrightarrow \Pi_{X_{(r-I^{\#}+1)}} \times_{G_K} \pi_1(\overline{\mathcal{M}}^{\log}_{0,I^{\#}})^{\underline{(l)}} \longrightarrow 1$$

via the natural inclusion $\Delta_{\mathbb{P}\setminus I^{\#}} \overset{\sim}{\to} \{1\} \times_{\{1\}} \Delta_{\mathbb{P}\setminus I^{\#}} \hookrightarrow \Pi_{X_{(r-I^{\#}+1)}} \times_{G_K} \pi_1(\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1})^{\underline{(l)}}$, we obtain an exact sequence

$$1 \longrightarrow \mathbb{Z}_l(1) \longrightarrow \mathfrak{P}_{X_{(r)}I} \longrightarrow \Delta_{\mathbb{P}\setminus I^{\#}} \longrightarrow 1\,,$$

where

$$\mathfrak{P}_{X_{(r)}I} \overset{\mathrm{def}}{=} \pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \times_{(\Pi_{X_{(r-I^{\#}+1)}} \times_{G_K} \pi_1(\overline{\mathcal{M}}^{\log}_{0,I^{\#}})^{\underline{(l)}})} \Delta_{\mathbb{P}\setminus I^{\#}}\,.$$

Now by considering the kernel of the morphism $\pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \to \pi_1(D^{\log}_{X_{(r)}I^{[i]}})^{\underline{(l)}}$

(induced by the composite $p^{\log}_{X_{(r-1)}:i} \circ \delta^{\log}_{X_{(r)}I}$), we obtain a section

$$\Delta_{\mathbb{P}\setminus I^{\#}} \longrightarrow \mathfrak{P}_{X_{(r)}I}$$

of the above exact sequence. We shall refer to this section $\Delta_{\mathbb{P}\setminus I^{\#}} \to \mathfrak{P}_{X_{(r)}I}$ of the above exact sequence as the *section of* $\mathfrak{P}_{X_{(r)}I} \to \Delta_{\mathbb{P}\setminus I^{\#}}$ *induced by* $p^{\log}_{X_{(r-1)}:i}$.

Definition 2.4. Let $r \geq 2$ be an integer, and I a subset of $\{1, 2, \cdots, r\}$ of cardinality $I^{\#} \geq 2$. Then we shall denote by $\mathfrak{I}_{X_{(r)}I}$ the kernel of the surjection

$$\mathfrak{D}_{X_{(r)}I} \twoheadrightarrow \Pi_{X_{(r-I^{\#}+1)}} \times_{G_K} \pi_1(\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1})^{\underline{(l)}}$$

obtained in the above argument. (Note that these subgroups are well-defined, up to conjugation in $\Pi_{X_{(r)}}$.) By the above argument, $\mathfrak{I}_{X_{(r+1)}\{i,j\}}$ is the inertia subgroup (well-defined, up to conjugation) associated to the cusp (of the geometric fiber of $p_{U_{X_{(r)}}:i} : U_{X_{(r+1)}} \to U_{X_{(r)}}$ at a geometric point of $U_{X_{(r)}}$) determined by the divisor $D_{X_{\underline{(r+1)}}\{i,j\}} \subseteq X_{\underline{(r+1)}}$.

Lemma 2.5. *In the above situation, the image via the section of* $\mathfrak{P}_{X_{(r)}I} \to \Delta_{\mathbb{P}\setminus I^{\#}}$ *induced by* $p^{\log}_{X_{(r-1)}:i}$ *of the* $(I^{\#}-1)$ *inertia subgroups of* $\Delta_{\mathbb{P}\setminus I^{\#}}$ (*well-defined, up to conjugation in* $\Delta_{\mathbb{P}\setminus I^{\#}}$) *corresponding to inertia subgroups associated to the cusps* (*of a geometric fiber* $\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1} \to \overline{\mathcal{M}}^{\log}_{0,I^{\#}}$ *obtained by forgetting the* i'*-th section*) *determined by the first* $(I^{\#}-1)$ *sections of* $\overline{\mathcal{M}}^{\log}_{0,I^{\#}+1} \to \overline{\mathcal{M}}^{\log}_{0,I^{\#}}$ *are conjugates of* $\mathfrak{I}_{X_{(r+1)}\{i,j\}}$ *in* $\Delta_{X_{(r)}}$, *where* $j \in I$.

Proof. Let $\overline{x}^{\log} \to D^{\log}_{X_{(r-1)}}$ be a strict geometric point of $D^{\log}_{X_{(r-1)}}$ (cf. [6], Definition 1.1, (i)) whose image is the generic point. First, we consider the log structure of $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log}$ (where the morphism $D^{\log}_{X_{(r)}I} \to D^{\log}_{X_{(r-1)}I^{[i]}}$ is the morphism induced by $p^{\log}_{X_{(r-1)}:i} \circ \delta^{\log}_{X_{(r)}I}$) and $\overline{x}^{\log}$. It is immediate that the log structure of $\overline{x}^{\log}$ has the chart:

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & k(\overline{x}) \\ n & \mapsto & 0^n . \end{array}$$

By the definitions, the underlying scheme of $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log}$ is the projective line $\mathbb{P}^1_{\overline{x}}$ over $\overline{x}$, and the log structure of $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log}$ has the following chart:

> Let $\overline{y} \to \mathbb{P}^1_{\overline{x}}$ be a geometric point of the underlying scheme $D_{X_{(r)}I} \times_{D_{X_{(r-1)}I^{[i]}}} \overline{x} \,(\simeq \mathbb{P}^1_{\overline{x}})$ of $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log}$. Then the following hold:
>
> (1) If the image of $\overline{y} \to \mathbb{P}^1_{\overline{x}}$ *does not lie* on the $D^{\log}_{X_{(r)}\{i,j\}}$'s (where $j \in I$), then the log structure of $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log}$ at $\overline{y} \to \mathbb{P}^1_{\overline{x}}$ is induced by
>
> $$\begin{array}{ccc} \mathbb{N} & \longrightarrow & k(\overline{y})[[t]] \\ n & \mapsto & 0^n . \end{array}$$

Moreover, the projection $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log} \to \overline{x}^{\log}$ has the chart:

$$\begin{array}{ccc} k(\overline{x}) & \longrightarrow & k(\overline{y})[[t]] \\ \uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{\mathrm{id}_{\mathbb{N}}} & \mathbb{N}\,. \end{array}$$

(2) If the image of $\overline{y} \to \mathbb{P}^1_{\overline{x}}$ lies on $D^{\log}_{X_{(r)}\{i,j\}}$ (where $j \in I$), then the log structure of $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log}$ at $\overline{y} \to \mathbb{P}^1_{\overline{x}}$ is induced by

$$\begin{array}{ccc} \mathbb{N}^{\oplus 2} & \longrightarrow & k(\overline{y})[[t]] \\ (n,m) & \mapsto & 0^n \cdot t^m\,. \end{array}$$

Moreover, the projection $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log} \to \overline{x}^{\log}$ has the chart:

$$\begin{array}{ccc} k(\overline{x}) & \longrightarrow & k(\overline{y})[[t]] \\ \uparrow & & \uparrow \\ \mathbb{N} & \longrightarrow & \mathbb{N}^{\oplus 2} \\ n & \mapsto & (n,0)\,. \end{array}$$

(3) If the image of $\overline{y} \to \mathbb{P}^1_{\overline{x}}$ lies on $D^{\log}_{X_{(r)}J}$ (where J is the subset of $\{1,2,\cdots,r\}$ which is uniquely determined by the condition that $J \subsetneq I$ and $J^{[i]} = I^{[i]}$), then the log structure of $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log}$ at $\overline{y} \to \mathbb{P}^1_{\overline{x}}$ is induced by

$$\begin{array}{ccc} \mathbb{N}^{\oplus 2} & \longrightarrow & k(\overline{y})[[t]] \\ (n,m) & \mapsto & 0^n \cdot t^m\,. \end{array}$$

Moreover, the projection $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log} \to \overline{x}^{\log}$ has the chart:

$$\begin{array}{ccc} k(\overline{x}) & \longrightarrow & k(\overline{y})[[t]] \\ \uparrow & & \uparrow \\ \mathbb{N} & \longrightarrow & \mathbb{N}^{\oplus 2} \\ n & \mapsto & (n,n)\,. \end{array}$$

Therefore, it is immediate that there exists a morphism $D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log} \to \mathbb{P}^{\log}_{\overline{x}}$ which is of type $\mathbb{N}$ (where $\mathbb{P}^{\log}_{\overline{x}}$ is the log scheme obtained by equipping $\mathbb{P}^1_{\overline{x}}$ with the log structure associated to the divisor determined by the divisors "$D^{\log}_{X_{(r)}I} \cap D^{\log}_{X_{(r)}\{i,j\}}$" [where $j \in I$] and "$D^{\log}_{X_{(r)}I} \cap D^{\log}_{X_{(r)}J}$" [where J is as in (3)]) which fits into a natural commutative diagram:

$$\begin{array}{ccc} D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log} & \longrightarrow & \mathbb{P}^{\log}_{\overline{x}} \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow \\ \overline{x}^{\log} & \longrightarrow & \overline{x}\,. \end{array}$$

This diagram induces a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_l(1) & \longrightarrow & \pi_1(D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log})^{(l)} & \longrightarrow & \pi_1(\mathbb{P}^{\log}_{\overline{x}})^{(l)} & \longrightarrow & 1 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle \text{via } \mathrm{pr}_2} & & \Big\downarrow & & \\
1 & \longrightarrow & \mathbb{Z}_l(1) & \longrightarrow & \pi_1(\overline{x}^{\log})^{(l)} & \longrightarrow & 1 & \longrightarrow & 1\,,
\end{array}$$

where the horizontal sequences are exact (cf. [6], Proposition 4.22; [7], Remark 2.8, (i)). By (1), the left-hand vertical arrow is an isomorphism, i.e., the right-hand square is *cartesian*. Thus, since the kernel of the middle vertical arrow $\pi_1(D^{\log}_{X_{(r)}I} \times_{D^{\log}_{X_{(r-1)}I^{[i]}}} \overline{x}^{\log})^{(l)} \overset{\text{via pr}_2}{\to} \pi_1(\overline{x}^{\log})^{(l)}$ is naturally isomorphic to the kernel of $\pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \to \pi_1(D^{\log}_{X_{(r)}I^{[i]}})^{\underline{(l)}}$, we conclude that the kernel of $\pi_1(D^{\log}_{X_{(r)}I})^{\underline{(l)}} \to \pi_1(D^{\log}_{X_{(r)}I^{[i]}})^{\underline{(l)}}$ *is naturally isomorphic to* $\pi_1(\mathbb{P}^{\log}_{\overline{x}})^{(l)}$; moreover, it follows from the definitions that this isomorphism *determines* the section of $\mathfrak{P}_{X_{(r)}I} \to \Delta_{\mathbb{P}\setminus I^{\#}} (\simeq \pi_1(\mathbb{P}^{\log}_{\overline{x}})^{(l)})$ induced by $p^{\log}_{X_{(r-1)}:i}$. Thus, Lemma 2.5 follows immediately from observations (2) and (3). $\square$

Proposition 2.6. *Let $r \geq 2$ be an integer. Then conjugates in $\Delta_{X_{(r+1)}}$ of the subgroups*

$$\mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2\}}\,;\ \mathfrak{D}^{\Delta}_{X_{(r+1)}\{2,3\}} \subseteq \Delta_{X_{(r+1)}}$$

topologically generate $\Delta_{X_{(r+1)}}$.

Proof. Since the composite

$$\mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Delta_{X_{(r+1)}} \overset{\text{via } p_{U_{X_{(r)}}:1}}{\longrightarrow} \Delta_{X_{(r)}}$$

is surjective, it is enough to show that the subgroup topologically genereated by the subgroups in question contains the kernel of the morphism $\Delta_{X_{(r+1)}} \to \Delta_{X_{(r)}}$ induced by $p_{U_{X_{(r)}}:1}$, i.e., $\Delta^{(1)}_{X_{(r+1)}/X_{(r)}}$. On the other hand, if let $\overline{x}^{\log} \to X^{\log}_{(r)}$ be a strict geometric point whose image is the generic point of the divisor $D^{\log}_{X_{(r)}\{1,2\}}$ of $X^{\log}_{(r)}$, then by [7], Proposition 1.7, the image of

$$\varprojlim \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}^{\log}_{\lambda})^{(l)} \overset{\text{via pr}_1}{\longrightarrow} \Delta_{X_{(r+1)}}$$

(where the projective limit is over all reduced covering points $\overline{x}^{\log}_{\lambda} \to \overline{x}^{\log}$) is $\Delta^{(1)}_{X_{(r+1)}/X_{(r)}}$. Moreover, since the irreducible components of the underlying scheme of $X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}^{\log}_{\lambda}$ $(= X^{\log}_{(r+1)} \times_{D^{\log}_{X_{(r)}\{1,2\}}} \overline{x}^{\log}_{\lambda})$ are the underlying schemes of $D^{\log}_{X_{(r+1)}\{2,3\}} \times_{D^{\log}_{X_{(r)}\{1,2\}}} \overline{x}^{\log}_{\lambda}$ and $D^{\log}_{X_{(r+1)}\{1,2,3\}} \times_{D^{\log}_{X_{(r)}\{1,2\}}} \overline{x}^{\log}_{\lambda}$ (cf. [7], Lemma 1.12, (iii)), by *the evident logarithmic version* of [19], Corollary 2.3.3 (cf. the proof of [19], Lemma 6.2.7), the group

$$\varprojlim \pi_1(X^{\log}_{(r+1)} \times_{X^{\log}_{(r)}} \overline{x}^{\log}_{\lambda})^{(l)}$$

is topologically generated by the images of the natural morphisms from

$$\varprojlim \pi_1(D^{\log}_{X_{(r+1)}\{2,3\}} \times_{D^{\log}_{X_{(r)}\{1,2\}}} \overline{x}^{\log}_{\lambda})^{(l)}$$

and

$$\varprojlim \pi_1(D^{\log}_{X_{(r+1)}\{1,2,3\}} \times_{D^{\log}_{X_{(r)}\{1,2\}}} \overline{x}^{\log}_{\lambda})^{(l)} .$$

Thus, it is enough to show that the subgroup topologically generated by the subgroups in question contains the image of the natural morphisms from

$$\varprojlim \pi_1(D^{\log}_{X_{(r+1)}\{2,3\}} \times_{D^{\log}_{X_{(r)}\{1,2\}}} \overline{x}^{\log}_{\lambda})^{(l)} \qquad (*_1)$$

and

$$\varprojlim \pi_1(D^{\log}_{X_{(r+1)}\{1,2,3\}} \times_{D^{\log}_{X_{(r)}\{1,2\}}} \overline{x}^{\log}_{\lambda})^{(l)} \qquad (*_2) .$$

Now since it is immediate that the natural strict morphism $D^{\log}_{X_{(r+1)}\{2,3\}} \times_{D^{\log}_{X_{(r)}\{1,2\}}} \overline{x}^{\log} \to X^{\log}_{(r+1)}$ factors through $D^{\log}_{X_{(r+1)}\{2,3\}} \otimes_K \overline{K}$, it thus follows that the image of the first group $(*_1)$ is contained in a conjugate of $\mathfrak{D}^{\Delta}_{X_{(r+1)}\{2,3\}}$. On the other hand, it follows immediately from Lemma 2.5 (together with observation (3) in the proof of Lemma 2.5), that the image of the second group $(*_2)$ is contained in the subgroup topologically generated by conjutages of the kernel of the composite

$$\mathfrak{D}^{\Delta}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Delta_{X_{(r+1)}} \overset{\text{via } p^{\log}_{X_{(r)}:1}}{\twoheadrightarrow} \Delta_{X_{(r+1)}}$$

and $\mathfrak{I}_{X_{(r+1)}\{1,2\}}$. This completes the proof of Proposition 2.6 . ☐

Lemma 2.7. *Let $r \geq 2$ and $1 \leq i < j \leq r$ be integers. Then the subgroup $\mathfrak{D}_{X_{(r)}\{i,j\}}$ (respectively, $\mathfrak{D}^{\Delta}_{X_{(r)}\{i,j\}}$) of $\Pi_{X_{(r)}}$ (respectively, $\Delta_{X_{(r)}}$) is the normalizer of $\mathfrak{I}_{X_{(r)}\{i,j\}}$ in $\Pi_{X_{(r)}}$ (respectively, $\Delta_{X_{(r)}}$).*

Proof. Since $\mathfrak{I}_{X_{(r)}\{i,j\}}$ is normal in $\mathfrak{D}_{X_{(r)}\{i,j\}}$ (respectively, $\mathfrak{D}^{\Delta}_{X_{(r)}\{i,j\}}$), the normalizer of $\mathfrak{I}_{X_{(r)}\{i,j\}}$ contains $\mathfrak{D}_{X_{(r)}\{i,j\}}$ (respectively, $\mathfrak{D}^{\Delta}_{X_{(r)}\{i,j\}}$). Moreover, we have a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathfrak{I}_{X_{(r)}\{i,j\}} \arrow[r] \arrow[d] & \mathfrak{D}_{X_{(r)}\{i,j\}} \arrow[r] \arrow[d] & \Pi_{X_{(r-1)}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \arrow[r] & \Pi_{X_{(r)}} \arrow[r, "\text{via } p^{\log}_{X_{(r-1)}:i}"'] & \Pi_{X_{(r-1)}} \arrow[r] & 1
\end{tikzcd}$$

(respectively,

$$\begin{tikzcd}
1 \arrow[r] & \mathfrak{I}_{X_{(r)}\{i,j\}} \arrow[r] \arrow[d] & \mathfrak{D}^{\Delta}_{X_{(r)}\{i,j\}} \arrow[r] \arrow[d] & \Delta_{X_{(r-1)}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \arrow[r] & \Delta_{X_{(r)}} \arrow[r, "\text{via } p^{\log}_{X_{(r-1)}:i}"'] & \Delta_{X_{(r-1)}} \arrow[r] & 1).
\end{tikzcd}$$

Therefore, it is enough to show that the normalizer of $\mathfrak{I}_{X_{(r)}\{i,j\}}$ in $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$ is $\mathfrak{I}_{X_{(r)}\{i,j\}}$. On the other hand, this is well-known (cf. e.g., [17], (2.3.1)). ☐

Remark 2.8. By a similar argument to the argument used in the proof of Lemma 2.7 (by replacing [17], (2.3.1) by [12], Lemma 1.3.12), we conclude that:

Let $r \geq 2$ and $1 \leq i < j \leq r$ be integers. Then the subgroup $\mathfrak{D}_{X_{(r)}\{i,j\}}$ (respectively, $\mathfrak{D}^{\Delta}_{X_{(r)}\{i,j\}}$) of $\Pi_{X_{(r)}}$ (respectively, $\Delta_{X_{(r)}}$) is the commensurator of $\mathfrak{I}_{X_{(r)}\{i,j\}}$ in $\Pi_{X_{(r)}}$ (respectively, $\Delta_{X_{(r)}}$).

Definition 2.9. Let $r \geq 2$ and $1 \leq i < j \leq r$ be integers.

(i) We shall denote by $U_{X_{(r)}\underline{\{i,j\}}}$ the fiber product of

$$\begin{tikzcd} & U_{X_{(r-1)}} \arrow[d, "p_{U_{X_{(r-2)}}:j-1}"] \\ U_{X_{(r-1)}} \arrow[r, "p_{U_{X_{(r-2)}}:i}"'] & U_{X_{(r-2)}} \end{tikzcd} .$$

Moreover, we shall denote by $p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:i}$ and $p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:j}$ the projections $U_{X_{(r)}\underline{\{i,j\}}} \to U_{X_{(r-1)}}$ such that $p_{U_{X_{(r-2)}}:j-1} \circ p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:i} = p_{U_{X_{(r-2)}}:i} \circ p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:j}$.

(ii) By the definition of $U_{X_{(r)}\underline{\{i,j\}}}$, the commutative diagram

$$\begin{tikzcd} U_{X_{(r)}} \arrow[r, "p_{U_{X_{(r-1)}}:i}"] \arrow[d, "p_{U_{X_{(r-1)}}:j}"'] & U_{X_{(r-1)}} \arrow[d, "p_{U_{X_{(r-2)}}:j-1}"] \\ U_{X_{(r-1)}} \arrow[r, "p_{U_{X_{(r-2)}}:i}"'] & U_{X_{(r-2)}} \end{tikzcd}$$

induces a morphism $U_{X_{(r)}} \to U_{X_{(r)}\underline{\{i,j\}}}$. We shall denote this morphism by $\iota_{U_{X_{(r)}}\underline{\{i,j\}}}$. By the definition of $\iota_{U_{X_{(r)}}\underline{\{i,j\}}}$, it is immediate that $\iota_{U_{X_{(r)}}\underline{\{i,j\}}} : U_{X_{(r)}} \to U_{X_{(r)}\underline{\{i,j\}}}$ is an open immersion, which is a "partial compactification", i.e., the natural open immersion $U_{X_{(r)}} \hookrightarrow X_{\underline{(r)}}$ factors through $\iota_{U_{X_{(r)}}\underline{\{i,j\}}}$; moreover,

$$U_{X_{(r)}\underline{\{i,j\}}} = X_{\underline{(r)}} \setminus \bigcup_{\{i',j'\} \neq \{i,j\}} D_{X_{\underline{(r)}}\{i',j'\}} .$$

(iii) We shall denote by $\Pi_{X_{(r)}\underline{\{i,j\}}}$ the geometrically pro-l fundamental group of $U_{X_{(r)}\underline{\{i,j\}}}$, and by $\Delta_{X_{(r)}\underline{\{i,j\}}}$ the kernel of the natural surjection

$$\Pi_{X_{(r)}\underline{\{i,j\}}} \twoheadrightarrow G_K .$$

Thus, we have an exact sequence

$$1 \longrightarrow \Delta_{X_{(r)}\underline{\{i,j\}}} \longrightarrow \Pi_{X_{(r)}\underline{\{i,j\}}} \longrightarrow G_K \longrightarrow 1 .$$

Lemma 2.10. *Let $r \geq 2$ and $1 \leq i < j \leq r$ be integers. Then the following diagram induced by the cartesian diagram which appears in the definition of*

$U_{X_{(r)}\underline{\{i,j\}}}$ *is cartesian:*

$$\begin{tikzcd}
\Pi_{X_{(r)}\underline{\{i,j\}}} \arrow[r, "\text{via } p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:i}"] \arrow[d, "\text{via } p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:j}"'] & \Pi_{X_{(r-1)}} \arrow[d, "\text{via } p_{U_{X_{(r-2)}}:j-1}"] \\
\Pi_{X_{(r-1)}} \arrow[r, "\text{via } p_{U_{X_{(r-2)}}:i}"'] & \Pi_{X_{(r-2)}} .
\end{tikzcd}$$

In particular, the kernel of the surjection $\Pi_{X_{(r)}\underline{\{i,j\}}} \overset{\text{via } p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:i}}{\twoheadrightarrow} \Pi_{X_{(r-1)}}$ *is naturally isomorphic to* $\Delta^{(i)}_{X_{(r-1)}/X_{(r-2)}}$.

Proof. This follows immediately from the fact that the sequence obtained as the base-change of

$$U_{X_{(r)}\underline{\{i,j\}}} \overset{p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:i}}{\longrightarrow} U_{X_{(r-1)}} \overset{p_{U_{X_{(r-2)}}:r-1}}{\longrightarrow} U_{X_{(r-2)}} \overset{p_{U_{X_{(r-3)}}:r-2}}{\longrightarrow} \cdots \overset{p_{U_{X_{(1)}}:2}}{\longrightarrow} X \longrightarrow \operatorname{Spec} K$$

from K to $\overline{K}$ is a successive extension of hyperbolic curves of product type (cf. Definition 1.12, (i)), together with Corollary 1.13, (i). $\square$

In the following, assume that

the field K is a finite field.

Let us denote by p_K (respectively, q_K) the characteristic (respectively, cardinality) of K. We shall fix an algebraic closure $\overline{K}$ of K. We shall denote by G_K the Galois group of $\overline{K}$ over K, and by $\mathrm{Fr}_K \in G_K$ the Frobenius element of G_K. Moreover, let L be a finite field whose characteristic (respectively, cardinality) we denote by p_L (respectively, q_L) such that l is invertible in L (i.e., $l \neq p_L$), $\overline{L}$ an algebraic closure of L, $G_L \overset{\text{def}}{=} \mathrm{Gal}(\overline{L}/L)$, Y a proper hyperbolic curve over L, and $\alpha_{(r)} : \Pi_{X_{(r)}} \overset{\sim}{\to} \Pi_{Y_{(r)}}$ an isomorphism. Then it follows from the "Riemann hypothesis for abelian varieties over finite fields" (cf. e.g., [16], p. 206) and the fact that $\mathbb{Z}_l(1)$ is "of weight 2" (since the eigenvalues of the action of "Fr_-" are "q_-") that the quotient $\Pi_{X_{(r)}} \twoheadrightarrow G_K$ (respectively, $\Pi_{Y_{(r)}} \twoheadrightarrow G_L$) arising from the structure morphism $U_{X_{(r)}} \to \operatorname{Spec} K$ (respectively, $U_{Y_{(r)}} \to \operatorname{Spec} L$) may be characterized as *the (unique) maximal ($\widehat{\mathbb{Z}}$-)free abelian quotient of* $\Pi_{X_{(r)}}$ (respectively, $\Pi_{Y_{(r)}}$). Therefore, the isomorphism $\alpha_{(r)}$ induces an isomorphism $\alpha_{(0)} : G_K \to G_L$.

Definition 2.11. We shall say that an isomorphism $\alpha_{(r)} : \Pi_{X_{(r)}} \overset{\sim}{\to} \Pi_{Y_{(r)}}$ is *Frobenius-preserving* if the isomorphism $\alpha_{(0)} : G_K \to G_L$ obtained as above maps the Frobenius element of G_K to the Frobenius element of G_L (cf. [14], Definition 1.18, (iii)).

Proposition 2.12. *Let $\alpha_{(r)} : \Pi_{X_{(r)}} \overset{\sim}{\to} \Pi_{Y_{(r)}}$ be an isomorphism. Then the following hold:*

(*i*) *There exists an element σ of the symmetric group on r letters such that for any integer $1 \leq i \leq r$, the isomorphism $\alpha_{(r)}$ maps the kernel $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$*

of the surjection $\Pi_{X_{(r)}} \twoheadrightarrow \Pi_{X_{(r-1)}}$ *induced by* $p^{\log}_{X_{(r-1)}:i}$ *bijectively onto the kernel* $\Delta^{(\sigma(i))}_{Y_{(r)}/Y_{(r-1)}}$ *of the surjection* $\Pi_{Y_{(r)}} \twoheadrightarrow \Pi_{Y_{(r-1)}}$ *induced by* $p^{\log}_{Y_{(r-1)}:\sigma(i)}$.

(*ii*) *Assume, moreover, that* $\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}}$ *is a Frobenius-preserving isomorphism* (*cf. Definition 2.11*). *Then, for a section* $G_K \to \Pi_{X_{(r)}}$ *of the natural morphism* $\Pi_{X_{(r)}} \to G_K$*, this section arises from a* K*-rational point of* $U_{X_{(r)}}$ *if and only if the section of the natural morphism* $\Pi_{Y_{(r)}} \to G_L$ *corresponding to the section* $G_K \to \Pi_{X_{(r)}}$ *under the isomorphism* $\alpha_{(r)}$ *arises from a* L*-rational point of* $U_{Y_{(r)}}$.

(*iii*) *Assume, moreover, that* $r \geq 2$. *Then, for any integers* $1 \leq i < j \leq r$*, the isomorphism* $\alpha_{(r)}$ *maps* $\mathfrak{I}_{X_{(r)}\{i,j\}}$ (*respectively,* $\mathfrak{D}_{X_{(r)}\{i,j\}}$) *bijectively onto a conjugate of* $\mathfrak{I}_{Y_{(r)}\{\sigma(i),\sigma(j)\}}$ (*respectively,* $\mathfrak{D}_{Y_{(r)}\{\sigma(i),\sigma(j)\}}$) *by an element of the kernel* $\Delta_{Y_{(r)}}$ *of the natural surjection* $\Pi_{Y_{(r)}} \to G_L$.

(*iv*) *Under the assumption in the statement of* (*iii*)*, for any integers* $1 \leq i < j \leq r$*, let us denote by*

$$\tau_{X_{(r-1)}\{i,j\}} : \Pi_{X_{(r)}}/\Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \xrightarrow{\sim} \Pi_{X_{(r)}}/\Delta^{(j)}_{X_{(r)}/X_{(r-1)}}$$

(*respectively,*

$$\tau_{Y_{(r-1)}\{i,j\}} : \Pi_{Y_{(r)}}/\Delta^{(i)}_{Y_{(r)}/Y_{(r-1)}} \xrightarrow{\sim} \Pi_{Y_{(r)}}/\Delta^{(j)}_{Y_{(r)}/Y_{(r-1)}})$$

the isomorphism obtained as the composite

$$\Pi_{X_{(r)}}/\Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \xrightarrow{\sim} \Pi_{X_{(r-1)}} \xleftarrow{\sim} \Pi_{X_{(r)}}/\Delta^{(j)}_{X_{(r)}/X_{(r-1)}}$$

(*respectively,*

$$\Pi_{Y_{(r)}}/\Delta^{(i)}_{Y_{(r)}/Y_{(r-1)}} \xrightarrow{\sim} \Pi_{Y_{(r-1)}} \xleftarrow{\sim} \Pi_{Y_{(r)}}/\Delta^{(j)}_{Y_{(r)}/Y_{(r-1)}}).$$

Then the following diagram commutes:

$$\begin{tikzcd}
\Pi_{X_{(r)}}/\Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \arrow[r, "\tau_{X_{(r-1)}\{i,j\}}"] \arrow[d, "\text{via } \alpha_{(r)}"'] & \Pi_{X_{(r)}}/\Delta^{(j)}_{X_{(r)}/X_{(r-1)}} \arrow[d, "\text{via } \alpha_{(r)}"] \\
\Pi_{Y_{(r)}}/\Delta^{(\sigma(i))}_{Y_{(r)}/Y_{(r-1)}} \arrow[r, "\tau_{Y_{(r-1)}\{\sigma(i),\sigma(j)\}}"'] & \Pi_{Y_{(r)}}/\Delta^{(\sigma(j))}_{Y_{(r)}/Y_{(r-1)}} .
\end{tikzcd}$$

Here, the vertical arrows are the isomorphisms induced by $\alpha_{(r)}$ (*cf.* (*i*)).

Proof. Assertion (i) follows from the fact that an isomorphism of $\Pi_{X_{(r)}}$ with $\Pi_{Y_{(r)}}$ induces an isomorphism of $\Delta_{X_{(r)}}$ with $\Delta_{Y_{(r)}}$, together with [15], Corollary 6.7.

Next, we prove assertion (ii). If $r = 1$, then this follows from [14], Remark 1.18.2. Thus, assume that $r \geq 2$. Then it is immediate that for a section $s : G_K \to \Pi_{X_{(r)}}$ of the natural morphism $\Pi_{X_{(r)}} \to G_K$, the section arises from a K-rational point of $U_{X_{(r)}}$ if and only if the composite of the section s and the morphism $\Pi_{X_{(r)}} \to \Pi_{X_{(r-1)}}$ induced by $p^{\log}_{X_{(r-1)}:r}$ arises from a K-rational point of $U_{X_{(r-1)}}$, and the section $G_K \to \Pi_{X_{(r)}} \times_{\Pi_{X_{(r-1)}}} G_K$ (where the morphism $\Pi_{X_{(r)}} \to \Pi_{X_{(r-1)}}$ is the morphism induced by $p^{\log}_{X_{(r-1)}:r}$, and $G_K \to \Pi_{X_{(r-1)}}$ is the composite) induced by the given section s arises from a K-rational point of the hyperbolic curve obtained as the fiber. Therefore, assertion (ii) follows from [14], Remark 1.18.2, together with induction on r.

Next, we prove assertion (iii). It is immediate that there exists an open subgroup of $G_{K'} \subseteq G_K$ and a section $G_{K'} \to \Pi_{X_{(r)}} \times_{G_K} G_{K'}$ such that this section arises from a K'-rational point of $U_{X_{(r)}}$. Thus, it follows from assertion (ii), the fact that $\mathfrak{I}_{X_{(r)}\{i,j\}}$ is an inertia subgroup of $\Pi_{X_{(r)}} \times_{\Pi_{X_{(r-1)}}} G_{K'}$ (where the morphism $\Pi_{X_{(r)}} \to \Pi_{X_{(r-1)}}$ is the morphism induced by $p^{\log}_{X_{(r-1)}:r}$, and $G_{K'} \to \Pi_{X_{(r-1)}}$ is the composite of the section and the morphism induced by $p^{\log}_{X_{(r-1)}:r}$) associated to a cusp of the hyperbolic curve obtained as the fiber, together with a similar argument to the argument used in the proof of [12], Lemma 1.3.9, that $\alpha_{(r)}$ maps $\mathfrak{I}_{X_{(r)}\{i,j\}}$ bijectively onto a conjugate (in $\Delta^{(\sigma(i))}_{Y_{(r)}/Y_{(r-1)}}$) of $\mathfrak{I}_{Y_{(r)}\{\sigma(i),\sigma(j)\}}$. On the other hand, the assertion that $\alpha_{(r)}$ maps $\mathfrak{D}_{X_{(r)}\{i,j\}}$ bijectively onto a conjugate (in $\Delta^{(\sigma(i))}_{Y_{(r)}/Y_{(r-1)}}$) of $\mathfrak{D}_{Y_{(r)}\{\sigma(i),\sigma(j)\}}$ follows from the fact that $\alpha_{(r)}$ maps $\mathfrak{I}_{X_{(r)}\{i,j\}}$ bijectively onto a conjugate (in $\Delta^{(\sigma(i))}_{Y_{(r)}/Y_{(r-1)}}$) of $\mathfrak{I}_{Y_{(r)}\{\sigma(i),\sigma(j)\}}$, together with Lemma 2.7. This completes the proof of assertion (iii).

Finally, we prove assertion (iv). By the discussion preceding Definition 2.3, we have commutative diagrams

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathfrak{I}_{X_{(r)}\{i,j\}} & \longrightarrow & \mathfrak{D}_{X_{(r)}\{i,j\}} & \longrightarrow & \Pi_{X_{(r-1)}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^{(i)}_{X_{(r)}/X_{(r-1)}} & \longrightarrow & \Pi_{X_{(r)}} & \xrightarrow[\text{via } p^{\log}_{X_{(r-1)}:i}]{} & \Pi_{X_{(r-1)}} & \longrightarrow & 1
\end{array}$$

and

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathfrak{I}_{X_{(r)}\{i,j\}} & \longrightarrow & \mathfrak{D}_{X_{(r)}\{i,j\}} & \longrightarrow & \Pi_{X_{(r-1)}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^{(j)}_{X_{(r)}/X_{(r-1)}} & \longrightarrow & \Pi_{X_{(r)}} & \xrightarrow[\text{via } p^{\log}_{X_{(r-1)}:j}]{} & \Pi_{X_{(r-1)}} & \longrightarrow & 1\,,
\end{array}$$

where the horizontal sequences are exact. In particular, the natural inclusion $\mathfrak{D}_{X_{(r)}\{i,j\}} \hookrightarrow \Pi_{X_{(r)}}$ induces isomorphisms

$$\mathfrak{D}_{X_{(r)}\{i,j\}}/\mathfrak{I}_{X_{(r)}\{i,j\}} \xrightarrow{\sim} \Pi_{X_{(r)}}/\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$$

and

$$\mathfrak{D}_{X_{(r)}\{i,j\}}/\mathfrak{I}_{X_{(r)}\{i,j\}} \xrightarrow{\sim} \Pi_{X_{(r)}}/\Delta^{(j)}_{X_{(r)}/X_{(r-1)}}\,.$$

Thus, the isomorphism $\tau_{X_{(r-1)}\{i,j\}}$ coincides with the composite

$$\Pi_{X_{(r)}}/\Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \xleftarrow{\sim} \mathfrak{D}_{X_{(r)}\{i,j\}}/\mathfrak{I}_{X_{(r)}\{i,j\}} \xrightarrow{\sim} \Pi_{X_{(r)}}/\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}\,.$$

Therefore, to verify the commutativity of the diagram in the statement of Proposition 2.12, (iv), it is enough to show that the isomorphism $\alpha_{(r)}$ maps $\mathfrak{D}_{X_{(r)}\{i,j\}}$ (respectively, $\mathfrak{I}_{X_{(r)}\{i,j\}}$) bijectively onto a conjugate of $\mathfrak{D}_{Y_{(r)}\{\sigma(i),\sigma(j)\}}$ (respectively, $\mathfrak{I}_{Y_{(r)}\{\sigma(i),\sigma(j)\}}$). On the other hand, this follows from (iii). □

Definition 2.13. Let $\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}}$ be an isomorphism.

(i) We shall denote by $\sigma_{\alpha_{(r)}}$ the element of the symmetric group on r letters defined in Proposition 2.12, (i).

(ii) We shall say that $\alpha_{(r)}$ is *order-preserving* if $\sigma_{\alpha_{(r)}}$ (defined in (i)) is the identity morphism. Note that by reordering the coordinates of $U_{Y_{(r)}}$, one can always assume that $\alpha_{(r)}$ is order-preserving.

Let $\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}}$ be a Frobenius-preserving and order-preserving isomorphism. Now by means of the isomorphisms $\tau_{X_{(r-1)}\{i,j\}}$ (respectively, $\tau_{Y_{(r-1)}\{i,j\}}$) defined in Proposition 2.12, (iv), we identify the quotients $\Pi_{X_{(r)}}/\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$ (respectively, $\Pi_{Y_{(r)}}/\Delta^{(i)}_{Y_{(r)}/Y_{(r-1)}}$), where $i = 1, \cdots, r$, of $\Pi_{X_{(r)}}$ (respectively, $\Pi_{Y_{(r)}}$) and denote by $\Pi_{X_{(r-1)}}$ (respectively, $\Pi_{Y_{(r-1)}}$) the quotient obtained from this identification, and by $\alpha_{(r-1)}$ the isomorphism of $\Pi_{X_{(r-1)}}$ with $\Pi_{Y_{(r-1)}}$ induced by $\alpha_{(r)}$ (cf. Proposition 2.12, (i)). Note that this isomorphism $\alpha_{(r-1)}$ is *independent* of i. Moreover, by a similar argument to this argument, for any positive integer $r' \leq r$, we obtain a quotient $\Pi_{X_{(r')}}$ (respectively, $\Pi_{Y_{(r')}}$) of $\Pi_{X_{(r)}}$ (respectively, $\Pi_{Y_{(r)}}$) and an isomorphism $\alpha_{(r')} : \Pi_{X_{(r')}} \xrightarrow{\sim} \Pi_{Y_{(r')}}$. Note that it follows immediately from the definition of the term "Frobenius-preserving" that the isomorphism $\alpha_{(r')} : \Pi_{X_{(r')}} \xrightarrow{\sim} \Pi_{Y_{(r')}}$ ($r' \leq r$) is also Frobenius-preserving.

3 Isomorphisms that preserve the fundamental groups of tripods

In this section, we define the notion of a *tripod-preserving isomorphism* (where we refer to the discussion entitled "*Curves*" in Introduction concerning the term "tripod").

In the following, let K (respectively, L) be a finite field whose cardinality we denote by q_K (respectively, q_L), $\overline{K}$ (respectively, $\overline{L}$) an algebraic closure of K (respectively, L), X (respectively, Y) a proper hyperbolic curve of genus $g_X \geq 2$ (respectively, $g_Y \geq 2$) over K (respectively, L), and l a prime number which is invertible in K and L. Let us write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$ and $G_L \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{L}/L)$. Moreover, let us denote by $\Pi_{\mathbb{P}_K}$ (respectively, $\Pi_{\mathbb{P}_L}$) the geometrically pro-l log fundamental group of the log scheme $\mathbb{P}_K^{\log}$ (respectively, $\mathbb{P}_L^{\log}$) obtained by equipping $\mathbb{P}_K^1$ (respectively, $\mathbb{P}_L^1$) with the log structure associated to the divisor $\{0, 1, \infty\}$, and by $\Delta_{\mathbb{P}_K}$ (respectively, $\Delta_{\mathbb{P}_L}$) the kernel of the natural surjection $\Pi_{\mathbb{P}_K} \twoheadrightarrow G_K$ (respectively, $\Pi_{\mathbb{P}_L} \twoheadrightarrow G_L$).

Write $E \stackrel{\mathrm{def}}{=} \mathbb{P}_K^1(\overline{K}) \setminus U_{\mathbb{P}}(\overline{K})$ (where $U_{\mathbb{P}} \subseteq \mathbb{P}_K^1$ is the interior of $\mathbb{P}_K^{\log}$, i.e., $U_{\mathbb{P}} = \mathbb{P}_K^1 \setminus \{0, 1, \infty\}$), and $\mathfrak{I}_e \subseteq \Delta_{\mathbb{P}_K}$ for an inertia subgroup associated to $e \in E$ (well-defined, up to conjugation in $\Delta_{\mathbb{P}_K}$). Then it is immediate that the composites

$$\mathfrak{I}_e \hookrightarrow \Delta_{\mathbb{P}_K} \twoheadrightarrow (\Delta_{\mathbb{P}_K})^{\mathrm{ab}}$$

induce an isomorphism

$$\mathfrak{I}_{e_1} \oplus \mathfrak{I}_{e_2} \xrightarrow{\sim} (\Delta_{\mathbb{P}_K})^{\mathrm{ab}},$$

where $e_1 \neq e_2$. Moreover, there exists a generator $\zeta_{e_i} \in \mathfrak{I}_{e_i}$ ($i = 1, 2, 3$; $e_i \neq e_j$ if $i \neq j$) such that the image of ζ_{e_3} via the composite

$$\mathfrak{I}_{e_3} \hookrightarrow \Delta_{\mathbb{P}_K} \twoheadrightarrow (\Delta_{\mathbb{P}_K})^{\mathrm{ab}} \xleftarrow{\sim} \mathfrak{I}_{e_1} \oplus \mathfrak{I}_{e_2}$$

is $(-\zeta_{e_1}, -\zeta_{e_2}) \in \mathfrak{I}_{e_1} \oplus \mathfrak{I}_{e_2}$, i.e., the image of the above composite is generated by $(\zeta_{e_1}, \zeta_{e_2}) \in \mathfrak{I}_{e_1} \oplus \mathfrak{I}_{e_2}$. Thus, if an automorphism $\overline{\phi}$ of $(\Delta_{\mathbb{P}_K})^{\mathrm{ab}}$ maps the image of $\mathfrak{I}_{e_i}$ in $(\Delta_{\mathbb{P}_K})^{\mathrm{ab}}$ ($i = 1, 2, 3$) bijectively onto the image of $\mathfrak{I}_{\sigma(e_i)}$ in $(\Delta_{\mathbb{P}_K})^{\mathrm{ab}}$ (where σ is an element of the group $\mathrm{Aut}(E)$ of automorphisms of E), then there exists a *unique* element $d_{\overline{\phi}} \in \mathbb{Z}_l^*$ such that

$$\overline{\phi}(\zeta_{e_i}) = d_{\overline{\phi}} \cdot \zeta_{\sigma(e_i)} \ \ (i = 1, 2, 3).$$

Let $\phi : \Pi_{\mathbb{P}_K} \xrightarrow{\sim} \Pi_{\mathbb{P}_K}$ be a Frobenius-preserving automorphism. Then the automorphism ϕ preserves the inertia subgroups up to conjugation. (Indeed, this follows from a similar argument to the argument used in the proof of [12], Lemma 1.3.9.) Therefore, by the above observation, we obtain an element $d_{\overline{\phi}} \in \mathbb{Z}_l^*$, where $\overline{\phi}$ is the automorphism of $(\Delta_{\mathbb{P}_K})^{\mathrm{ab}}$ induced by ϕ.

Next, let $\phi : \Pi_{\mathbb{P}_K} \xrightarrow{\sim} \Pi_{\mathbb{P}_L}$ be a Frobenius-preserving isomorphism. Then it follows from the existence of such an isomorphism that $q_K = q_L$ (by considering the action of the respective Frobenius elements on $(\Delta_{\mathbb{P}_K})^{\mathrm{ab}}$ and $(\Delta_{\mathbb{P}_L})^{\mathrm{ab}}$). In particular, the fields K and L are *isomorphic*. By means of some isomorphism of fields $K \xrightarrow{\sim} L$, we obtain an isomorphism $\Pi_{\mathbb{P}_K} \xrightarrow{\sim} \Pi_{\mathbb{P}_L}$.

In summary, we obtain a composite map

$$\mathrm{Isom}_{\mathrm{Frob}}(\Pi_{\mathbb{P}_K}, \Pi_{\mathbb{P}_L})/\mathrm{Inn}(\Delta_{\mathbb{P}_L}) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Frob}}(\Pi_{\mathbb{P}_K})/\mathrm{Inn}(\Delta_{\mathbb{P}_K})$$

$$
\begin{array}{cccc}
\longrightarrow & \mathrm{Aut}_{\mathrm{Iner}}((\Delta_{\mathbb{P}_K})^{\mathrm{ab}}) & \longrightarrow & \mathbb{Z}_l^* \\
& \overline{\phi} & \mapsto & d_{\overline{\phi}},
\end{array}
$$

where $\mathrm{Isom}_{\mathrm{Frob}}(\Pi_{\mathbb{P}_K}, \Pi_{\mathbb{P}_L})$ (respectively, $\mathrm{Aut}_{\mathrm{Frob}}(\Pi_{\mathbb{P}_K})$) is the set of Frobenius-preserving isomorphisms (respectively, automorphisms) of $\Pi_{\mathbb{P}_K}$ with $\Pi_{\mathbb{P}_L}$ (respectively, of $\Pi_{\mathbb{P}_K}$), $\mathrm{Aut}_{\mathrm{Iner}}((\Delta_{\mathbb{P}_K})^{\mathrm{ab}})$ is the set of automorphisms of $(\Delta_{\mathbb{P}_K})^{\mathrm{ab}}$ which preserve the images of the three inertia subgroups in $(\Delta_{\mathbb{P}_K})^{\mathrm{ab}}$, and the first arrow is the bijection induced by some isomorphism of fields $K \xrightarrow{\sim} L$. Note that this composite *depends* on the choice of an isomorphism of K with L; however, the image of this composite is *independent* of the choice of an isomorphism of K with L.

Definition 3.1. We shall refer to the image of this composite

$$
\mathrm{Isom}_{\mathrm{Frob}}(\Pi_{\mathbb{P}_K}, \Pi_{\mathbb{P}_L})/\mathrm{Inn}(\Delta_{\mathbb{P}_L}) \xrightarrow{\sim} \mathrm{Aut}_{\mathrm{Frob}}(\Pi_{\mathbb{P}_K})/\mathrm{Inn}(\Delta_{\mathbb{P}_K})
$$

$$
\begin{array}{cccc}
\longrightarrow & \mathrm{Aut}_{\mathrm{Iner}}((\Delta_{\mathbb{P}_K})^{\mathrm{ab}}) & \longrightarrow & \mathbb{Z}_l^* \\
& \overline{\phi} & \mapsto & d_{\overline{\phi}},
\end{array}
$$

as the *set of tripod-degrees* (*over* K). We shall refer to an element of the set of tripod-degrees (over K) as a *tripod-degree* (*over* K).

Remark 3.2.

(i) The set of tripod-degrees (over K) only depends on K ($\simeq L$) and l.

(ii) Since the image of the composite

$$
\mathrm{Isom}(\mathbb{P}_K^{\log}, \mathbb{P}_L^{\log}) \longrightarrow \mathrm{Isom}_{\mathrm{Frob}}(\Pi_{\mathbb{P}_K}, \Pi_{\mathbb{P}_L})/\mathrm{Inn}(\Delta_{\mathbb{P}_L}) \longrightarrow \mathbb{Z}_l^*
$$

(where $\mathrm{Isom}(\mathbb{P}_K^{\log}, \mathbb{P}_L^{\log})$ is the set of isomorphisms of $\mathbb{P}_K^{\log}$ with $\mathbb{P}_L^{\log}$ [as log schemes], the first arrow is the morphism induced by the functoriality of the functor of taking the log fundamental group, and the second arrow is the morphism defining the set of tripod-degrees) is the set $\langle q_K \rangle$ generated by $q_K \in \mathbb{Z}_l^*$, the set of tripod-degrees (over K) contains $\langle q_K \rangle \subseteq \mathbb{Z}_l^*$. In particular, if $\langle q_K \rangle = \mathbb{Z}_l^*$, then any element of $\mathbb{Z}_l^*$ is a tripod-degree (over K).

(iii) By an unpublished result of Akio Tamagawa, in general, the set of tripod-degrees (over K) is a *proper* subset of $\mathbb{Z}_l^*$.

Next, let $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ be a Frobenius-preserving isomorphism. Then it follows from the existence of the isomorphism α that $q_K = q_L$ (by considering the actions of the respective Frobenius elements on $H^2(\Delta_X, \mathbb{Z}_l)$ and $H^2(\Delta_Y, \mathbb{Z}_l)$). In particular, the fields K and L are *isomorphic*. By means of some isomorphism of fields $K \xrightarrow{\sim} L$, we obtain an isomorphism $\mathbb{P}_K^{\log} \xrightarrow{\sim} \mathbb{P}_L^{\log}$. Now by considering the composite of $m_{X_{(r+1)}\{1,2\}} : M_X \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{Z}_l}(H^2(\Delta_X, \mathbb{Z}_l), \mathbb{Z}_l)) \xrightarrow{\sim} \mathfrak{I}_{X_{(r+1)}\{1,2\}}$ (respectively, $m_{Y_{(r+1)}\{1,2\}} : M_Y \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{Z}_l}(H^2(\Delta_Y, \mathbb{Z}_l), \mathbb{Z}_l)) \xrightarrow{\sim} \mathfrak{I}_{Y_{(r+1)}\{1,2\}}$) (cf. Definition 4.7 below) and the isomorphism of $\mathfrak{I}_{X_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{I}_{Y_{(r+1)}\{1,2\}}$) with an inertia subgroup of $\Delta_{\mathbb{P}_K \setminus 3}$ (respectively, $\Delta_{\mathbb{P}_L \setminus 3}$) obtained in Lemma 2.5, we obtain a natural isomorphism of M_X (respectively, M_Y) with an inertia subgroup of $\Delta_{\mathbb{P}_K \setminus 3}$ (respectively, $\Delta_{\mathbb{P}_L \setminus 3}$) (cf. the discussion following

Definition 2.3). Thus, by means of the isomorphism $\mathbb{P}_K^{\log} \xrightarrow{\sim} \mathbb{P}_L^{\log}$, we obtain an isomorphism $M_X \xrightarrow{\sim} M_Y$ (cf. Remark 3.3 below). Therefore, we obtain a composite map

$$\mathrm{Isom}_{\mathrm{Frob}}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_Y) \longrightarrow \mathrm{Isom}(M_X, M_Y) \xrightarrow{\sim} \mathrm{Aut}(M_X) \xrightarrow{\sim} \mathbb{Z}_l^* ,$$

where $\mathrm{Isom}_{\mathrm{Frob}}(\Pi_X, \Pi_Y)$ is the set of Frobenius-preserving isomorphisms of Π_X with Π_Y, the second arrow is the bijection induced by some isomorphism of fields $K \xrightarrow{\sim} L$. Note that this composite *depends* on the choice of an isomorphism of K with L; however, the image of this composite is *independent* of the choice of an isomorphism of K with L.

Remark 3.3.

(i) Note that the isomorphism $M_X \xrightarrow{\sim} M_Y$ (obtained as above) is *independent* of α (by construction); moreover, this isomorphism is "geometric", i.e., it arises from an isomorphism $\mathbb{P}_K^{\log} \xrightarrow{\sim} \mathbb{P}_L^{\log}$.

(ii) The morphism

$$\mathrm{Isom}(M_X, M_Y) \xrightarrow{\sim} \mathrm{Aut}(M_X)$$

(appearing in the above composite map) may be interpreted as a certain "automorphization" of isomorphisms of M_X with M_Y by means of the "geometric" isomorphism of (i), that is independent of α.

Definition 3.4. Let $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ be a Frobenius-preserving isomorphism.

(i) We shall denote by $\deg(\alpha) \in \mathbb{Z}_l^*$ the image of α via the composite

$$\mathrm{Isom}_{\mathrm{Frob}}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_Y) \longrightarrow \mathrm{Isom}(M_X, M_Y) \xrightarrow{\sim} \mathrm{Aut}(M_X) \xrightarrow{\sim} \mathbb{Z}_l^* .$$

Note that $\deg(\alpha)$ *depends* on the choice of an isomorphism of K with L.

(ii) We shall say that $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ is *tripod-preserving* if $\deg(\alpha)$ is a tripod-degree (over K). Note that this condition is *independent* of the choice of an isomorphism of K with L.

Next, let $\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}}$ be a Frobenius-preserving and order-preserving isomorphism.

Definition 3.5. We shall say that $\alpha_{(r)}$ is *tripod-preserving* if the isomorphism $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$ induced by $\alpha_{(r)}$ (cf. the discussion following Definition 2.13) is tripod-preserving (cf. Definition 3.4).

Lemma 3.6. *If $r \geq 3$, then $\alpha_{(r)}$ is tripod-preserving.*

Proof. Now it is immediate that there exists an open subgroup $G_{K'} \subseteq G_K$, and a section $G_{K'} \to \Pi_{X_{(r-2)}}$ which arises from a K'-rational point of $U_{X_{(r-2)}}$. By base-chaging this section via the composite

$$\mathfrak{D}_{X_{(r-1)}\{1,2\}} \hookrightarrow \Pi_{X_{(r-1)}} \xrightarrow{\text{via } p^{\log}_{X_{(r-2)}:1}} \Pi_{X_{(r-2)}} ,$$

we obtain a morphism

$$s : G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}} \xrightarrow{\mathrm{pr}_2} \mathfrak{D}_{X_{(r-1)}\{1,2\}} \hookrightarrow \Pi_{X_{(r-1)}} .$$

It is immediate that this morphism arises from a "strict log K'-rational point" of $X^{\log}_{(r-1)}$ (i.e., a K'-rational point of $X_{(r-1)}$ equipped with the log structure induced by the log structure of $X^{\log}_{(r-1)}$) for which the image of the underlying morphism of schemes lies on the open subscheme of $D_{X_{(r-1)}\{1,2\}}$ on which the stalk of the characteristic sheaf (cf. the discussion entitled "*Log schemes*" in Introduction) of $D^{\log}_{X_{(r-1)}\{1,2\}}$ is isomorphic to $\mathbb{N}$. Thus, the fiber product

$$(G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}}) \times_{\Pi_{X_{(r-1)}}} \Pi_{X_{(r)}}$$

(where the morphism $G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}} \to \Pi_{X_{(r-1)}}$ is s, and $\Pi_{X_{(r)}} \to \Pi_{X_{(r-1)}}$ is the morphism induced by $p^{\log}_{X_{(r-1)}:1}$) is isomorphic to the geometrically pro-l log fundamental group of the log scheme obtained as the fiber of $p^{\log}_{X_{(r-1)}:1}$ at the "strict log K'-rational point" of $X^{\log}_{(r-1)}$ corresponding to s, and the morphism

$$(G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}}) \times_{\Pi_{X_{(r-1)}}} \Pi_{X_{(r)}} \xrightarrow{\mathrm{pr}_1} G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}}$$

coincides with the morphism induced by the structure morphism of the log scheme (obtained as the fiber of $p^{\log}_{X_{(r-1)}:1}$ at the "strict log K'-rational point" of $X^{\log}_{(r-1)}$). Now it is immediate that the underlying scheme of the log scheme obtained as such a fiber has exactly two irreducible components of genera 0 and g_X; moreover, if we denote by H the closed subgroup of

$$(G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}}) \times_{\Pi_{X_{(r-1)}}} \Pi_{X_{(r)}}$$

(well-defined, up to conjugation) obtained as the image of the morphism induced on geometrically pro-l log fundamental groups by the strict closed immersion from the irreducible component of genus 0, then the kernel H^{Δ} of the composite

$$H \hookrightarrow (G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}}) \times_{\Pi_{X_{(r-1)}}} \Pi_{X_{(r)}} \xrightarrow{\mathrm{pr}_1} G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}}$$

is isomorphic to $\Delta_{\mathbb{P}_K}$ (cf. the discussion following Definition 2.3). On the other hand, it follows that the outer representation

$$G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}} \xrightarrow{\rho_H} \mathrm{Out}(H^{\Delta})$$

determined by the exact sequence

$$1 \longrightarrow H^{\Delta} \longrightarrow H \longrightarrow G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}} \longrightarrow 1$$

factors through $G_{K'} \times_{\Pi_{X_{(r-2)}}} \mathfrak{D}_{X_{(r-1)}\{1,2\}} \xrightarrow{\mathrm{pr}_1} G_{K'}$, and the profinte group

$$H^{\Delta} \overset{\mathrm{out}}{\rtimes} G_{K'}$$

(where $G_{K'} \to \mathrm{Out}(H^{\Delta})$ is the morphism induced by ρ_H) is isomorphic to the geometrically pro-l fundamental group of $\mathbb{P}^1_{K'} \setminus \{0, 1, \infty\}$ (cf. observations (1), (2), and (3) in the proof of Lemma 2.5). Therefore, Lemma 3.6 follows from Proposition 2.12, (ii), (iii); [13], Corollary 2.8. $\square$

4 The reconstruction of the fundamental group of the configuration space

In this section, we reconstruct the geometrically pro-l fundamental group of the higher dimensional configuration space.

Let K be a finite field whose characteristic (respectively, cardinality) we denote by p_K (respectively, q_K), and l a prime number that is invertible in K. We shall fix an algebraic closure $\overline{K}$ of K. We shall denote by G_K the Galois group of $\overline{K}$ over K, and by $\mathrm{Fr}_K \in G_K$ the Frobenius element of G_K. Moreover, in the following, let X be a proper hyperbolic curve of genus $g_X \geq 2$ over K.

Definition 4.1. Let r be a natural number.

(i) We shall denote by

$$\{\Delta_{X_{(r)}}(n)\}$$

the central filtration of $\Delta_{X_{(r)}}$ defined in Definition 1.12, (ii), associated to the successive extension of hyperbolic curves of product type obtained as the base-change of

$$U_{X_{(r)}} \xrightarrow{p_{U_{X_{(r-1)}}}:r} U_{X_{(r-1)}} \xrightarrow{p_{U_{X_{(r-2)}}}:r-1} \cdots \xrightarrow{p_{U_{X_{(1)}}}:2} X \longrightarrow \operatorname{Spec} K$$

from K to $\overline{K}$ i.e., the central filtration with respect to the natural surjection

$$\Delta_{X_{(r)}} \twoheadrightarrow \Delta^{\mathrm{ab}}_{\underline{X_{(r)}}},$$

and by

$$\{\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}(n)\}$$

the central filtration of $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$ defined in the discussion following Lemma 1.2, associated to the family of smooth curves

$$U_{X_{(r)}} \otimes_K \overline{K} \xrightarrow{\text{via } p_{U_{X_{(r-1)}}}:i} U_{X_{(r-1)}} \otimes_K \overline{K},$$

i.e., the central filtration with respect to the natural surjection

$$\Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \twoheadrightarrow \Delta^{\mathrm{ab}}_{X}.$$

(ii) The sequence obtained as the base-change of

$$U_{X_{(r)}\underline{\{i,j\}}} \xrightarrow{p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}}:i} U_{X_{(r-1)}} \xrightarrow{p_{U_{X_{(r-2)}}}:r-1} U_{X_{(r-2)}} \xrightarrow{p_{U_{X_{(r-3)}}}:r-2} \cdots \xrightarrow{p_{U_{X_{(1)}}}:2} X \longrightarrow \operatorname{Spec} K$$

from K to $\overline{K}$ is a successive extension of hyperbolic curves of product type. We shall denote by

$$\{\Delta_{X_{(r)}\underline{\{i,j\}}}(n)\}$$

the central filtration of $\Delta_{X_{(r)}\underline{\{i,j\}}}$ defined in Definition 1.12, (ii), associated to this successive extension of hyperbolic curves of product type, i.e., the central filtration with respect to the natural surjection

$$\Delta_{X_{(r)}\underline{\{i,j\}}} \twoheadrightarrow \Delta^{\mathrm{ab}}_{\underline{X_{(r)}}}.$$

Proposition 4.2. *Let r be a natural number.*

(*i*) *The sequence of graded Lie algebras*

$$1 \longrightarrow \mathrm{Gr}(\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}) \longrightarrow \mathrm{Gr}(\Delta_{X_{(r+1)}}) \overset{\text{via } p_{U_{X_{(r)}}:i}}{\longrightarrow} \mathrm{Gr}(\Delta_{X_{(r)}}) \longrightarrow 1$$

induced by the exact sequence obtained in Definition 2.1, (*iv*), *is exact. In particular, the graded Lie algebra* $\mathrm{Gr}(\Delta_{X_{(r)}})$ *is center-free.*

(*ii*) *There exist* $2g_X$ *elements*

$$\alpha^{(i)}_{X,1};\cdots;\alpha^{(i)}_{X,g_X};\beta^{(i)}_{X,1};\cdots;\beta^{(i)}_{X,g_X} \in \Delta^{(i)}_{X_{(r+1)}/X_{(r)}} \setminus \Delta^{(i)}_{X_{(r+1)}/X_{(r)}}(2)$$

and r elements

$$\zeta^{(i)}_{X,1};\cdots;\zeta^{(i)}_{X,i-1};\zeta^{(i)}_{X,i+1};\cdots;\zeta^{(i)}_{X,r+1} \in \Delta^{(i)}_{X_{(r+1)}/X_{(r)}}(2) \setminus \Delta^{(i)}_{X_{(r+1)}/X_{(r)}}(3)$$

such that the graded Lie algebra $\mathrm{Gr}(\Delta^{(i)}_{X_{(r+1)}/X_{(r)}})$ *is generated by the image of these elements, and, moreover,* $\zeta^{(i)}_{X,k}$ (*where* $i \neq k$) *topologically generates the inertia subgroup* (*well-defined, up to conjugation*) *associated to the cusp* (*of the geometric fiber of* $p_{U_{X_{(r)}}:i} : U_{X_{(r+1)}} \to U_{X_{(r)}}$ *at a geometric point of* $U_{X_{(r)}}$) *determined by the divisor* $D_{\underline{X_{(r+1)}}\{i,k\}} \subseteq \underline{X_{(r+1)}}$. *Moreover, the graded Lie algebra* $\mathrm{Gr}(\Delta^{(i)}_{X_{(r+1)}/X_{(r)}})$ *is isomorphic to the graded Lie algebra generated by these elements subject to the following relation:*

$$\sum_{j=1}^{g_X}[\overline{\alpha}^{(i)}_{X,j},\overline{\beta}^{(i)}_{X,j}] + \sum_{k\neq i}\overline{\zeta}^{(i)}_{X,k} = 0\,.$$

(*iii*) *The graded Lie algebra* $\mathrm{Gr}(\Delta_{X_{(r+1)}})$ *is isomorphic to the graded Lie algebra generated by the image of*

$$\alpha^{(i)}_{X,1};\cdots;\alpha^{(i)}_{X,g_X};\beta^{(i)}_{X,1};\cdots;\beta^{(i)}_{X,g_X} \in \Delta^{(i)}_{X_{(r+1)}/X_{(r)}} \subseteq \Delta_{X_{(r+1)}} \ \ (1 \leq i \leq r{+}1)\,,$$

together with

$$\zeta^{(i)}_{X,1};\cdots;\zeta^{(i)}_{X,i-1};\zeta^{(i)}_{X,i+1};\cdots;\zeta^{(i)}_{X,r+1} \in \Delta^{(i)}_{X_{(r+1)}/X_{(r)}} \subseteq \Delta_{X_{(r+1)}} \ \ (1 \leq i \leq r{+}1)$$

in (*ii*) *subject to the following relations:*

(R_1) $\sum_{j=1}^{g_X}[\overline{\alpha}^{(i)}_{X,j},\overline{\beta}^{(i)}_{X,j}] + \sum_{k\neq i}\overline{\zeta}^{(i)}_{X,k} = 0 \ \ (1 \leq i \leq r+1)$;

(R_2) $\overline{\zeta}^{(i)}_{X,k} = \overline{\zeta}^{(k)}_{X,i}$;

(R_3) $[\overline{\zeta}^{(i)}_{X,k},\overline{\zeta}^{(i')}_{X,k'}] = 0 \ \ (\text{if } \{i,k\}\cap\{i',k'\} = \emptyset)$;

(R_4) $[\overline{\zeta}^{(i)}_{X,k},\overline{\alpha}^{(i')}_{X,j}] = [\overline{\zeta}^{(i)}_{X,k},\overline{\beta}^{(i')}_{X,j}] = 0 \ \ (\text{if } i \neq i' \text{ and } k \neq i')$;

(R_5) $[\overline{\alpha}^{(i)}_{X,j},\overline{\alpha}^{(i')}_{X,j'}] = [\overline{\beta}^{(i)}_{X,j},\overline{\beta}^{(i')}_{X,j'}] = 0 \ \ (\text{if } i \neq i')$;

(R_6) $[\overline{\alpha}^{(i)}_{X,j},\overline{\beta}^{(i')}_{X,j'}] = \begin{cases} \overline{\zeta}^{(i')}_{X,i} & (\text{if } j = j' \text{ and } i \neq i') \\ 0 & (\text{if } j \neq j' \text{ and } i \neq i') \end{cases}$

Proof. Assertion (i) follows from [11], Proposition 3.2, (i). Assertion (ii) follows from [8], Proposition 1. Assertion (iii) follows from [18], (2.8.2). $\square$

Lemma 4.3. *Let $1 \leq i < j \leq r$ be integers.*

(*i*) *The following diagram induced by the cartesian diagram defining $U_{(r)\underline{\{i,j\}}}$ (cf. Definition 2.9, (i)) is cartesian:*

$$\begin{tikzcd}
\mathrm{Gr}(\Delta_{X_{(r)}\underline{\{i,j\}}}) \arrow[r, "\text{via } p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:i}"] \arrow[d, "\text{via } p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:j}"'] & \mathrm{Gr}(\Delta_{X_{(r-1)}}) \arrow[d, "\text{via } p_{U_{X_{(r-2)}}:j-1}"] \\
\mathrm{Gr}(\Delta_{X_{(r-1)}}) \arrow[r, "\text{via } p_{U_{X_{(r-1)}}:i}"'] & \mathrm{Gr}(\Delta_{X_{(r-2)}})\,.
\end{tikzcd}$$

(*ii*) *The kernel of the morphism* $\mathrm{Gr}(\Delta_{X_{(r)}}) \overset{\text{via } \iota_{U_{X_{(r)}\underline{\{i,j\}}}}}{\to} \mathrm{Gr}(\Delta_{X_{(r)}\underline{\{i,j\}}})$ *is the ideal generated by $\overline{\zeta}^{(j)}_{X,i} = \overline{\zeta}^{(i)}_{X,j}$ (cf. the statement of Proposition 4.2, (iii)). In particular, the set*

$$\{\overline{\zeta}^{(j)}_{X,i}\}$$

is a base (over $\mathbb{Q}_l$) of the kernel of

$$\mathrm{Gr}(\Delta_{X_{(r)}})(2/3) \overset{\text{via } \iota_{U_{X_{(r)}\underline{\{i,j\}}}}}{\to} \mathrm{Gr}(\Delta_{X_{(r)}\underline{\{i,j\}}})(2/3)\,.$$

Proof. Assertion (i) follows from Lemma 2.10, together with Corollary 1.13, (ii). Assertion (ii) follows from the fact that the kernel of morphism $\mathrm{Gr}(\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}) \to \mathrm{Gr}(\Delta^{(i)}_{X_{(r-1)}/X_{(r-2)}})$ induced by the left-hand vertical arrow in the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \Delta^{(i)}_{X_{(r)}/X_{(r-1)}} \arrow[r] \arrow[d] & \Delta_{X_{(r)}} \arrow[r, "\text{via } p_{U_{X_{(r-1)}}:i}"] \arrow[d] & \Delta_{X_{(r-1)}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \Delta^{(i)}_{X_{(r-1)}/X_{(r-2)}} \arrow[r] & \Delta_{X_{(r)}\underline{\{i,j\}}} \arrow[r, "\text{via } p_{U^{\underline{\{i,j\}}}_{X_{(r-1)}}:i}"'] & \Delta_{X_{(r-1)}} \arrow[r] & 1\,,
\end{tikzcd}$$

is the ideal generated by $\overline{\zeta}^{(j)}_{X,i} = \overline{\zeta}^{(i)}_{X,j}$. (This follows easily from an observation concerning the generators of the graded Lie algebras $\Delta^{(i)}_{X_{(r)}/X_{(r-1)}}$ and $\Delta^{(i)}_{X_{(r-1)}/X_{(r-2)}}$ given in [8], Proposition 1.) $\square$

Let $1 \leq i < j \leq r+1$ be integers. Next, let us fix *choices of the inertia subgroups*

$$\mathfrak{I}_{X_{(r+1)}\{i,j\}} \subseteq \Pi_{X_{(r+1)}}$$

(among the various conjugates of $\mathfrak{I}_{X_{(r+1)}\{i,j\}}$) for $1 \leq i < j \leq r+1$. (Note that, by Lemma 2.7, these choices induce *choices of the subgroups*

$$\mathfrak{D}_{X_{(r+1)}\{i,j\}} \subseteq \Pi_{X_{(r+1)}}\,;$$

moreover, by considering the images of these subgroups via the surjection induced by $p^{\log}_{X_{(r)}:k}$ [where $1 \le k \le r+1$], these choices induce $r+1$ *respective choices of the subgroups*

$$\mathfrak{I}_{X_{(r)}\{i,j\}} \subseteq \Pi_{X_{(r)}}$$

and

$$\mathfrak{D}_{X_{(r)}\{i,j\}} \subseteq \Pi_{X_{(r)}} .)$$

Lemma 4.4. *Let $1 \le i < j \le r+1$ be integers. Let*

$$I_{\{i,j\}} = \begin{cases} \{i-1, j-1\} & (\text{if } i \neq 1) \\ \{1, j-1\} & (\text{if } i = 1 \text{ and } j \neq 2) \\ \{1,2\} & (\text{if } i = 1 \text{ and } j = 2) \end{cases} \qquad k_{\{i,j\}} = \begin{cases} 1 & (\text{if } i \neq 1) \\ 2 & (\text{if } i = 1 \text{ and } j \neq 2) \\ 3 & (\text{if } i = 1 \text{ and } j = 2) \end{cases}$$

$$l_{\{i,j\}} = \begin{cases} 1 & (\text{if } i \neq 1) \\ 2 & (\text{if } i = 1) \end{cases} \qquad m_{\{i,j\}} = \begin{cases} i-1 & (\text{if } i \neq 1) \\ 1 & (\text{if } i = 1) . \end{cases}$$

Then the commutative diagram

$$\begin{tikzcd} D^{\log}_{X_{(r+1)}\{i,j\}} \arrow[r, "\delta^{\log}_{X_{(r+1)}\{i,j\}}"] \arrow[d] & X^{\log}_{(r+1)} \arrow[r, "p^{\log}_{X_{(r)}:i}"] \arrow[d, "p^{\log}_{X_{(r)}:k_{\{i,j\}}}"'] & X^{\log}_{(r)} \arrow[d, "p^{\log}_{X_{(r-1)}:l_{\{i,j\}}}"] \\ D^{\log}_{X_{(r)}I_{\{i,j\}}} \arrow[r, "\delta^{\log}_{X_{(r)}I_{\{i,j\}}}"'] & X^{\log}_{(r)} \arrow[r, "p^{\log}_{X_{(r-1)}:m_{\{i,j\}}}"'] & X^{\log}_{(r-1)} \end{tikzcd}$$

induces the following cartesian diagram:

$$\begin{tikzcd} \mathfrak{D}_{X_{(r+1)}\{i,j\}} \arrow[r, "\text{via } p^{\log}_{X_{(r)}:i} \circ \delta^{\log}_{X_{(r+1)}\{i,j\}}"] \arrow[d, "\text{via } p^{\log}_{X_{(r)}:k_{\{i,j\}}}"'] & \Pi_{X_{(r)}} \arrow[d, "\text{via } p^{\log}_{X_{(r-1)}:l_{\{i,j\}}}"] \\ \mathfrak{D}_{X_{(r)}I_{\{i,j\}}} \arrow[r, "\text{via } p^{\log}_{X_{(r-1)}:m_{\{i,j\}}} \circ \delta^{\log}_{X_{(r)}I_{\{i,j\}}}"'] & \Pi_{X_{(r-1)}} . \end{tikzcd}$$

Proof. By the definitions, the commutative diagram

$$\begin{tikzcd} D^{\log}_{X_{(r+1)}\{i,j\}} \arrow[r, "\delta^{\log}_{X_{(r+1)}\{i,j\}}"] \arrow[d] & X^{\log}_{(r+1)} \arrow[r, "p^{\log}_{X_{(r)}:i}"] \arrow[d, "p^{\log}_{X_{(r)}:k_{\{i,j\}}}"'] & X^{\log}_{(r)} \arrow[d, "p^{\log}_{X_{(r-1)}:l_{\{i,j\}}}"] \\ D^{\log}_{X_{(r)}I_{\{i,j\}}} \arrow[r, "\delta^{\log}_{X_{(r)}I_{\{i,j\}}}"'] & X^{\log}_{(r)} \arrow[r, "p^{\log}_{X_{(r-1)}:m_{\{i,j\}}}"'] & X^{\log}_{(r-1)} \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{I}_{X_{(r+1)}\{i,j\}} \arrow[r] \arrow[d] & \mathfrak{D}_{X_{(r+1)}\{i,j\}} \arrow[r, "\text{via } p^{\log}_{X_{(r)}:i} \circ \delta^{\log}_{X_{(r+1)}\{i,j\}}"] \arrow[d, "\text{via } p^{\log}_{X_{(r)}:k_{\{i,j\}}}"'] & \Pi_{X_{(r)}} \arrow[r] \arrow[d, "\text{via } p^{\log}_{X_{(r-1)}:l_{\{i,j\}}}"] & 1 \\ 1 \arrow[r] & \mathfrak{I}_{X_{(r)}I_{\{i,j\}}} \arrow[r] & \mathfrak{D}_{X_{(r)}I_{\{i,j\}}} \arrow[r, "\text{via } p^{\log}_{X_{(r-1)}:m_{\{i,j\}}} \circ \delta^{\log}_{X_{(r)}I_{\{i,j\}}}"'] & \Pi_{X_{(r-1)}} \arrow[r] & 1 , \end{tikzcd}$$

where the horizontal sequences are exact. Now since the restriction of the morphism $D^{\log}_{X_{(r+1)}\{i,j\}} \to D^{\log}_{X_{(r)}I_{\{i,j\}}}$ induced by $p^{\log}_{X_{(r)}:k_{\{i,j\}}}$ to the generic point of $D^{\log}_{X_{(r+1)}\{i,j\}}$ is *strict*, we conclude that the left-hand vertical arrow is an *isomorphism*. This completes the proof of Lemma 4.4. $\square$

Remark 4.5. Note that by Lemma 4.4, the commutative diagram

$$\begin{tikzcd} \mathfrak{D}_{X_{(r+1)}\{i,j\}} \arrow[rr, "\text{via } p^{\log}_{X_{(r)}:i}\circ\delta^{\log}_{X_{(r+1)}\{i,j\}}"] \arrow[d, "\text{via } p^{\log}_{X_{(r)}:k_{\{i,j\}}}"'] & & \Pi_{X_{(r)}} \arrow[d, "\text{via } p^{\log}_{X_{(r-1)}:l_{\{i,j\}}}"] \\ \mathfrak{D}_{X_{(r)}I_{\{i,j\}}} \arrow[rr, "\text{via } p^{\log}_{X_{(r-1)}:m_{\{i,j\}}}\circ\delta_{X_{(r)}I_{\{i,j\}}}"'] & & \Pi_{X_{(r-1)}} . \end{tikzcd}$$

is cartesian; however, the commutative diagram

$$\begin{tikzcd} D^{\log}_{X_{(r+1)}\{i,j\}} \arrow[rr, "p^{\log}_{X_{(r)}:i}\circ\delta^{\log}_{X_{(r+1)}\{i,j\}}"] \arrow[d, "p^{\log}_{X_{(r)}:k_{\{i,j\}}}"'] & & X^{\log}_{(r)} \arrow[d, "p^{\log}_{X_{(r-1)}:l_{\{i,j\}}}"] \\ D^{\log}_{X_{(r)}I_{\{i,j\}}} \arrow[rr, "p^{\log}_{X_{(r-1)}:m_{\{i,j\}}}\circ\delta_{X_{(r)}I_{\{i,j\}}}"'] & & X^{\log}_{(r-1)} \end{tikzcd}$$

which induces the above diagram is *not cartesian*. Indeed, this follows from the fact that the pull-back of the invertible sheaf corresponding to the morphism of type $\mathbb{N}$ (cf. [6], Theorem 4.13)

$$D^{\log}_{X_{(r)}I_{\{i,j\}}} \stackrel{p^{\log}_{X_{(r-1)}:m_{\{i,j\}}}\circ\delta_{X_{(r)}I_{\{i,j\}}}}{\longrightarrow} X^{\log}_{(r-1)}$$

via $p^{\log}_{X_{(r-1)}:l_{\{i,j\}}}$ is *not isomorphic* to the invertible sheaf corresponding to the morphism of type $\mathbb{N}$

$$D^{\log}_{X_{(r+1)}\{i,j\}} \stackrel{p^{\log}_{X_{(r)}:i}\circ\delta^{\log}_{X_{(r+1)}\{i,j\}}}{\longrightarrow} X^{\log}_{(r)} .$$

However, the *restrictions* to $U_{X_{(r)}}$ of these two invertible sheaves are *isomorphic* (cf. [7], Lemma 1.21).

Moreover, let us fix a *section* $s'_0 : G_K \to \Pi_{X_{(r)}}$ of the morphism $\Pi_{X_{(r)}} \to G_K$ induced by the structure morphism of $U_{X_{(r)}}$ and a *lifting* s_0 of s'_0 to $G_K \to \mathfrak{D}_{X_{(r+1)}\{1,2\}}$, i.e., a morphism $G_K \to \mathfrak{D}_{X_{(r+1)}\{1,2\}}$ such that the composite of the morphism and $\mathfrak{D}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Pi_{X_{(r+1)}} \stackrel{\text{via } p^{\log}_{X_{(r)}:1}}{\to} \Pi_{X_{(r)}}$ coincides with s'_0. Note that since G_K is *free*, such a lifting always exists. Then the section s_0 of the natural morphism $\Pi_{X_{(r+1)}} \to G_K$ determines natural actions of G_K (by conjugation) on $\Delta_{X_{(r+1)}}$, and on $\Delta_{X_{(r+1)}\underline{\{i,j\}}}$, hence also on

$$\mathrm{Lin}_{X_{(r+1)}}(a/b) \stackrel{\text{def}}{=} \mathrm{Lin}(\Delta_{X_{(r+1)}}(a/b))(\mathbb{Q}_l)\,;$$

$$\mathrm{Lin}_{X_{(r+1)}\underline{\{i,j\}}}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Lin}(\Delta_{X_{(r+1)}\underline{\{i,j\}}}(a/b))(\mathbb{Q}_l)\,;$$

$$\mathrm{Lie}_{X_{(r+1)}}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Lie}(\Delta_{X_{(r+1)}}(a/b))\,;\ \ \mathrm{Lie}_{X_{(r+1)}\underline{\{i,j\}}}(a/b) \stackrel{\mathrm{def}}{=} \mathrm{Lie}(\Delta_{X_{(r+1)}\underline{\{i,j\}}}(a/b))\,;$$

$$\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{(r+1)}})(a/b)\,;\ \ \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{(r+1)}\underline{\{i,j\}}})(a/b)$$

for a, $b \in \mathbb{Z}$ such that $1 \leq a \leq b$.

Proposition 4.6. *Let $1 \leq i < j \leq r+1$ be integers, and a, $b \in \mathbb{Z}$ such that $1 \leq a \leq b$.*

(*i*) *The eigenvalues of the action of* Fr_K *on* $\mathrm{Lie}_{X_{(r+1)}}(a/a+1)$ (*respectively,* $\mathrm{Lie}_{X_{(r+1)}\underline{\{i,j\}}}(a/a+1)$) *are algebraic numbers all of whose complex absolute values are equal to $q_K^{a/2}$.*

(*ii*) *There is a unique G_K-equivariant isomorphism of Lie algebra*

$$\mathrm{Lie}_{X_{(r+1)}}(a/b) \stackrel{\sim}{\longrightarrow} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{(r+1)}})(a/b)$$

$$(\text{respectively,}\quad \mathrm{Lie}_{X_{(r+1)}\underline{\{i,j\}}}(a/b) \stackrel{\sim}{\longrightarrow} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{(r+1)}\underline{\{i,j\}}})(a/b)\,)$$

which induces the identity isomorphism

$$\mathrm{Lie}_{X_{(r+1)}}(c/c+1) \stackrel{\sim}{\longrightarrow} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{(r+1)}})(c/c+1)$$

$$(\text{respectively,}\quad \mathrm{Lie}_{X_{(r+1)}\underline{\{i,j\}}}(c/c+1) \stackrel{\sim}{\longrightarrow} \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{X_{(r+1)}\underline{\{i,j\}}})(c/c+1)\,)$$

for all $c \in \mathbb{Z}$ such that $a \leq c \leq b-1$.

Proof. Assertion (i) follows immediately from the "Riemann hypothesis for abelian varieties over finite fields" (cf. e.g., [16], p. 206). Assertion (ii) follows formally from assertion (i) by considering the eigenspaces with respect to the action of Fr_K. ☐

Definition 4.7.

(i) We shall write

$$M_X \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{Z}_l}(H^2(\Delta_X, \mathbb{Z}_l), \mathbb{Z}_l)$$

(cf. the discussion prededing [14], Remark 1.2.0). Note that M_X is (non-canonically) isomorphic to $\mathbb{Z}_l(1)$ as a G_K-module.

(ii) Let $1 \leq i < j \leq r+1$ be integers. Then there exists a natural isomorphism $M_X \stackrel{\sim}{\to} \mathfrak{I}_{X_{(r+1)}\{i,j\}}$ (cf. the statement of [14], Proposition 2.1). We shall denote this isomorphism by $m_{X_{(r+1)}\{i,j\}}$.

(iii) The cup product on the group cohomology of Δ_X

$$\bigwedge^2 H^1(\Delta_X, M_X) \longrightarrow H^2(\Delta_X, M_X \otimes_{\mathbb{Z}_l} M_X)$$

determines an isomorphism

$$\mathrm{Hom}(\Delta_X^{\mathrm{ab}}, M_X) \stackrel{\sim}{\longrightarrow} \Delta_X^{\mathrm{ab}}\,,$$

hence a natural G_K-equivariant injection

$$M_X \hookrightarrow \bigwedge^2 \Delta_X^{\mathrm{ab}}$$

(cf. the disucussion preceding [14], Definition 3.2). We shall denote this G_K-equivariant injection by i_X^{cup}.

(iv) The isomorphism

$$\mathrm{Hom}(\Delta_X^{\mathrm{ab}}, M_X) \xrightarrow{\sim} \Delta_X^{\mathrm{ab}},$$

in (iii) determines a homomorphism

$$\bigwedge^2 \Delta_X^{\mathrm{ab}} \longrightarrow M_X .$$

We shall denote by $a \cup_X a'$ the image of $a \wedge a'$ via this homomorphism.

Proposition 4.8. *Let us write*

$$V_{U_{X_{(r+1)}}} \stackrel{\mathrm{def}}{=} \bigoplus_{i<j} (\mathfrak{I}_{X_{(r+1)}\{i,j\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \oplus \mathrm{Lie}_{X_{(r+1)}}(1/2) .$$

(*Note that by applying the natural isomorphisms* $m_{X_{(r+1)}\{i,j\}} : M_{X\{i,j\}} \xrightarrow{\sim} \mathfrak{I}_{X_{(r+1)}\{i,j\}}$ *and the identity morphism* $\mathrm{Lie}_{X_{(r+1)}}(1/2) \xrightarrow{\sim} (\bigoplus_{k=1}^{r+1} \Delta_X^{\mathrm{ab}\,(k)}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, *one obtains a natural isomorphism of* $V_{U_{X_{(r+1)}}}$ *with the* $\mathbb{Q}_l$*-vector space obtained by tensoring the free* $\mathbb{Z}_l$*-module*

$$\bigoplus_{i<j} M_{X\{i,j\}} \oplus \bigoplus_{k=1}^{r+1} \Delta_X^{\mathrm{ab}\,(k)}$$

[*where* $M_{X\{i,j\}}$ *is the copy of* M_X *indexed by* $\{i,j\}$, *and* $\Delta_X^{\mathrm{ab}\,(k)}$ *is the copy of* Δ_X^{ab} *indexed by* k] *with* $\mathbb{Q}_l$.) *Then the first isomorphism in Proposition 4.6, (ii), together with the natural inclusions* $\mathfrak{I}_{X_{(r+1)}\{i,j\}} \hookrightarrow \Delta_{X_{(r+1)}}$, *determine a* G_K*-equivariant morphism*

$$V_{U_{X_{(r+1)}}} \longrightarrow \mathrm{Lie}_{X_{(r+1)}}(1/\infty)$$

which exhibits, in a G_K*-equivariant fashion,* $\mathrm{Lie}_{X_{(r+1)}}(1/\infty)$ *as the quotient of the completion with respect to the filtration topology of the free Lie algebra* $\mathfrak{Lie}(V_{U_{X_{(r+1)}}})$ *generated by* $V_{U_{X_{(r+1)}}}$ *equipped with a natural grading, hence also a filtration, by taking the* $\mathfrak{I}_{X_{(r+1)}\{i,j\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ *to be of weight* 2, $\mathrm{Lie}_{X_{(r+1)}}(1/2)$ *to be of weight* 1, *by the relations determined by the images of the morphisms*

(R_1') $M_X \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$

$$\xrightarrow{\oplus_j m_{X_{(r+1)}\{i,j\}} \oplus i_X^{\mathrm{cup}}} \Big(\bigoplus_{j=1,\cdots,r+1;\ j\neq i} \mathfrak{I}_{X_{(r+1)}\{i,j\}} \oplus \bigwedge^2 \Delta_X^{\mathrm{ab}\,(i)} \Big) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

$$\xrightarrow{\mathrm{incl.}\oplus[\,,\,]} \mathfrak{Lie}(V_{U_{X_{(r+1)}}})(2/3) \ \ (1 \leq i \leq r+1) ;$$

(R'_3) $(\mathfrak{I}_{X_{(r+1)}\{i,k\}} \otimes_{\mathbb{Z}_l} \mathfrak{I}_{X_{(r+1)}\{i',k'\}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \xrightarrow{[\,,]} \mathfrak{Lie}(V_{U_{X_{(r+1)}}})(4/5)$
$(\{i,k\} \cap \{i',k'\} = \emptyset)$;

(R'_4) $(\mathfrak{I}_{X_{(r+1)}\{i,k\}} \otimes_{\mathbb{Z}_l} \Delta_X^{\mathrm{ab}\,(i')}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \xrightarrow{[\,,]} \mathfrak{Lie}(V_{U_{X_{(r+1)}}})(3/4)$
$(i \neq i' \ , \ \ k \neq i')$;

$$(R'_{5\,\mathrm{and}\,6})\quad \begin{array}{ccc} (\bigotimes^2_{\mathbb{Z}_l} \Delta_X^{\mathrm{ab}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l & \longrightarrow & (\Delta_X^{\mathrm{ab}\,(i)} \otimes_{\mathbb{Z}_l} \Delta_X^{\mathrm{ab}\,(i')} \oplus \mathfrak{I}_{X_{(r+1)}\{i',i\}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \\ a \otimes a' & \mapsto & (a \otimes a',\, -m_{X_{(r+1)}\{i',i\}}(a \cup_X a')) \end{array}$$

$\xrightarrow{[\,,]\oplus\mathrm{incl.}} \mathfrak{Lie}(V_{U_{X_{(r+1)}}})(2/3)$ $(i \neq i')$

(*cf. the relations in the statement of Proposition 4.2, (iii)), where "*incl*" is the natural inclusion morphism.*

Proof. This follows from Propositions 4.2, (iii); 4.6, (ii). □

Definition 4.9. Let $1 \leq i < j \leq r$ be integers, and $a, b \in \mathbb{Z}$ such that $1 \leq a \leq b$.

(i) Now we have natural G_K-equivariant surjections:

$\mathrm{Lin}_{X_{(r+1)}}(a/b) \twoheadrightarrow \mathrm{Lin}_{X_{(r+1)}\underline{\{1,2\}}}(a/b)$; $\mathrm{Lie}_{X_{(r+1)}}(a/b) \twoheadrightarrow \mathrm{Lie}_{X_{(r+1)}\underline{\{1,2\}}}(a/b)$.

We shall denote by

$$\mathrm{Lin}^{\mathrm{iner}}_{X_{(r+1)}}(a/b)\,;\ \ \mathrm{Lie}^{\mathrm{iner}}_{X_{(r+1)}}(a/b)$$

the respective kernels of these surjections.

(ii) Now we have a natural G_K-equivariant morphism:

$$\Delta_{X_{(r+1)}\underline{\{1,2\}}} \longrightarrow \mathrm{Lin}_{X_{(r+1)}\underline{\{1,2\}}}(1/\infty)$$

(cf. Definition 1.1, (ii)). We shall write

$$\Delta^{\mathrm{Lie}}_{X_{(r+1)}} \stackrel{\mathrm{def}}{=} \Delta_{X_{(r+1)}\underline{\{1,2\}}} \times_{\mathrm{Lin}_{X_{(r+1)}\underline{\{1,2\}}}(1/\infty)} \mathrm{Lin}_{X_{(r+1)}}(1/\infty)\,.$$

(We regard $\mathrm{Lin}_{X_{(r+1)}\underline{\{1,2\}}}(1/\infty)$ and $\mathrm{Lin}_{X_{(r+1)}}(1/\infty)$ as being equipped with the topology determined by the respective natural *l-adic topologies* of $\mathrm{Lin}_{X_{(r+1)}\underline{\{1,2\}}}(1/b)$ and $\mathrm{Lin}_{X_{(r+1)}}(1/b)$ [where b is a positive integer]; moreover, we regard $\Delta^{\mathrm{Lie}}_{X_{(r+1)}}$ as being equipped with the topology determined by the profinite topology of $\Delta_{X_{(r+1)}\underline{\{1,2\}}}$ and the topologies of $\mathrm{Lin}_{X_{(r+1)}\underline{\{1,2\}}}(1/\infty)$ and $\mathrm{Lin}_{X_{(r+1)}}(1/\infty)$.) Moreover, we shall denote by

$$\mathrm{Int}^{\Delta}_{X_{(r+1)}} : \Delta_{X_{(r+1)}} \longrightarrow \Delta^{\mathrm{Lie}}_{X_{(r+1)}}$$

the G_K-equivariant morphism induced by the morphism

$$\Delta_{X_{(r+1)}} \longrightarrow \Delta_{X_{(r+1)}\underline{\{1,2\}}}$$

induced by $\iota_{U_{X_{(r+1)}\underline{\{1,2\}}}}$ (cf. Definition 2.9, (ii)) and the natural G_K-equivariant morphism

$$\Delta_{X_{(r+1)}} \longrightarrow \mathrm{Lin}_{X_{(r+1)}}(1/\infty)\,.$$

(iii) Now we have a natural G_K-equivariant injection

$$\mathrm{Lin}^{\mathrm{iner}}_{X_{(r+1)}}(b+1/\infty) \xrightarrow{\sim} \{1\} \times_{\{1\}} \mathrm{Lin}^{\mathrm{iner}}_{X_{(r+1)}}(b+1/\infty)$$

$$\longrightarrow \Delta_{X_{(r+1)}\underline{\{1,2\}}} \times_{\mathrm{Lin}_{X_{(r+1)}\underline{\{1,2\}}}(1/\infty)} \mathrm{Lin}_{X_{(r+1)}}(1/\infty) \xrightarrow{\sim} \Delta^{\mathrm{Lie}}_{X_{(r+1)}}$$

whose image forms a closed normal subgroup of $\Delta^{\mathrm{Lie}}_{X_{(r+1)}}$. We shall denote by

$$\Delta^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}}$$

the quotient of $\Delta^{\mathrm{Lie}}_{X_{(r+1)}}$ by this normal subgroup.

(iv) We shall write

$$\Pi^{\mathrm{Lie}}_{X_{(r+1)}} \stackrel{\mathrm{def}}{=} \Delta^{\mathrm{Lie}}_{X_{(r+1)}} \rtimes G_K \ ; \ \Pi^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}} \stackrel{\mathrm{def}}{=} \Delta^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}} \rtimes G_K \,,$$

where the action of G_K on $\Delta^{\mathrm{Lie}}_{X_{(r+1)}}$ and $\Delta^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}}$ is the action induced by the section s_0. Moreover, we shall denote by

$$\mathrm{Int}^{\Pi}_{X_{(r+1)}} : \Pi_{X_{(r+1)}} \longrightarrow \Pi^{\mathrm{Lie}}_{X_{(r+1)}}$$

the morphism induced by $\mathrm{Int}^{\Delta}_{X_{(r+1)}}$.

(v) Now we have a natural morphism:

$$\Pi_{X_{(r+1)}} \xrightarrow{\mathrm{Int}^{\Pi}_{X_{(r+1)}}} \Pi^{\mathrm{Lie}}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}}$$

$$(\text{respectively, } \mathfrak{D}_{X_{(r+1)}\{i,j\}} \hookrightarrow \Pi_{X_{(r+1)}} \xrightarrow{\mathrm{Int}^{\Pi}_{X_{(r+1)}}} \Pi^{\mathrm{Lie}}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}} \ ;$$

$$\text{respectively, } \mathfrak{I}_{X_{(r+1)}\{i,j\}} \hookrightarrow \Pi_{X_{(r+1)}} \xrightarrow{\mathrm{Int}^{\Pi}_{X_{(r+1)}}} \Pi^{\mathrm{Lie}}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}} \ ).$$

We shall denote the image of this composite by

$$\Pi^{\leq b}_{X_{(r+1)}} \ (\text{respectively, } \mathfrak{D}^{\leq b}_{X_{(r+1)}\{i,j\}} \ ; \ \text{respectively, } \mathfrak{I}^{\leq b}_{X_{(r+1)}\{i,j\}});$$

moreover, we shall write

$$\Delta^{\leq b}_{X_{(r+1)}} \stackrel{\mathrm{def}}{=} \Pi^{\leq b}_{X_{(r+1)}} \cap \Delta^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}} \ ;$$

$$\mathfrak{D}^{\Delta\,\leq b}_{X_{(r+1)}\{i,j\}} \stackrel{\mathrm{def}}{=} \mathfrak{D}^{\leq b}_{X_{(r+1)}\{i,j\}} \cap \Delta^{\mathrm{Lie}\,\leq b}_{X_{(r+1)}} \,.$$

Proposition 4.10. *For each $\alpha \in \mathrm{Lin}_{X_{(r+1)}}(1/\infty)$, there exists a unique element $\beta \in \mathrm{Lin}_{X_{(r+1)}}(1/\infty)$ such that*

$$\mathrm{Fr}_K \circ \mathrm{Inn}(\alpha) = \mathrm{Inn}(\beta) \circ \mathrm{Fr}_K \circ \mathrm{Inn}(\beta^{-1})$$

(where "$\mathrm{Inn}(-)$" denotes the inner automorphism of $\mathrm{Lin}_{X_{(r+1)}}(1/\infty)$ defined by conjugation by the element "$-$"). Moreover, when α lies in the subgroup obtained by tensoring the image of $\mathfrak{I}_{X_{(r+1)}\{1,2\}}$ via $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$ with $\mathbb{Q}_l$, β also lies in the subgroup obtained by tensoring the image of $\mathfrak{I}_{X_{(r+1)}\{1,2\}}$ via $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$ with $\mathbb{Q}_l$.

Proof. The assertion follows from Proposition 4.6, (i), and successive approximation of β with respect to the natural filtration $\mathrm{Lin}_{X_{(r)}}(a/\infty) \subseteq \mathrm{Lin}_{X_{(r)}}(1/\infty)$. $\square$

Remark 4.11. Observe that changing the *choice of a lifting*

$$s_0 : G_K \longrightarrow \mathfrak{D}_{X_{(r+1)}\{1,2\}}$$

of s_0' affects the image of the element $\mathrm{Fr}_K \in G_K$ via the composite of the inclusion $G_K \hookrightarrow \Pi_{X_{(r+1)}}$ with the morphism $\mathrm{Int}^{\Pi}_{X_{(r+1)}} : \Pi_{X_{(r+1)}} \to \Pi^{\mathrm{Lie}}_{X_{(r+1)}}$ by conjugation by an element of the subgroup obtained by tensoring the image of $\mathfrak{I}_{X_{(r+1)}\{1,2\}}$ via $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$ with $\mathbb{Q}_l$ (cf. Proposition 4.10). In particular, it follows that changing the choice of a lifting $G_K \to \mathfrak{D}_{X_{(r+1)}\{1,2\}}$ of s_0' affects the *Galois invariant splitting* of Proposition 4.6, (ii), by conjugation by an element of the subgroup obtained by tensoring the image of $\mathfrak{I}_{X_{(r+1)}\{1,2\}}$ via $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$ with $\mathbb{Q}_l$. Put another way, if we identify the "$\mathrm{Lin}_{X_{(r+1)}}(1/\infty)$", "$\mathrm{Lin}_{X_{(r+1)}\underline{\{1,2\}}}(1/\infty)$" portion of $\Delta^{\mathrm{Lie}}_{X_{(r+1)}}$ (cf. Definition 4.9, (ii)) with the topological groups formed the $\mathbb{Q}_l$-valued points of the pro-algebraic groups corresponding to the (completion of the) corresponding graded objects "$\mathrm{Gr}(-)(1/\infty)$" via the Galois invariant splitting of Proposition 4.6, (ii), then the following holds:

> *Changing the choice of a lifting $s_0 : G_K \to \mathfrak{D}_{X_{(r+1)}\{1,2\}}$ of s_0' affects the images of the morphism*
>
> $$\mathrm{Int}^{\Pi}_{X_{(r+1)}} : \Pi_{X_{(r+1)}} \longrightarrow \Pi^{\mathrm{Lie}}_{X_{(r+1)}}$$
>
> *by conjugation by an element of the subgroup obtained by tensoring the image of $\mathfrak{I}_{X_{(r+1)}\{1,2\}}$ via $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$ with $\mathbb{Q}_l$.*

Lemma 4.12. $\mathrm{Int}^{\Delta}_{X_{(r+1)}}$ *is an injection.*

Proof. This follows from induction on r, Corollary 1.13, (ii), together with the fact that the central filtration

$$\{\Delta_{X/S}(n)\}$$

defined in the discussion following Lemma 1.2 satisfies that

$$\bigcap_{n\geq 1} \Delta_{X/S}(n) = 1\,.$$

$\square$

Lemma 4.13. *Let $r \geq 2$ be an integer. Then conjugates (in $\Delta^{\leq b}_{X_{(r+1)}}$) of the subgroups $\mathfrak{D}^{\Delta\leq b}_{X_{(r+1)}\{1,2\}}$ and $\mathfrak{D}^{\Delta\leq b}_{X_{(r+1)}\{2,3\}}$ of $\Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ topologically generate the subgroup $\Delta^{\leq b}_{X_{(r+1)}} \subseteq \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$.*

Proof. This follows immediately from Proposition 2.6 and Lemma 4.12. $\square$

Lemma 4.14. *Let $1 \leq i < j \leq r$ be integers, and a, $b \in \mathbb{Z}$ such that $1 \leq a \leq b$.*

(*i*) $\Delta^{\mathrm{Lie}\leq 1}_{X_{(r+1)}}$ *is naturally isomorphic to* $\Delta_{X_{(r+1)}\underline{\{1,2\}}}$.

(*ii*) *The kernel of the natural projection* $\Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ *is isomorphic to*

$$\mathrm{Lin}^{\mathrm{iner}}_{X_{(r+1)}}(b+1/b+2)\,.$$

In particular, the kernel of the natural projection $\mathfrak{D}^{\leq b+1}_{X_{(r+1)}} \twoheadrightarrow \mathfrak{D}^{\leq b}_{X_{(r+1)}}$ *is isomorphic to*

$$\begin{cases} 1 & (\text{if } b \neq 1 \text{ or } \{i,j\} \neq \{1,2\}) \\ \mathfrak{I}_{X_{(r+1)}\{i,j\}} & (\text{if } b = 1 \text{ and } \{i,j\} = \{1,2\})\,. \end{cases}$$

Therefore, for $2 \leq b$*, the natural projection* $\mathfrak{D}^{\leq b+1}_{X_{(r+1)}} \twoheadrightarrow \mathfrak{D}^{\leq b}_{X_{(r+1)}}$ *is an isomorphism.*

Proof. This follows immediately from Lemma 4.4; Definition 4.9. □

In the following, let us consider some assumptions on the section $s_0 : G_K \to \Pi_{X_{(r+1)}}$ fixed in the discussion preceding Proposition 4.6:

Definition 4.15. Let $r \geq 2$ be an integer. Then we shall say that the section $s_0 : G_K \to \Pi_{X_{(r+1)}}$ (fixed in the discussion preceding Proposition 4.6) satisfies the condition $(\dagger_{\mathbb{P}})$ (respectively, $(\dagger_S)$ for a set $S = \{x_1, \cdots, x_r\}$ of K-rational points of X of cardinality $= r$ with an ordering) if the following holds:

The image of the section $s_0 : G_K \to \Pi_{X_{(r+1)}}$ *is contained in*

$$\mathfrak{D}_{X_{(r+1)}\{1,2\}} \cap \mathfrak{D}_{X_{(r+1)}\{1,2,3\}} \subseteq \Pi_{X_{(r+1)}}$$

(*respectively, the section* $s_0' : G_K \to \Pi_{X_{(r)}}$ *arises from the* K*-rational point of* $U_{X_{(r)}}$ *corresponding to* $(x_1, \cdots, x_r)$).

Note that since G_K is *free*, and $D_{X_{(r+1)}\{1,2\}} \cap D_{X_{(r+1)}\{1,2,3\}}$ is *non-empty*, a section which satisfies the condition $(\dagger_{\mathbb{P}})$ always exists.

By the discussion following of Definition 2.3, we have an exact sequence

$$1 \longrightarrow \mathfrak{I}_{X_{(r+1)}\{1,2,3\}} \longrightarrow \mathfrak{P}_{X_{(r+1)}\{1,2,3\}} \longrightarrow \Delta_{\mathbb{P}_K} \longrightarrow 1\,;$$

moreover, we also have a section of this sequence which is refered as *the section of* $\mathfrak{P}_{X_{(r+1)}\{1,2,3\}} \twoheadrightarrow \Delta_{\mathbb{P}_K}$ *induced by* $p^{\log}_{X_{(r+1)}:i}$ (i = 1, 2, 3). Let us denote by $\Delta_{\mathbb{P}_K}\{i\}$ the image of the section of $\mathfrak{P}_{X_{(r+1)}\{1,2,3\}} \twoheadrightarrow \Delta_{\mathbb{P}_K}$ induced by $p^{\log}_{X_{(r+1)}:i}$. Note that then the subgroup $\Delta_{\mathbb{P}_K}\{i\} \subseteq \mathfrak{D}_{X_{(r+1)}\{1,2,3\}}$ of $\mathfrak{D}_{X_{(r+1)}\{1,2,3\}}$ is *normal* by the definition of the section of $\mathfrak{P}_{X_{(r+1)}\{1,2,3\}} \twoheadrightarrow \Delta_{\mathbb{P}_K}$ induced by $p^{\log}_{X_{(r+1)}:i}$.

Definition 4.16. Since the subgroup

$$\Delta_{\mathbb{P}_K}\{i\} \subseteq \mathfrak{D}_{X_{(r+1)}\{1,2,3\}}$$

of $\mathfrak{D}_{X_{(r+1)}\{1,2,3\}}$ is *normal*, if the section s_0 satisfies the condition $(\dagger_{\mathbb{P}})$ (cf. Definition 4.15), then the action of G_K on $\mathfrak{D}_{X_{(r+1)}\{1,2,3\}}$ induced via conjugation induces an action of G_K on $\Delta_{\mathbb{P}_K}\{i\}$. Therefore, we obtain a subgroup

$$\Delta_{\mathbb{P}_K}\{i\} \rtimes G_K \subseteq \mathfrak{D}_{X_{(r+1)}\{1,2,3\}}\,.$$

We shall write $\Pi_{\mathbb{P}_K}\{i\} \stackrel{\mathrm{def}}{=} \Delta_{\mathbb{P}_K}\{i\} \rtimes G_K$.

Lemma 4.17. *The group* $\Pi_{\mathbb{P}_K}\{i\}$ *is isomorphic to* $\Pi_{\mathbb{P}_K}$.

Proof. This follows immediately from the fact that the subgroup $\mathfrak{I}_{X_{(r+1)}\{1,2,3\}} \subseteq \mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2,3\}}$ of $\mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2,3\}}$ is contained in the *center* of $\mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2,3\}}$, together with the fact that any element of the subgroup $\Delta_{X_{(r-1)}} \times \{1\} \subseteq \Delta_{X_{(r-1)}} \times \Delta_{\mathbb{P}_K} \simeq \mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2,3\}}/\mathfrak{I}_{X_{(r+1)}\{1,2,3\}}$ of $\mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2,3\}}/\mathfrak{I}_{X_{(r+1)}\{1,2,3\}}$ *commutes* with any element of the subgroup $\{1\}\times\Delta_{\mathbb{P}_K} \subseteq \Delta_{X_{(r-1)}}\times\Delta_{\mathbb{P}_K} \simeq \mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2,3\}}/\mathfrak{I}_{X_{(r+1)}\{1,2,3\}}$ of $\mathfrak{D}^{\Delta}_{X_{(r+1)}\{1,2,3\}}/\mathfrak{I}_{X_{(r+1)}\{1,2,3\}}$. ☐

Definition 4.18.

(i) We shall denote by

$$\{\Delta_{\mathbb{P}_K}\{2\}(n)\}$$

the central filtration of $\Delta_{\mathbb{P}_K}\{2\}$ with respect to the surjection

$$\Delta_{\mathbb{P}_K}\{2\} \twoheadrightarrow 1$$

(cf. Definition 1.1, (i)). Then it follows immediately from Lemma 2.5 that $\mathrm{Lin}(\Delta_{\mathbb{P}_K}\{2\}(2/3))(\mathbb{Q}_l)$ is naturally isomorphic to

$$(\mathfrak{I}_{X_{(r+1)}\{1,2\}} \oplus \mathfrak{I}_{X_{(r+1)}\{2,3\}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\,.$$

Now we shall write

$$\Delta^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \stackrel{\mathrm{def}}{=} \mathrm{Lin}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty))(\mathbb{Q}_l) \times_{(\mathfrak{I}_{X_{(r+1)}\{2,3\}}\otimes_{\mathbb{Z}_l}\mathbb{Q}_l)} \mathfrak{I}_{X_{(r+1)}\{2,3\}}\,,$$

where the morphism $\mathrm{Lin}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty))(\mathbb{Q}_l) \to \mathfrak{I}_{X_{(r+1)}\{2,3\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ (respectively, $\mathfrak{I}_{X_{(r+1)}\{2,3\}} \to \mathfrak{I}_{X_{(r+1)}\{2,3\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$) is the composite

$$\mathrm{Lin}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty))(\mathbb{Q}_l) \twoheadrightarrow \mathrm{Lin}(\Delta_{\mathbb{P}_K}\{2\}(2/3))(\mathbb{Q}_l)$$

$$\xrightarrow{\sim} (\mathfrak{I}_{X_{(r+1)}\{1,2\}} \oplus \mathfrak{I}_{X_{(r+1)}\{2,3\}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \stackrel{\mathrm{pr}_2}{\twoheadrightarrow} \mathfrak{I}_{X_{(r+1)}\{2,3\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

(respectively, the natural inclusion). Then by the definition of $\Delta^{\mathrm{Lie}}_{X_{(r+1)}}$, the natural morphism $\mathrm{Lin}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty))(\mathbb{Q}_l) \to \mathrm{Lin}_{X_{(r+1)}}(1/\infty)$ (induced by $\Delta_{\mathbb{P}_K}\{2\} \hookrightarrow \Delta_{X_{(r+1)}}$) and the the natural inclusion $\mathfrak{I}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Delta_{X_{(r+1)}\underline{\{1,2\}}}$ induce a natural morphism

$$\Delta^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \longrightarrow \Delta^{\mathrm{Lie}}_{X_{(r+1)}}\,.$$

Now let us assume that the section s_0 fixed in the disucussion preceding Proposition 4.6 satisfies the condition ($\dagger_{\mathbb{P}}$) (cf. Definition 4.15). Then we shall write

$$\Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \stackrel{\mathrm{def}}{=} \Delta^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \rtimes G_K\,,$$

where the action of G_K on $\Delta^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\}$ is the action obtained via conjugate. Now it follows that the morphism $\Delta^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \to \Delta^{\mathrm{Lie}}_{X_{(r+1)}}$ induces a morphism

$$\Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \longrightarrow \Pi^{\mathrm{Lie}}_{X_{(r+1)}}\,;$$

moreover, the following diagram commutes

$$\begin{tikzcd}
\Pi_{\mathbb{P}_K}\{2\} \arrow[r] \arrow[d] & \Pi_{X_{(r+1)}} \arrow[d, "\mathrm{Int}^{\Pi}_{X_{(r+1)}}"] \\
\Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \arrow[r] & \Pi^{\mathrm{Lie}}_{X_{(r+1)}} ,
\end{tikzcd}$$

where the left-hand vertical arrow is the morphism obtained by a similar way to the way to define $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$.

(ii) Let $S = \{x_1, \cdots, x_r\} \subseteq X(K)$ be a subset of $X(K)$ of cardinality $= r$ with an ordering. Then we shall denote by $U_S \subseteq X$ (respectively, $U_{\underline{S}} \subseteq X$) be the open subscheme obtained as the complement of S (respectively, $\underline{S} \overset{\mathrm{def}}{=} S \setminus \{x_1\}$) in X, and by

$$\{\Delta_{U_S}(n)\} \text{ (respectively, } \{\Delta_{U_{\underline{S}}}(n)\})$$

the central filtration of the pro-l fundamental group Δ_{U_S} (respectively, $\Delta_{U_{\underline{S}}}$) of U_S (respectively, $U_{\underline{S}}$) with respect to the natural surjection

$$\Delta_{U_S} \twoheadrightarrow \Delta_X^{\mathrm{ab}} \text{ (respectively, } \Delta_{U_{\underline{S}}} \twoheadrightarrow \Delta_X^{\mathrm{ab}}) .$$

Now we shall write

$$\Delta^{\mathrm{Lie}}_{U_S} \overset{\mathrm{def}}{=} \mathrm{Lin}(\Delta_{U_S}(1/\infty))(\mathbb{Q}_l) \times_{\mathrm{Lin}(\Delta_{U_{\underline{S}}}(1/\infty))(\mathbb{Q}_l)} \Delta_{U_{\underline{S}}} .$$

Let us denote by $\mathfrak{D}_{x_1}$ the decomposition subgroup associated to x_1 of the geometrically pro-l fundamental group Π_{U_S} of U_S (well-defined, up to conjugate). Then $\mathfrak{D}_{x_1}$ fits into an exact sequence

$$1 \longrightarrow \mathfrak{I}_{x_1} \longrightarrow \mathfrak{D}_{x_1} \longrightarrow G_K \longrightarrow 1 ,$$

where $\mathfrak{I}_{x_1}$ is the inertia subgroup associated to x_1 of Π_{U_S} (well-defined, up to conjugate). Let us fix a *section* $s_0^S : G_K \to \mathfrak{D}_{x_1}$ of this exact sequence. Then we obtain actions of G_K on Π_{U_S}, and on the geometrically pro-l fundamental group $\Pi_{U_{\underline{S}}}$ of $U_{\underline{S}}$ (via conjugate), hence also on

$$\mathrm{Lin}(\Delta_{U_S}(a/b))(\mathbb{Q}_l) ; \ \mathrm{Lin}(\Delta_{U_{\underline{S}}}(a/b))(\mathbb{Q}_l) ;$$

$$\mathrm{Lie}(\Delta_{U_S}(a/b)) ; \ \mathrm{Lie}(\Delta_{U_{\underline{S}}}(a/b)) ;$$

$$\mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{U_S}(a/b)) ; \ \mathrm{Gr}_{\mathbb{Q}_l}(\Delta_{U_{\underline{S}}}(a/b)) ; \ \Delta^{\mathrm{Lie}}_{U_S}$$

for a, $b \in \mathbb{Z}$ such that $1 \leq a \leq b$. Then we shall write

$$\Pi^{\mathrm{Lie}}_{U_S} \overset{\mathrm{def}}{=} \Delta^{\mathrm{Lie}}_{U_S} \rtimes G_K .$$

Proposition 4.19.

(i) If the section s_0 satisfies the condition $(\dagger_{\mathbb{P}})$, then there exsits a unique G_K-equivariant isomorphism of Lie algebras

$$\mathrm{Lie}(\Delta_{\mathbb{P}_K}\{2\}(a/b)) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}}(\Delta_{\mathbb{P}_K}\{2\})(a/b)$$

(*where* $a \leq b$ *are integers*). *Now let us write*

$$V_{\mathbb{P}_K}\{2\} \stackrel{\text{def}}{=} (\mathfrak{I}_{X_{(r+1)}\{1,2\}} \oplus \mathfrak{I}_{X_{(r+1)}\{2,3\}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\,.$$

(*Note that, by applying the natural isomorphisms* $m_{X_{(r+1)}\{i,j\}} : M_{X\{i,j\}} \xrightarrow{\sim} \mathfrak{I}_{X_{(r+1)}\{i,j\}}$ [*cf. Definition 4.7, (ii)*], *one obtains a natural isomorphism of* $V_{\mathbb{P}_K}\{2\}$ *with the* $\mathbb{Q}_l$*-vector space obtained by tensoring the free* $\mathbb{Z}_l$*-module*

$$M_{X\{1,2\}} \oplus M_{X\{2,3\}}$$

[*where* $M_{X\{i,j\}}$ *is the copy of* M_X *indexed by* $\{i,j\}$] *with* $\mathbb{Q}_l$.) *Then the natural inclusions* $\mathfrak{I}_{X_{(r+1)}\{i,j\}} \hookrightarrow \Delta_{\mathbb{P}_K}\{2\}$, *together with the unique* G_K*-equivariant isomorphism of Lie algebras*

$$\mathrm{Lie}(\Delta_{\mathbb{P}_K}\{2\}(a/b)) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}}(\Delta_{\mathbb{P}_K}\{2\})(a/b)$$

determine a G_K*-equivariant morphism*

$$V_{\mathbb{P}_K}\{2\} \longrightarrow \mathrm{Lie}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty))$$

which exhibits, in a G_K*-equivariant fashion,* $\mathrm{Lie}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty))$ *as the completion with respect to the filtration topology of the free Lie algebra* $\mathfrak{Lie}(V_{\mathbb{P}_K}\{2\})$ *generated by* $V_{\mathbb{P}_K}\{2\}$ *equipped with a natural grading, hence also a filtration, by taking the* $\mathfrak{I}_{X_{(r+1)}\{i,j\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ *to be of weight* 2. *Moreover, the morphism of Lie algebras* $\mathrm{Lie}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty)) \to \mathrm{Lie}_{X_{(r+1)}}(1/\infty)$ *corresponding to the morphism* $\mathrm{Lin}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty))(\mathbb{Q}_l) \to \mathrm{Lin}_{X_{(r+1)}}(1/\infty)$ *disucussed in Definition 4.18,* (*i*), *coincides with the morphism induced by the natural inclusion*

$$V_{\mathbb{P}_K}\{2\} \hookrightarrow V_{U_{X_{(r+1)}}}$$

(*cf. Proposition 4.8*).

(*ii*) *Let* $S = \{x_1, \cdots, x_r\} \subseteq X(K)$ *be a subset of* $X(K)$ *of cardinality* $= r$ *equipped with an ordering. Then there exsits a unique* G_K*-equivariant isomorphism of Lie algebras*

$$\mathrm{Lie}(\Delta_{U_S}(a/b)) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}}(\Delta_{U_S})(a/b)$$

$$(\text{respectively, } \mathrm{Lie}(\Delta_{U_{\underline{S}}}(a/b)) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}}(\Delta_{U_{\underline{S}}})(a/b))$$

(*where* $a \leq b$ *are integers*). *Now let us write*

$$V_{U_S} \stackrel{\text{def}}{=} \bigoplus_{1 \leq i \leq r} (M_{X\{i\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \oplus \mathrm{Lie}(\Delta_{U_S})(1/2)$$

$$(\text{respectively, } V_{U_{\underline{S}}} \stackrel{\text{def}}{=} \bigoplus_{2 \leq i \leq r} (M_{X\{i\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \oplus \mathrm{Lie}(\Delta_{U_S})(1/2))$$

where $M_{X\{i\}}$ *is the copy of* M_X *indexed by* $\{i\}$. [*Note that, by applying the identity morphism* $\mathrm{Lie}(\Delta_{U_S})(1/2) \xrightarrow{\sim} \Delta_X^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, *one obtains a natural isomorphism of* V_{U_S} *with the* $\mathbb{Q}_l$*-vector space obtained by tensoring the free* $\mathbb{Z}_l$*-module*

$$\bigoplus_{1 \leq i \leq r} M_{X\{i\}} \oplus \Delta_X^{\mathrm{ab}} \quad (\text{respectively, } \bigoplus_{2 \leq i \leq r} M_{X\{i\}} \oplus \Delta_X^{\mathrm{ab}})$$

with $\mathbb{Q}_l$.] Then the isomorphism

$$\mathrm{Lie}(\Delta_{U_S}(a/b)) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}}(\Delta_{U_S})(a/b)$$

$$(\text{respectively, } \mathrm{Lie}(\Delta_{U_{\underline{S}}}(a/b)) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Q}}(\Delta_{U_{\underline{S}}})(a/b)),$$

together with the composite of the natural isomorphism $M_{X\{i\}} \xrightarrow{\sim} \mathfrak{I}_{x_i}[U_S]$ (respectively, $M_{X\{i\}} \xrightarrow{\sim} \mathfrak{I}_{x_i}[U_{\underline{S}}]$) (cf. Definition 4.7, (ii)) and the natural inclusions $\mathfrak{I}_{x_i}[U_S] \hookrightarrow \Delta_{U_S}$ (respectively, $\mathfrak{I}_{x_i}[U_{\underline{S}}] \hookrightarrow \Delta_{U_{\underline{S}}}$) [where $\mathfrak{I}_{x_i}[U_S]$ (respectively, $\mathfrak{I}_{x_i}[U_{\underline{S}}]$) is the inertia subgroup of Δ_{U_S} (respectively, $\Delta_{U_{\underline{S}}}$) associated to $x_i \in S$ (respectively, $x_i \in \underline{S}$)], determine a G_K-equivariant morphism

$$V_{U_S} \longrightarrow \mathrm{Lie}(\Delta_{U_S}(1/\infty))$$

$$(\text{respectively, } V_{U_{\underline{S}}} \longrightarrow \mathrm{Lie}(\Delta_{U_{\underline{S}}}(1/\infty)))$$

which exhibits, in a G_K-equivariant fashion, $\mathrm{Lie}(\Delta_{U_S}(1/\infty))$ (respectively, $\mathrm{Lie}(\Delta_{U_{\underline{S}}}(1/\infty))$) as the quotient of the completion with respect to the filtration topology of the free Lie algebra $\mathfrak{Lie}(V_{U_S})$ (respectively, $\mathfrak{Lie}(V_{U_{\underline{S}}})$) generated by V_{U_S} (respectively, $V_{U_{\underline{S}}}$) equipped with a natural grading, hence also a filtration, by taking the $M_{X\{i\}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ to be of weight 2, $\mathrm{Lie}(\Delta_{U_S}(1/2))$ (respectively, $\mathrm{Lie}(\Delta_{U_{\underline{S}}}(1/2))$) to be of weight 1, by the relations determined by the image of the morphism:

$$M_X \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \overset{\bigoplus \mathrm{id}_{M_X} \oplus i_X^{\mathrm{cup}}}{\rightarrow} (\bigoplus_{i=1}^{r} M_{X\{i\}} \oplus \bigwedge^2 \Delta_X^{\mathrm{ab}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \overset{\mathrm{incl.}\oplus[\,,\,]}{\rightarrow} \mathfrak{Lie}(V_{U_S})(2/3)$$

$$(\text{respectively, } M_X \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \overset{\bigoplus \mathrm{id}_{M_X} \oplus i_X^{\mathrm{cup}}}{\rightarrow} (\bigoplus_{i=2}^{r} M_{X\{i\}} \oplus \bigwedge^2 \Delta_X^{\mathrm{ab}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

$$\overset{\mathrm{incl.}\oplus[\,,\,]}{\rightarrow} \mathfrak{Lie}(V_{U_{\underline{S}}})(2/3)).$$

Proof. This follows from a similar argument to the argument used in the proof of Proposition 4.8, together with [14] Proposition 3.3, (i). $\square$

In the following, let L be a finite field whose characteristic (respectively, cardinality) we denote by p_L (respectively, q_L) such that l is invertible in L (i.e., $l \neq p_L$), $\overline{L}$ an algebraic closure of L, $G_L \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{L}/L)$, and Y a proper hyperbolic curve over L. Moreover, let $\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}}$ be a Frobenius-preserving and order-preserving isomorphism, and $t'_0 : G_L \to \Pi_{Y_{(r)}}$ the section of the natural morphism $\Pi_{Y_{(r)}} \to G_L$ corresponding to s'_0 under the isomorphism $\alpha_{(r)}$. Then by Lemma 4.20 below, we obtain an isomorphism

$$\alpha^{\mathrm{Lie}}_{(r+1)} : \Pi^{\mathrm{Lie}}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathrm{Lie}}_{Y_{(r+1)}}.$$

Note that by the construction of $\alpha^{\mathrm{Lie}}_{(r+1)}$, together with the assumption on the section s_0 which is fixed in the discussion preceding Proposition 4.6, we may assume that

> $\alpha^{\mathrm{Lie}}_{(r+1)}$ *maps the image of* $\mathfrak{I}_{X_{(r+1)}\{1,2\}}$ *via* $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$ *bijectively onto the image of* $\mathfrak{I}_{Y_{(r+1)}\{1,2\}}$ *via* $\mathrm{Int}^{\Pi}_{Y_{(r+1)}}$.

Lemma 4.20. *Let $\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}}$ be a Frobenius-preserving and order-preserving isomorphism. Then, for any integer $1 \leq r' \leq r+1$, there exists a unique isomorphism*

$$\alpha^{\mathrm{Lie}}_{(r')} : \Pi^{\mathrm{Lie}}_{X_{(r')}} \xrightarrow{\sim} \Pi^{\mathrm{Lie}}_{Y_{(r')}}$$

$$(\text{respectively, } \ \alpha^{\mathrm{Lie}\leq b}_{(r')} : \Pi^{\mathrm{Lie}\leq b}_{X_{(r')}} \xrightarrow{\sim} \Pi^{\mathrm{Lie}\leq b}_{Y_{(r')}} \text{ for any } b \geq 1)$$

which, for any integer $1 \leq i \leq r'$, fits into the following commutative diagrams:

$$\begin{tikzcd} \Pi^{\mathrm{Lie}}_{X_{(r')}} \arrow[r, "\alpha^{\mathrm{Lie}}_{(r')}"] \arrow[d] & \Pi^{\mathrm{Lie}}_{Y_{(r')}} \arrow[d] \\ \Pi^{\mathrm{Lie}\leq b}_{X_{(r')}} \arrow[r, "\alpha^{\mathrm{Lie}\leq b}_{(r')}"'] & \Pi^{\mathrm{Lie}\leq b}_{Y_{(r')}} \end{tikzcd} \qquad \begin{tikzcd} \Pi^{\mathrm{Lie}}_{X_{(r')}} \arrow[r, "\alpha^{\mathrm{Lie}}_{(r')}"] \arrow[d, "\text{via } p_{U_{X_{(r'-1)}}:i}"'] & \Pi^{\mathrm{Lie}}_{Y_{(r')}} \arrow[d, "\text{via } p_{U_{Y_{(r'-1)}}:i}"] \\ \Pi^{\mathrm{Lie}}_{X_{(r'-1)}} \arrow[r, "\alpha^{\mathrm{Lie}}_{(r'-1)}"'] & \Pi^{\mathrm{Lie}}_{Y_{(r'-1)}} \end{tikzcd}.$$

Moreover, if $r' \leq r$, then $\alpha^{\mathrm{Lie}}_{(r')}$ fits into the following commutative diagram:

$$\begin{tikzcd} \Pi_{X_{(r')}} \arrow[r, "\alpha_{(r')}"] \arrow[d, "\mathrm{Int}^{\Pi}_{X_{(r')}}"'] & \Pi_{Y_{(r')}} \arrow[d, "\mathrm{Int}^{\Pi}_{Y_{(r')}}"] \\ \Pi^{\mathrm{Lie}}_{X_{(r')}} \arrow[r, "\alpha^{\mathrm{Lie}}_{(r')}"'] & \Pi^{\mathrm{Lie}}_{Y_{(r')}} \end{tikzcd}.$$

Proof. By the discussion following Definition 2.13, $\alpha_{(r)}$ induces a Frobenius-preserving isomorphism $\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$. Thus, the existence of an isomorphism

$$\alpha^{\mathrm{Lie}}_{(r')} : \Pi^{\mathrm{Lie}}_{X_{(r')}} \xrightarrow{\sim} \Pi^{\mathrm{Lie}}_{Y_{(r')}}$$

$$(\text{respectively, } \ \alpha^{\mathrm{Lie}\leq b}_{(r')} : \Pi^{\mathrm{Lie}\leq b}_{X_{(r')}} \xrightarrow{\sim} \Pi^{\mathrm{Lie}\leq b}_{Y_{(r')}})$$

which satisfies the condition in the statement of Lemma 4.20 follows from Proposition 4.8. Now let $r' \leq r$. Then the isomorphism $\alpha_{(r')} : \Pi_{X_{(r')}} \xrightarrow{\sim} \Pi_{Y_{(r')}}$ (obtained in the discussion following Definition 2.13) induces an isomorphism $\Pi^{\mathrm{Lie}}_{X_{(r')}} \xrightarrow{\sim} \Pi^{\mathrm{Lie}}_{Y_{(r')}}$ which fits into the commutative diagram

$$\begin{tikzcd} \Pi_{X_{(r')}} \arrow[r, "\alpha_{(r')}"] \arrow[d, "\mathrm{Int}^{\Pi}_{X_{(r')}}"'] & \Pi_{Y_{(r')}} \arrow[d, "\mathrm{Int}^{\Pi}_{Y_{(r')}}"] \\ \Pi^{\mathrm{Lie}}_{X_{(r')}} \arrow[r] & \Pi^{\mathrm{Lie}}_{Y_{(r')}} \end{tikzcd},$$

by the definitions of $\Pi^{\mathrm{Lie}}_{X_{(r')}}$ and $\Pi^{\mathrm{Lie}}_{Y_{(r')}}$. Thus, to prove Lemma 4.20, it is enough to show that this isomorphism of $\Pi^{\mathrm{Lie}}_{X_{(r')}}$ with $\Pi^{\mathrm{Lie}}_{Y_{(r')}}$ coincides with the isomorphism $\alpha^{\mathrm{Lie}}_{(r')}$. On the other hand, this follows from Proposition 4.6, (ii), by considering the eigenspaces with respect to the action of the Frobenius element (cf. Proposition 4.6, (ii)). ∎

Lemma 4.21. *Let $r \geq 2$ be an integer. Then if the section s_0 satisfies the condition $(\dagger_{\mathbb{P}})$, then the following conditions are equivalent:*

(i) $\alpha_{(r)}$ *is tripod-preserving.*

(ii) *The isomorphism* $\alpha^{\mathrm{Lie}}_{(r+1)}$ *maps the image of* $\Pi_{\mathbb{P}_K}\{2\}$ *via* $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$ *bijectively onto the image of* $\Pi_{\mathbb{P}_L}\{2\}$ *via* $\mathrm{Int}^{\Pi}_{Y_{(r+1)}}$.

In particular, if $\alpha_{(r)}$ *is tripod-preserving, then* $\alpha^{\mathrm{Lie}}_{(r+1)}$ *maps the image (via* $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$*) of the decomposition subgroup* $\mathfrak{D}^{\mathbb{P}}_{X_{(r+1)}\{2,3\}}$ *of* $\Pi_{\mathbb{P}_K}\{2\}$ *such that* $\mathfrak{D}^{\mathbb{P}}_{X_{(r+1)}\{2,3\}} \cap \Delta_{X_{(r+1)}}$ *coincides with* $\mathfrak{I}_{X_{(r+1)}\{2,3\}}$ *bijectively onto a* $\Pi_{Y_{(r+1)}}$*-conjugate of the image (via* $\mathrm{Int}^{\Pi}_{Y_{(r+1)}}$*) of the decomposition subgroup* $\mathfrak{D}^{\mathbb{P}}_{Y_{(r+1)}\{2,3\}}$ *of* $\Pi_{\mathbb{P}_L}\{2\}$ *such that* $\mathfrak{D}^{\mathbb{P}}_{Y_{(r+1)}\{2,3\}} \cap \Delta_{Y_{(r+1)}}$ *coincides with* $\mathfrak{I}_{Y_{(r+1)}\{2,3\}}$.

Proof. It follows immediately from the definition of the term "tripod-preserving" that condition (ii) implies condition (i) (cf. Lemma 4.17). We prove the assertion that condition (i) implies condition (ii). Since $\alpha_{(r)}$ is tripod-preserving, there exists an isomorphism $\alpha_{\mathbb{P}} : \Pi_{\mathbb{P}_K}\{2\} \xrightarrow{\sim} \Pi_{\mathbb{P}_L}\{2\}$ such that the composite $M_X^{\oplus 2} \xrightarrow{\sim} M_Y^{\oplus 2}$ of the natural isomorphism $M_X^{\oplus 2} \xrightarrow{\sim} \Delta^{\mathrm{ab}}_{\mathbb{P}_K}$ (cf. Definition 4.7, (ii)), the isomorphism $\Delta^{\mathrm{ab}}_{\mathbb{P}_K} \xrightarrow{\sim} \Delta^{\mathrm{ab}}_{\mathbb{P}_L}$ induced by $\alpha_{\mathbb{P}}$, and the natural isomorphism $\Delta^{\mathrm{ab}}_{\mathbb{P}_L} \xrightarrow{\sim} M_L^{\oplus 2}$ coincides with the isomorphism obtained by the isomorphism $M_X \to M_Y$ obtained by $\alpha_{(r)}$; moreover, it follows from the definitions of $\Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\}$ and $\Pi^{\mathrm{Lie}}_{\mathbb{P}_L}\{2\}$ that $\alpha_{\mathbb{P}}$ induces an isomorphism $\alpha^{\mathrm{Lie}}_{\mathbb{P}} : \Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \xrightarrow{\sim} \Pi^{\mathrm{Lie}}_{\mathbb{P}_L}\{2\}$ which fits into a commutative diagram

$$\begin{array}{ccc}
\Pi_{\mathbb{P}_K}\{2\} & \xrightarrow{\alpha_{\mathbb{P}}} & \Pi_{\mathbb{P}_L}\{2\} \\
\downarrow & & \downarrow \\
\Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} & \xrightarrow[\alpha^{\mathrm{Lie}}_{\mathbb{P}}]{} & \Pi^{\mathrm{Lie}}_{\mathbb{P}_L}\{2\}.
\end{array}$$

On the other hand, by Proposition 4.19, (i), the isomorphism $M_X \xrightarrow{\sim} M_Y$ induced by $\alpha_{(r)}$ induces an isomorphism $\Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} \xrightarrow{\sim} \Pi^{\mathrm{Lie}}_{\mathbb{P}_L}\{2\}$ which fits into a commutative diagram

$$\begin{array}{ccc}
\Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\} & \longrightarrow & \Pi^{\mathrm{Lie}}_{\mathbb{P}_L}\{2\} \\
\downarrow & & \downarrow \\
\Pi^{\mathrm{Lie}}_{X_{(r+1)}} & \xrightarrow[\alpha^{\mathrm{Lie}}_{(r+1)}]{} & \Pi^{\mathrm{Lie}}_{Y_{(r+1)}},
\end{array}$$

where the vertical arrows are the morphism obtained in Definition 4.18, (i). Thus, to prove Lemma 4.21, it is enough to show that this isomorphism of $\Pi^{\mathrm{Lie}}_{\mathbb{P}_K}\{2\}$ with $\Pi^{\mathrm{Lie}}_{\mathbb{P}_L}\{2\}$ coincides with the isomorphism $\alpha^{\mathrm{Lie}}_{\mathbb{P}}$. On the other hand, this follows from the fact that $\mathrm{Lie}(\Delta_{\mathbb{P}_K}\{2\}(1/\infty))$ (respectively, $\mathrm{Lie}(\Delta_{\mathbb{P}_L}\{2\}(1/\infty))$) is generated by the image of $V_{\mathbb{P}_K}\{2\}$ (respectively, $V_{\mathbb{P}_L}\{2\}$), by considering the eigenspaces with respect to the action of the respective Frobenius elements (cf. Proposition 4.19, (i)). ☐

Lemma 4.22. *Let* $S = \{x_1, \cdots, x_r\}$ *be a subset of* $X(K)$ *of cardinality* $= r$ *with an ordering,* $s_S : G_K \to \Pi_{X_{(r)}}$ *the section of the natural morphism* $\Pi_{X_{(r)}} \to G_K$ *corresponding to the* K*-rational point* $(x_1, \cdots, x_r) \in U_{X_{(r)}}(K)$*, and* $(y_1, \cdots, y_r) \in U_{Y_{(r)}}(L)$ *an* L*-rational point of* $U_{Y_{(r)}}$ *to which the section*

t_T obtained as the composite $G_L \overset{\alpha_{(0)}}{\overset{\sim}{\leftarrow}} G_K \overset{s_S}{\to} \Pi_{X_{(r)}} \overset{\alpha_{(r)}}{\overset{\sim}{\to}} \Pi_{Y_{(r)}}$ of the natural morphism $\Pi_{Y_{(r)}} \to G_L$ corresponds (cf. Proposition 2.12, (ii)). Let us write $T \overset{\text{def}}{=} \{y_1, \cdots, y_r\}$. Let $\alpha_{S,T} : \Pi_{U_S} \overset{\sim}{\to} \Pi_{V_T}$ be an isomorphism of the geometrically pro-l fundamental group Π_{U_S} of $U_S \overset{\text{def}}{=} X \setminus S$ with the geometrically pro-l fundamental group Π_{V_T} of $V_T \overset{\text{def}}{=} Y \setminus T$ such that the isomorphism $\Pi_X \overset{\sim}{\to} \Pi_Y$ induced by $\alpha_{S,T}$ coincides with the isomorphism $\alpha_{(1)} : \Pi_X \overset{\sim}{\to} \Pi_Y$ induced by $\alpha_{(r)}$. Let us assume that the section s'_0 (respectively, t'_0) of the natural morphism $\Pi_{X_{(r)}} \to G_K$ (respectively, $\Pi_{Y_{(r)}} \to G_L$) (fixed in the discussion preceding Proposition 4.6) coincides with s_S (respectively, t_T). [In particular, the section $G_K \to \Pi_{X_{(r+1)}}$ (respectively, $G_L \to \Pi_{Y_{(r+1)}}$) fixed in the discussion preceding Proposition 4.6 satisfies the condition ($\dagger_S$) (respectively, ($\dagger_T$)).] Then there exist morphisms $\Pi_{U_S} \to \Pi^{\mathrm{Lie}}_{X_{(r+1)}}$ and $\Pi_{V_T} \to \Pi^{\mathrm{Lie}}_{Y_{(r+1)}}$ which fit into a commutative diagram

$$\begin{tikzcd}
\Pi_{U_S} \arrow[r, "\alpha_{S,T}"] \arrow[d] & \Pi_{V_T} \arrow[d] \\
\Pi^{\mathrm{Lie}}_{X_{(r+1)}} \arrow[r, "\alpha^{\mathrm{Lie}}_{(r+1)}"'] & \Pi^{\mathrm{Lie}}_{Y_{(r+1)}}
\end{tikzcd}$$

and satisfy the following condition: The quotient of Π_{U_S} (respectively, Π_{V_T}) determined by the composite

$$\Pi_{U_S} \longrightarrow \Pi^{\mathrm{Lie}}_{X_{(r+1)}} \twoheadrightarrow \Pi_{X_{(r+1)}\underline{\{1,2\}}}$$

$$(\text{respectively, } \Pi_{V_T} \longrightarrow \Pi^{\mathrm{Lie}}_{Y_{(r+1)}} \twoheadrightarrow \Pi_{Y_{(r+1)}\underline{\{1,2\}}})$$

coincides with the natural quotient $\Pi_{U_S} \twoheadrightarrow \Pi_{U_{\underline{S}}}$ (respectively, $\Pi_{V_T} \twoheadrightarrow \Pi_{V_{\underline{T}}}$), where $\underline{S} \overset{\text{def}}{=} \{x_2, \cdots, x_r\}$ (respectively, $\underline{T} \overset{\text{def}}{=} \{y_2, \cdots, y_r\}$); moreover, this composite determines an isomorphism $\Pi_{U_{\underline{S}}} \overset{\sim}{\to} \Pi_{X_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{X_{(r)}}} G_K$ (respectively, $\Pi_{V_{\underline{T}}} \overset{\sim}{\to} \Pi_{Y_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{Y_{(r)}}} G_L$), where the morphism $\Pi_{X_{(r+1)}\underline{\{1,2\}}} \to \Pi_{X_{(r)}}$ (respectively, $\Pi_{Y_{(r+1)}\underline{\{1,2\}}} \to \Pi_{Y_{(r)}}$) is the morphism induced by $U_{X_{(r+1)}\underline{\{1,2\}}} \overset{p_{U^{\{1,2\}}_{X_{(r)}}:2}}{\to} U_{X_{(r)}}$ (respectively, $U_{Y_{(r+1)}\underline{\{1,2\}}} \overset{p_{U^{\{1,2\}}_{Y_{(r)}}:2}}{\to} U_{Y_{(r)}}$), and the morphism $G_K \to \Pi_{X_{(r)}}$ (respectively, $G_L \to \Pi_{Y_{(r)}}$) is s_S (respectively, t_T). In particular, if we denote by $\mathfrak{D}^S_{X_{(r+1)}\{2,3\}}$ (respectively, $\mathfrak{D}^T_{Y_{(r+1)}\{2,3\}}$) the decomposition subgroup of Π_{U_S} (respectively, Π_{V_T}) associated to x_2 (respectively, y_2) such that $\mathfrak{D}^S_{X_{(r+1)}\{2,3\}} \cap \Delta_{X_{(r+1)}}$ (respectively, $\mathfrak{D}^T_{Y_{(r+1)}\{2,3\}} \cap \Delta_{Y_{(r+1)}}$) coincides wtih $\mathfrak{I}_{X_{(r+1)}\{2,3\}}$ (respectively, $\mathfrak{I}_{Y_{(r+1)}\{2,3\}}$), then the isomorphism $\alpha^{\mathrm{Lie}}_{(r+1)}$ maps the image of $\mathfrak{D}^S_{X_{(r+1)}\{2,3\}}$ via $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$ bijectively onto a $\Pi_{Y_{(r+1)}}$-conjugate of the image of $\mathfrak{D}^T_{Y_{(r+1)}\{2,3\}}$ via $\mathrm{Int}^{\Pi}_{X_{(r+1)}}$.

Proof. By the assumption on $\alpha_{S,T}$, $\alpha_{S,T}$ induces an isomorphism $\alpha_{\underline{S},\underline{T}} : \Pi_{U_{\underline{S}}} \overset{\sim}{\to} \Pi_{V_{\underline{T}}}$. On the other hand, by the definitions of $\Pi_{X_{(r+1)}\underline{\{1,2\}}}$ and $\Pi_{Y_{(r+1)}\underline{\{1,2\}}}$, the isomorphism $\alpha_{(r)} : \Pi_{X_{(r)}} \overset{\sim}{\to} \Pi_{Y_{(r)}}$ induces an isomorphism $\Pi_{X_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{X_{(r)}}} G_K \overset{\sim}{\to} \Pi_{Y_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{Y_{(r)}}} G_L$, where these fiber product is as in the statement of

Lemma 4.22; moreover, it follows from the assumption on s_S (respectively, t_T) that the profinite group $\Pi_{X_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{X_{(r)}}} G_K$ (respectively, $\Pi_{Y_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{Y_{(r)}}} G_L$) is naturally isomorphic to the geometrically pro-l fundamental group of $U_{\underline{S}}$ (respectively, $V_{\underline{T}}$). Let us fix isomorphisms $\Pi_{U_{\underline{S}}} \xrightarrow{\sim} \Pi_{X_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{X_{(r)}}} G_K$ and $\Pi_{V_{\underline{T}}} \xrightarrow{\sim} \Pi_{Y_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{Y_{(r)}}} G_L$. Then it follows from Proposition 4.23 below that by composition with a cuspidally inner automorphism of $\Pi_{Y_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{Y_{(r)}}} G_L$ (relative to $\Pi_{Y_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{Y_{(r)}}} G_L \twoheadrightarrow \Pi_Y$) if necessary, we may assume that the following diagram commutes:

$$\begin{tikzcd}
\Pi_{U_{\underline{S}}} \arrow[r, "\alpha_{\underline{S},\underline{T}}"] \arrow[d, "\wr"'] & \Pi_{V_{\underline{T}}} \arrow[d, "\wr"] \\
\Pi_{X_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{X_{(r)}}} G_K \arrow[r, "\text{via } \alpha_{(r)}"'] & \Pi_{Y_{(r+1)}\underline{\{1,2\}}} \times_{\Pi_{Y_{(r)}}} G_L .
\end{tikzcd}$$

In particular, it follows from Proposition 4.19, (ii), together with the fact that the isomorphism of $\Pi_X \xrightarrow{\sim} \Pi_Y$ induced by $\alpha_{S,T}$ coincides with the isomorphism $\alpha_{(1)}$, that we obtain a commutative diagram

$$\begin{tikzcd}
\Pi^{\mathrm{Lie}}_{U_S} \arrow[r] \arrow[d] & \Pi^{\mathrm{Lie}}_{V_T} \arrow[d] \\
\Pi^{\mathrm{Lie}}_{X_{(r+1)}} \arrow[r, "\alpha^{\mathrm{Lie}}_{(r+1)}"'] & \Pi^{\mathrm{Lie}}_{Y_{(r+1)}} .
\end{tikzcd}$$

On the other hand, by the definitions of $\Pi^{\mathrm{Lie}}_{U_S}$ and $\Pi^{\mathrm{Lie}}_{V_T}$, $\alpha_{S,T}$ induces an isomorphism $\alpha^{\mathrm{Lie}}_{S,T} : \Pi^{\mathrm{Lie}}_{U_S} \xrightarrow{\sim} \Pi^{\mathrm{Lie}}_{V_T}$ which fits into the following commutative diagram:

$$\begin{tikzcd}
\Pi_{U_S} \arrow[r, "\alpha_{S,T}"] \arrow[d] & \Pi_{V_T} \arrow[d] \\
\Pi^{\mathrm{Lie}}_{U_S} \arrow[r, "\alpha^{\mathrm{Lie}}_{S,T}"'] & \Pi^{\mathrm{Lie}}_{V_T} .
\end{tikzcd}$$

Therefore, Lemma 4.22 follows from a similar argument to the argument used in the proof of Lemma 4.21, together with Proposition 4.19, (ii). $\square$

Proposition 4.23. *Let K be a finite field, S a connected scheme which is locally of finite type over K, $f : X \to S$ a hyperbolic curve, $s : S \to X$ a section of f, $U \subseteq X$ the open subscheme of X obtained as the complement of the (scheme-theoretic) image of s, and $f^U : U \to S$ the restriction of f to U. Let*

$$\alpha : \Pi_U \xrightarrow{\sim} \Pi_U$$

be an automorphism of the geometrically pro-l fundamental group Π_U of U which fits into a commutative diagram

$$\begin{tikzcd}
\Pi_U \arrow[r, "\alpha"] \arrow[d] & \Pi_U \arrow[d] \\
\Pi_X \arrow[r, equal] & \Pi_X ,
\end{tikzcd}$$

where Π_X is the geometrically pro-l fundamental group of X, and the vertical arrows are the surjections induced by the natural open immersion $U \hookrightarrow X$. Then α is a cuspidally inner automorphism, i.e., there exists an element γ of the kernel of the natural surjection $\Pi_U \twoheadrightarrow \Pi_X$ such that $\alpha = \mathrm{Inn}(\gamma)$.

Proof. If S is isomorphic to the spectrum of a finite extension field of K, then Proposition 4.23 follows from a similar argument to the argument used in the proof of the uniqueness of "α_∞" in [14], Theorem 3.10. Therefore, Proposition 4.23 follows from Lemma 4.24 below, together with the slimness of the kernel of the surjection induced by f^U on geometrically pro-l fundamental groups. $\square$

Lemma 4.24. *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

be an exact sequence of profinite groups, and ϕ an automorphism of Π which induces the identity morphisms of Δ and G. Assume that Δ is slim. Then ϕ is the identity morphism.

Proof. By the slimness of Δ, we have a natural isomorphism

$$\Pi \xrightarrow{\sim} \mathrm{Aut}(\Delta) \times_{\mathrm{Out}(\Delta)} G\,.$$

Now it is easily verified that if ϕ is an automorphism of Π which preserves the subgroup $\Delta \subseteq \Pi$, then the automorphism of $\mathrm{Aut}(\Delta) \times_{\mathrm{Out}(\Delta)} G$ corresponding to ϕ (under the above isomorphism) is given by

$$\begin{array}{ccc} \mathrm{Aut}(\Delta) \times_{\mathrm{Out}(\Delta)} G & \xrightarrow{\sim} & \mathrm{Aut}(\Delta) \times_{\mathrm{Out}(\Delta)} G \\ (f, g) & \mapsto & (\phi\,|_\Delta \circ f \circ \phi^{-1}\,|_\Delta, \overline{\phi}(g))\,, \end{array}$$

where $\overline{\phi}$ is the automorphism of G induced by ϕ. Thus, the assertion is immediate. $\square$

In the following, we assume that

$$r \geq 2\,.$$

In the rest of this section, we reconstruct the geometrically pro-l fundamental group of the higher dimensional configuration space.

Lemma 4.25. *The image of the diagonal morphism*

$$\Pi_{X_{(r)}} \longrightarrow \Pi_{X_{(r)}} \times_{\Pi_{X_{(r-1)}}} \Pi_{X_{(r)}} \xleftarrow{\sim} \Pi_{X_{(r+1)}\underline{\{1,2\}}}$$

(cf. Lemma 2.10) is a conjugate of $\mathfrak{D}^{\leq 1}_{X_{(r+1)}\{1,2\}}$ in $\Pi_{X_{(r+1)}\underline{\{1,2\}}} \simeq \Pi^{\mathrm{Lie}\,\leq 1}_{X_{(r+1)}}$ (cf. Lemma 4.14, (i)).

Proof. This follows from the definitions of $\mathfrak{D}^{\leq 1}_{X_{(r+1)}\{1,2\}}$ and $\Pi_{X_{(r+1)}\underline{\{1,2\}}}$. $\square$

Lemma 4.26.

(*i*) *The diagrams*

$$\begin{array}{ccc} \mathfrak{D}^{\leq 2}_{X_{(r+1)}\{1,2\}} & \longrightarrow & \mathfrak{D}^{\leq 1}_{X_{(r+1)}\{1,2\}} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{D}^{\leq 2}_{X_{(r)}\{1,2\}} & \longrightarrow & \mathfrak{D}^{\leq 1}_{X_{(r)}\{1,2\}} \end{array} \qquad \begin{array}{ccc} \Pi^{\leq 2}_{X_{(r+1)}} & \longrightarrow & \Pi^{\leq 1}_{X_{(r+1)}} \\ \big\downarrow & & \big\downarrow \\ \Pi^{\leq 2}_{X_{(r)}} & \longrightarrow & \Pi^{\leq 1}_{X_{(r)}} \end{array}$$

induced by the diagram

$$\begin{array}{ccc} \Pi^{\mathrm{Lie}\leq 2}_{X_{(r+1)}} & \longrightarrow & \Pi^{\mathrm{Lie}\leq 1}_{X_{(r+1)}} \\ {\scriptstyle \text{via } p^{\log}_{X_{(r)}:3}}\big\downarrow & & \big\downarrow{\scriptstyle \text{via } p^{\log}_{X_{(r)}:3}} \\ \Pi^{\mathrm{Lie}\leq 2}_{X_{(r)}} & \longrightarrow & \Pi^{\mathrm{Lie}\leq 1}_{X_{(r)}} \end{array}$$

are cartesian.

(*ii*) *The subgroup of* $\Pi^{\mathrm{Lie}\leq 2}_{X_{(r+1)}}$ *obtained as the intersection of the inverse image of* $\mathfrak{D}^{\leq 1}_{X_{(r+1)}\{1,2\}}$ (*respectively,* $\Pi^{\leq 1}_{X_{(r+1)}}$) *via the natural projection*

$$\Pi^{\mathrm{Lie}\leq 2}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq 1}_{X_{(r+1)}}$$

and the inverse image of $\mathfrak{D}^{\leq 2}_{X_{(r)}\{1,2\}}$ (*respectively,* $\Pi^{\leq 2}_{X_{(r)}}$) *via*

$$\Pi^{\mathrm{Lie}\leq 2}_{X_{(r+1)}} \overset{\text{via } p^{\log}_{X_{(r)}:3}}{\twoheadrightarrow} \Pi^{\mathrm{Lie}\leq 2}_{X_{(r)}}$$

coincides with $\mathfrak{D}^{\leq 2}_{X_{(r+1)}\{1,2\}}$ (*respectively,* $\Pi^{\leq 2}_{X_{(r+1)}}$).

Proof. Assertion (i) follows immediately from Lemmas 4.4; 4.14, (ii). Assertion (ii) follows from assertion (i), Lemma 4.27 below, together with the fact that the homomorphism

$$\Pi^{\mathrm{Lie}\leq 2}_{X_{(r+1)}} \xrightarrow{\text{via } p^{\log}_{X_{(r)}:3}} \Pi^{\mathrm{Lie}\leq 2}_{X_{(r)}}$$

induces an *isomorphism* of the kernel of the natural projection

$$\Pi^{\mathrm{Lie}\leq 2}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq 1}_{X_{(r+1)}}$$

with the kernel of the natural projection

$$\Pi^{\mathrm{Lie}\leq 2}_{X_{(r)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq 1}_{X_{(r)}}$$

(cf. Lemma 4.3, (ii); Proposition 4.6). ☐

Lemma 4.27. *Let*

$$\begin{array}{ccc} G_1 & \xrightarrow{f_2} & G_2 \\ {\scriptstyle f_3}\big\downarrow & & \big\downarrow \\ G_3 & \longrightarrow & G_4 \end{array}$$

be a commutative diagram of groups, and $H_1 \subseteq G_1$ a subgroup of G_1. Write H_2 (respectively, H_3) for the image of H_1 via f_2 (respectively, f_3). Then if the morphism

$$H_1 \longrightarrow H_2 \times_{G_4} H_3$$

induced by f_2 and f_3 is an isomorphism, and the intersection

$$\mathrm{Ker}\, f_2 \cap \mathrm{Ker}\, f_3$$

is trivial, then the natural inclusion morphism

$$H_1 \hookrightarrow f_2^{-1}(H_2) \cap f_3^{-1}(H_3)$$

is an isomorphism.

Proof. Observe that the morphisms f_2 and f_3 induce a morphism

$$f_2^{-1}(H_2) \cap f_3^{-1}(H_3) \longrightarrow H_2 \times_{G_4} H_3 .$$

Since the composite

$$H_1 \hookrightarrow f_2^{-1}(H_2) \cap f_3^{-1}(H_3) \longrightarrow H_2 \times_{G_4} H_3$$

of the natural inclusion $H_1 \hookrightarrow f_2^{-1}(H_2) \cap f_3^{-1}(H_3)$ and this morphism is an *isomorphism* by our assumption, we conclude that this morphism is *surjective*. Moreover, since $\mathrm{Ker}\, f_2 \cap \mathrm{Ker}\, f_3$ is trivial, this morphism is an *isomorphism*. Then the assertion is immediate. $\square$

Lemma 4.28. *The composite*

$$\mathfrak{D}^{\leq 2}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Pi^{\leq 2}_{X_{(r+1)}} \overset{\text{via } p^{\log}_{X_{(r)}}:3}{\twoheadrightarrow} \Pi^{\leq 2}_{X_{(r)}}$$

coincides with the composite

$$\mathfrak{D}^{\leq 2}_{X_{(r+1)}\{2,3\}} \overset{\text{via } p^{\log}_{X_{(r)}}:3}{\twoheadrightarrow} \Pi_{X_{(r)}} \twoheadrightarrow \Pi^{\leq 2}_{X_{(r)}} .$$

In particular, the morphism

$$\mathfrak{D}^{\leq 2}_{X_{(r+1)}\{2,3\}} \xrightarrow{\sim} \mathfrak{D}^{\leq 1}_{X_{(r+1)}\{2,3\}} \longrightarrow \Pi^{\leq 1}_{X_{(r+1)}} \times_{\Pi^{\leq 1}_{X_{(r)}}} \Pi^{\leq 2}_{X_{(r)}} \xleftarrow{\sim} \Pi^{\leq 2}_{X_{(r+1)}}$$

(*cf. Lemmas 4.14, (ii); 4.26, (i)*) *determined by the natural inclusion $\mathfrak{D}^{\leq 1}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Pi^{\leq 1}_{X_{(r+1)}}$ and the composite $\mathfrak{D}^{\leq 1}_{X_{(r+1)}\{2,3\}} \overset{\text{via } p^{\log}_{X_{(r)}}:3}{\twoheadrightarrow} \Pi_{X_{(r)}} \twoheadrightarrow \Pi^{\leq 2}_{X_{(r)}}$ coincides with the natural inclusion $\mathfrak{D}^{\leq 2}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Pi^{\leq 2}_{X_{(r+1)}}$.*

Proof. This follows immediately from Lemma 4.14, (ii). $\square$

Lemma 4.29. *Let $1 \leq i < j \leq r+1$ and $b \geq 2$ be integers. Then any two liftings of the natural inclusion $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{i,j\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ to inclusions $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{i,j\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ differ by conjugation in $\Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ by a unique element of the kernel of the surjection $\Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$.*

Proof. By Lemma 4.14, (ii), it is enough to show that

$$H^i(\mathfrak{D}^{\leq b}_{X_{(r+1)}\{i,j\}}, \mathrm{Lin}^{\mathrm{iner}}_{X_{(r)}}(b+1/b+2)) = 0$$

for $i = 0, 1$. Since the action of $\mathfrak{D}^{\Delta\leq b}_{X_{(r+1)}\{i,j\}}$ on $\mathrm{Lin}^{\mathrm{iner}}_{X_{(r)}}(b+1/b+2)$ is trivial, it thus suffices to observe (by considering the Hochschild-Serre spectral sequence associated to the surjection $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{i,j\}} \twoheadrightarrow G_K$) that the action of Fr_K on $\mathrm{Lin}^{\mathrm{iner}}_{X_{(r)}}(b+1/b+2)$ is "of weight $b+1 \geq 3$", while the action of Fr_K on $(\mathfrak{D}^{\Delta\leq b}_{X_{(r+1)}\{i,j\}})^{\mathrm{ab}}$ is "of weight ≤ 2" (cf. Proposition 4.6, (i)). This completes the proof of assertion. $\square$

Lemma 4.30. *Let*

$$\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}}$$

be a Frobenius-preserving and order-preserving isomorphism which is either tripod-preserving or the following condition $(*)$ *holds:*

$(*)$*: There exist*

- (i) *a subset $S = \{x_1, \cdots, x_r\}$ (respectively, $T \overset{\mathrm{def}}{=} \{y_1, \cdots, y_r\}$) of $X(K)$ (respectively, $Y(L)$) of cardinality $= r$ with an ordering such that if a section $s_S : G_K \to \Pi_{X_{(r)}}$ of the natural morphism $\Pi_{X_{(r)}} \to G_K$ corresponds to the K-rational point $(x_1, \cdots, x_r) \in U_{X_{(r)}}(K)$, then the section $t_T : G_L \to \Pi_{Y_{(r)}}$ of the natural morphism $\Pi_{Y_{(r)}} \to G_L$ corresponding to s_S (under the isomorphism $\alpha_{(r)}$) coincides with the section arising from the L-rational point $(y_1, \cdots, y_r) \in U_{Y_{(r)}}(L)$ (cf. Proposition 2.12, (ii)), and*
- (ii) *an isomorphism $\alpha_{S,T} : \Pi_{U_S} \xrightarrow{\sim} \Pi_{V_T}$ of the geometrically pro-l fundamental group Π_{U_S} of $U_S \overset{\mathrm{def}}{=} X \setminus S$ with the geometrically pro-l fundamental group Π_{V_T} of $V_T \overset{\mathrm{def}}{=} Y \setminus T$ such that the isomorphism $\Pi_X \xrightarrow{\sim} \Pi_Y$ induced by $\alpha_{S,T}$ coincides with the isomorphism $\alpha_{(1)} : \Pi_X \xrightarrow{\sim} \Pi_Y$ induced by $\alpha_{(r)}$.*

Then there exists an isomorphism

$$\alpha_{(r+1)} : \Pi_{X_{(r+1)}} \xrightarrow{\sim} \Pi_{Y_{(r+1)}}$$

well-defined up to composition with a $\mathrm{Ker}\,(\Pi_{Y_{(r+1)}} \twoheadrightarrow \Pi_{Y_{(r+1)}\underline{\{1,2\}}})$*-inner automorphism such that, for $i = 1, \cdots, r+1$, the following diagram commutes:*

$$\begin{array}{ccc} \Pi_{X_{(r+1)}} & \xrightarrow{\alpha_{(r+1)}} & \Pi_{Y_{(r+1)}} \\ {\scriptstyle \text{via } p_{U_{X_{(r)}}:i}} \downarrow & & \downarrow {\scriptstyle \text{via } p_{U_{Y_{(r)}}:i}} \\ \Pi_{X_{(r)}} & \xrightarrow[\alpha_{(r)}]{} & \Pi_{Y_{(r)}} \end{array} .$$

Proof. If $\alpha_{(r)}$ is tripod-preserving (respectively, satisfies the condition $(*)$), then we assume that the section s_0 satisfies the condition $(\dagger_{\mathbb{P}})$ (respectively, $(\dagger_S)$).

Then since $\alpha_{(r)}$ is Frobenius-preserving, it follows immediately from the naturality of our construction that $\alpha_{(r)}$ induces, for each positive integer b, isomorphisms

$$\alpha^{\mathrm{Lie}\leq b}_{(r+1)} : \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}} \xrightarrow{\sim} \Pi^{\mathrm{Lie}\leq b}_{Y_{(r+1)}} \ ; \ \alpha^{\mathrm{Lie}\leq b}_{(r)} : \Pi^{\mathrm{Lie}\leq b}_{X_{(r)}} \xrightarrow{\sim} \Pi^{\mathrm{Lie}\leq b}_{Y_{(r)}}$$

that fit into the following commutative diagrams:

$$\begin{tikzcd}
\Pi^{\mathrm{Lie}}_{X_{(r+1)}} \arrow[r] \arrow[d, "\alpha^{\mathrm{Lie}}_{(r+1)}"'] & \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}} \arrow[r] \arrow[d, "\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}"] & \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}} \arrow[d, "\alpha^{\mathrm{Lie}\leq b}_{(r+1)}"] \\
\Pi^{\mathrm{Lie}}_{Y_{(r+1)}} \arrow[r] & \Pi^{\mathrm{Lie}\leq b+1}_{Y_{(r+1)}} \arrow[r] & \Pi^{\mathrm{Lie}\leq b}_{Y_{(r+1)}}
\end{tikzcd}$$

$$\begin{tikzcd}
\Pi_{X_{(r)}} \arrow[r, "\mathrm{Int}^{\Pi}_{X_{(r)}}"] \arrow[d, "\alpha_{(r)}"'] & \Pi^{\mathrm{Lie}}_{X_{(r)}} \arrow[r] \arrow[d, "\alpha^{\mathrm{Lie}}_{(r)}"'] & \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r)}} \arrow[r] \arrow[d, "\alpha^{\mathrm{Lie}\leq b+1}_{(r)}"] & \Pi^{\mathrm{Lie}\leq b}_{X_{(r)}} \arrow[d, "\alpha^{\mathrm{Lie}\leq b}_{(r)}"] \\
\Pi_{Y_{(r)}} \arrow[r, "\mathrm{Int}^{\Pi}_{Y_{(r)}}"'] & \Pi^{\mathrm{Lie}}_{Y_{(r)}} \arrow[r] & \Pi^{\mathrm{Lie}\leq b+1}_{Y_{(r)}} \arrow[r] & \Pi^{\mathrm{Lie}\leq b}_{Y_{(r)}}
\end{tikzcd}$$

$$\begin{tikzcd}
\Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}} \arrow[r, "\text{via } p_{U_{X_{(r)}}:3}"] \arrow[d, "\alpha^{\mathrm{Lie}\leq b}_{(r+1)}"'] & \Pi^{\mathrm{Lie}\leq b}_{X_{(r)}} \arrow[d, "\alpha^{\mathrm{Lie}\leq b}_{(r)}"] \\
\Pi^{\mathrm{Lie}\leq b}_{Y_{(r+1)}} \arrow[r, "\text{via } p_{U_{Y_{(r)}}:3}"'] & \Pi^{\mathrm{Lie}\leq b}_{Y_{(r)}}
\end{tikzcd}$$

(cf. Lemma 4.20).

Moreover, $\alpha^{\mathrm{Lie}\leq b}_{(r+1)}$ is compatible with the Frobenius elements on either side, and (by the assumption on the section s_0 fixed in the discussion preceding Proposition 4.6) $\alpha^{\mathrm{Lie}\leq b}_{(r+1)}$ maps $\mathfrak{I}^{\leq b}_{X_{(r+1)}\{1,2\}}$ bijectively onto $\mathfrak{I}^{\leq b}_{Y_{(r+1)}\{1,2\}}$. In particular, $\alpha^{\mathrm{Lie}\leq b}_{(r+1)}$ maps

$$\mathfrak{I}^{\leq b}_{X_{(r+1)}\{1,2\}} \rtimes G_K$$

bijectively onto

$$\mathfrak{I}^{\leq b}_{Y_{(r+1)}\{1,2\}} \rtimes G_L$$

[where we note that, by the assumption on the section s_0, $\mathfrak{I}^{\leq b}_{X_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{I}^{\leq b}_{Y_{(r+1)}\{1,2\}}$) is stable under the action of G_K (respectively, G_L) on $\Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ (respectively, $\Pi^{\mathrm{Lie}\leq b}_{Y_{(r+1)}}$)].

On the other hand, if $\alpha_{(r+1)}$ is tripod-preserving (respectively, satisfies the condition $(*)$), then it follows from Lemma 4.21 (respectively, Lemma 4.22), $\alpha^{\mathrm{Lie}\leq b}_{(r+1)}$ maps $\mathfrak{D}^{\mathbb{P}\leq b}_{X_{(r+1)}\{2,3\}}$ (respectively, $\mathfrak{D}^{S\leq b}_{X_{(r+1)}\{2,3\}}$) bijectively onto a $\Pi_{Y_{(r+1)}}$-conjugate of $\mathfrak{D}^{\mathbb{P}\leq b}_{Y_{(r+1)}\{2,3\}}$ (respectively, $\mathfrak{D}^{T\leq b}_{Y_{(r+1)}\{2,3\}}$) [cf. the notation of Lemma 4.21 (respectively, Lemma 4.22)], where for "$-$"$=\mathbb{P}$, S, or T, and "$-'$"$= X$ or Y, $\mathfrak{D}^{-\leq b}_{-'_{(r+1)}\{2,3\}}$ is the image of the composite

$$\mathfrak{D}^{-}_{-'_{(r+1)}\{2,3\}} \hookrightarrow \Pi_{-'_{(r+1)}} \xrightarrow{\mathrm{Int}^{\Pi}_{-'_{(r+1)}}} \Pi^{\mathrm{Lie}}_{-'_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b}_{-'_{(r+1)}} .$$

First, I claim that the isomorphism $\alpha^{\mathrm{Lie}\leq 1}_{(r+1)}$ of $\Pi_{X_{(r+1)}\underline{\{1,2\}}}$ with $\Pi_{Y_{(r+1)}\underline{\{1,2\}}}$ (cf. Lemma 4.14, (i)) induces a bijection between the set of $\Pi_{X_{(r+1)}\underline{\{1,2\}}}$-conjugates of $\mathfrak{D}^{\leq 1}_{X_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\leq 1}_{X_{(r+1)}\{2,3\}}$) and the set of $\Pi_{Y_{(r+1)}\underline{\{1,2\}}}$-conjugates of $\mathfrak{D}^{\leq 1}_{Y_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\leq 1}_{Y_{(r+1)}\{2,3\}}$). Indeed, this follows from Lemma 4.25 (respectively, a similar argument to the argument used in the proof of Proposition 2.12, (iii)).

Next, I claim that the isomorphism $\alpha^{\mathrm{Lie}\leq 2}_{(r+1)}$ induces a bijection between the set of $\Pi^{\leq 2}_{X_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq 2}_{X_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\leq 2}_{X_{(r+1)}\{2,3\}}$) and the set of $\Pi^{\leq 2}_{Y_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq 2}_{Y_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\leq 2}_{Y_{(r+1)}\{2,3\}}$). Indeed, this follows from the claim just verified above, together with Lemma 4.26, (ii) (respectively, Lemma 4.26, (ii), together with Lemma 4.28).

Next, I claim that the isomorphism $\alpha^{\mathrm{Lie}\leq b}_{(r+1)}$ induces a bijection between the set of $\Pi^{\leq b}_{X_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{2,3\}}$) and the set of $\Pi^{\leq b}_{Y_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq b}_{Y_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\leq b}_{Y_{(r+1)}\{2,3\}}$) for each positive integer b. To verify this claim, we apply induction on b. The case where $b = 1$ or 2 is verified above. Thus, we assume that $b \geq 2$, and that the claim has been verified for "b" that are $\leq$ the b under consideration. Now observe that it follows from Lemma 4.29 that any two liftings of the natural inclusion $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ (respectively, $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$) to inclusions $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ (respectively, $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$) differ by conjugation in $\Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ by a unique element of the kernel of the surjection $\Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$; moreover, it follows from the definition that the kernel of the surjection $\Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ is contained in the *center* of $\Delta^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$. Therefore, the restriction to $\mathfrak{D}^{\Delta\leq b}_{X_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\Delta\leq b}_{X_{(r+1)}\{2,3\}}$) of any lifting of the natural inclusion $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ (respectively, $\mathfrak{D}^{\leq b+1}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$) to an inclusion $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ (respectively, $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$) coincides with the natural inclusion $\mathfrak{D}^{\Delta\leq b+1}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ (respectively, $\mathfrak{D}^{\Delta\leq b+1}_{X_{(r+1)}\{2,3\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$). Thus, it follows that the isomorphism $\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}$ induces a bijection between the set of $\Delta^{\leq b+1}_{X_{(r+1)}}$-conjugates of $\mathfrak{D}^{\Delta\leq b+1}_{X_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\Delta\leq b+1}_{X_{(r+1)}\{2,3\}}$) and the set of $\Delta^{\leq b+1}_{Y_{(r+1)}}$-conjugates of $\mathfrak{D}^{\Delta\leq b+1}_{Y_{(r+1)}\{1,2\}}$ (respectively, $\mathfrak{D}^{\Delta\leq b+1}_{Y_{(r+1)}\{2,3\}}$); in particular, since $\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}$ is compatible with the Frobenius elements on either side, it follows from Lemma 4.13 that *the isomorphism* $\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}$ *maps* $\Pi^{\leq b+1}_{X_{(r+1)}}$ *bijectively onto* $\Pi^{\leq b+1}_{Y_{(r+1)}}$. Moreover, observe that the subgroup $\mathfrak{D}^{\leq b+1}_{X_{(r+1)}\{1,2\}} \subseteq \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ is a subgroup which is *uniquely* determined by the condition that it be a

the image of a lifting of the natural inclusion $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ *to an inclusion* $\mathfrak{D}^{\leq b}_{X_{(r+1)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ *whose image contains*

$$\mathfrak{I}^{\leq b+1}_{X_{(r+1)}\{1,2\}} \rtimes G_K\,.$$

(Indeed, the assertion that this condition uniquely determines the subgroup

$\mathfrak{D}^{\leq b+1}_{X_{(r+1)}\{1,2\}} \subseteq \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ may be verified follows: First, let us observe that the isomorphism $\mathfrak{D}^{\leq b+1}_{X_{(r+1)}\{1,2\}} \xrightarrow{\sim} \mathfrak{D}^{\leq b}_{X_{(r+1)}\{1,2\}}$ induced by the natural projection $\Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ [cf. Lemma 4.14, (ii)] induces an isomorphism $\mathfrak{I}^{\leq b+1}_{X_{(r+1)}\{1,2\}} \rtimes G_K \xrightarrow{\sim} \mathfrak{I}^{\leq b}_{X_{(r+1)}\{1,2\}} \rtimes G_K$. Thus, any two liftings of the natural inclusion $\mathfrak{D}^{\leq b}_{X_{(r)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$ to inclusions $\mathfrak{D}^{\leq b}_{X_{(r)}\{1,2\}} \hookrightarrow \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$ whose images contain

$$\mathfrak{I}^{\leq b+1}_{X_{(r+1)}\{1,2\}} \rtimes G_K \subseteq \Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}}$$

[since $b \geq 2$] in fact coincide on $\mathfrak{I}^{\leq b}_{X_{(r+1)}\{1,2\}} \rtimes G_K \subseteq \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}$. Therefore, by Lemma 4.29, it is enough to verify that the submodule of Fr_K-invariants of

$$\mathrm{Ker}\,(\Pi^{\mathrm{Lie}\leq b+1}_{X_{(r+1)}} \twoheadrightarrow \Pi^{\mathrm{Lie}\leq b}_{X_{(r+1)}}) = \mathrm{Lin}^{\mathrm{iner}}_{X_{(r+1)}}(b+1/b+2)$$

[cf. Lemma 4.14, (ii)] is zero. However, this follows immediately from Proposition 4.6, (i).) Now by considering a similar condition for $\mathfrak{D}^{\leq b}_{Y_{(r+1)}\{1,2\}} \subseteq \Pi^{\mathrm{Lie}\leq b+1}_{Y_{(r+1)}}$, the claim that the isomorphism $\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}$ induces a bijection between the set of $\Pi^{\leq b+1}_{X_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq b+1}_{X_{(r+1)}\{1,2\}}$ and the set of $\Pi^{\leq b+1}_{Y_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq b+1}_{Y_{(r+1)}\{1,2\}}$ follows from the fact that the isomorphism $\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}$ maps $\mathfrak{I}^{\leq b+1}_{X_{(r+1)}\{1,2\}} \rtimes G_K$ bijectively onto $\mathfrak{I}^{\leq b+1}_{Y_{(r+1)}\{1,2\}} \rtimes G_L$, together with the fact that the isomorphism $\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}$ maps $\Pi^{\leq b+1}_{X_{(r+1)}}$ bijectively onto $\Pi^{\leq b+1}_{Y_{(r+1)}}$. On the other hand, by replacing $\{1,2\}$ by $\{2,3\}$, $\mathfrak{I}^{\leq b+1}_{X_{(r+1)}\{1,2\}} \rtimes G_K$ by

$$\begin{cases} \mathfrak{D}^{\mathbb{P}\leq b+1}_{X_{(r+1)}\{2,3\}} & (\text{if } \alpha_{(r)} \text{ is tripod-preserving}) \\ \mathfrak{D}^{S\leq b+1}_{X_{(r+1)}\{2,3\}} & (\text{if } \alpha_{(r)} \text{ satisfies } (*)), \end{cases}$$

and $\mathfrak{I}^{\leq b+1}_{Y_{(r+1)}\{1,2\}} \rtimes G_L$ by

$$\begin{cases} \mathfrak{D}^{\mathbb{P}\leq b+1}_{Y_{(r+1)}\{2,3\}} & (\text{if } \alpha_{(r)} \text{ is tripod-preserving}) \\ \mathfrak{D}^{T\leq b+1}_{Y_{(r+1)}\{2,3\}} & (\text{if } \alpha_{(r)} \text{ satisfies } (*)), \end{cases}$$

it follows from a similar argument to the argument used in the proof of the assertion that the isomorphism $\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}$ induces a bijection between the set of $\Pi^{\leq b+1}_{X_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq b+1}_{X_{(r+1)}\{1,2\}}$ and the set of $\Pi^{\leq b+1}_{Y_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq b+1}_{Y_{(r+1)}\{1,2\}}$ that the isomorphism $\alpha^{\mathrm{Lie}\leq b+1}_{(r+1)}$ induces a bijection between the set of $\Pi^{\leq b+1}_{X_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq b+1}_{X_{(r+1)}\{2,3\}}$ and the set of $\Pi^{\leq b+1}_{Y_{(r+1)}}$-conjugates of $\mathfrak{D}^{\leq b+1}_{Y_{(r+1)}\{2,3\}}$.

By the various claims verified above, by taking the projective limit, we thus conclude that the isomorphism $\alpha^{\mathrm{Lie}}_{(r+1)}$ induces an isomorphism of $\Pi_{X_{(r+1)}}$ with $\Pi_{Y_{(r+1)}}$ by Lemma 4.13.

Finally, we note that the *indeterminacy*, referred to in the statement of Lemma 4.30, of the isomorphism $\alpha_{(r+1)}$ up to composition with a cuspidally inner automorphism arises precisely from the *indeterminacy of the choice of the subgroups* $\mathfrak{I}_{X_{(r+1)}\{i,j\}} \subseteq \Pi_{X_{(r+1)}}$, $\mathfrak{I}_{Y_{(r+1)}\{i',j'\}} \subseteq \Pi_{Y_{(r+1)}}$ ($1 \leq i < j \leq r+1$, $1 \leq i' < j' \leq r+1$) *and the sections* of the natural morphisms $\Pi_{X_{(r+1)}} \to G_K$ and $\Pi_{Y_{(r+1)}} \to G_L$ (cf. Remark 4.11) with respect to cuspidally inner automorphisms of $\Pi_{X_{(r+1)}}$, $\Pi_{Y_{(r+1)}}$, respectively. $\square$

Lemma 4.31. *Any Frobenius-preserving isomorphisms of* Π_X *with* Π_Y *are tripod-preserving.*

Proof. Let α be a Frobenius-preserving isomorphism of Π_X with Π_Y. Note that since replacing the base field by a finite extension field of the base field does not affect the validity of the assertion that α is tripod-preserving, we may assume that $X(K)$ is *non-empty.* Then it follows from [14], Theorem 3.10, that there exists an isomorphism $\alpha_{(2)}$ of $\Pi_{X_{(2)}}$ with $\Pi_{Y_{(2)}}$ which fits into a commutative diagram

$$\begin{tikzcd} \Pi_{X_{(2)}} \arrow[r, "\alpha_{(2)}"] \arrow[d] & \Pi_{Y_{(2)}} \arrow[d] \\ \Pi_{\underline{X_{(2)}}} \arrow[r, "\alpha\times_{G_K}\alpha"] \arrow[d, "\text{via pr}_1"'] & \Pi_{\underline{Y_{(2)}}} \arrow[d, "\text{via pr}_1"] \\ \Pi_X \arrow[r, "\alpha"'] & \Pi_Y , \end{tikzcd}$$

where the left-hand (respectively, right-hand) top vertical arrow is the morphism induced by the natural open immersion $U_{X_{(2)}} \hookrightarrow X \times_K X$ (respectively, $U_{Y_{(2)}} \hookrightarrow Y \times_L Y$). By base-changing the above diagram via the section $G_K \to \Pi_X$ arising from a K-rational point x of X and the section of the natural morphism $\Pi_Y \to G_L$ corresponding to the section $G_K \to \Pi_X$ (under the isomorphism α), we obtain a commutative diagram

$$\begin{tikzcd} \Pi_{X\setminus\{x\}} \arrow[r, "\alpha_\circ"] \arrow[d] & \Pi_{Y\setminus\{y\}} \arrow[d] \\ \Pi_X \arrow[r, "\alpha"] \arrow[d] & \Pi_Y \arrow[d] \\ G_K \arrow[r, "\alpha_{(0)}"'] & G_L , \end{tikzcd}$$

where y is an L-rational point of Y such that the section arising from y coincides with the section of the natural morphism $\Pi_Y \to G_L$ corresponding to the section $G_K \to \Pi_X$ arising from a K-rational point x of X (cf. Proposition 2.12, (ii)). Let $X' \to X$ be a *non-trivial* Galois covering over X, and $Y' \to Y$ the Galois covering over Y corresponding to $X' \to X$ (under the isomorphism α). Then by base-changing the above diagram via the natural inclusions $\Pi_{X'} \hookrightarrow \Pi_X$ and $\Pi_{Y'} \hookrightarrow \Pi_Y$, we obtain a commutative diagram

$$\begin{tikzcd} \Pi_{X'\setminus\{x'_1,\cdots,x'_d\}} \arrow[r, "\alpha'_{\circ'}"] \arrow[d] & \Pi_{Y'\setminus\{y'_1,\cdots,y'_d\}} \arrow[d] \\ \Pi_{X'} \arrow[r, "\alpha'"'] & \Pi_{Y'} , \end{tikzcd}$$

where d is the degree of the covering $X' \to X$, and $\{x'_1, \cdots, x'_d\}$ (respectively, $\{y'_1, \cdots, y'_d\}$) is the subset of $X'(\overline{K})$ (respectively, $Y'(\overline{L})$) obtained as the inverse image of $\{x\}$ (respectively, $\{y\}$) via the morphism $X'(\overline{K}) \to X(\overline{K})$ (respectively, $Y'(\overline{L}) \to Y(\overline{L})$). Now by extending the base fields, we may assume that $\{x'_1, \cdots, x'_d\}$ (respectively, $\{y'_1, \cdots, y'_d\}$) is a subset of $X'(K)$ (respectively, $Y'(L)$). In particular, we obtain a commutative diagram:

$$\begin{array}{ccc} \Pi_{X' \setminus \{x'_1, x'_2\}} & \xrightarrow{\alpha'_\circ} & \Pi_{Y' \setminus \{y'_1, y'_2\}} \\ \downarrow & & \downarrow \\ \Pi_{X'} & \xrightarrow[\alpha']{} & \Pi_{Y'} . \end{array}$$

Now we assume that the decomposition subgroup $\mathfrak{D}_{x'_i} \subseteq \Pi_{X' \setminus \{x'_1, x'_2\}}$ associated to x'_i (well-definied, up to conjugate) corresponds to the decomposition subgroup $\mathfrak{D}_{y'_i} \subseteq \Pi_{Y' \setminus \{y'_1, y'_2\}}$ associated to y'_i (well-definied, up to conjugate) under the isomorphism $\alpha'_\circ$.

Now I *claim* that the section of the natural morphism $\Pi_{X'_{(2)}} \to G_K$ arising from $(x'_1, x'_2) \in U_{X'_{(2)}}(K)$ corresponds to the section of the natural morphism $\Pi_{Y'_{(2)}} \to G_L$ arising from $(y'_1, y'_2) \in U_{Y'_{(2)}}(L)$ under the isomorphism $\alpha'_{(2)}$ of $\Pi_{X'_{(2)}}$ with $\Pi_{Y'_{(2)}}$ obtained from α' (cf. [14], Theorem 3.10). Indeed, it follows from Proposition 4.23 that we may assume that the top horizontal arrow in the diagram

$$\begin{array}{ccc} \Pi_{X'_{(2)}} \times_{\Pi_{X'}} G_K & \longrightarrow & \Pi_{Y'_{(2)}} \times_{\Pi_{Y'}} G_L \\ \downarrow & & \downarrow \\ G_K & \longrightarrow & G_L \end{array}$$

obtained by base-changing the diagram

$$\begin{array}{ccc} \Pi_{X'_{(2)}} & \xrightarrow{\alpha'_{(2)}} & \Pi_{Y'_{(2)}} \\ {\scriptstyle \text{via } p_{U_{X'_{(1)}}:1}} \downarrow & & \downarrow {\scriptstyle \text{via } p_{U_{Y'_{(1)}}:1}} \\ \Pi_{X'} & \xrightarrow[\alpha']{} & \Pi_{Y'} \end{array}$$

via the morphism $G_K \to \Pi_{X'}$ induced by the composite $\mathfrak{D}_{x'_1} \hookrightarrow \Pi_{X' \setminus \{x'_1, x'_2\}} \twoheadrightarrow \Pi_{X'}$ and the morphism $G_L \to \Pi_{Y'}$ induced by the composite $\mathfrak{D}_{y'_1} \hookrightarrow \Pi_{Y' \setminus \{y'_1, y'_2\}} \twoheadrightarrow \Pi_{Y'}$ *coincides* with the isomorphism of $\Pi_{X' \setminus \{x'_1\}}$ with $\Pi_{Y' \setminus \{y'_1\}}$ induced by $\alpha'_\circ$. Thus, the *claim* follows from the fact that the composite $\mathfrak{D}_{x'_2} \hookrightarrow \Pi_{X' \setminus \{x'_1, x'_2\}} \twoheadrightarrow \Pi_{X' \setminus \{x'_1\}}$ is compatible with the composite $\mathfrak{D}_{y'_2} \hookrightarrow \Pi_{Y' \setminus \{y'_1, y'_2\}} \twoheadrightarrow \Pi_{Y' \setminus \{y'_1\}}$ under the isomorphism of $\Pi_{X' \setminus \{x'_1\}}$ with $\Pi_{Y' \setminus \{y'_1\}}$ induced by $\alpha'_\circ$.

By the above *claim* just verified, the isomorphism $\alpha'_{(2)}$ satisfies the condition $(*)$ in the statement of Lemma 4.30; in particular, $\alpha'_{(2)}$ extends to an isomorphism of $\Pi_{X'_{(3)}}$ with $\Pi_{Y'_{(3)}}$. Thus, it follows from Lemma 3.6 that α', hence also α is tripod-preserving (cf. [14], Remark 1.2.0). $\square$

The main result of this paper is as follows:

Theorem 4.32. *Let X (respectively, Y) be a proper hyperbolic curve over a finite field K (respectively, L). Let*

$$\alpha_{(1)} : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be a Frobenius-preserving isomorphism. Then, for any positive integer r, there exists a unique isomorphism

$$\alpha_{(r)} : \Pi_{X_{(r)}} \xrightarrow{\sim} \Pi_{Y_{(r)}} ,$$

well-defined up to composition with a cuspidally inner automorphism (i.e., a $\mathrm{Ker}\,(\Pi_{Y_{(r)}} \twoheadrightarrow \Pi_{\underline{Y_{(r)}}})$*-inner automorphism), which is compatible with the natural respective actions of the symmetric group on r letters such that, for $i = 1, \cdots, r+1$, the following diagram commutes:*

$$\begin{tikzcd}
\Pi_{X_{(r+1)}} \arrow[r, "\alpha_{(r+1)}"] \arrow[d, "\text{via } p_{U_{X_{(r)}}:i}"'] & \Pi_{Y_{(r+1)}} \arrow[d, "\text{via } p_{U_{Y_{(r)}}:i}"] \\
\Pi_{X_{(r)}} \arrow[r, "\alpha_{(r)}"'] & \Pi_{Y_{(r)}} .
\end{tikzcd}$$

Proof. This follows immediately from Proposition 4.23; Lemmas 4.30; 4.31; [14], Theorem 3.10. ☐

The following Corollary follows immediately from Theorem 4.32, together with the fact that a hyperbolic curve over a finite field is $\mathfrak{Prime}$-*separated* (cf. [14], definition 1.18, (i); Proposition 2.3, (ii)).

Corollary 4.33. *Let X (respectively, Y) be a proper hyperbolic curve over a finite field K (respectively, L).*

(*i*) *Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be a Frobenius-preserving isomorphism of the geometrically pro-l fundamental group of X with the geometrically pro-l fundamental group of Y, and r a positive integer. Let x_i (where $1 \leq i \leq r$) be an element of the set $(X \setminus \{x_1, \cdots, x_{i-1}\})(K)$ of K-rational points of $X \setminus \{x_1, \cdots, x_{i-1}\}$. Then there exist an element y_i of the set $(Y \setminus \{y_1, \cdots, y_{i-1}\})(L)$ and an isomorphism

$$\alpha^{\mathrm{new}} : \Pi_{X\setminus\{x_1,\cdots,x_r\}} \xrightarrow{\sim} \Pi_{Y\setminus\{y_1,\cdots,y_r\}}$$

of the geometrically pro-l fundamental group of $X \setminus \{x_1, \cdots, x_r\}$ with the geometrically pro-l fundamental group of $Y \setminus \{y_1, \cdots, y_r\}$ which is compatible with α. Moreover, such an isomorphism α^{new} is uniquely determined up to composition with a cuspidally inner automorphism.

(*ii*) *Let*

$$\alpha : \pi_1(X) \xrightarrow{\sim} \pi_1(Y)$$

be a Frobenius-preserving isomorphism of the (profinite) fundamental group of X with the (profinite) fundamental group of Y, and r a positive integer. Let x_i be an element of $X(K)$ (where $1 \leq i \leq r$), and y_i the element of $Y(L)$ whose decomposition subgroup of $\pi_1(Y)$ (well-defined, up to conjugate) corresponds to the decomposition subgroup associated to x_i (well-defined, up to conjugate) via α. Then there exists an isomorphism

$$\alpha^{\mathrm{cp}(l)} : \pi_1(X \setminus \{x_1, \cdots, x_r\})^{\mathrm{cp}(l)} \xrightarrow{\sim} \pi_1(Y \setminus \{y_1, \cdots, y_r\})^{\mathrm{cp}(l)}$$

of the maximal cuspidally pro-l quotient of $\pi_1(X \setminus \{x_1, \cdots, x_r\})$ *(relative to* $\pi_1(X \setminus \{x_1, \cdots, x_r\}) \twoheadrightarrow \pi_1(X)$*) with the maximal cuspidally pro-l quotient of* $\pi_1(Y \setminus \{y_1, \cdots, y_r\})$ *(relative to* $\pi_1(Y \setminus \{y_1, \cdots, y_r\}) \twoheadrightarrow \pi_1(Y)$*) which is compatible with* α*. Moreover, such an isomorphism* $\alpha^{\mathrm{cp}(l)}$ *is uniquely determined up to composition with a cuspidally inner automorphism.*

Remark 4.34.

(i) Since a hyperbolic curve over a finite field is *not l-separated* in general (cf. Remark 4.35 below), the "y_i's" (hence also "α^{new}") in the statement of Corollary 4.33, (i), *depend*, unlike the case with Corollary 4.33, (ii), *on the ordering* of $\{x_1, \cdots, x_r\}$.

(ii) In the notation of Corollary 4.33, (ii), since it follows from [14], Theorem 3.12, that there exists a unique isomorphism (of schemes) of $\phi : X \xrightarrow{\sim} Y$ such that the isomorphism $\pi_1(X) \xrightarrow{\sim} \pi_1(Y)$ induced on fundamental groups by ϕ coincides with α, it follows immediately that there exists an isomorphism $\pi_1(X \setminus \{x_1, \cdots, x_r\}) \xrightarrow{\sim} \pi_1(Y \setminus \{y_1, \cdots, y_r\})$ which is compatible with α. On the other hand, Corollary 4.33, (ii), provides a *direct way* to construct such an isomorphism between the fundamental groups, i.e., a way to construct such an isomorphism *without passing through "the world of schemes"*.

Remark 4.35. In general, a hyperbolic curve over a finite field is *not l-separated.* The following example of this phenomenon was given by Akio Tamagawa:

Let X be a hyperbolic curve over a finite field K of characteristic p, and $\overline{K}$ an algebraic closure of K. Let us denote by Π_X the geometrically pro-l fundamental group of X (where l is a prime number such that $l \neq p$), by Δ_X the pro-l fundamental group of $X \otimes_K \overline{K}$, and by G_K the Galois group of $\overline{K}$ over K. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \wr\downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Delta_X) & \longrightarrow & \mathrm{Aut}(\Delta_X) & \longrightarrow & \mathrm{Out}(\Delta_X) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact, and the left-hand vertical arrow is an isomorphism; in particular, the right-hand square is *cartesian.* It follows from [1], Corollary 7, that $\mathrm{Out}(\Delta_X)$ is *almost pro-l* (i.e., there exists a non-trivial open subgroup which is *pro-l*), by replacing G_K by an open subgroup of G_K, we assume that the right-hand vertical arrow $G_K \to \mathrm{Out}(\Delta_X)$ in the above diagram *factors through a pro-l quotient of* G_K. Then since the right-hand square is *cartesian*, we conclude that Π_X is isomorphic to

$$(\Delta_X \overset{\mathrm{out}}{\rtimes} G_K^{(l)}) \times G_K^{(\neq l)},$$

where $G_K^{(\neq l)}$ is the maximal pro-$(\mathfrak{Prime} \setminus \{l\})$ quotient of G_K; thus, $G_K^{(\neq l)}$ is isomorphic to the product of $\mathbb{Z}_{l'}$'s (where $l' \in \mathfrak{Prime} \setminus \{l\}$). Let $L \subseteq \overline{K}$ be a finite extension field of K of degree $[L : K]$ prime to l such that $X(K) \neq X(L)$. (In fact, it follows from the "Weil conjecture for curves over finite fields" [cf. e.g., [5], Chapter V, Exercise 1.10] that such an extension field exists.) Let

$x \in X(L)\backslash X(K)$, $x' \in X(L)$ obtained as the conjugate of x via a generator of the Galois group of the extension L/K, and $x_L \in X_L(L)$ (respectively, $x'_L \in X_L(L)$) the L-rational point of $X_L \stackrel{\text{def}}{=} X \otimes_K L$ determined by x (respectively, x'). Then it follows from the fact $x \notin X(K)$ that $x_L \neq x'_L$; however, it follows from the fact that Π_X is isomorphic to $(\Delta_X \stackrel{\text{out}}{\rtimes} G_K^{(l)}) \times G_K^{(\neq l)}$ that the Π_{X_L}-conjugacy class (where Π_{X_L} is the geometrically pro-l fundamental group of X_L) of the section of $\Pi_{X_L} \to G_L$ corresponding to x_L *coincides with* the Π_{X_L}-conjugacy class of the section of $\Pi_{X_L} \to G_L$ corresponding to x'_L. Therefore, X_L is *not* l*-separated*.

Moreover, it follows immediately from the existence of the isomorphism

$$\Pi_X \simeq (\Delta_X \stackrel{\text{out}}{\rtimes} G_K^{(l)}) \times G_K^{(\neq l)}$$

that there exist automorphisms of Π_X which are *not Frobenius-preserving*.

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