THETA CONSTANT IDENTITIES AT PERIODS OF COVERINGS OF DEGREE 3

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Received 31 May 2007
Accepted 4 June 2007

We derive relations between theta functions evaluated at period matrices of cyclic covers of order 3 ramified above 3k points.

Keywords: Theta functions; cyclic covers; symmetric groups; identities.

Mathematics Subject Classification 2000: 37k20, 14H42

1. Introduction

Let $R$ be a Riemann Surface defined by the equation:

$$y^3 = \prod_{i=1}^{i=3n} (z - \lambda_i).$$

We find relations that are satisfied by theta constants with rational characteristics evaluated at $\tau_R$, the period matrix of $R$. Special type identities for period matrices are known in the case of a general Riemann Surface (Schottky–Jung identities). According to Mumford, for hyperelliptic curves there are vanishing theta constants of even characteristics that characterize the associated period matrix. Special relations among non vanishing theta constants evaluated at period matrices of hyperelliptic curves were obtained by Frobenius.

The original Schottky problem seeks special relations among theta constants that characterize the entire moduli space of algebraic curves of genus $g$. In this note, we seek special relations that are satisfied by more specialized sets of curves such as cyclic covers of the sphere of degree $n$. When $n = 2$, cyclic covers constitute the set of hyperelliptic curves. The next case is $n = 3$. Here, we find relations satisfied by theta constants with rational characteristics evaluated at the period matrices of such curves. These identities are consequence of the Thomae formula for cyclic $n$ sheeted covers of the sphere. This formula expresses powers of such theta constants evaluated at the period matrix $\tau_R$ by polynomial expressions in the $\lambda_i$. A
relation between these polynomials products a relation between the corresponding
theta constants. Using the representation theory of $S_{3m}$, the symmetric group of
degree $3m$, we find a basis for the space of these polynomials, as a result we obtain
relations between the corresponding theta constants.

For the simplest case of 6 branch points, our results overlap those of Matsumoto
[7]. In his paper, Matsumoto finds the explicit action of $S_6$ on theta constants
evaluated at $\tau_R$. He obtains the branch points $\lambda_i$ as quotients of theta constants.
He also obtains identities between cubic powers of these constants, which coincide
with those in the last section of our note. Using the representation theory of $S_6$, we
see that the space generated by theta constants is 5 dimensional. This seems to be
a new result even when $g = 4$.

2. The Thomae Formula for Cyclic Covers and Relations Between
Theta Constants
Following Nakayashiki [9], we explain the general Thomae formula for an algebraic
curve $R$ satisfying the equation:

$$ y^3 = \prod_{i=1}^{i=3m}(z - \lambda_i). $$

Let $f : R \mapsto \mathbb{C}P^1$ be the map given by $(z, y) \mapsto z$. Now, let \{\infty, \omega_1, \omega_3\} =
$f^{-1}\{\emptyset\}$ and let $Q_i = f^{-1}(\lambda_i)$ be the unique branch point on $R$ that is the pre
image of $\lambda_i$. Fix a canonical homology basis $a_1, a_2, \ldots, a_{3m-2}, b_1, b_2, \ldots, b_{3m-2}$ on $R$.
Thus, $a_i a_j = 0 = b_i b_j, i \neq j$ and $a_i b_j = -1$. Let $v_1 \cdots v_{3m-2}$ be a basis of normalized holomorphic differentials with respect to the basis $a_1, a_2, \ldots, a_{3m-2}, b_1, \ldots, b_{3m-2}$.
Thus, $\int_a v_j = \delta_{ij}$ and $\int_b v_j = \tau_{ij}$. The $g \times g$ matrix, $\tau = \tau_{ij}$ is symmetric and Im$\tau$
defines positive definite quadratic form. Fix an ordering of the $\lambda_i$. This ordering
induces an ordering of the branch points \{Q_1, Q_2, \ldots, Q_{3m}\}. We abuse notation
by identifying $Q_i$ with its branch point image. Thus $\lambda_1$ will correspond to $Q_1$, $\lambda_2$
to $Q_2$, etc.

Let $\phi$ be the automorphism of order 3 defined by $(z, y) \mapsto (z, \omega y)$ with $\omega^3 =
1, \omega = e^{i\frac{2\pi}{3}}$. For divisors $\alpha$ and $\beta$ on $R$, define $\alpha \equiv \beta$ if there exists a meromorphic function $g : R \mapsto \mathbb{C}P^1$ with divisor div$(g) = \alpha - \beta$. The group Div$^0/\equiv$ is $Jac(R)$, the Jacobian of $R$. (Div$^0$ — divisors of degree 0.) Let $\psi$ be the mapping $\psi : \text{Div} \mapsto \text{Div}/\equiv$. Then, the following lemma is true:

Lemma 2.1. Let $P_1, P_2 \in R, P_1 \neq P_2$ and

$$ D_i = P_i + \phi(P_1) + \phi^2(P_1), \quad i = 1, 2 $$

then $\psi(D_1) \equiv \psi(D_2).

Proof. For $P_1, P_2$ as above, define $f_1(P) = (f(P) - f(P_1)/f(P) - f(P_2))$, then
$\text{div}(f_1) = D_1 - D_2$. Define $D = \psi(P_1 + \phi(P_1) + \phi^2(P_1))$ to be the equivalence class in the Jacobian.
Lemma 2.2. Let $K$ be the canonical divisor of the a holomorphic differential. Define $\Delta$ to be a divisor such that $2\Delta = K$, then the following holds:

(1) $D \equiv 3Q_1 \equiv \infty^1 + \infty^2 + \infty^3$,
(2) $K \equiv (2m - 2)D$,
(3) $\sum_{i=1}^{3m} Q_i \equiv mD$.

Proof. The first item follows exactly as in the proof of the previous lemma. To show the rest, note that $z(dz/w^2)$ is a holomorphic differential with divisor $Q_1^{6m-6}$.

Now let $\Lambda = \Lambda_0, \Lambda_1, \Lambda_2$ be a partition of $\{1, 2, 3, 4, 5, \ldots 3m\}$ with each $|\Lambda_i| = m$. For each subset $S$ of $\{1, 2 \ldots 3m\}$ we set,

$$X_S = \sum_{Q_j \in S} Q_j.$$

We are interested in the following divisor $e_\Lambda$ associated with the partition:

$$e_\Lambda = X_{\Lambda_0} + 2X_{\Lambda_1} - D - \Delta.$$

Choose a base point $P_0$ on $R$ and for each $P$ consider the mapping $\Phi_{P_0}(P) = (\int_{P_0}^P v_1 \cdots \int_{P_0}^P v_{3m-2})$. Using the definition of divisors, we see that $\Phi_{P_0}$ extends to the period map $\Phi_{P_0} : \text{Div}(R) \to \mathbb{C}^{3m-2}$. The final definition will be of theta constants with characteristics:

Definition 2.3. Let $\mathbb{H}_g$ be the collection of $g \times g$ symmetric matrices, $\tau$ such that the imaginary part of $\tau$ forms a positive definite form. For $[\varepsilon, \varepsilon']$, real vectors $g$ vectors and $\tau \in \mathbb{H}_g$, we define theta constant $\Theta[\varepsilon, \varepsilon'](\tau)$ with characteristics $[\varepsilon]$ as an infinite series given by:

$$\Theta[\varepsilon, \varepsilon'](\tau) = \sum_{l \in \mathbb{Z}^{2g}} \exp(2\pi i \left( \frac{l + \varepsilon}{2} \tau (l + \varepsilon) + (l + \frac{\varepsilon}{2})^t \varepsilon' + \frac{\varepsilon}{2} \right)).$$

These series are uniformly and absolutely convergent on compact subsets of $\mathbb{H}_g$. For each $w \in \mathbb{C}^{3m-2}$, associate a unique vector $2g$ vector $[w_1, w_2]$ and $w_1, w_2$ are unique vectors real $g$ dimensional vectors such that: $w = w_1 + \tau w_2$. Composing with the map $p_{P_0}$, we associate theta constants with characteristics to divisors. Nakayashiki [9] proves the following theorem of Bershadsky and Radul [2]:

Theorem 2.4. Let $\theta[e_\Lambda](\tau)$ be a theta constant associated with $\Lambda$ through the period map $p_{P_0}$. Then, $e_\Lambda$ is a point of order 6 on the Jacobian and

$$\theta[e_\Lambda](\tau_R) = C_\Lambda (\det A)^3 ((\Lambda_0 \Lambda_0)(\Lambda_1 \Lambda_1)(\Lambda_2 \Lambda_2))^3 (\Lambda_0 \Lambda_1)(\Lambda_1 \Lambda_2)(\Lambda_0 \Lambda_2).$$

Here, $A$ is the matrix of certain holomorphic 1 forms integrated with respect to $a_i$ and $C_\Lambda$ is a constant depending only on the partition. Moreover,

$$(\Lambda_i \Lambda_i) = \prod_{k<l} (\lambda_{ik} - \lambda_{il}), \quad (\Lambda_i \Lambda_j) = \prod_{k=1, l=1}^{m} (\lambda_{ik} - \lambda_{jl}).$$
As explained in the introduction, one of the problems in the theory of complex algebraic curves is to understand the set of period matrices associated with certain families of curves. For example, we can seek certain algebraic relations satisfied by theta constants evaluated at period matrices of curves belonging to such families. Because theta constants are modular forms for subgroups of \( Sp(g, \mathbb{Z}) \) results of this type may have number theoretic significance.

We apply Theorem 2.4 to generate special relations between theta functions with characteristics \( e_i \), evaluated at \( \tau_R \). For each partition, \( \Lambda \), we denote the polynomial on the right-hand side of the equation above by \( p_{\Lambda} \). To obtain identities, we expand the polynomials and search for identities between them. The key observation on the polynomials: First, choose \( \Lambda = \{(1, 2, \ldots , m), \{m + 1, \ldots , 2m\}, \{2m, \ldots , 3m\}\} \). Then, by the definition of \( p_{\Lambda} \) we have:

\[
p_{\Lambda} = \left( \prod_{i<j, j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j, i=m+1, j=m+2}^{j=2m} (\lambda_i - \lambda_j) \prod_{i<j, j=2m+1}^{j=3m} (\lambda_i - \lambda_j) \right)^3
\]

\[
\times \prod_{i=1, j=m+1}^{i=m, j=2m} (\lambda_i - \lambda_j) \prod_{i=m+1, j=2m+1}^{i=2m, j=3m} (\lambda_i - \lambda_j) \prod_{i=1, j=2m+1}^{i=3m} (\lambda_i - \lambda_j).
\]

Now, write

\[
\left( \prod_{i<j, j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j, i=m+1, j=m+2}^{j=2m} (\lambda_i - \lambda_j) \prod_{i<j, j=2m+1}^{j=3m} (\lambda_i - \lambda_j) \right)^3
\]

\[
= \left( \prod_{i<j, j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j, i=m+1, j=m+2}^{j=2m} (\lambda_i - \lambda_j) \prod_{i<j, j=2m+1}^{j=3m} (\lambda_i - \lambda_j) \right)^2
\]

\[
\times \left( \prod_{i<j, j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j, i=m+1, j=m+2}^{j=2m} (\lambda_i - \lambda_j) \prod_{i<j, i=2m, j=2m+1}^{j=3m} (\lambda_i - \lambda_j) \right).
\]

We see that we can rewrite \( p_{\Lambda} \) for this partition as:

\[
p_{\Lambda} = \left( \prod_{i<j, j=2}^{j=m} (\lambda_i - \lambda_j) \prod_{i<j, i=m+1, j=m+2}^{j=2m} (\lambda_i - \lambda_j) \prod_{i<j, j=2m+1}^{j=3m} (\lambda_i - \lambda_j) \right)^2
\]

\[
\times \text{disc}(\lambda_1 \cdots \lambda_{3m}),
\]

where

\[
\text{disc}(\lambda_1 \cdots \lambda_{3m}) = \prod_{i\neq j, 1 \leq i, j \leq 3m} (\lambda_i - \lambda_j).
\]
Observe that this holds for every partition. That is,

\[
(\Lambda_0 \Lambda_0)(\Lambda_1 \Lambda_1)(\Lambda_2 \Lambda_2)^2(\Lambda_0 \Lambda_1)(\Lambda_1 \Lambda_2)(\Lambda_0 \Lambda_2)
= (\Lambda_0 \Lambda_0)(\Lambda_1 \Lambda_1)(\Lambda_2 \Lambda_2)^2(\Lambda_0 \Lambda_0)(\Lambda_1 \Lambda_1)(\Lambda_2 \Lambda_2)(\Lambda_0 \Lambda_1)(\Lambda_1 \Lambda_2)(\Lambda_0 \Lambda_2)
= ((\Lambda_0 \Lambda_0)(\Lambda_1 \Lambda_1)(\Lambda_2 \Lambda_2))^2 \text{disc}(\lambda_1, \ldots, \lambda_{3m}).
\]  

(4)

Since the factor \text{disc}(\lambda_1, \ldots, \lambda_{3m}) is independent of the partition \Lambda_0, \Lambda_1, \Lambda_2, we conclude that identities between \( \pm \theta^3[e_\Lambda] \) are equivalent to identities between the polynomials:

\[
((\Lambda_0 \Lambda_0)(\Lambda_1 \Lambda_1)(\Lambda_2 \Lambda_2)).
\]

The group \( S_{3m} \) acts naturally on the polynomials \((\Lambda_0 \Lambda_0)(\Lambda_1 \Lambda_1)(\Lambda_2 \Lambda_2))\) by its action on the partitions of \( \{1 \cdots 3m\} \). Thus, \( \text{Span}((\Lambda_0 \Lambda_0)(\Lambda_1 \Lambda_1)(\Lambda_2 \Lambda_2)) \), the vector space spanned by these polynomials, provides us with a representation of \( S_{3m} \).

We exhibit a basis for this space of polynomials in the next section.

3. Explicit Basis

In this section, we provide an explicit basis for the space of polynomials defined in Sec. 2. We do this by following the process described in [5] to construct a basis for the irreducible representations of \( S_n \). For the complex numbers, these representations are completely classified. We describe the construction for any representation of the symmetric group and obtain the relevant case for cyclic covers as an immediate corollary. At this point, we must assume that the reader is familiar with some notions from the representation theory of \( S_n \).

Let \( n \) be a natural number and let \( k_1 \cdots k_m \) be a partition of \( n \). That is, \( \sum_{i=1}^{m} k_i = n \) and \( k_1 \geq k_2 \geq k_3 \cdots \geq k_m \).

**Definition 3.1.** A Young diagram associated to a partition consists of \( m \) rows such that the \( i \)th row has \( k_i \) elements of integers.

**Definition 3.2.** Let \( Y \) be a Young diagram. A tableau is obtained by arranging the numbers \( \{1 \cdots n\} \) within the \( m \) rows of \( Y \) so that:

- Each row contains exactly \( k_i \) elements,
- The numbers distributed in each row are arranged in increasing order.

Assume that \( \Lambda = \{\Lambda_0, \ldots, \Lambda_k\} \) are a tableau of \( n \). Choose an ordering for \( 1, \ldots, n \). For each member of the tableau \( \Lambda_i = i_1 < \cdots < i_l \) define the polynomials:

\[
(\Lambda_i \Lambda_i) = \prod_{i_k < i_l, \{i_k, i_l\} \in \Lambda_i} (\lambda_{i_k} - \lambda_{i_l}),
\]

and,

\[
p_\Lambda = \prod_{i=1}^{k} (\Lambda_i \Lambda_i).
\]
$S_n$ acts on $\Lambda$ and, therefore acts on the polynomials $p_\Lambda$. We are interested in finding a basis for the linear span of $p_\Lambda$. For this purpose, we use a modification of the Garnier relation (7.1) given in [5].

Arrange the tableau in columns (i.e. the first column will be elements of $\Lambda_0$ the second column elements of $\Lambda_1$ etc). Overall, we have $k$ columns for $\Lambda$.

Let $\sigma_1 \cdots \sigma_k$ be coset representatives for

$S_X \times S_Y$

in $S_{X \cup Y}$. Then, we have the Garnier relations:

**Theorem 3.3.** Let $\mu_i$ denote the number of elements in the $i$th column of $\Lambda$. If $|X \cup Y| > \mu_i$, then

$$\sum_{m=1}^{k} \text{sign} \sigma_m (p_{\sigma_m \Lambda}) = 0.$$ 

**Proof.** If $|X \cup Y| > \mu_i$, then by the pigeon hole principle there exists an involution $\delta$ under which $\sigma_m \Lambda$ is invariant. Thus,

$$\sum_{m=1}^{k} \text{sign} \sigma_m (p_{\sigma_m \Lambda}) = \sum_{m=1}^{k} \text{sign} \sigma_m (p_{\delta \sigma_m \Lambda})$$

$$= - \sum_{m=1}^{k} \text{sign} \sigma_m (p_{\sigma_m \Lambda}) = 0. \quad \Box$$

In order to exhibit an explicit basis, we define a standard Young tableau.

**Definition 3.4.** A standard tableau is a tableau whose the rows and the columns are arranged in increasing order.

**Definition 3.5.** Let $\Lambda^1$ and $\Lambda^2$ be tableaux. Then, we set $\Lambda^1 < \Lambda^2$ if there is an $i$ such that:

- if $j > i$, then $j$ is in the same column of $\Lambda^1, \Lambda^2$,
- $i$ is in a column further left in $\Lambda^1$ than $\Lambda^2$.

It is easy to see that the ordering defined above imposes total ordering on the tableaux.

**Theorem 3.6.** Let $\Lambda^1 \cdots \Lambda^k$ be the collection of standard tableaux for a given partition. Then, $p_{\Lambda^1} \cdots p_{\Lambda^k}$ is a basis for the vector space spanned by $\Lambda$.

**Proof.** We follow the proof given in [5]. We show that $p_{\Lambda^k}$ spans any other polynomial corresponding to our partition. Let $t$ be a tableau and suppose by induction that the theorem is proved for each tableau $t_1$, such that $t_1 < t$. If $t$ is non standard there exists adjacent columns $a_1 < \cdots < a_q < \cdots < a_r$ and

*We were not able to find a reference to our approach for constructing Specht modules, though we are confident it’s a folklore.*
We consider the case of 6 branch points, so the genus of the surface is 4. We enumerate the 15 partitions and the polynomials corresponding to them:

1. \( b_1 < b_2 < \cdots < b_q < \cdots b_s \) such that \( a_q > b_q \). Apply the Garnier relations for \( X = \{a_1, \ldots, a_r\}, Y = \{b_1, \ldots, b_q\} \). For each \( \sigma \), a representative of \( S_X \times S_Y \) in \( S_X \cup Y \) we have \( |\sigma| < t \) by the definition of the order \( < \). The result follows immediately from the induction hypothesis.

**Definition 3.7.** For an element \( k \) of the tableau \( t \), the hook \( h_k \) are the elements of the tableau \( t \) to the right of it, including the element itself, and the elements in \( t \) below \( k \).

It is well known ([5]) that the number of standard tableaus equals:

\[
\frac{n!}{\prod_k h_k}
\]

**4. The Ideal of Theta Identities**

We apply the theory of the previous paragraph to cyclic covers of order 3. According to the theory, the hooks of the partitions correspond to tableau with 3 rows and \( m \) elements in each row. Our first corollary is:

**Corollary 4.1.** The dimension of the space of polynomials \( p_\Lambda \) (and hence \( \theta^3[e_\Lambda](\tau_R) \) corresponding to them) is:

\[
\frac{(3m)! \times 2}{(m + 2)!(m + 1)!m!}
\]

Hence, the corresponding space of \( \theta^3[e_\Lambda](\tau_R) \) also has this dimension. We can also give for this space:

**Corollary 4.2.** Let \( \Lambda_S \) be a standard partition then \( \theta^3[e_{\Lambda_S}](\tau_R) \) is a basis for the vector space spanned by theta constants, \( \theta^3[e_\Lambda](\tau_R) \) and \( \Lambda = \Lambda_0, \Lambda_1, \Lambda_2 \) is a partition of \( \{1 \cdots 3k\} \) such that \( |\Lambda_i| = k \).

**5. Example**

We consider the case of 6 branch points, so the genus of the surface is 4. We enumerate the 15 partitions and the polynomials corresponding to them:

1. \( \Lambda = \{(1, 2), (3, 4), (5, 6)\} \): \( p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_6) \),

2. \( \Lambda = \{(1, 2), (3, 5), (4, 6)\} \): \( p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_6) \),

3. \( \Lambda = \{(1, 2), (3, 6), (4, 5)\} \): \( p_\Lambda = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_6)(\lambda_4 - \lambda_5) \),

4. \( \Lambda = \{(1, 3), (2, 4), (5, 6)\} \): \( p_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_5 - \lambda_6) \),

5. \( \Lambda = \{(1, 3), (2, 5), (4, 6)\} \): \( p_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_5)(\lambda_4 - \lambda_6) \),

6. \( \Lambda = \{(1, 3), (2, 6), (4, 5)\} \): \( p_\Lambda = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_6)(\lambda_4 - \lambda_5) \),

7. \( \Lambda = \{(1, 4), (2, 5), (3, 6)\} \): \( p_\Lambda = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_5)(\lambda_3 - \lambda_6) \),

8. \( \Lambda = \{(1, 4), (2, 6), (3, 5)\} \): \( p_\Lambda = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_6)(\lambda_3 - \lambda_5) \),
Let $\theta^3[e_{\Lambda}]$ be an odd integral theta characteristics in genus $3m - 2$. Then for any $\tau \in \mathbb{H}_{3m-2}$:

$$\sum_{0 \leq \nu, \mu \leq 3} (-1)^{\sum_{i=1}^{m-2} \mu_i \nu_i} \theta^3 \left[ \frac{\mu}{\mu' + \frac{\nu}{3}} \right] (\tau) = 0$$

where $\mathbb{H}_{3m-2}$ is the Siegel upper half space of genus $3m - 2$.

It is plausible that the vanishing of theta constants with rational characteristics of order 3 at $\tau_R$ will produce a new proof of the special identities obtained in
this note using the Thomae formula. Finally, note that for all the identities the coefficients in (4) are ±1. We conjecture that this is a general phenomenon.

6. Conclusion

There exists a large literature about Schottky–Jung identities and identities for hyperelliptic curves. In this note we obtained special identities for other classes of algebraic curves. We plan to pursue these themes in future notes.

Acknowledgment

This work was partially done during a visit to the TAMU math department and the author thanks the department for the invitation and kind hospitality. He also thanks Samuel Grushevsky, Mike Fried and Hershel Farkas for constructive remarks about this note. He especially thanks the referee for carefully checking the manuscript and pointing out numerous corrections in the mathematical and stylistic content of this note.

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