

UNIFORM BOUNDEDNESS OF p -PRIMARY TORSION OF ABELIAN SCHEMES

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ABSTRACT. Let k be a field finitely generated over \mathbb{Q} and denote by Γ_k its absolute Galois group. For an abelian variety A over k , a prime p and a character $\chi : \Gamma_k \rightarrow \mathbb{Z}_p^*$, define $A[p^\infty](\chi)$ to be the module of p -primary torsion of $A(\bar{k})$ on which Γ_k acts as χ -multiplication. Assume that χ does not appear as a subrepresentation of the p -adic representation associated with an abelian variety over k . Then $A[p^\infty](\chi)$ is always finite, but the exponent of $A[p^\infty](\chi)$ may depend on A , a priori. Our main result is about the uniform boundedness of $A[p^\infty](\chi)$ when A varies in a 1-dimensional family. More precisely, if S is a curve over k and A is an abelian scheme over S , then there exists an integer $N := N(A, S, k, p, \chi)$, such that $A_s[p^\infty](\chi) \subset A_s[p^N]$ holds for any $s \in S(k)$. Such a result has been widely open, even when χ is trivial, except for the case of elliptic curves.

This arithmetic result is obtained as a corollary of the following geometric result on the p -primary torsion of abelian varieties over function fields of curves, combined with Mordell's conjecture (Faltings' theorem). Let K be the function field of a curve over an algebraically closed field of characteristic 0 and A an abelian variety over K . Assume for simplicity that A contains no nontrivial isotrivial subvariety. Then, for any $c \geq 0$, there exists an integer $N := N(c, A, S, k, p) \geq 0$ such that $A[p^\infty](K') \subset A[p^N]$ for all finite extension K'/K with K' of genus $\leq c$. A key ingredient of the proof of this geometric result is a certain result on the number of points on reduction modulo p^n of p -adic analytic homogeneous spaces.

Our uniform boundedness result when χ is the p -adic cyclotomic character, together with certain descent methods, yields a proof of the 1-dimensional case of (a generalized variant of) the modular tower conjecture, which was, actually, the original motivation for this work. This is a conjecture arising from the regular inverse Galois problem, whose original form was posed by M. Fried in the early 1990s.

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1. INTRODUCTION

Fix a prime p and let k be a field of characteristic $q \neq p$ and let S be a separated, smooth, geometrically connected curve over k of type (g, r) (that is such that there exists a proper, smooth, geometrically connected, genus g curve \bar{S} over k and a degree r etale divisor $D \subset \bar{S}$ such that $S = \bar{S} \setminus D$). Denote by $\text{Spec}(k(S)) \rightarrow S$ the generic point of S .

Given a geometric generic point $s : \text{Spec}(\Omega) \rightarrow S$ denote by $\Pi := \pi_1(S, s)$ the etale fundamental group of S . Recall that, in our situation, it is simply the Galois group of the maximal algebraic extension of $k(S)$ in Ω which is unramified everywhere on S ; in particular, it is a quotient of the absolute Galois group $\Gamma_{k(S)} := \text{Gal}(k(S)^{sep}/k(S))$, where $k(S)^{sep}$ stands for the separable closure of $k(S)$ in Ω .

Now, fix an abelian scheme $A \rightarrow S$ with generic fiber $A_\eta \rightarrow \text{Spec}(k(S))$ of dimension d . (We refer to section 2.1 for the notation used for abelian varieties.) Since $A \rightarrow S$ is an abelian scheme and $q \neq p$, the natural action of $\Gamma_{k(S)}$ on the Tate module $T_p(A_\eta)$ factors through Π . This, in turn, defines an action of Π over $A_\eta[p^n]$, $n \geq 0$ hence over $A_\eta[p^\infty]$. Finally, for any $v \in A_\eta[p^\infty]$, write Π_v for the stabilizer of v under Π ; it is a closed subgroup of Π of index $\leq |\langle v \rangle|^{2d}$, hence it is also open in Π , and, by Galois theory, corresponds to a connected finite etale cover $S_v \rightarrow S$ (defined over a finite extension k_v/k). Similarly, write $\Pi_{\langle v \rangle}$ for the stabilizer of $\langle v \rangle$ under Π and $S_{\langle v \rangle} \rightarrow S$ for the resulting connected finite etale cover (defined over a finite extension $k_{\langle v \rangle}/k$). The inclusion of open subgroups

$\Pi_v \subset \Pi_{\langle v \rangle} \subset \Pi$ yields a commutative diagram of finite etale covers:

$$\begin{array}{ccc} S_v & \longrightarrow & S_{\langle v \rangle} \\ \downarrow & \searrow & \\ S & & \end{array}$$

We will write g_v and $g_{\langle v \rangle}$ for the genus of the smooth compactifications of $S_v \times_{k_v} \bar{k}$ and $S_{\langle v \rangle} \times_{k_{\langle v \rangle}} \bar{k}$, respectively.

Eventually, denote by $(A_\eta)_0$ the largest abelian subvariety of A_η which is isogenous to an isotrivial abelian variety (see section 2.1).

We can now state the main geometric result of this paper:

Theorem 1.1. *Assume that k is algebraically closed. Then, for any $c \geq 0$, there exists an integer $N := N(c, A, S, k, p) \geq 0$ such that for all $v \in A_\eta[p^\infty]$ either $g_v \geq c$ or $p^N v \in (A_\eta)_0$. The same statement holds if g_v is replaced by $g_{\langle v \rangle}$.*

Say that a character $\chi : \Gamma_k \rightarrow \mathbb{Z}_p^*$ is *non-Tate* if it does not appear as a subrepresentation of the p -adic representation associated with an abelian variety over k . This holds, for instance, if χ is the trivial or the p -adic cyclotomic character, provided k is finitely generated over the prime field. When χ is non-Tate, $A[p^\infty](\chi)$ is always finite. Theorem 1.1 yields the following corollary about the uniform boundedness of $A[p^\infty](\chi)$ when A varies in a 1-dimensional family.

Corollary 1.2. *Assume that k is finitely generated over \mathbb{Q} and let $\chi : \Gamma_k \rightarrow \mathbb{Z}_p^*$ be a non-Tate character. Then there exists an integer $N := N(A, S, k, p, \chi)$, such that, for any $s \in S(k)$, the Γ_k -module $A_s[p^\infty](\chi) := \{T \in A_s[p^\infty] \mid \sigma T = \chi(\sigma)T, \sigma \in \Gamma_k\}$ is contained in $A_s[p^N]$.*

The strategy of the proof of theorem 1.1 is as follows. First, we show that it is enough to prove the following statement (Theorem 4.1): *Assume that A_η contains no nontrivial abelian subvariety isogenous to an isotrivial abelian variety, and, for any $v \in T_p(A_\eta)^*$, set $v_n := v \bmod p^n T_p(A_\eta) \in A_\eta[p^n]^*$, $n \geq 0$. Then $g_{v_n} \rightarrow \infty$ and $g_{\langle v_n \rangle} \rightarrow \infty$ ($n \rightarrow \infty$).*

Then, to any $v \in T_p(A_\eta)^*$ associate the Π -module $M := \mathbb{Q}_p[\Pi]v \cap T_p(A_\eta)$ and denote by $\Pi(n)$ the kernel of the reduction morphism $\mathrm{GL}(M) \rightarrow \mathrm{GL}(M/p^n)$. The inclusion of open subgroups $\Pi(n) \subset \Pi_{v_n} \subset \Pi$ corresponds to a sequence of finite etale covers $S(n) \rightarrow S_{v_n} \rightarrow S$, with $S(n) \rightarrow S$ Galois, which makes it easier to handle. We first prove that the genus $g(n)$ of $S(n)$ goes to infinity with n . This is achieved by using the fact that the image of Π in $\mathrm{GL}(M)$ cannot be almost-abelian (section 2.3). Then, using the Riemann-Hurwitz formula, we compare $g(n)$ and g_{v_n} and show that the “increasing rate” of g_{v_n} is “close enough” to that of $g(n)$ for $n \gg 0$. This relies on the asymptotic bound given by J. Oesterlé for the number of points on reduction modulo p^n of p -adic analytic subspaces of \mathbb{Z}_p^m . The proof of the variant for $g_{\langle v \rangle}$ is along the same lines but working with the projectivization of the representation of Π over M .

To deduce corollary 1.2 from theorem 1.1, we introduce a sequence of (disconnected) finite etale covers $\mathcal{S}_{n+1, \chi} \rightarrow \mathcal{S}_{n, \chi}$ of S with the property that k -rational points $s_n : \mathrm{Spec}(k) \rightarrow \mathcal{S}_{n, \chi}$ lying above $s : \mathrm{Spec}(k) \rightarrow S$ correspond to elements of order exactly p^n in $A_s[p^\infty](\chi)$. It is thus enough to prove that $\mathcal{S}_{n, \chi}(k) = \emptyset$ for $n \gg 0$. According to theorem 1.1, $\mathcal{S}_{n, \chi}$ can be written as a disjoint union $\mathcal{S}_{n, \chi} = \mathcal{S}_{n, \chi}^{(1)} \amalg \mathcal{S}_{n, \chi}^{(2)}$, where $\mathcal{S}_{n, \chi}^{(1)}$ corresponds to the “isotrivial part” of $A_\eta[p^n]$ and $\mathcal{S}_{n, \chi}^{(2)}$ to the “non-isotrivial” one. On one hand, we show that $\mathcal{S}_{n, \chi}^{(1)}(k) = \emptyset$ for $n \gg 0$, and, on the other hand, from theorem 1.1, the genus of each component of $\mathcal{S}_{n, \chi}^{(2)}$ is larger than 2 for $n \gg 0$. Applying the Mordell conjecture (Faltings’ finiteness theorem) and the definition of non-Tate character then yields the result.

Our original motivation for corollary 1.2 was to prove the 1-dimensional case of the modular tower conjecture. This is a conjecture arising from regular inverse Galois theory, whose original form was posed by M. Fried in the early 1990s (cf. [Fr95]). A generalized variant of this conjecture roughly states that if $(H_{n+1} \rightarrow H_n)$ is a projective system of moduli spaces classifying G -covers of genus g curves equipped with r points, with group G_n , whose ramification only occurs above the given r points, then, for any field k finitely generated over \mathbb{Q} , $H_n(k) = \emptyset$ for n large enough, provided the projective limit of $(G_{n+1} \rightarrow G_n)$ contains an open subgroup which admits a quotient isomorphic to \mathbb{Z}_p . As the H_n have dimension $3g - 3 + r$ (under the hyperbolicity condition $2 - 2g - r < 0$), the 1-dimensional case corresponds to $(g, r) = (0, 4), (1, 1)$. (The 0-dimensional case corresponds to $(g, r) = (0, 3)$ and is easy to treat. See lemma 5.10.) In that case, we thus have (as a special case of theorem 5.11): *The modular tower conjecture holds for $(g, r) = (0, 4), (1, 1)$.* A typical example of the modular tower conjecture is when one takes for G_n the dihedral group D_{2p^n} of order $2p^n$ and $(g, r) = (0, 2s)$. Then, roughly speaking, the component of H_n defined by the condition that the corresponding r ramification indices are all 2 classifies jacobians of genus $s - 1$ hyperelliptic curve with a torsion point of order exactly p^n . In particular, for $s = 2$, $(H_{n+1} \rightarrow H_n)$ is just the projective system of modular curves $(Y_1(p^{n+1}) \rightarrow Y_1(p^n))$ and, in that special case, the modular tower conjecture can be deduced from the Mordell conjecture (Faltings' finiteness theorem) and the fact that $\varprojlim Y_1(p^n)(k) = \emptyset$. This is the basic idea (essentially due to Manin [Ma69]) hidden behind our proof of corollary 1.2 and its application to the modular tower conjecture. In the general case, one has to distinguish between the fine and coarse moduli situations. If H_n is a fine moduli scheme for $n \geq 0$, then one can apply directly corollary 1.2 to the p -adic cyclotomic character and the jacobian $\text{Pic}_{\mathcal{C}_0|H_0}^0 \rightarrow H_0$ of the universal curve $\mathcal{C}_0 \rightarrow H_0$. When H_n is a mere coarse moduli scheme for $n \geq 0$, the idea is, roughly, to use descent techniques (see section 5.2) to construct an auxiliary tower $(\tilde{H}_{n+1} \rightarrow \tilde{H}_n)$ of fine moduli schemes with the property that $\tilde{H}_n(k) \neq \emptyset$ as soon as $H_n(k) \neq \emptyset$.

The paper is organized as follows. In section 2, we list the notation used for abelian varieties (section 2.1) and collect technical preliminaries about non-Tate characters (section 2.2) and about almost-abelian p -adic representations (section 2.3). In section 3, we prove a result (theorem 3.1) concerning reduction modulo p^n of certain p -adic analytic spaces, which is the technical core of our proofs of the main results. In section 4, the proofs of theorem 1.1 and corollary 1.2 are carried out in sections 4.1 and 4.2, respectively. Eventually, in section 5, we convey the proof of the 1-dimensional modular tower conjecture. After recalling the basic notion about G -curves and their stacks (section 5.1), we describe carefully our descent techniques (section 5.2) and apply them together with corollary 1.2 (section 5.3).

Remark 1.3. In [Fr06], Fried outlined a possible proof, which is different from ours, of the original form of the modular tower conjecture for $(g, r) = (0, 4)$.

2. PRELIMINARIES ON ABELIAN SCHEMES

2.1. Notation for abelian varieties. Fix a prime p and let k be a field of characteristic $q \neq p$. Given an abelian variety A over k , we will use the following (classical) notation:

$$\begin{aligned} A[p^n] &:= \ker([p^n] : A(\bar{k}) \rightarrow A(\bar{k})) \text{ for the kernel of the multiplication-by-} p^n \text{ endomorphism;} \\ A[p^\infty] &:= \bigcup_n A[p^n] \text{ for the } p\text{-primary torsion of } A(\bar{k}); \\ T_p(A) &:= \varprojlim_{n \geq 0} A[p^n] \text{ for the } p\text{-adic Tate module of } A; \\ A[p^n]^* &:= A[p^n] \setminus A[p^{n-1}] \quad (A[p^0]^* := A[p^0]); \\ T_p(A)^* &:= T_p(A) \setminus pT_p(A) = \varprojlim A[p^n]^*; \\ V_p(A) &:= T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \end{aligned}$$

Next, given an abelian variety A defined over the function field K of a k -curve S , we will write A_0 for the largest abelian subvariety of A which is isogenous to an isotrivial abelian variety. (Here we say that an abelian variety B over K is isotrivial, if $B \times_K \bar{K}$ descends to an abelian variety over \bar{k} .) Equivalently, A_0 is the largest abelian subvariety of A which is a homomorphic image of an isotrivial abelian variety. Indeed, if $A_1 \rightarrow A_2$ is an epimorphism of abelian varieties, then, by Poincaré's complete reducibility theorem, A_2 is isogenous to an abelian subvariety of A_1 . But as any abelian subvariety of an isotrivial abelian variety is again isotrivial, if A_1 is isotrivial then A_2 is isogenous to an isotrivial abelian subvariety of A_1 . In particular, if $A_i \subset A$ is isogenous to an isotrivial abelian variety, $i = 1, 2$, then so is $A_1 + A_2 \subset A$ (as a homomorphic image of $A_1 \times A_2$); thus A_0 is well-defined. (When k has characteristic 0, any abelian variety isogenous to an isotrivial abelian variety is again isotrivial, so A_0 is simply the largest isotrivial subvariety of A .) We define A^0 to be the quotient abelian variety A/A_0 . Then (again by Poincaré's complete reducibility theorem) we have $(A^0)_0 = 0$.

2.2. Non-Tate characters. Fix a prime p and let k be a field of characteristic $q \neq p$. Let $\chi : \Gamma_k \rightarrow \mathbb{Z}_p^*$ be a character. Say that χ is *non-Tate* if it does not appear as a subrepresentation of the p -adic representation associated with an abelian variety over k . When k is finitely generated over its prime field, the trivial character and the p -adic cyclotomic character are typical examples of non-Tate characters.

Lemma 2.1. *For any finitely generated extension k' of k , $\chi : \Gamma_k \rightarrow \mathbb{Z}_p^*$ is non-Tate if and only if $\chi|_{\Gamma_{k'}} : \Gamma_{k'} \rightarrow \mathbb{Z}_p^*$ is non-Tate.*

Proof. First, the 'if' part is trivial. Indeed, if χ appears in $T_p(A)$ for some abelian variety A over k , $\chi|_{\Gamma_{k'}}$ appears in $T_p(A \times_k k')$. Next, to show the 'only if' part, suppose that $\chi|_{\Gamma_{k'}}$ appears in $T_p(A')$ for some abelian variety A' over k' .

Assume first that k' is a finite extension of k . In this case, one can easily reduce the problem to the following two cases: k'/k is separable; and k'/k is purely inseparable (for $q > 0$). In the former case, define A to be the Weil restriction $Res_{k'/k}(A')$, which is an abelian variety over k with dimension $[k' : k]\dim(A')$. Then the adjoint property of Weil restriction implies that χ appears in $T_p(A)$. Indeed, more specifically, let M_n denote the finite étale commutative group scheme over k that corresponds to the Galois module $\mathbb{Z}/p^n(\chi_n)$ (i.e., the module \mathbb{Z}/p^n on which Γ_k acts via χ_n). Then

$$\mathrm{Hom}_{\Gamma_k}(\mathbb{Z}_p(\chi), T_p(A)) = \varprojlim \mathrm{Hom}_k(M_n, A) = \varprojlim \mathrm{Hom}_{k'}(M_n \times_k k', A') = \mathrm{Hom}_{\Gamma_{k'}}(\mathbb{Z}_p(\chi|_{\Gamma_{k'}}), T_p(A')).$$

In the latter case, take $N \geq 0$ such that $(k')^{q^N} \subset k$ and set $A := A' \times_{k'} k$, where $k' \rightarrow k$ is given by $a' \mapsto (a')^{q^N}$. Now, since $\chi|_{\Gamma_{k'}}$ appears in $T_p(A')$ by assumption, $\chi = (\chi|_{\Gamma_{k'}})|_{\Gamma_k}$ appears in $T_p(A)$. (Here the equality follows from the fact that the q^N th power map $k \rightarrow k$, $a \mapsto a^{q^N}$ induces the identity on Γ_k .)

Finally, assume that k' is an arbitrary finitely generated extension of k and that $\chi|_{\Gamma_{k'}}$ appears in $T_p(A')$ for some abelian variety A' over k' . Then, taking a model of $A' \rightarrow \mathrm{Spec}(k')$ over the spectrum of a finitely generated k -subalgebra of k' whose fraction field coincides with k' and specializing it at a closed point, one can show that there exist a finite extension k'' of k and an abelian variety A'' over k'' such that $\chi|_{\Gamma_{k''}}$ appears in $T_p(A'')$. Thus, by the preceding argument, χ appears in $T_p(A)$ for some abelian variety over k . \square

2.3. Almost-abelian p -adic representations. Fix a prime p . Let V be a finite-dimensional \mathbb{Q}_p -vector space. For a property P of subgroups of $\mathrm{Aut}_{\mathbb{Q}_p}(V)$ (e.g., trivial, abelian, unipotent, pro- p , etc.), say that a subgroup G of $\mathrm{Aut}_{\mathbb{Q}_p}(V)$ is *almost- P* if there exists a closed subgroup $H \subset G$ of finite index such that H has P . Further, given a group Π and a representation $\rho : \Pi \rightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V)$, say that $\rho : \Pi \rightarrow \mathrm{Aut}_{\mathbb{Q}_p}(V)$ is almost- P if the image of ρ is almost- P (though the standard terminology for 'almost-unipotent' may be 'quasi-unipotent').

Let k be an algebraically closed field of characteristic $q \neq p$ and let S be a connected normal scheme

of finite type over k . Denote by $\eta : \text{Spec}(k(S)) \rightarrow S$ the generic point of S . Let $A \rightarrow S$ be an abelian scheme with generic fiber $A_\eta \rightarrow \text{Spec}(k(S))$, and consider the natural p -adic representation on the Tate module $\rho : \pi_1(S) \rightarrow \text{Aut}_{\mathbb{Z}_p}(T_p(A_\eta)) \subset \text{Aut}_{\mathbb{Q}_p}(V_p(A_\eta))$.

Proposition 2.2. *Let M be a $\pi_1(S)$ -submodule of $T_p(A_\eta)$ and consider the corresponding representation $\rho_M : \pi_1(S) \rightarrow \text{Aut}_{\mathbb{Z}_p}(M) \subset \text{Aut}_{\mathbb{Q}_p}(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Then the following are all equivalent.*

- (1) ρ_M is almost-trivial (i.e., ρ_M has finite image).
- (2) ρ_M is almost-abelian.
- (3) ρ_M is almost-unipotent.
- (4) $M \subset T_p((A_\eta)_0)$.

Proof. (1) \Rightarrow (2). Clear.

(2) \Rightarrow (3). First observe $\text{Aut}_{\mathbb{Z}_p}(M) \simeq \text{GL}_m(\mathbb{Z}_p)$, where m is the rank of the free \mathbb{Z}_p -module M . Thus, from the short exact sequence

$$1 \rightarrow 1 + pM_m(\mathbb{Z}_p) \rightarrow \text{GL}_m(\mathbb{Z}_p) \rightarrow \text{GL}_m(\mathbb{Z}/p) \rightarrow 1,$$

the group $\text{Aut}_{\mathbb{Z}_p}(M)$ is almost-pro- p . This, together with the assumption that ρ_M is almost-abelian, assures that one may assume that $\rho_M(\pi_1(S))$ is an abelian pro- p group, up to replacing S by a finite etale cover.

Since S is of finite type over k , there exists a subfield k_1 of k , finitely generated over its prime field, such that $A \rightarrow S \rightarrow \text{Spec}(k)$ admits a model $A_1 \rightarrow S_1 \rightarrow \text{Spec}(k_1)$, where S_1 is a geometrically connected, geometrically normal scheme of finite type over k_1 and $A_1 \rightarrow S_1$ is an abelian scheme. Denote by $\eta_1 : \text{Spec}(k_1(S_1)) \rightarrow S_1$ the generic point of S_1 and by $A_{1,\eta_1} \rightarrow \text{Spec}(k_1(S_1))$ the generic fiber of $A_1 \rightarrow S_1$. One has a canonical surjection $\pi_1(S) \twoheadrightarrow \pi_1((S_1)_{\overline{k_1}})$ ([SGA1, Exp. X, proof of Cor. 1.6]). At the level of Tate modules, one has a canonical isomorphism $T_p(A_\eta) \xrightarrow{\sim} T_p(A_{1,\eta_1})$ which can be taken to be compatible with the actions of $\pi_1(S)$ and $\pi_1((S_1)_{\overline{k_1}})$. Thus, one may assume that $k = \overline{k_1}$. Here, observe that $T_p(A_\eta) = T_p(A_{1,\eta_1})$ admits a natural action of $\pi_1(S_1)$, extending the action of $\pi_1(S) = \pi_1((S_1)_k)$.

Denote by Δ the maximal abelian pro- p quotient of $\pi_1(S)$ and by N the kernel of the natural surjection $\pi_1(S) \twoheadrightarrow \Delta$. Set $\Pi := \pi_1(S_1)/N$. Thus, one gets the following exact sequence

$$1 \rightarrow \Delta \rightarrow \Pi \rightarrow \Gamma_{k_1} \rightarrow 1.$$

Since Δ is abelian, this exact sequence induces a natural action of Γ_{k_1} on Δ .

Lemma 2.3. *For any finite extension k'_1 of k_1 , the coinvariant quotient $\Delta_{\Gamma_{k'_1}}$ is finite.*

Proof. This fact, together with several ways of proving it, is essentially widely known. For example, by the theory of (generalized) Albanese varieties (see, e.g., [Mo08, Appendix]), we may reduce the problem to the case that S_1 is a semi-abelian variety over k_1 . Then, taking a good model of S_1 (as a semi-abelian scheme) over the spectrum of a finitely generated \mathbb{Z} -subalgebra R of k_1 whose fraction field coincides with k_1 ; defining R' to be the integral closure of R in k'_1 ; choosing a closed point on $\text{Spec}(R')$ whose residue field has characteristic $\neq p$; and considering the action on Δ of the Frobenius element at this point, we deduce the desired finiteness. \square

As $\rho_M(\pi_1(S))$ is an abelian pro- p group, $\rho_M : \pi_1(S) \rightarrow \text{Aut}_{\mathbb{Z}_p}(M)$ factors through Δ , or, equivalently, M is contained in $T_p(A_\eta)^N$. Now, up to replacing M by $T_p(A_\eta)^N$, one may assume that $M \subset T_p(A_\eta)$ is a $\pi_1(S_1)$ -submodule and that the resulting representation $\rho_M : \pi_1(S_1) \rightarrow \text{Aut}_{\mathbb{Z}_p}(M)$ factors through Π .

Now, define $\overline{\Delta}$ and $\overline{\Pi}$ to be the images of Δ and Π in $\text{Aut}_{\mathbb{Z}_p}(M)$. Set $R_0 := \mathbb{Z}_p[\overline{\Delta}] \subset \text{End}_{\mathbb{Z}_p}(M)$ and $R := R_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. As R_0 is a free \mathbb{Z}_p -module, the canonical morphism $R_0 \rightarrow R$ is injective, hence so is $R_0^{\text{red}} \rightarrow R^{\text{red}}$, where $R_0^{\text{red}} = R_0/\sqrt{0_{R_0}}$ and $R^{\text{red}} := R/\sqrt{0_R}$. We shall denote the natural surjection $R \twoheadrightarrow R^{\text{red}}$ by $f \mapsto f^{\text{red}}$. The action of $\overline{\Pi}$ by conjugation on $\overline{\Delta}$ extends to one on R_0 by \mathbb{Z}_p -linearity and to one on R by \mathbb{Q}_p -linearity. Note that $\overline{\Pi}$ acts on R not only as \mathbb{Q}_p -module automorphisms but also as \mathbb{Q}_p -algebra automorphisms. In particular, the action of $\overline{\Pi}$ on R induces one on R^{red} . Denote

by $i : \Pi \rightarrow \overline{\Pi} \rightarrow \text{Aut}_{\mathbb{Q}_p\text{-alg}}(R)$ and $i^{red} : \overline{\Pi} \rightarrow \text{Aut}_{\mathbb{Q}_p\text{-alg}}(R^{red})$ the resulting representations. Observe that i (hence also i^{red}) factors through $\Pi \rightarrow \Gamma_{k_1}$, since Δ is abelian.

Since R^{red} is a finite reduced commutative \mathbb{Q}_p -algebra, it is isomorphic to a product $\prod_{1 \leq i \leq r} L_i$ of finite field extensions L_i/\mathbb{Q}_p , $i = 1, \dots, r$ and its \mathbb{Q}_p -algebra automorphism group is finite. In particular, i^{red} has finite image. So, up to replacing k_1 by a finite extension, one may assume that i^{red} is trivial. Then, if we denote the image of Δ in $(R_0^{red})^* \subset (R^{red})^*$ by $\overline{\Delta}^{red}$, it follows that the natural surjection $\Delta \rightarrow \overline{\Delta}^{red}$ factors through $\Delta_{\Gamma_{k_1}}$. Thus, by lemma 2.3, $\overline{\Delta}^{red}$ is finite.

Now, up to replacing S by a finite etale cover, one may assume that Δ has trivial image in $(R^{red})^*$, or, equivalently, has unipotent image in $R^* \subset \text{Aut}(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, as desired.

(3) \Rightarrow (4). Up to replacing S by a finite etale cover, one may assume that ρ_M is unipotent. Denote by M^0 the image of M in $T_p(A_\eta)/T_p((A_\eta)_0) = T_p((A_\eta)^0)$, and suppose $M^0 \neq \{0\}$. Then $\rho_{M^0} : \pi_1(S) \rightarrow \text{Aut}(M^0)$ is also unipotent. From this, there exists a $v \in M^0 \setminus \{0\}$ such that $\pi_1(S)$ acts trivially on v . In particular, this forces the p -primary torsion of $(A_\eta)^0(k(S))$ to be infinite. But this contradicts the fact that $(A_\eta)^0(k(S))$ is a finitely generated abelian group by the Lang-Néron theorem ([LN59]). Thus, $M^0 = \{0\}$, as desired.

(4) \Rightarrow (1). Immediate from the definition of $(A_\eta)_0$. \square

Corollary 2.4. *The following are all equivalent.*

- (1) $T_p(A_\eta)$ admits no nonzero $\pi_1(S)$ -submodule M with ρ_M almost-trivial.
- (2) $T_p(A_\eta)$ admits no nonzero $\pi_1(S)$ -submodule M with ρ_M almost-abelian.
- (3) $T_p(A_\eta)$ admits no nonzero $\pi_1(S)$ -submodule M with ρ_M almost-unipotent.
- (4) $(A_\eta)_0 = 0$.

Proof. (4) is equivalent to $T_p((A_\eta)_0) = \{0\}$, hence to saying that $T_p(A_\eta)$ admits no nonzero $\pi_1(S)$ -submodule M with $M \subset T_p((A_\eta)_0)$. Thus, corollary 2.4 immediately follows from proposition 2.2. \square

Corollary 2.5. *Consider the following conditions:*

- (1) ρ is almost-trivial.
- (2) ρ is almost-abelian.
- (3) ρ is almost-unipotent.
- (4) $(A_\eta)_0 = A_\eta$.
- (5) A_η is isotrivial.

Then we have (1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftarrow (5). (In particular, if $\pi_1(S)$ itself is almost-abelian (i.e., admits an abelian open subgroup), then (4) automatically holds.)

If, moreover, $q = 0$, (1)–(5) are all equivalent.

Proof. Applying proposition 2.2 to $M = T_p(A_\eta)$, one gets the equivalence of (1)–(4). For (5) \Rightarrow (4) and (4) \Rightarrow (5) for $q = 0$, see 2.1. \square

Remark 2.6. In corollary 2.5, the equivalences (1) \Leftrightarrow (4) and (1) \Leftrightarrow (5) for $q = 0$ directly follow from [G66, Prop. 4.3 and Prop. 4.4].

Remark 2.7. Let $X \rightarrow S$ be a proper, smooth, geometrically connected curve and apply corollary 2.5 to $A = \text{Pic}_{X/S}^0$, the jacobian of $X \rightarrow S$. Further, consider the following condition: (6) X_η is isotrivial. Then we have (5) \Leftrightarrow (6). Indeed, (6) \Rightarrow (5) is clear and (5) \Rightarrow (6) essentially follows from a version of Torelli's theorem. We omit the details.

3. REDUCTION MODULO p^n OF p -ADIC ANALYTIC HOMOGENEOUS SPACES

The aim of this section is to prove theorem 3.1 concerning reduction modulo p^n of certain p -adic analytic homogeneous spaces, which is the technical core of our proofs of the main results in §4. The readers who would like to start the proofs of the main results quickly may skip this section, after glancing at the statement of theorem 3.1.

Let p be a prime. Let M be a free \mathbb{Z}_p -module of rank $m \geq 1$ and set $M_n := M/p^n$, $n \geq 0$, and $W := M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Set $\mathrm{GL}(M) := \mathrm{Aut}_{\mathbb{Z}_p}(M)$ and $\mathrm{GL}(M_n) := \mathrm{Aut}_{\mathbb{Z}/p^n}(M_n)$, $n \geq 0$. Let $M \rightarrow M_n$, $v \mapsto v_n$ denote reduction modulo $p^n M$ and $\mathrm{GL}(M) \rightarrow \mathrm{GL}(M_n)$, $g \mapsto g_n$ the induced morphism. As projective versions of these, set $\mathbb{P}(M) := (M \setminus pM)/\mathbb{Z}_p^*$ and $\mathbb{P}(M_n) := (M_n \setminus pM_n)/(\mathbb{Z}/p^n)^*$, $n \geq 0$. Set $\mathrm{PGL}(M) := \mathrm{GL}(M)/\mathbb{Z}_p^* \mathrm{Id}$ and $\mathrm{PGL}(M_n) := \mathrm{GL}(M_n)/(\mathbb{Z}/p^n)^* \mathrm{Id}$, $n \geq 0$. Eventually, denote by $M \setminus pM \rightarrow \mathbb{P}(M)$, $v \mapsto \bar{v}$; $M_n \setminus pM_n \rightarrow \mathbb{P}(M_n)$, $v_n \mapsto \bar{v}_n$; $\mathrm{GL}(M) \rightarrow \mathrm{PGL}(M)$, $\gamma \mapsto \bar{\gamma}$; and $\mathrm{GL}(M_n) \rightarrow \mathrm{PGL}(M_n)$, $\gamma_n \mapsto \bar{\gamma}_n$ the canonical projectivization morphisms.

Let G be a closed subgroup of $\mathrm{GL}(M)$, which is automatically a compact p -adic analytic Lie group. Consider an element $v \in M \setminus pM$ with the following property:

$$(*) \text{ For any open subgroup } H \subset G \text{ one has } W = \mathbb{Q}_p[H]v.$$

Then:

Theorem 3.1. *For any closed subgroup $I \subset G$, one has:*

$$(1) \lim_{n \rightarrow \infty} \frac{|I_n \setminus G_n v_n|}{|G_n v_n|} = \frac{1}{|I|};$$

$$(2) \lim_{n \rightarrow \infty} \frac{|\bar{I}_n \setminus G_n \bar{v}_n|}{|G_n \bar{v}_n|} = \frac{1}{|\bar{I}|},$$

where $\frac{1}{\infty} := 0$.

To prove theorem 3.1, we start with:

Theorem 3.2. *For any closed subgroup $J \subset G$, one has:*

$$(1) \lim_{n \rightarrow \infty} \frac{|(G_n v_n)^J|}{|G_n v_n|} = 0, \text{ unless } J \text{ is trivial};$$

$$(2) \lim_{n \rightarrow \infty} \frac{|(G_n \bar{v}_n)^{\bar{J}}|}{|G_n \bar{v}_n|} = 0, \text{ unless } \bar{J} \text{ is trivial}.$$

Proof. We begin with a general lemma about projective limits of finite sets.

Lemma 3.3. *Let $A := (A_{n+1} \xrightarrow{\phi_{n+1,n}} A_n)_{n \geq 0}$ be a projective system of nonempty finite sets such that*

$$(\#) \text{ for all } n \geq 0, \text{ there exists } d(n) \geq 0, \text{ such that for all } x_n \in A_n, |\phi_{n+1,n}^{-1}(x_n)| = d(n)$$

and let $B := (B_{n+1} \xrightarrow{\phi_{n+1,n}} B_n)_{n \geq 0}$ be a projective subsystem of A . Set $B_\infty := \varprojlim B_n$ and $B_{\infty,n} :=$

$p_n(B_\infty) \subset B_n$, where $p_n : B_\infty \rightarrow B_n$ is the canonical projection. In particular, $(B_{\infty,n+1} \xrightarrow{\phi_{n+1,n}} B_{\infty,n})_{n \geq 0}$ is a projective subsystem of B such that $B_\infty = \varprojlim B_{\infty,n}$. Then

$$(1) m(B) := \lim_{n \rightarrow \infty} \frac{|B_n|}{|A_n|} \text{ and } m_\infty(B) := \lim_{n \rightarrow \infty} \frac{|B_{\infty,n}|}{|A_n|} \text{ exist and belong to } [0, 1].$$

$$(2) m(B) = m_\infty(B).$$

Proof of lemma 3.3. For (1), set $\delta(n) := \max\{|B_{n+1} \cap \phi_{n+1,n}^{-1}(x_n)|\}_{x_n \in B_n} \leq d(n)$. Then, by (#), one gets $\frac{|B_{n+1}|}{|A_{n+1}|} \leq \frac{\delta(n)}{d(n)} \frac{|B_n|}{|A_n|} \leq \frac{|B_n|}{|A_n|}$. Thus, $m(B)$ exist and belongs to $[0, 1]$ since, by definition, $\frac{|B_n|}{|A_n|} \in [0, 1]$, $n \geq 0$. The same argument obviously works for $m_\infty(B)$.

For (2), the inequality $m_\infty(B) \leq m(B)$ follows from the inclusion $B_{\infty,n} \subset B_n$, $n \geq 0$. For the opposite inequality, observe that for any $n \geq 0$ there exists $m(n) \geq n$ such that $\phi_{m(n),n}(B_{m(n)}) = B_{\infty,n}$. Indeed, else, set $\mathcal{B}_{m,n} := \phi_{m,n}^{-1}(B_{\infty,n}) \subset B_m$ and $\mathcal{B}'_{m,n} := B_m \setminus \mathcal{B}_{m,n}$ for $m \geq n$. Then $(\mathcal{B}'_{m+1,n} \rightarrow \mathcal{B}'_{m,n})_{m \geq n}$ gives a well-defined projective subsystem of B . So, if $\mathcal{B}'_{m,n} \neq \emptyset$ for all $m \geq n$, then

$$\emptyset \neq \varprojlim_m \mathcal{B}'_{m,n} \subset (\varprojlim B_m) \setminus (\varprojlim B_{n,\infty}) = B_\infty \setminus B_\infty = \emptyset,$$

which is a contradiction. Thus, one has

$$m(B) \leq \frac{|B_{m(n)}|}{|A_{m(n)}|} \leq \frac{\delta(m(n)-1) \cdots \delta(n)}{d(m(n)-1) \cdots d(n)} \frac{|\phi_{m(n),n}(B_{m(n)})|}{|A_n|} \leq \frac{|\phi_{m(n),n}(B_{m(n)})|}{|A_n|} = \frac{|B_{\infty,n}|}{|A_n|}.$$

Hence $m(B) \leq m_\infty(B)$. \square

Now, observe that $A := (G_{n+1}v_{n+1} \rightarrow G_n v_n)_{n \geq 0}$ satisfies assumption $(\#)$. Indeed, for any $x_n = g_n v_n \in G_n v_n$ and any $x_{n+1} = g_{n+1} v_{n+1} \in \phi_{n+1,n}^{-1}(x_n) \subset G_{n+1} v_{n+1}$ one has

$$|\phi_{n+1,n}^{-1}(x_n)| = [\text{Stab}_{G_{n+1}}(x_n) : \text{Stab}_{G_{n+1}}(x_{n+1})] = [\text{Stab}_{G_{n+1}}(v_n) : \text{Stab}_{G_{n+1}}(v_{n+1})],$$

which only depends on n but not on $x_n \in G_n v_n$.¹

So, to deduce (1) of theorem 3.2 from lemma 3.3, take $B := ((G_{n+1}v_{n+1})^J \rightarrow (G_n v_n)^J)_{n \geq 0}$. Then $B_\infty = (Gv)^J$ and $B_{\infty,n} = ((Gv)^J)_n$, $n \geq 0$. From lemma 3.3, it is enough to show that

$\lim_{n \rightarrow \infty} \frac{|((Gv)^J)_n|}{|G_n v_n|} = 0$. Denote by δ and δ_J the dimension (as p -adic analytic space) of Gv and $(Gv)^J$ respectively. By [O82, Th. 2], there exist real (even rational) numbers $\mu_\delta > 0$ and $\mu_{\delta_J} \geq 0$ such that $\lim_{n \rightarrow \infty} p^{-n\delta} |G_n v_n| = \mu_\delta$ and $\lim_{n \rightarrow \infty} p^{-n\delta_J} |((Gv)^J)_n| = \mu_{\delta_J}$, respectively. (Observe that $(Gv)_n = G_n v_n$.) So, it is enough to prove that $\delta_J < \delta$. Suppose otherwise, then there would exist an open subgroup $H \subset G$ and an element $g \in G$ such that $gHv \subset (Gv)^J$. But, from assumption $(*)$, this implies that J acts trivially on W hence on M , as desired.

Similarly, to deduce (2) of theorem 3.2, define $B_n \subset G_n v_n$, $n \geq 0$, to be the inverse image of $(G_n \bar{v}_n)^J \subset G_n \bar{v}_n$ under the canonical projection $G_n v_n \twoheadrightarrow G_n \bar{v}_n$. Then $B_\infty = \{x \in Gv \mid \gamma \bar{x} = \bar{x}, \gamma \in J\}$ and, by lemma 3.3, it is enough to show that $\lim_{n \rightarrow \infty} \frac{|B_{\infty,n}|}{|G_n v_n|} = 0$. Since J is a compact p -adic Lie group, it is finitely generated. So, fixing a finite set $\gamma^{(1)}, \dots, \gamma^{(r)}$ of generators, $B_\infty = \bigcap_{1 \leq i \leq r} B_\infty^{(i)}$, where

$B_\infty^{(i)} := \{x \in Gv \mid \gamma^{(i)} \bar{x} = \bar{x}\}$. Now, $B_\infty^{(i)}$ is the finite disjoint union of p -adic analytic closed subspaces indexed by the set $\text{spec}(\gamma^{(i)})$ of eigenvalues of $\gamma^{(i)}$ in W :

$$B_\infty^{(i)} = \coprod_{\lambda \in \text{spec}(\gamma^{(i)})} K(\gamma^{(i)}, \lambda),$$

where $K(\gamma^{(i)}, \lambda) := \{x \in Gv \mid \gamma^{(i)} x = \lambda x\}$. Since $\text{spec}(\gamma^{(i)})$ is finite, the $K(\gamma^{(i)}, \lambda)$ are also open in $B_\infty^{(i)}$. As in the proof of (1), by [O82, Th. 2], it is enough to prove that $\dim(B_\infty) < \delta$. Suppose otherwise, i.e., $\dim(B_\infty) = \delta$, then $\dim(B_\infty^{(i)}) = \delta$ must hold for all $i = 1, \dots, r$. Thus, for each $i = 1, \dots, r$, there exists $\lambda^{(i)} \in \text{spec}(\gamma^{(i)})$, such that $K(\gamma^{(i)}, \lambda^{(i)})$ has dimension δ , hence contains an open subset gHv (for some open subgroup $H \subset G$ and some element $g \in G$). But then assumption $(*)$ yields that $\gamma^{(i)} = \lambda^{(i)} Id$ on W , hence on M . This means that \bar{J} is trivial, as desired. \square

Proof of theorem 3.1. Given any finite group F , write $\mathcal{M}(F)$ for the (finite) set of nontrivial minimal subgroups of F . Equivalently, $\mathcal{M}(F)$ is the set of cyclic subgroups of F with prime order. (Note that $\mathcal{M}(F) = \emptyset$ if and only if $F = \{1\}$.)

We first prove (1) by using (1) of theorem 3.2. Set $(G_n v_n)' := \bigcup_{J \in \mathcal{M}(I_n)} (G_n v_n)^J$, then one has

$$\frac{1}{|I_n|} \left(1 - \frac{|(G_n v_n)'|}{|G_n v_n|} \right) \leq \frac{|I_n \setminus G_n v_n|}{|G_n v_n|} \leq \frac{1}{|I_n|} \left(1 - \frac{|(G_n v_n)'|}{|G_n v_n|} \right) + \frac{|(G_n v_n)'|}{|G_n v_n|}.$$

¹Since $Gv \subset M$ is a smooth closed p -adic analytic subspace, say, of dimension δ , it even follows from [Se81, Rem. 1, p.148] that the canonical projections $G_{n+1}v_{n+1} \rightarrow G_n v_n$ are p^δ -to-1 for $n \gg 0$.

Since $|I_n| \rightarrow |I|$, the only thing to prove is $\frac{|(G_nv_n)'|}{|G_nv_n|} \rightarrow 0$.

If I is finite, then I is isomorphic to I_n for $n \gg 0$, so

$$0 \leq \frac{|(G_nv_n)'|}{|G_nv_n|} \leq \sum_{J \in \mathcal{M}(I)} \frac{|(G_nv_n)^J|}{|G_nv_n|} \rightarrow 0 \quad (n \rightarrow \infty)$$

by (1) of theorem 3.2.

If I is infinite, then I admits an infinite pro- p cyclic subgroup. Indeed, set $K_p := Id + p^{i(p)}M_m(\mathbb{Z}_p)$, where $i(p) = 1$ for $p \neq 2$ and $i(2) = 2$. Then the p -adic exponential defines a homeomorphism $exp : p^{i(p)}M_m(\mathbb{Z}_p) \simeq K_p$ such that $exp(aM) = (exp(M))^a$, $M \in p^{i(p)}M_m(\mathbb{Z}_p)$, $a \in \mathbb{Z}$. So, K_p is a torsion-free open normal pro- p subgroup of $GL_m(\mathbb{Z}_p)$ and $I \cap K_p$ is an infinite pro- p torsion-free subgroup of $GL_m(\mathbb{Z}_p)$. Now, for any $g \in (I \cap K_p) \setminus \{Id\}$, one has $\langle g \rangle \simeq \mathbb{Z}_p$, as desired.

So, up to replacing I by such a subgroup, one may assume $I \simeq \mathbb{Z}_p$. Fix any $N \geq 0$ and set $J = I^{p^N} \subset I$. Then, for any $x_n \in G_nv_n$ with stabilizer I_{x_n} under I , note that $x_n \notin (G_nv_n)^J$ if and only if $J \not\subset I_{x_n}$. But, since closed subgroups of $I \simeq \mathbb{Z}_p$ are totally ordered for \subset , the latter is also equivalent to $I_{x_n} \subsetneq J = I^{p^N}$, or $I_{x_n} \subset I^{p^{N+1}}$. So, $x_n \notin (G_nv_n)^J$ implies $|Ix_n| = [I : I_{x_n}] \geq [I : I^{p^{N+1}}] = p^{N+1}$. Now,

$$0 \leq \frac{|I_n \setminus G_nv_n|}{|G_nv_n|} \leq \frac{|I_n \setminus (G_nv_n \setminus (G_nv_n)^J)|}{|G_nv_n|} + \frac{|(G_nv_n)^J|}{|G_nv_n|} \leq \frac{1}{p^{N+1}} + \frac{|(G_nv_n)^J|}{|G_nv_n|}.$$

By (1) of theorem 3.2, one has $\frac{|(G_nv_n)^J|}{|G_nv_n|} \rightarrow 0$, so

$$0 \leq \lim_{n \rightarrow \infty} \frac{|I_n \setminus G_nv_n|}{|G_nv_n|} \leq \frac{1}{p^{N+1}},$$

which yields the desired conclusion as $N \geq 0$ is arbitrary.

Finally, the proof for the projective situation (2) is exactly similar: use (2) (instead of (1)) of theorem 3.2. \square

the last inequality from the induction hypothesis (C_{n-1}).

4. PROOFS

4.1. Proof of theorem 1.1. ² Denote the canonical epimorphism $A_\eta \twoheadrightarrow (A_\eta)^0 = A_\eta / (A_\eta)_0$ by $v \mapsto v^0$. It induces a canonical epimorphism of Π -module $A_\eta[p^\infty] \twoheadrightarrow (A_\eta)^0[p^\infty]$ hence, for any $v \in A_\eta[p^\infty]$ an inclusion of open subgroups $\Pi_{\langle v \rangle} \subset \Pi_{\langle v^0 \rangle} \subset \Pi$ corresponding to a commutative diagram of finite etale covers

$$\begin{array}{ccc} S_{\langle v \rangle} & \longrightarrow & S_{\langle v^0 \rangle} \\ \downarrow & \searrow & \\ S & & \end{array} .$$

But then, by the Riemann-Hurwitz formula, $g_{\langle v \rangle} \geq g_{\langle v^0 \rangle}$. The same observation obviously works for v, v^0 as well instead of $\langle v \rangle, \langle v^0 \rangle$. So we can restrict to the case when $(A_\eta)_0$ is trivial (that is, A_η contains no nontrivial abelian subvariety isogenous to an isotrivial abelian variety), which we assume from now on and till the end of this section.

In particular, to prove theorem 1.1, it is enough to prove the following.

²As $g_v \geq g_{\langle v \rangle}$, the assertion for g_v is implied by the one for $g_{\langle v \rangle}$. However, here we treat both cases in parallel, partly because the proof for g_v is somewhat simpler than the one for $g_{\langle v \rangle}$, and the former may help the readers to understand the latter.

Theorem 4.1. *Assume that A_η contains no nontrivial abelian subvariety isogenous to an isotrivial abelian variety. Let $v \in T_p(A_\eta)^*$ and set $v_n := v \bmod p^n T_p(A_\eta) \in A_\eta[p^n]^*$, $n \geq 0$. Then $g_{v_n} \rightarrow \infty$ and $g_{\langle v_n \rangle} \rightarrow \infty$ ($n \rightarrow \infty$).*

Indeed, assume that, say, the assertion for $g_{\langle v \rangle}$ in theorem 1.1 does not hold. That is, there exists a $c \geq 0$ such that for all $n \geq 0$ there exists $v_n \in A_\eta[p^\infty]$ with $g_{\langle v_n \rangle} \leq c$ but $p^n v_n \neq 0$. From the inclusion of open subgroups $\Pi_{\langle v_n \rangle} \subset \Pi_{\langle p^{-n} \langle v_n \rangle | v_n \rangle} \subset \Pi$, one gets $g_{\langle p^{-n} \langle v_n \rangle | v_n \rangle} \leq g_{\langle v_n \rangle} \leq c$ so, one gets:

$$A_\eta[p^n]_c^* := \{v \in A_\eta[p^n]^* \mid g_{\langle v \rangle} \leq c\} \neq \emptyset.$$

Furthermore, for any $v \in A_\eta[p^n]_c^*$ the inclusion of open subgroups $\Pi_{\langle v \rangle} \subset \Pi_{\langle pv \rangle} \subset \Pi$ yields, again, $g_{\langle pv \rangle} \leq g_{\langle v \rangle}$. So $\{A_\eta[p^n]_c^*\}_{n \geq 0}$ forms a projective subsystem of nonempty (finite) subsets of $(A_\eta[p^{n+1}]^* \rightarrow A_\eta[p^{n+1}]^*)_{n \geq 0}$ hence:

$$\emptyset \neq \varprojlim A_\eta[p^n]_c^* \subset \varprojlim A_\eta[p^n]^* = T_p(A_\eta)^*,$$

which contradicts theorem 4.1 for $g_{\langle v_n \rangle}$. The case of g_{v_n} is just similar.

So, the rest of this section will be devoted to proving theorem 4.1.

4.1.1. *The Π -module $W(v)$.* For any $v \in T_p(A_\eta)^*$, define:

$$W(v) := \mathbb{Q}_p[\Pi]v \subset V_p(A_\eta).$$

Then $W(v)$ contains the \mathbb{Q}_p -submodule:

$$W_\infty(v) := \bigcap_{H \subset \Pi} \mathbb{Q}_p[H]v,$$

where the intersection is over all open subgroups $H \subset \Pi$. By definition $v \in W_\infty(v)$ so $W_\infty(v)$ is nonzero. Also, one can write $W_\infty(v) = \mathbb{Q}_p[H_0]v$ for some open subgroup $H_0 \subset \Pi$. Indeed, for any open subgroups $H_1, H_2 \subset \Pi$ observe that $\mathbb{Q}_p[H_1]v \cap \mathbb{Q}_p[H_2]v \supset \mathbb{Q}_p[H_1 \cap H_2]v$. So any open subgroup $H_0 \subset \Pi$ such that $\mathbb{Q}_p[H_0]v$ has minimal \mathbb{Q}_p -dimension works.

In particular, the closed subgroup

$$\Pi_\infty(v) := \{\gamma \in \Pi \mid \gamma \cdot W_\infty(v) = W_\infty(v)\} \subset \Pi$$

contains H_0 hence is also open in Π .

Furthermore, from the inclusion:

$$\bigcap_{n \geq 0} \Pi_{\langle v_n \rangle} = \Pi_{\langle v \rangle} \subset \Pi_\infty(v),$$

one obtains:

$$\Pi = \Pi_\infty(v) \bigcup_{n \geq 0} (\Pi \setminus \Pi_{\langle v_n \rangle}).$$

So it follows from the compactness of Π that there exists an integer $N \geq 0$ such that:

$$\Pi = \Pi_\infty(v) \bigcup_{n \leq N} (\Pi \setminus \Pi_{\langle v_n \rangle}),$$

hence:

$$\Pi_{\langle v_N \rangle} = \bigcap_{n \leq N} \Pi_{\langle v_n \rangle} \subset \Pi_\infty(v).$$

In particular $\Pi_{\langle v_n \rangle} \subset \Pi_\infty(v)$, $n \gg 0$. Similarly, $\Pi_{v_n} \subset \Pi_\infty(v)$, $n \gg 0$.

But, since we are interested in estimating $g_{v_n}, g_{\langle v_n \rangle}$ for $n \gg 0$, one may replace Π by $\Pi_\infty(v)$ and $W(v)$ by $W_\infty(v)$. So, from now on we assume that:

(*) *For any open subgroup $H \subset \Pi$, one has $W(v) = \mathbb{Q}_p[H]v$.*

4.1.2. *The full level- p^n structure $S(n) \rightarrow S$.* Set $M := W(v) \cap T_p(A_\eta) (\simeq \mathbb{Z}_p^m)$. We retain the notation of section 3 and denote by $\rho: \Pi \rightarrow \mathrm{GL}(M)$ the corresponding representation, and by $G \subset \mathrm{GL}(M)$ its image. Then G is a compact subgroup of $\mathrm{GL}_m(\mathbb{Z}_p)$. So it is an infinite almost-pro- p compact p -adic Lie group. (Here the infiniteness follows from corollary 2.4.)

Denote by $\Pi(n) \triangleleft \Pi$ and $G_n \simeq \Pi/\Pi(n) \subset \mathrm{GL}(M_n)$ the kernel and the image of $\Pi \xrightarrow{f} \mathrm{GL}(M) \rightarrow \mathrm{GL}(M_n)$ respectively. Finally, set $G(n) := \rho(\Pi(n)) \subset G$ and $G_\# := \rho(\Pi_\#)$ ($\# = v, \langle v \rangle$, etc). The inclusion of open subgroups $\Pi(n) \subset \Pi_{v_n} \subset \Pi_{\langle v_n \rangle} \subset \Pi$ corresponds to the sequence of finite etale covers

$$S(n) \rightarrow S_{v_n} \rightarrow S_{\langle v_n \rangle} \rightarrow S$$

where $S(n) \rightarrow S$ is Galois with group G_n . Write $g(n)$ for the genus of the smooth compactification of $S(n)$.

4.1.3. $g(n) \rightarrow \infty$. As already noticed, $|G_n| \rightarrow \infty$ by corollary 2.4, so, by the Riemann-Hurwitz formula, $\sup\{g(n)\} < \infty$ if and only if $\sup\{g(n)\} \leq 1$.

If $\sup\{g(n)\} = 1$, then there exists $n_0 \geq 0$ such that $g(n) = 1$ for $n \geq n_0$. Then the smooth compactification $\overline{S(n_0)}$ of $S(n_0)$ is an elliptic curve and that $G(n_0) = \varprojlim (\Pi(n_0)/\Pi(n))$ is a quotient of $\pi_1(\overline{S(n_0)}) (\leftarrow \hat{\mathbb{Z}}^2)$, which contradicts corollary 2.4.

If $\sup\{g(n)\} = 0$, then $S(n) \rightarrow S$ is a Galois cover of genus 0 curve with degree $|G_n| \rightarrow \infty$. Set $i(p) = 1$ for $p \neq 2$ and $i(2) = 2$. Then, replacing S by $S(i(p))$ if necessary, we may assume: (i) G_n is a p -group; and (ii) any element of $G_n = G/G(n)$ of order p is contained in $G(n-1)/G(n)$, hence in the center of G_n . Indeed, (i) is clear and (ii) follows from a standard argument involving the p -adic exponential, as in the proof of theorem 3.1. Now, we resort to the classification of finite subgroups of $\mathrm{PGL}_2(k) = \mathrm{Aut}(\mathbb{P}_k^1)$. More specifically, it follows from [Su82, Th. 6.17, Case I] and condition (i) above that either G_n is a cyclic p -group or $p = 2$ and G_n is a dihedral group of order 2^{m+1} . Moreover, condition (ii) above forces $m \in \{0, 1\}$ in the latter case. So, as $|G_n| \rightarrow \infty$, $(G_{n+1} \rightarrow G_n)_{n \geq 0}$ must be a projective system of cyclic p -groups. Thus, G is abelian, which contradicts corollary 2.4. \square

Remark 4.2. When $k = \mathbb{C}$ and M is the whole p -adic Tate module $T_p(A_\eta)$ (with A_η principally polarized), J.-M. Hwang and W.-K. To proved that a uniform bound (i.e., depending only on $\dim(A_\eta)$) for the growth of $g(n)$ exists [HT06]. By classical arguments (Zarhin's trick and specialization), such a uniform bound for $T_p(A_\eta)$ also exists only under the assumption that k has characteristic 0.

4.1.4. $g_{v_n} \rightarrow \infty$. First, for any connected finite etale cover $T \rightarrow S$ of degree d , set $\lambda(T \rightarrow S) := \frac{2g(T) - 2}{d}$, where $g(T)$ stands for the genus of the smooth compactification of T . Then, if $U \rightarrow T \rightarrow S$ are connected finite etale covers, it follows from the Riemann-Hurwitz formula for $U \rightarrow T$ that $\lambda(U) \geq \lambda(T)$.

Now, set $\lambda_n := \lambda(S(n) \rightarrow S)$ and $\lambda_{v_n} := \lambda(S_{v_n} \rightarrow S)$. Thus, one has $\lambda_{n+1} \geq \lambda_n$, $\lambda_{v_{n+1}} \geq \lambda_{v_n}$ and $\lambda_n \geq \lambda_{v_n}$, $n \geq 0$. In particular, $\lambda := \lim_{n \rightarrow \infty} \lambda_n$ and $\lambda_v := \lim_{n \rightarrow \infty} \lambda_{v_n}$ exist and $\lambda \geq \lambda_v$.

As already noticed, $|G_n| \rightarrow \infty$ (by corollary 2.4), and, as well, $|G_n v_n| \rightarrow \infty$. Indeed, else, since $G_n v_n \simeq G/G_{v_n}$, Π_v would be a closed subgroup of finite index in Π , hence open in Π . Thus, by assumption (*), $W(v) = \mathbb{Q}_p[\Pi_v]v = \mathbb{Q}_p v$ is 1-dimensional, which contradicts corollary 2.4. So, $g(n) \rightarrow \infty$ if and only if $\lambda > 0$, and $g_{v_n} \rightarrow \infty$ if and only if $\lambda_v > 0$. But from paragraph 4.1.3, $\lambda > 0$ holds. Thus, it is enough to prove that $\lambda = \lambda_v$.

To do this, we shall rewrite λ_n and λ_{v_n} in group-theoretic terms, by means of the Riemann-Hurwitz formula. More specifically, for $i = 1, \dots, r$, write $I_{i,n} \subset G_n$ for the image of the inertia group at P_i in G_n and denote by $d_{i,n}$ and $e_{i,n}$ (resp. $d_n(P)$ and $e_n(P)$) the exponent of the different and the ramification index of any place of $S(n)$ (resp. of the place P of S_{v_n}) above P_i in $S(n) \rightarrow S$ (resp. in $S_{v_n} \rightarrow S$). Eventually, for any place P of S_{v_n} , write $d^m(P)$ and $e^n(P)$ for the exponent of the different

and the ramification index of any place of $S(n)$ above P in $S(n) \rightarrow S_{v_n}$. Recall that these data are linked by the relations

$$e_{i,n} = e^n(P)e_n(P), \quad d_{i,n} = e^n(P)d_n(P) + d^n(P).$$

Now, by the Riemann-Hurwitz formula, one has:

$$\lambda_n := 2g(0) - 2 + \sum_{1 \leq i \leq r} \frac{d_{i,n}}{e_{i,n}}, \quad \lambda_{v_n} = 2g(0) - 2 + \frac{1}{|G_n v_n|} \sum_{1 \leq i \leq r} \sum_{P \in S_{v_n}, P|P_i} d_n(P).$$

Set $\epsilon_n := \lambda_n - \lambda_{v_n} \geq 0$. Then observe

$$\epsilon_n = \sum_{1 \leq i \leq r} \epsilon_{i,n}, \quad \epsilon_{i,n} := \frac{1}{|G_n v_n|} \sum_{P \in S_{v_n}, P|P_i} \frac{d^n(P)}{e^n(P)}.$$

Moreover, it is classically known ([Se68, Chap. IV, Prop. 4]) that

$$\epsilon_{i,n} = \frac{1}{|G_n v_n|} \sum_{P \in S_{v_n}, P|P_i} \sum_{j \geq 0} \frac{e^n(P)_j - 1}{e^n(P)},$$

where $e^n(P)_j$ denotes the order of the j th ramification group $I_n(P)_j$ (with “lower numbering”) of any place of $S(n)$ above P in $S(n) \rightarrow S_{v_n}$, $j \geq 0$. (Thus, $e^n(P)_0 = e^n(P)$, and $e^n(P)_j = 1$ for $j \gg 0$.)

As $G(1)/G(n) = \ker(G_n \rightarrow G_1)$ is a p -group, so is $I_{i,n}(1) := I_{i,n} \cap (G(1)/G(n)) = \ker(I_{i,n} \rightarrow I_{i,1})$. Thus, (since $p \neq \text{char}(k)$) the natural surjection $I_{i,n} \rightarrow I_{i,1}$ induces an isomorphism $(I_{i,n})_+ \rightarrow (I_{i,1})_+$, where $(I_{i,n})_+$ denotes the wild inertia subgroup of $I_{i,n}$, i.e., $(I_{i,n})_+$ is the trivial (resp. unique q -Sylow) subgroup of $I_{i,n}$ if the characteristic of k is 0 (resp. q). Now, for integers $j \geq 0$ and $n \geq 0$, define a subgroup $(\tilde{I}_{i,n})_j$ of $I_{i,n}$ as follows: $(\tilde{I}_{i,n})_0 := I_{i,n}$ and, for $j > 0$, $(\tilde{I}_{i,n})_j$ is the inverse image of the j th ramification group $(I_{i,1})_j$ of $I_{i,1}$ under the above isomorphism $(I_{i,n})_+ \rightarrow (I_{i,1})_+$. (Observe, in particular, that there exists $j_0 \geq 0$ such that $(\tilde{I}_{i,n})_j = \{1\}$ for $j > j_0$.) For $j > 0$, the j th ramification group $(I_{i,n})_j$ of $I_{i,n}$ maps surjectively onto $(I_{i,1})_{[j/e_{i,n}(1)]}$ by the above isomorphism $(I_{i,n})_+ \rightarrow (I_{i,1})_+$, where $e_{i,n}(1) := |I_{i,n}(1)|$ ([Se68, Chap. IV, Lem. 5]). From this, one obtains $(I_{i,n})_j = (\tilde{I}_{i,n})_{[j/e_{i,n}(1)]}$ for $j \geq 0$. (Note that this is clear for $j = 0$.)

Moreover, define $(\tilde{I}_i)_j$ to be the projective limit of $\{(\tilde{I}_{i,n})_j\}_{n \geq 0}$. Thus, $(\tilde{I}_i)_0$ coincides with the inertia subgroup $I_i \subset G$ at P_i , while, for $j > 0$, $(\tilde{I}_i)_j$ is a finite, normal q -subgroup of I_i which is projected isomorphically onto $(\tilde{I}_{i,n})_j$ for each $n > 0$.

Lemma 4.3. *Let I be a finite group and J a subgroup of I . Let X be a finite set on which I acts. Consider the natural surjection of quotient sets: $J \backslash X \rightarrow I \backslash X$. Then, for any $x \in X$, the fiber of this map at $Ix \in I \backslash X$ consists of Jsx , where s runs over the representatives of $J \backslash I/I_x$. (Here I_x stands for the stabilizer of x in I .) In particular, if J is normal in I , then the cardinality of this fiber coincides with $\frac{|I||I_x \cap J|}{|J||I_x|}$.*

Proof. Easy. \square

Now, there exists a natural bijection $I_{i,n} \backslash G_n \simeq \{Q \in S(n) \mid Q|P_i\}$, hence one obtains

$$I_{i,n} \backslash G_n v_n \simeq I_{i,n} \backslash G_n / (G_n)_{v_n} \simeq \{P \in S_{v_n} \mid P|P_i\}.$$

More explicitly, fix $Q_0 \in S(n)$ with $Q_0|P_i$ such that $I_{i,n}$ is the inertia group at Q_0 . Then, for $s \in G_n$, The orbit $I_{i,n} s v_n$ corresponds to the double coset $I_{i,n} s (G_n)_{v_n}$, which corresponds to the image $P(s)$ of Q_0^s in S_{v_n} . (Here, we have adopted the action from the right.) For any normal subgroup J of $I_{i,n}$,

consider the natural surjection $J \backslash G_n v_n \rightarrow I_{i,n} \backslash G_n v_n$. Then, by Lemma 4.3, the cardinality of the fiber of this surjection at the orbit $I_{i,n} s v_n$ is equal to

$$\frac{|I_{i,n}| |J \cap (G_n)_{s v_n}|}{|J| |I_{i,n} \cap (G_n)_{s v_n}|} = \frac{|I_{i,n}| |s^{-1} J s \cap (G_n)_{v_n}|}{|J| |s^{-1} I_{i,n} s \cap (G_n)_{v_n}|}.$$

If $J = \{1\}$, this coincides with $\frac{e_{i,n}}{e^n(P(s))}$, while, if $J = (I_{i,n})_j$, it coincides with $\frac{e_{i,n}}{(e_{i,n})_j} \cdot \frac{e^n(P(s))_j}{e^n(P(s))}$, where $(e_{i,n})_j := |(I_{i,n})_j|$. From these, one obtains

$$\sum_{P \in S_{v_n}, P|P_i} \frac{e_{i,n}}{e^n(P)} = \sum_{s \in I_{i,n} \backslash G_n / (G_n)_{v_n}} \frac{e_{i,n}}{e^n(P(s))} = |G_n v_n|$$

and

$$\sum_{P \in S_{v_n}, P|P_i} \frac{e_{i,n}}{(e_{i,n})_j} \cdot \frac{e^n(P)_j}{e^n(P)} = \sum_{s \in I_{i,n} \backslash G_n / (G_n)_{v_n}} \frac{e_{i,n}}{(e_{i,n})_j} \cdot \frac{e^n(P(s))_j}{e^n(P(s))} = |(I_{i,n})_j \backslash G_n v_n|.$$

Therefore, one has

$$\sum_{P \in S_{v_n}, P|P_i} \frac{e^n(P)_j - 1}{e^n(P)} = \frac{(e_{i,n})_j}{e_{i,n}} |(I_{i,n})_j \backslash G_n v_n| - \frac{1}{e_{i,n}} |G_n v_n|$$

for each $j \geq 0$ and

$$\epsilon_{i,n} = \sum_{j \geq 0} \left(\frac{(e_{i,n})_j}{e_{i,n}} \frac{|(I_{i,n})_j \backslash G_n v_n|}{|G_n v_n|} - \frac{1}{e_{i,n}} \right).$$

Now, since

$$(I_{i,n})_j = (\tilde{I}_{i,n})_{[j/e_{i,n}(1)]} = \begin{cases} (\tilde{I}_{i,n})_0, & j = 0, \\ (\tilde{I}_{i,n})_k, & (k-1)e_{i,n}(1) \leq j \leq k e_{i,n}(1) \quad (0 < \exists k \leq j_0), \\ \{1\}, & j > j_0 e_{i,n}(1). \end{cases}$$

one has

$$\begin{aligned} \epsilon_{i,n} &= \left(\frac{|(\tilde{I}_{i,n})_0 \backslash G_n v_n|}{|G_n v_n|} - \frac{1}{e_{i,n}} \right) + e_{i,n}(1) \sum_{0 < k \leq j_0} \left(\frac{|(\tilde{I}_{i,n})_k| |(\tilde{I}_{i,n})_k \backslash G_n v_n|}{e_{i,n} |G_n v_n|} - \frac{1}{e_{i,n}} \right) \\ &= \left(\frac{|I_{i,n} \backslash G_n v_n|}{|G_n v_n|} - \frac{1}{|I_{i,n}|} \right) + \frac{1}{e_{i,1}} \sum_{0 < k \leq j_0} |(\tilde{I}_i)_k| \left(\frac{|(\tilde{I}_{i,n})_k \backslash G_n v_n|}{|G_n v_n|} - \frac{1}{|(\tilde{I}_i)_k|} \right). \end{aligned}$$

Now, by applying theorem 3.1 to I_i and $(\tilde{I}_i)_k$, $0 < k \leq j_0$, one has $\epsilon_{i,n} \rightarrow 0$ ($n \rightarrow \infty$), as desired. \square

4.1.5. $g_{(v_n)} \rightarrow \infty$. Here, the outline of proof is similar to the case of g_{v_n} , but the details are slightly more complicated. With the notation of section 3, one has the following canonical epimorphism of short exact sequences.

$$(1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}_p^* & \longrightarrow & \mathrm{GL}(M) & \longrightarrow & \mathrm{PGL}(M) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & (\mathbb{Z}/p^n)^* & \longrightarrow & \mathrm{GL}(M_n) & \longrightarrow & \mathrm{PGL}(M_n) \longrightarrow 1. \end{array}$$

Consider the projectivizations $\bar{\rho} : \Pi \rightarrow \mathrm{PGL}(M)$, $\bar{\rho}_n : \Pi \rightarrow \mathrm{PGL}(M_n)$ of ρ and ρ_n respectively. Denote by $\bar{G} \subset \mathrm{PGL}(M)$ and $\bar{\Pi}(n) \triangleleft \Pi$, $\bar{G}_n \subset \mathrm{PGL}(M_n)$ the image of $\bar{\rho}$ and $\bar{\rho}_n$ respectively. Finally,

set $\overline{G}(n) := \overline{\rho}(\overline{\Pi}(n)) \subset \overline{G}$.

With this notation, (1) induces the epimorphism of short exact sequences

$$(2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}_p^* \cap G & \longrightarrow & G & \longrightarrow & \overline{G} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & (\mathbb{Z}/p^n)^* \cap G_n & \longrightarrow & G_n & \longrightarrow & \overline{G}_n \longrightarrow 1. \end{array}$$

The inclusion of open subgroups :

$$(3) \quad \begin{array}{ccc} \Pi(n) & \subset & \overline{\Pi}(n) \\ \cap & & \cap \\ \Pi_{v_n} & \subset & \Pi_{\langle v_n \rangle} \subset \Pi \end{array}$$

corresponds to a commutative diagram of finite etale covers:

$$(4) \quad \begin{array}{ccc} S(n) & \longrightarrow & \overline{S}(n) \\ \downarrow & & \downarrow \\ S_{v_n} & \longrightarrow & S_{\langle v_n \rangle} \longrightarrow S, \end{array}$$

where $\overline{S}(n) \rightarrow S$ is Galois with group \overline{G}_n and $S(n) \rightarrow \overline{S}(n)$ is Galois with group $(\mathbb{Z}/p^n)^* \cap G_n$. Write $\overline{g}(n)$ for the genus of $\overline{S}(n)$.

4.1.5.1. $\overline{g}(n) \rightarrow \infty$. First, observe that $|\overline{G}_n| \rightarrow \infty$. (Indeed, else, $\overline{G} = \varprojlim \overline{G}_n$ would be finite hence G would be almost-abelian, which contradicts corollary 2.4.) So, by the Riemann-Hurwitz formula, $\sup\{\overline{g}(n)\} < \infty$ if and only if $\sup\{\overline{g}(n)\} \leq 1$.

If $\sup\{\overline{g}(n)\} = 1$, then, up to replacing Π by $\overline{\Pi}(n)$, S by $\overline{S}(n)$ for some $n \geq 0$ large enough, one may assume that $\overline{g}(n) = 1$, $n \geq 0$, hence that \overline{S} is an elliptic curve and that \overline{G} is a quotient of $\pi_1(\overline{S})$ ($\leftarrow \hat{\mathbb{Z}}^2$). In particular the normal subgroup $I \subset G$ generated by the inertia subgroups lies in $\ker(G \twoheadrightarrow \overline{G}) = \mathbb{Z}_p^* \cap G$. But, by the semistable reduction theorem [SGA7, Exp. IX, Th.3.6], each inertia group is almost-unipotent, hence, being also scalar, it is finite. Thus, I is also finite, as an abelian group generated by a finite number of finite abelian groups. Now, consider the short exact sequence

$$1 \rightarrow I \rightarrow G \rightarrow G/I \rightarrow 1.$$

Since G/I is a quotient of $\hat{\mathbb{Z}}^2$ and I is finite, G is almost-abelian. (Consider any open subgroup U of G such that $U \cap I = \{1\}$.) This contradicts corollary 2.4.

If $\sup\{\overline{g}(n)\} = 0$, then $\overline{S}(n) \rightarrow S$ is a Galois cover of genus 0 curve with degree $|\overline{G}_n| \rightarrow \infty$. So, according to the consideration of paragraph 4.1.3, up to replacing S by a finite Galois cover, one may assume $\overline{G} \simeq \mathbb{Z}_p$. Now, consider the short exact sequence

$$1 \rightarrow \mathbb{Z}_p^* \cap G \rightarrow G \rightarrow \overline{G} \rightarrow 1.$$

Since $\overline{G} \simeq \mathbb{Z}_p$ is a projective profinite group, this exact sequence must split. Then, since $\mathbb{Z}_p^* \cap G$ lies in the center of G and \overline{G} is abelian, G is also abelian. This contradicts corollary 2.4.

4.1.5.2. $g_{\langle v_n \rangle} \rightarrow \infty$. The proof is exactly similar to the one of paragraph 4.1.4 resorting to (2) of theorem 3.1 instead of (1).

4.2. **Proof of corollary 1.2.** For each $n \geq 1$, set $\chi_n := \chi \bmod p^n \mathbb{Z}_p : \Gamma_k \rightarrow (\mathbb{Z}/p^n)^*$. Set $i(p) = 1$ for $p \neq 2$ and $i(2) = 2$. Then, up to replacing k by the fixed field of $\ker(\chi_{i(p)})$ in \overline{k} , one may assume that $\chi_{i(p)} : \Gamma_k \rightarrow (\mathbb{Z}/p^{i(p)})^*$ is trivial. (Here, we have used lemma 2.1.) This technical reduction ensures

that $\text{Im}(\chi_n) \subset (\mathbb{Z}/p^n)^*$ is contained in the order $p^{n-i(p)}$ cyclic subgroup $1 + p^{i(p)}\mathbb{Z}/p^n\mathbb{Z}$ of $(\mathbb{Z}/p^n)^*$ (for $n \geq i(p)$); this will be used in the proof of lemma 4.6.

4.2.1. *The curves $S_{v_n, \chi}$.* For each $v_n \in A_\eta[p^n]^*$, consider the restriction morphism $pr_{\langle v_n \rangle} : \pi_1(S_{\langle v_n \rangle}) \rightarrow \Gamma_k$ (whose image coincides with $\Gamma_{k_{\langle v_n \rangle}}$) and the natural representation $\rho_{\langle v_n \rangle} : \pi_1(S_{\langle v_n \rangle}) \rightarrow \text{Aut}_{\mathbb{Z}/p^n}(\langle v_n \rangle)$. These define together with $\chi_n := \chi \bmod p^n \mathbb{Z}_p$, a representation

$$\rho_{v_n, \chi} : \pi_1(S_{\langle v_n \rangle}) \rightarrow \text{Aut}_{\mathbb{Z}/p^n}(\langle v_n \rangle), \quad \gamma \mapsto \chi_n(pr_{\langle v_n \rangle}(\gamma))^{-1} \rho_{\langle v_n \rangle}(\gamma).$$

Then let $S_{v_n, \chi} \rightarrow S_{\langle v_n \rangle}$ be the etale Galois cover (defined over a finite extension $k_{v_n, \chi}/k$) corresponding to the open normal subgroup $\ker(\rho_{v_n, \chi}) \subset \pi_1(S_{\langle v_n \rangle})$, and denote by $g_{v_n, \chi}$ the genus of the smooth compactification of $S_{v_n, \chi} \times_{k_{v_n, \chi}} \bar{k}$.

Lemma 4.4. *For any character $\chi : \Gamma_k \rightarrow \mathbb{Z}_p^*$ and $v_n \in A_\eta[p^n]^*$ the finite etale cover $S_{v_n, \chi} \rightarrow S$ has the following properties:*

- (1) $S_{v_n, \chi} \times_{k_{v_n, \chi}} \bar{k} \simeq S_{v_n} \times_{k_{v_n}} \bar{k}$, and, in particular, $g_{v_n, \chi} = g_{v_n}$ is independent of χ .
- (2) For any k -rational point $s : \text{Spec}(k) \rightarrow S$, consider the specialization isomorphism of $\pi_1(S)$ -modules

$$sp_s : A_\eta[p^\infty] \xrightarrow{\sim} A_s[p^\infty].$$

Then $sp_s(v_n) \in A_s[p^\infty](\chi)$ if and only if $s : \text{Spec}(k) \rightarrow S$ lifts to a k -rational point:

$$\begin{array}{ccc} S_{v_n, \chi} & & \\ \downarrow & \swarrow^{s_{v_n, \chi}} & \\ S & \longleftarrow_s & \text{Spec}(k) \end{array}$$

Proof. For (1), just observe that $\pi_1(S_{v_n, \chi} \times_{k_{v_n, \chi}} \bar{k}) = \ker(pr_{\langle v_n \rangle}) \cap \ker(\rho_{v_n, \chi}) = \ker(\rho_{v_n, \chi})|_{\ker(pr_{\langle v_n \rangle})} = \ker(\rho_{v_n}) \cap \ker(pr_{\langle v_n \rangle}) = \pi_1(S_{v_n} \times_{k_{v_n}} \bar{k})$

For (2), denote again by s the section $\Gamma_k \hookrightarrow \pi_1(S)$ of $\pi_1(S) \twoheadrightarrow \Gamma_k$ induced (up to conjugacy) by $s : \text{Spec}(k) \rightarrow S$, which identifies Γ_k with the decomposition group at s . Then the existence of the lift $s_{v_n, \chi} : \text{Spec}(k) \rightarrow S_{v_n, \chi}$ of $s : \Gamma_k \hookrightarrow \pi_1(S)$ is equivalent to the inclusion $s(\Gamma_k) \subset \pi_1(S_{v_n, \chi}) (= \ker(\rho_{v_n, \chi}))$, which can be rewritten as $s(\sigma) \cdot v_n = \chi(\sigma)v_n$ ($\sigma \in \Gamma_k$) or, applying the specialization isomorphism, $\sigma \cdot sp_s(v_n) = \chi(\sigma)sp_s(v_n)$. \square

4.2.2. *The projective system $(\mathcal{S}_{n+1, \chi} \rightarrow \mathcal{S}_{n, \chi})$.* For each $n \geq 0$, define

$$\mathcal{S}_{n, \chi} := \coprod_{v_n \in A_\eta[p^n]^*} S_{v_n, \chi}.$$

From theorem 1.1, there exists an integer $N \geq 0$ such that for all $v \in A_\eta[p^\infty]$ either $p^N v^0 = 0$ or $g_{v, \chi} = g_v \geq 2$. So $\mathcal{S}_{n, \chi}$ can be written as a disjoint union $\mathcal{S}_{n, \chi} = \mathcal{S}_{n, \chi}^{(1)} \coprod \mathcal{S}_{n, \chi}^{(2)}$, where

$$\mathcal{S}_{n, \chi}^{(1)} := \coprod_{v_n \in A_\eta[p^n]^*, p^N v_n^0 = 0} S_{v_n, \chi} \quad \text{and} \quad \mathcal{S}_{n, \chi}^{(2)} := \coprod_{v_n \in A_\eta[p^n]^*, p^N v_n^0 \neq 0} S_{v_n, \chi}.$$

Then $(\mathcal{S}_{n+1, \chi}(k) \rightarrow \mathcal{S}_{n, \chi}(k))_{n \geq 0}$ is a projective system with transition maps induced by the canonical morphisms $S_{v_n, \chi} \rightarrow S_{pv_n, \chi}$. They behave with respect to the decomposition $\mathcal{S}_{n, \chi} = \mathcal{S}_{n, \chi}^{(1)} \coprod \mathcal{S}_{n, \chi}^{(2)}$ as

follows.

$$(5) \quad \begin{array}{ccc} \mathcal{S}_{n+1,\chi}^{(1)}(k) & & \mathcal{S}_{n+1,\chi}^{(2)}(k) \\ \downarrow & \swarrow (*) & \downarrow \\ \mathcal{S}_{n,\chi}^{(1)}(k) & & \mathcal{S}_{n,\chi}^{(2)}(k) \end{array}$$

Namely, $\{\mathcal{S}_{n,\chi}^{(1)}(k)\}_{n \geq 0}$ forms a projective subsystem of $(\mathcal{S}_{n+1,\chi}(k) \rightarrow \mathcal{S}_{n,\chi}(k))_{n \geq 0}$, while $\{\mathcal{S}_{n,\chi}^{(2)}(k)\}_{n \geq 0}$ may not (since some element of $\mathcal{S}_{n+1,\chi}^{(2)}(k)$ may map into $\mathcal{S}_{n,\chi}^{(1)}(k)$, as shown by the diagonal arrow with $(*)$ in the above diagram).

Now, suppose that corollary 1.2 does not hold. Then, for all $n \geq 0$, there exists $s_n \in S(k)$ and there exists $\nu_n \in A_{s_n}[p^\infty](\chi)$ such that $p^n \nu_n \neq 0$. Up to replacing ν_n by $|\langle \nu_n \rangle| p^{-n} \nu_n$ one may thus assume that $\mathcal{S}_{n,\chi}(k) \neq \emptyset$, $n \geq 0$. But this yields a contradiction.

Indeed, by the non-Tate assumption on χ and lemma 4.4 (ii), one has

$$\lim_{\leftarrow} \mathcal{S}_{n,\chi}(k) = \emptyset.$$

On the other hand, assuming lemma 4.5 below, the arrows $(*)$ in (5) disappear for $n \gg 0$. So, the projective system $(\mathcal{S}_{n+1,\chi}(k) \rightarrow \mathcal{S}_{n,\chi}(k))_{n \geq 0}$ restricts to a projective system $(\mathcal{S}_{n+1,\chi}^{(2)}(k) \rightarrow \mathcal{S}_{n,\chi}^{(2)}(k))_{n \gg 0}$. But, from the Mordell conjecture for finitely generated fields of characteristic 0 [FW92, Chap. VI, Th.3], $\mathcal{S}_{v_n,\chi}^{(2)}(k)$ is finite, $n \geq 0$. Hence $\mathcal{S}_{v_n,\chi}(k)$ is finite, $n \gg 0$, which yields the contradiction, using the fact that a projective system of nonempty finite sets is nonempty.

Lemma 4.5. $\mathcal{S}_{n,\chi}^{(1)}(k) = \emptyset$, $n \gg 0$.

Proof. Assume that for all $n \geq 0$ there exists $v_n \in A_\eta[p^n]^*$ such that $p^N v_n^0 = 0$ and $\mathcal{S}_{v_n,\chi}(k) \neq \emptyset$. Then, up to replacing v_n by $p^N v_{n+N}$, one may assume that $v_n \in (A_\eta)_0[p^n]^*$, $n \geq 0$.

Also, denote by $s_n : \text{Spec}(k) \rightarrow S$ the image of a k -rational point $\text{Spec}(k) \rightarrow \mathcal{S}_{v_n,\chi}$; it yields a splitting $s_n : \Gamma_k \hookrightarrow \pi_1(S)$ of the restriction epimorphism $\pi_1(S) \rightarrow \Gamma_k$.

Let Δ , Γ and Σ_{s_n} denote the images in $\text{Aut}_{\mathbb{Z}_p}(T_p((A_\eta)_0))$ of $\pi_1(S \times_k \bar{k})$, $\pi_1(S)$ and $s_n(\Gamma_k)$, respectively, under ρ . Since $\pi_1(S) = s_n(\Gamma_k) \cdot \pi_1(S \times_k \bar{k})$, we have $\Gamma = \Sigma_{s_n} \cdot \Delta$.

As $(A_\eta)_0$ is (isogenous to) an isotrivial abelian variety, Δ is finite. Now, $\Gamma \subset \text{Aut}_{\mathbb{Z}_p}(T_p((A_\eta)_0)) \simeq \text{GL}_{2d}(\mathbb{Z}_p)$ ($d := \dim((A_\eta)_0)$) is a compact p -adic Lie group, hence, in particular, it is finitely generated. Since a finitely generated profinite group admits only finitely many open subgroups of given bounded index and since $[\Gamma : \Sigma_{s_n}] \leq |\Delta| < \infty$, there are only finitely many possibilities for the $\Sigma_{s_n} \subset \Gamma$, $n \geq 0$. Thus, there exists $s \in S(k)$ such that $\Sigma_{s_n} = \Sigma_s$ for infinitely many $n \geq 0$. Write $|\Delta| = p^a m$ with $p \nmid m$. Then we have:

Lemma 4.6. *Let $s, t \in S(k)$ with $\Sigma_s = \Sigma_t$. If $sp_t(v_n) \in A_t[p^n](\chi)$, then $sp_s(p^a v_n) \in A_s[p^n](\chi)$.*

Proof of lemma 4.6. The statement is trivial for $n \leq a$, so assume that $n > a$ and write $\delta(\sigma) := \rho(s(\sigma)t(\sigma)^{-1}) \in \Delta$. Also, since $\rho(s(\sigma)) \in \Sigma_s = \Sigma_t$, there exists $\tau = \tau_\sigma \in \Gamma_k$ such that $\rho(s(\sigma)) = \rho(t(\tau))$. As a result, one obtains $\delta(\sigma)v_n = \chi_n(\tau)\chi_n(\sigma^{-1})v_n$. In particular, the order of $\chi_n(\tau)\chi_n(\sigma^{-1}) \in (\mathbb{Z}/p^n)^*$ divides the order of $\delta(\sigma) \in \Delta$, hence divides the order $p^a m$ of Δ . On the other hand, by the assumption on χ put at the beginning of section 4.2, $\chi_n(\tau)\chi_n(\sigma^{-1})$ lies in the order $p^{n-i(p)}$ cyclic subgroup $1 + p^{i(p)}\mathbb{Z}/p^n\mathbb{Z}$ of $(\mathbb{Z}/p^n)^*$. Thus, it follows that $\chi_n(\tau)\chi_n(\sigma^{-1}) \in 1 + p^{n-a}\mathbb{Z}/p^n$. Thus, we have

$$\rho(s(\sigma))p^a v_n = \chi_n(\tau)p^a v_n = \chi_n(\sigma)p^a v_n,$$

which completes the proof. \square

It follows from lemma 4.6 that, up to replacing v_n by $p^a v_{n+a}$, one may assume that $s : \mathrm{Spec}(k) \rightarrow S$ lifts to a k -rational point $s_{v_n} : \mathrm{Spec}(k) \rightarrow S_{v_n, \chi}$ for infinitely many $n \geq 0$, hence $\lim_{\leftarrow} S_{v_n, \chi}(k) \neq \emptyset$: a contradiction. \square

Note that the above proof of corollary 1.2 only resorts to the (weaker) statement for g_v in theorem 1.1. Using the (stronger) statement for $g_{\langle v \rangle}$, one gets the following variant of corollary 1.2.

Corollary 4.7. *Assume that k is finitely generated over \mathbb{Q} and that the set $S^\#$ of all $s \in S(k)$ such that $T_p(A_s)$ admits a rank 1 Γ_k -submodule is finite. Then there exists $N = N(A, S, k, p) \geq 0$ such that, for all $s \in S(k) \setminus S^\#$ and $T \in A_s[p^\infty]$, if $\langle T \rangle$ is Γ_k -stable then $p^N T = 0$.*

Proof. It follows the lines of the proof of corollary 1.2. Up to replacing $A \rightarrow S$ by $A \times_S (S \setminus S^\#) \rightarrow S \setminus S^\#$, one can assume that $S^\# = \emptyset$. As above, for each $n \geq 0$, define

$$\mathcal{S}_n := \coprod_{v_n \in A_\eta[p^n]^*} S_{\langle v_n \rangle}.$$

From theorem 1.1, there exists an integer $N \geq 0$ such that, for all $v \in A_\eta[p^\infty]$, either $p^N v^0 = 0$ or $g_{\langle v \rangle} \geq 2$. So \mathcal{S}_n can be written as a disjoint union $\mathcal{S}_n = \mathcal{S}_n^{(1)} \coprod \mathcal{S}_n^{(2)}$, where

$$\mathcal{S}_n^{(1)} := \coprod_{v_n \in A_\eta[p^n]^*, p^N v^0 = 0} S_{\langle v_n \rangle} \text{ and } \mathcal{S}_n^{(2)} := \coprod_{v_n \in A_\eta[p^n]^*, p^N v^0 \neq 0} S_{\langle v_n \rangle}.$$

Then $(\mathcal{S}_{n+1}(k) \rightarrow \mathcal{S}_n(k))_{n \geq 0}$ is a projective system with transition maps induced by the canonical morphisms $S_{\langle v_n \rangle} \rightarrow S_{\langle p v_n \rangle}$. They behave with respect to the decomposition $\mathcal{S}_n = \mathcal{S}_n^{(1)} \coprod \mathcal{S}_n^{(2)}$ as follows.

$$(6) \quad \begin{array}{ccc} \mathcal{S}_{n+1}^{(1)}(k) & & \mathcal{S}_{n+1}^{(2)}(k) \\ \downarrow & \swarrow (*) & \downarrow \\ \mathcal{S}_n^{(1)}(k) & & \mathcal{S}_n^{(2)}(k) \end{array}$$

Suppose that corollary 4.7 does not hold. Then for all $n \geq 0$ there exists $s_n \in S(k)$ and there exists $\nu_n \in A_{s_n}[p^\infty]$ such that $p^n \nu_n \neq 0$ but $\langle \nu_n \rangle$ is Γ_k -stable. Up to replacing ν_n by $|\langle \nu_n \rangle| p^{-n} \nu_n$ one may thus assume that $\mathcal{S}_n(k) \neq \emptyset$, $n \geq 0$. Since $S^\# = \emptyset$, one has

$$\lim_{\leftarrow} \mathcal{S}_n(k) = \emptyset.$$

Then, assuming that $\mathcal{S}_n^{(1)}(k) = \emptyset$, $n \gg 0$, one obtains, by the same argument as in the proof of corollary 1.2 (i.e., resorting to theorem 1.1 and the Mordell conjecture), that

$$\mathcal{S}_n(k) = \emptyset, \quad n \gg 0,$$

as desired.

To prove that $\mathcal{S}_n^{(1)}(k) = \emptyset$, $n \gg 0$, suppose, on the contrary, that, for all $n \geq 0$, there exists $v_n \in A_\eta[p^n]^*$ such that $p^N v_n^0 = 0$ and $S_{\langle v_n \rangle}(k) \neq \emptyset$. Then, up to replacing v_n by $p^N v_{n+N}$, one may assume that $v_n \in (A_\eta)_0[p^n]^*$, $n \geq 0$. Also, denote by $s_n : \mathrm{Spec}(k) \rightarrow S$ the image of a k -rational point $\mathrm{Spec}(k) \rightarrow S_{\langle v_n \rangle}$; it yields a section $s_n : \Gamma_k \hookrightarrow \pi_1(S)$ of the restriction epimorphism $\pi_1(S) \twoheadrightarrow \Gamma_k$. Following the notations of the proof of lemma 4.5 for Δ , Γ , Σ_{s_n} , etc., one can choose, as in the proof of lemma 4.5, a k -rational point $s : \mathrm{Spec}(k) \rightarrow S$ such that, for infinitely many $n \geq 0$, $\overline{\Sigma}_{s_n} = \overline{\Sigma}_s$. Then $\langle v_n \rangle$ is stable under $\overline{\Sigma}_s = \overline{\Sigma}_{s_n}$, which implies that $sp_s(\langle v_n \rangle) = \langle sp_s(v_n) \rangle$ is stable under Γ_k . Thus, $s : \mathrm{Spec}(k) \rightarrow S$ itself lifts to a k -rational point $\mathrm{Spec}(k) \rightarrow S_{\langle v_n \rangle}$, hence $\lim_{\leftarrow} \mathcal{S}_n^{(1)}(k) \neq \emptyset$: a contradiction. \square

Remark 4.8. (*Variant in positive characteristic*) To state a variant of corollary 1.2 in characteristic $q > 0$, $q \neq p$, one has to take care of the fact that Samuel's theorem [Sa66], which is to play the part of the Mordell conjecture, involves a certain non-isotriviality assumption. For more details, see the subsequent paper [CT08].

Remark 4.9. For a prime p , a field k of characteristic $q \neq p$, a scheme S of finite type over k and an abelian scheme A over S , say that a character $\chi : \Gamma_k \rightarrow \mathbb{Z}_p^*$ is A/S -non-Tate, if, for any finite extension k' of k and any $s' \in S(k')$, $\chi|_{\Gamma_{k'}}$ does not appear in $T_p(A_{s'})$. By lemma 2.1, a non-Tate character is A/S -non-Tate, a fortiori. Also, the proof of lemma 2.1 shows that χ is A/S -non-Tate if and only if, for any finitely generated extension k' of k and any $s' \in S(k')$, $\chi|_{\Gamma_{k'}}$ does not appear in $T_p(A_{s'})$. Now, in the statement of corollary 1.2, replace the assumption that χ is non-Tate by the weaker assumption that χ is A/S -non-Tate. Then the conclusion of corollary 1.2 is still valid, as the proof shows.

5. APPLICATION TO THE 1-DIMENSIONAL MODULAR TOWER CONJECTURE

5.1. G-curves and their stacks.

5.1.1. *Notation.* The main reference for this preliminary section is [BR07, §§1-6]. Let S be a connected scheme. An S -curve of genus g is a smooth, projective, geometrically connected S -scheme of dimension 1 the geometric fibers of which have genus g .

Given a finite group G of order prime to the characteristics of S , an S - G -curve with group G is a pair (Y, α) , where Y is an S -curve and $\alpha : G \hookrightarrow \text{Aut}_S(Y)$ is a group monomorphism. Two S - G -curves (Y_i, α_i) , $i = 1, 2$ with the same group G are S - G -isomorphic if there exists an S -scheme isomorphism $u : Y_1 \rightarrow Y_2$ such that $u\alpha_1(g)u^{-1} = \alpha_2(g)$, $g \in G$. An S - G -cover with group G is a pair $(f : Y \rightarrow X, \alpha)$, where $f : Y \rightarrow X$ is a Galois cover of S -curves and $\alpha : G \rightarrow \text{Aut}_X(Y)$ is a group isomorphism. Two S - G -covers $(Y_i \rightarrow X, \alpha_i)$, $i = 1, 2$ of a given S -curve $X \rightarrow S$ with the same group G are S - G -isomorphic if there exists an X -scheme isomorphism $u : Y_1 \rightarrow Y_2$ such that $u\alpha_1(g)u^{-1} = \alpha_2(g)$, $g \in G$. Two S - G -covers $(Y_i \rightarrow X_i, \alpha_i)$, $i = 1, 2$ with the same group G are weakly S - G -isomorphic if there exists an S -scheme isomorphism $v : X_1 \rightarrow X_2$ such that the S - G -covers $(v \circ f_1 : Y_1 \rightarrow X_2, \alpha_1)$ and $(f_2 : Y_2 \rightarrow X_2, \alpha_2)$ are S - G -isomorphic. The groupoid of S - G -curves with group G and S - G -isomorphisms is then equivalent to the groupoid of S - G -covers with group G and weak S - G -isomorphisms. In the following, we will drop the α in our notation though it remains part of the data.

Fix a finite group G and denote by $\mathcal{C}(G)$ the set of conjugacy classes of G . Given an element $C \in \mathcal{C}(G)$, write $o(C)$ for the common order of the elements in C . Define also two maps

$$\begin{aligned} \text{deg} : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))} &\rightarrow \mathbb{Z}_{\geq 0}, \mathbf{C} \mapsto \sum_{C \in \mathcal{C}(G)} \mathbf{C}(C), \\ \delta \text{eg} : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))} &\rightarrow \mathbb{Z}_{\geq 0}, \mathbf{C} \mapsto \sum_{C \in \mathcal{C}(G)} \mathbf{C}(C) \frac{|G|}{o(C)}, \end{aligned}$$

and, for any integer $g \geq 0$, a map

$$\gamma_g : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))} \rightarrow \frac{1}{2}\mathbb{Z}, \mathbf{C} \mapsto 1 + |G|(g-1) + \frac{1}{2}(|G|\text{deg}(\mathbf{C}) - \delta \text{eg}(\mathbf{C})).$$

Given any integer $g \geq 0$ and $\mathbf{C} \in \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$ such that $2 - 2g - \text{deg}(\mathbf{C}) < 0$, one can consider the category fibered in groupoids $\mathcal{H}_{g,G,\mathbf{C}} \rightarrow \text{Spec}(\mathbb{Z}[\frac{1}{|G|}])$ of genus $\gamma_g(\mathbf{C})$ G -curves Y with group G such that the resulting G -cover $Y \rightarrow Y/G$ has inertia canonical invariant \mathbf{C} . (In case $\gamma_g(\mathbf{C}) \notin \mathbb{Z}_{\geq 0}$, set $\mathcal{H}_{g,G,\mathbf{C}} = \emptyset$.) Equivalently, $\mathcal{H}_{g,G,\mathbf{C}} \rightarrow \text{Spec}(\mathbb{Z}[\frac{1}{|G|}])$ is the category fibered in groupoids of G -covers of genus g curves with group G and inertia canonical invariant \mathbf{C} . (More precisely, the genus g curve is assumed to be equipped with an étale divisor of degree $\text{deg}(\mathbf{C})$ and the inertia canonical invariant \mathbf{C}

is assumed to be one for the points on this etale divisor. This precision is crucial when $\mathbf{C}(\{1\}) > 0$. Finally, for any integer $r \geq 0$ such that $2 - 2g - r < 0$, write $\mathcal{H}_{g,G,r}$ for the disjoint union of the $\mathcal{H}_{g,G,\mathbf{C}}$, $\deg(\mathbf{C}) = r$.

Proposition 5.1. *$\mathcal{H}_{g,G,\mathbf{C}}$ is a Deligne-Mumford stack (with finite diagonal), smooth and of finite type over $\mathbb{Z}[\frac{1}{|G|}]$. Its coarse moduli space $\mathbb{H}_{g,G,\mathbf{C}}$ is a scheme normal and of finite type over $\mathbb{Z}[\frac{1}{|G|}]$.*

5.1.2. *Functoriality properties.* Any group epimorphism $p : \tilde{E} \twoheadrightarrow E$ defines a morphism of monoids $\mu : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(\tilde{E}))} \rightarrow \mathbb{Z}_{\geq 0}^{(\mathcal{C}(E))}$, sending $\tilde{C} \in \mathcal{C}(\tilde{E})$ to $p(\tilde{C}) \in \mathcal{C}(E)$. Observe that $\deg(\mu(\tilde{\mathbf{C}})) = \deg(\tilde{\mathbf{C}})$. Geometrically, if $\tilde{f} : Y \rightarrow X$ is a G -cover with group \tilde{E} and inertia canonical invariant $\tilde{\mathbf{C}}$ then $\mu(\tilde{\mathbf{C}})$ is the inertia canonical invariant of the G -cover $Y/\ker(p) \rightarrow X$ with group E .

Similarly, any group monomorphism $i : G \hookrightarrow \tilde{E}$ defines a morphism of monoids $\nu : \mathbb{Z}_{\geq 0}^{(\mathcal{C}(\tilde{E}))} \rightarrow \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$ as follows. Consider the canonical map $c_G : G \rightarrow \mathcal{C}(G)$, sending $g \in G$ to its conjugacy class $c_G(g)$ in G and let $s : \mathcal{C}(G) \hookrightarrow G$, $C \mapsto s(C)$ a section of it. Then ν sends $\tilde{C} \in \mathcal{C}(\tilde{E})$ to $\sum_{\tilde{e} \in G \setminus \tilde{E}/(s(\tilde{C}))} c_G(\tilde{e}s(\tilde{C}) \frac{o(\tilde{C})}{|Gn_{\tilde{e}}(s(\tilde{C}))\tilde{e}^{-1}} \tilde{e}^{-1}) \in \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$. Observe that $\deg(\tilde{\mathbf{C}}) \leq \deg(\nu(\tilde{\mathbf{C}})) \leq [\tilde{E} : G]\deg(\tilde{\mathbf{C}})$.

Geometrically, if $\tilde{f} : Y \rightarrow X$ is a G -cover with group \tilde{E} and inertia canonical invariant $\tilde{\mathbf{C}}$ then $\nu(\tilde{\mathbf{C}})$ is the inertia canonical invariant of the G -cover $Y \rightarrow Y/G$ with group G .

In the following, given an extension of finite groups $1 \rightarrow K \rightarrow F \xrightarrow{p} Q \rightarrow 1$, we will say that Q acts trivially on K if there exists a set-theoretic section $s : Q \rightarrow F$ of p such that $s(q)ks(q)^{-1} = k$, $q \in Q$, $k \in K$. (This is equivalent to requiring the canonical representation $Q \rightarrow \text{Out}(K)$ be trivial.) Then note that for an extension of finite groups $1 \rightarrow G \xrightarrow{i} \tilde{E} \xrightarrow{p} E \rightarrow 1$ where E acts trivially on G and for any $\tilde{C} \in \mathcal{C}(\tilde{E})$, the set $\{g^{o(p(\tilde{C}))} \mid g \in \tilde{C}\}$ forms a conjugacy class of G , which we denote by $\tilde{C}^{o(p(\tilde{C}))}$. Then $\nu(\tilde{C})$ is simply $\frac{|E|}{o(p(\tilde{C}))} \tilde{C}^{o(p(\tilde{C}))} \in \mathbb{Z}_{\geq 0}^{(\mathcal{C}(G))}$.

Proposition 5.2. *With the above notation, one has:*

- (1) μ induces a stack morphism $\mathcal{H}_{g,\tilde{E},\tilde{\mathbf{C}}} \rightarrow \mathcal{H}_{g,E,\mu(\tilde{\mathbf{C}})} \times_{\mathbb{Z}[\frac{1}{|E|}]} \mathbb{Z}[\frac{1}{|E|}]$ (again denoted by μ), which is proper and etale³ (but representable if and only if $Z(\tilde{E})$ injects in $Z(E)$). In particular, μ sends each connected component surjectively onto a connected component. Further, μ induces a stack morphism $\mathcal{H}_{g,\tilde{E},\tilde{r}} \rightarrow \mathcal{H}_{g,E,\tilde{r}} \times_{\mathbb{Z}[\frac{1}{|E|}]} \mathbb{Z}[\frac{1}{|E|}]$ (again denoted by μ).
- (2) ν induces a stack morphism $\mathcal{H}_{g,\tilde{E},\tilde{\mathbf{C}}} \rightarrow \mathcal{H}_{\gamma_g(\mu(\tilde{\mathbf{C}})),G,\nu(\tilde{\mathbf{C}})} \times_{\mathbb{Z}[\frac{1}{|G|}]} \mathbb{Z}[\frac{1}{|E|}]$ (again denoted by ν), which is representable, finite and unramified. Further, ν induces a stack morphism $\mathcal{H}_{g,\tilde{E},\tilde{r}} \rightarrow \prod_{\tilde{r} \leq r \leq [\tilde{E}:G]\tilde{r}} \mathcal{H}_{g,G,r} \times_{\mathbb{Z}[\frac{1}{|G|}]} \mathbb{Z}[\frac{1}{|E|}]$ (again denoted by ν).

Proof. Standard and similar to [BR07, Prop. 6.14]. \square

In particular, corresponding to $p : G \twoheadrightarrow \{1\}$ and $i : \{1\} \hookrightarrow G$, one gets a canonical morphisms $\mu : \mathcal{H}_{g,G,\mathbf{C}} \rightarrow \mathcal{M}_{g,|\deg(\mathbf{C})|} \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}]$ and $\nu : \mathcal{H}_{g,G,\mathbf{C}} \rightarrow \mathcal{M}_{\gamma_g(\mathbf{C}),[\deg(\mathbf{C})]} \times_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|G|}]$ respectively, where $\mathcal{M}_{g,[r]}$ denotes the stack of genus g proper smooth curves equipped with degree r etale divisors. (In case $g \notin \mathbb{Z}_{\geq 0}$, set $\mathcal{M}_{g,[r]} = \emptyset$.) From the former morphism, one obtains in particular that $\dim(\mathbb{H}_{g,G,r}) = 3g - 3 + r$, unless $\mathbb{H}_{g,G,r} = \emptyset$. Thus, the only cases when $\mathbb{H}_{g,G,r}$ is a curve correspond to $(g, r) = (0, 4), (1, 1)$.

5.2. The representability problem. So far, the base scheme has been (an open subscheme of) the spectrum of \mathbb{Z} . From now on, we take the spectrum of a field k of characteristic ≥ 0 as a base scheme. In particular, various moduli stacks are considered to be ones over k .

³For the notion of (not necessarily representable) proper and etale morphisms between Deligne-Mumford stacks, see [DM69, §4].

5.2.1. *Automorphism group of objects in $\mathcal{H}_{g,G,r}$.* Throughout this subsection, let $f : Y \rightarrow X$ be a G -cover corresponding to a geometric point $x : \text{Spec}(\Omega) \rightarrow \mathcal{H}_{g,G,r}$, where Ω is a separably closed field. For each $v \in \text{Aut}(X)$, $v \circ f$ gives another G -cover $Y \rightarrow X$ with group G . So, let E_f denote the stabilizer of the G -isomorphism class of f in $\text{Aut}(X)$ and refer to it as the *base group* of f . Set $B := X/E_f$ and consider the resulting cover $\tilde{f} : Y \xrightarrow{f} X \xrightarrow{q} B$. Then, by the definition of E_f , one sees that \tilde{f} is a G -cover, say, with group \tilde{E}_f . Let \overline{R}_f be the image in B of the r marked points in X (given as a part of data of $x : \text{Spec}(\Omega) \rightarrow \mathcal{H}_{g,G,r}$), and R_q the ramification locus of the G -cover $q : X \rightarrow B$ in B . Moreover, define R to be the union of \overline{R}_f and R_q in B . If we define r_B (resp. \overline{r}_f , resp. r_q) to be the cardinality of R (resp. \overline{R}_f , resp. R_q), we have a trivial estimate: $r_B \leq \overline{r}_f + r_q \leq r + r_q$. We shall regard R, \overline{R}_f, R_q as reduced closed subschemes of B .

There exists a natural exact sequence $1 \rightarrow G \rightarrow \tilde{E}_f \xrightarrow{p} E_f \rightarrow 1$, where E_f acts trivially on G . Indeed, by the definition of E_f , any $e \in E_f$ admits a lift ϵ of e to Y which is compatible with $\text{Aut}(f) \xrightarrow{\alpha^{-1}} G$ that is, $\alpha^{-1}(\epsilon g \epsilon^{-1}) = \alpha^{-1}(g)$, $g \in \text{Aut}(f)$ and, hence, $\epsilon g \epsilon^{-1} = g$, $g \in G$. In other words, \tilde{E}_f can be identified canonically with the subgroup of $\text{Aut}(Y)$ generated by G and the centralizer of G in $\text{Aut}(Y)$.

Since $\mathcal{H}_{g,G,r}$ are Deligne-Mumford stacks, the representability condition can be explicitly stated in terms of the automorphism groups of the geometric objects.

Lemma 5.3. *A nonempty open substack $\mathcal{U} \hookrightarrow \mathcal{H}_{g,G,r}$ is representable if and only if G has trivial center and, for any geometric point $f : Y \rightarrow X$ of \mathcal{U} , the corresponding E_f is trivial.*

In general, $\mathcal{H}_{g,G,r}$ is not representable, and the (rational) points on the coarse moduli space $H_{g,G,r}$ are not in one-to-one correspondence with the geometric objects classified by $\mathcal{H}_{g,G,r}$. However, we can associate each rational point with a certain geometric object, as follows.

Let K be a field (containing k), and assume that $\Omega = K^{sep}$. Assume, moreover, that the K^{sep} -rational point of $H_{g,G,r}$ defined by x is K -rational.

Lemma 5.4. (1) $(B; \overline{R}_f, R_q, R)$ admits a natural model $(B_K; \overline{R}_{f,K}, R_{q,K}, R_K)$ defined over K .
 (2) B, X and Y are Galois over B_K . (Note that $\text{Aut}(B/B_K) = \Gamma_K$.) In particular, if we set $\tilde{E}_{f,K} := \text{Aut}(Y/B_K)$ and $E_{f,K} := \text{Aut}(X/B_K)$, we obtain the following natural commutative diagram:

$$\begin{array}{ccc} \rho : \tilde{E}_{f,K} & \rightarrow & \text{Aut}(G) \\ & \downarrow & \downarrow \\ \overline{\rho} : E_{f,K} & \rightarrow & \text{Out}(G) \end{array}$$

via conjugation.

(3) In fact, $\overline{\rho}$ is trivial.

Proof. (1) The proof of the existence of a natural model B_K of B is just similar to [DE99, proof of Th. 3.1] (cf. Rem. 3.2(b), *loc.cit.*) and done via descent theory. The descent of \overline{R}_f, R_q and R is automatically done at the same time.

(2) $B = B_K \times_K K^{sep}$ is clearly Galois over B_K with Galois group Γ_K . The assertion that X and Y are Galois over B_K follows essentially from the fact that K is the field of moduli of f as a G -cover.

(3) Again, this follows essentially from the fact that K is the field of moduli of f as a G -cover. \square

We shall refer to B_K (resp. $(B_K; \overline{R}_{f,K}, R_{q,K}, R_K)$) as the canonical model of B (resp. $(B; \overline{R}_f, R_q, R)$) with respect to the G -cover f . These models have the following functorial property:

Lemma 5.5. *For each $i = 0, 1$, let $f_i : Y_i \rightarrow X$ be a G -cover with group G_i (corresponding to a K^{sep} -rational point on $H_{g,G_i,r}$), and assume that we are given an X -morphism $g : Y_1 \rightarrow Y_0$. (Observe that then g naturally induces a surjection $G_1 \rightarrow G_0$ compatible with g .) Moreover, set $B_i := X/E_{f_i}$ and let $B_{i,K}$ be the canonical model of B_i with respect to the G -cover f_i . Then we have an inclusion $E_{f_1} \subset E_{f_0}$, and the natural K^{sep} -morphism $B_1 \rightarrow B_0$ induced by this inclusion descends to a K -morphism $B_{1,K} \rightarrow B_{0,K}$.*

If, moreover, $E_{f_1} = E_{f_0}$, we have $B_{1,K} = B_{0,K}$.

Proof. Immediate from the various definitions. \square

5.2.2. *The abelian case.* We continue the investigation in the previous subsection in the case that G is abelian.

Lemma 5.6. *In the situation of lemma 5.4, assume moreover that G is abelian. In the G -cover $\tilde{f} : Y \rightarrow B$ with group \tilde{E}_f , consider the subcover $Y^{ab} \rightarrow B$ corresponding to the quotient $\tilde{E}_f \rightarrow (\tilde{E}_f)^{ab}$. Moreover, for each positive integer b , consider the subcover $Z_b \rightarrow B$ of $Y^{ab} \rightarrow B$ corresponding to the quotient $(\tilde{E}_f)^{ab} \rightarrow (\tilde{E}_f)^{ab}/(\tilde{E}_f)^{ab}[b] =: N_b$. Set $e := |E_f|$ and define e' to be the exponent of the abelian group $(E_f)^{ab}$. Then:*

- (1) N_b admits a subquotient isomorphic to $G/G[eb]$.
- (2) If b is divisible by e' , then N_b is isomorphic to a subquotient of G and the natural action of Γ_K on N_b (via conjugation) is trivial.
- (3) If b is divisible by e , the ramification locus in B of the G -cover $Z_b \rightarrow B$ is contained in \overline{R}_f .
- (4) Assume, moreover, that $|G|$ is a power of a prime p and that the image of the cyclotomic character $\chi : \Gamma_K \rightarrow \mathbb{Z}_p^*$ has finite index c in \mathbb{Z}_p^* . Set $\nu := \nu(p, c, r) := \lceil \log_p(cr p / (p - 1)) \rceil$. Then, if b is divisible by both e and $p^\nu e'$, then the G -cover $Z_b \rightarrow B$ is unramified everywhere.

Proof. By applying the Hochschild-Serre spectral sequence to the exact sequence

$$1 \rightarrow G \rightarrow \tilde{E}_f \rightarrow E_f \rightarrow 1$$

of finite groups, one gets a natural exact sequence

$$0 \rightarrow H^1(E_f, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(\tilde{E}_f, \mathbb{Q}/\mathbb{Z}) \rightarrow H^1(G, \mathbb{Q}/\mathbb{Z})^{E_f} \rightarrow H^2(E_f, \mathbb{Q}/\mathbb{Z}),$$

of finite abelian groups. By the duality between abelian groups M and their character groups $M^\vee := \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$ (or, equivalently, by the Pontryagin duality), one gets a natural exact sequence

$$H^2(E_f, \mathbb{Q}/\mathbb{Z})^\vee \rightarrow G \rightarrow (\tilde{E}_f)^{ab} \rightarrow (E_f)^{ab} \rightarrow 0.$$

(Here, we have used the fact that G is abelian and that the action of E_f on G is trivial.) Thus, for the natural morphism $G \rightarrow (\tilde{E}_f)^{ab}$, the cokernel is killed by e' and the kernel is killed by e .

(1) Let \overline{G} denote the image of G in $(\tilde{E}_f)^{ab}$. By the preceding argument, the quotient \overline{G} is bigger than $G/G[e]$, that is, we have $\overline{G} \twoheadrightarrow G/G[e]$ as quotients of G . Now,

$$N_b = (\tilde{E}_f)^{ab}/(\tilde{E}_f)^{ab}[b] \supset \overline{G}/\overline{G}[b] \twoheadrightarrow (G/G[e])/(G/G[e])[b] = G/G[eb],$$

as desired.

(2) By the preceding argument, if b is divisible e' , we have $b(\tilde{E}_f)^{ab} \subset \overline{G} \leftarrow G$. On the other hand, the b -multiplication map induces a natural isomorphism $N_b = (\tilde{E}_f)^{ab}/(\tilde{E}_f)^{ab}[b] \simeq b(\tilde{E}_f)^{ab}$. Thus, the first assertion follows. Since all the construction is natural and Γ_K -equivariant, the second assertion follows from lemma 5.4(2)(3).

(3) The ramification locus in X of the G -cover $f : Y \rightarrow X$ with group G is contained in R_f . Thus, for the G -cover $\tilde{f} : Y \rightarrow B$ with group \tilde{E}_f , the inertia group $I \subset \tilde{E}_f$ at any point of $B \setminus \overline{R}_f$ is injectively mapped into E_f . In particular, any element of such I is killed by $e = |E_f|$. So, if we denote the image of I in $(\tilde{E}_f)^{ab}$ by \overline{I} , we have $\overline{I} \subset (\tilde{E}_f)^{ab}[e]$, as desired.

(4) Take any (closed) point P in $\overline{R}_{f,K} \subset B_K$. Then we have $[K(P) : K] \leq \overline{r}_f \leq r$, hence

$$[\mathbb{Z}_p^* : \chi(\Gamma_{K(P)})] = [\mathbb{Z}_p^* : \chi(\Gamma_K)][\chi(\Gamma_K) : \chi(\Gamma_{K(P)})] \leq [\mathbb{Z}_p^* : \chi(\Gamma_K)][K : K(P)] \leq cr.$$

Suppose, moreover, that $\chi(\text{mod } p^n) : \Gamma_{K(P)} \rightarrow (\mathbb{Z}/p^n)^*$ is trivial for some $n > 0$. Then we have

$$cr \geq [\mathbb{Z}_p^* : \chi(\Gamma_{K(P)})] \geq |(\mathbb{Z}/p^n)^*| = (p-1)p^{n-1},$$

hence $n \leq \nu$. (Observe that this last estimate is also available for $n = 0$.)

Now, consider the inertia group $I_b \subset N_b$ at P for the abelian G -cover $Z_b \rightarrow X$. (In particular, $I_1 \subset N_1 = (\bar{E}_f)^{ab}$.) I_b admits a natural action of $\Gamma_{K(P)}$, which is via the cyclotomic character. So, by (2), the $\Gamma_{K(P)}$ -action on $I_{e'}$ is via the p -adic cyclotomic character on the one hand and trivial on the other hand. By the preceding argument, this implies that $I_{e'}$ is killed by p^ν , hence I_1 is killed by $e'p^\nu$. This, together with (3), proves the assertion. \square

Corollary 5.7. *In the situation of lemma 5.6, fix a prime p and assume that the image of the cyclotomic character $\chi : \Gamma_K \rightarrow \mathbb{Z}_p^*$ has finite index c in \mathbb{Z}_p^* . Then there exist integers $\beta := \beta(p, c, r, e) \geq 1$ and $\delta := \delta(p, c, r, e) \geq 0$, such that, if G is cyclic of order p^m , then N_β is cyclic of order p^μ with $\mu \geq m - \delta$; $Z_\beta \rightarrow B$ is a G -cover with group N_β and unramified everywhere; Z_β is Galois over B_K ; and the natural action of Γ_K on N_β is trivial.*

Proof. This is just summarizing (a part of) lemma 5.6. More specifically, set $\beta := p^\nu e$ and define δ to be the largest integer that satisfies $p^\delta | e\beta$. Then, N_β is isomorphic to a subquotient of G by (2), hence it is a cyclic p -group. So, set $|N_\beta| = p^\mu$. N_β admits a subquotient isomorphic to $G/G[e\beta] = G/G[p^\delta]$ by (1), hence $\mu \geq m - \delta$. $Z_\beta \rightarrow B$ is a G -cover with group N_β by definition, and it is unramified everywhere by (4). Z_β is Galois over B_K by construction, and, finally, the natural action of Γ_K on N_β is trivial by (2). \square

5.3. The 1-dimensional modular tower conjecture.

5.3.1. *Statements.* For Fried's modular towers and related conjectures, see [Fr95], [FK97], [D06]. Let p be a prime, and (g, r) a pair of non-negative integers with $2 - 2g - r < 0$. Let $\mathbf{G} = \{G_{n+1} \rightarrow G_n\}_{n \geq 0}$ be a projective system of finite groups and assume that $G := \varprojlim G_n$ is p -obstructed in the sense that G contains an open subgroup that admits a quotient isomorphic to \mathbb{Z}_p . Denote by Σ_G the set of prime numbers which divide the order of (some finite quotient of) G . By [C07, Cor. 3.6] (see also [BF02], [K04]), one has:

Lemma 5.8. *For any field k finitely generated over the prime field of characteristic $q \notin \Sigma_G$ (hence, in particular, $q \neq p$),*

$$\varprojlim H_{g, G_n, r}(k) = \emptyset.$$

Now, a generalized variant of Fried's modular tower conjecture is the following ⁴.

Conjecture 5.9. *For any field k finitely generated over the prime field of characteristic $q \notin \Sigma_G$, there exists an integer $N := N(p, k, g, \mathbf{G}, r)$ such that $H_{g, G_n, r}(k) = \emptyset$, $n \geq N$.*

We formulate the d -dimensional version of this conjecture in characteristic $q \geq 0$, as follows.

(MT _{d} (q)) For any field k finitely generated over the prime field of characteristic $q \notin \Sigma_G$, of characteristic 0, any k -scheme S of finite type with $\dim(S) \leq d$ and any k -morphism $\xi : S \rightarrow H_{g, G_0, r}$, there exists an integer $N := N(p, k, g, \mathbf{G}, r, S, \xi)$ such that $S_n(k) = \emptyset$, $n \geq N$. Here, we set $S_n := S \times_{H_{g, G_0, r}} H_{g, G_n, r}$.

First, we have the following corollary of lemma 5.8:

Corollary 5.10. *(MT₀(q)) holds for all $q \geq 0$. In particular, conjecture 5.9 holds for $(g, r) = (0, 3)$.*

Proof. Since S_n is of finite type over k and 0-dimensional, $S_n(k)$ is a finite set for any $n \geq 0$. Thus, the assertion follows from lemma 5.8. \square

Now, the main result in this section is:

Theorem 5.11. *(MT₁(0)) holds. In particular, conjecture 5.9 holds for $(g, r) = (0, 4), (1, 1)$ and $q = 0$.*

⁴This is a weak version in the sense that p is fixed. For the formulation of a version of the modular tower conjecture which allows p to vary, see [Fr06, §6]

5.3.2. *Proof.* To clarify the structure of the proof of theorem 5.11, we shall also introduce the d -dimensional generalization of corollary 1.2 in characteristic $q \geq 0$:

(UB $_d(q)$) Let k be a field finitely generated over the prime field of characteristic q , S a scheme of finite type over k with $\dim(S) \leq d$, and A an abelian scheme over S . Let p be a prime $\neq q$ and $\chi : \Gamma_k \rightarrow \mathbb{Z}_p^*$ a non-Tate character. Then there exists an integer $N := N(A, S, k, p, \chi)$, such that $A_s[p^\infty](\chi) \subset A_s[p^N]$ holds for any $s \in S(k)$.

Theorem 5.12. *(UB $_d(0)$) holds for $d \leq 1$.*

Proof. First, (UB $_0(q)$) for any $q \geq 0$ follows from the definition of non-Tate characters. (Indeed, if S is of finite type over k and $\dim(S) = 0$, $S(k)$ is a finite set. So, we may treat only finitely many abelian varieties A_s ($s \in S(k)$.) Next, to prove (UB $_1(0)$), assume that S is of finite type over k and $\dim(S) = 1$. By replacing S by S^{red} , we may assume that S is reduced. By (UB $_0(0)$), we may replace S by an open dense subscheme freely. So, we may assume that S is regular and separated, hence smooth and separated over k . Finally, treating S componentwise, we may assume that S is connected. Suppose that S is not geometrically connected. Then $S(k) = \emptyset$ and there remains nothing to prove. Thus, in summary, we may assume that S is a smooth, separated, geometrically connected curve over k . Now, the assertion follows from corollary 1.2. \square

Now, theorem 5.11 formally follows from corollary 5.10, theorem 5.12 and the following (for $d = 1$ and $q = 0$):

Proposition 5.13. *Assume that (UB $_d(q)$) and (MT $_{d-1}(q)$) hold. Then (MT $_d(q)$) holds.*

Proof. Denote by $G(n)$ the kernel of $G \rightarrow G_n$. Since $G = \varprojlim G_n$ is p -obstructed, there exists an open subgroup U of G which admits a quotient isomorphic to \mathbb{Z}_p . For some $n_0 \geq 0$, $G(n_0)$ is contained in U . As $G(n_0)$ is open in U , its image in the quotient is open, hence $G(n_0)$ also admits a quotient Z isomorphic to \mathbb{Z}_p . For each $n \geq 0$, denote by $Z(n)$ the image of $G(n + n_0)$ in Z , and set $Z_n := Z/Z(n)$. Thus, $Z_n \simeq \mathbb{Z}/p^{m(n)}\mathbb{Z}$, where $0 \leq m(0) \leq m(1) \leq \dots \leq m(n) \leq \dots \rightarrow \infty$. Now, by applying the functoriality properties in 5.1.2 (especially, proposition 5.2) to

$$G_{n+n_0} = G/G(n + n_0) \leftarrow G(n_0)/G(n + n_0) \rightarrow Z/Z(n) = Z_n,$$

we may reduce the problem to the case where G is isomorphic to \mathbb{Z}_p and G_n is isomorphic to $\mathbb{Z}/p^{m(n)}$.

As in the assertion of (MT $_d(q)$), let S be a scheme of finite type over k and $\dim(S) \leq d$. Let S'_n be the closure of $S_n(k)$ in S_n . Denote by \mathcal{I}_n the (finite) set of irreducible components of S'_n of dimension d . Since $S_{n+1} \rightarrow S_n$ is finite, $\{\mathcal{I}_n\}_{n \geq 0}$ forms a projective system. Thus, one gets the following dichotomy: either $\mathcal{I}_n = \emptyset$ for $n \gg 0$ or $\varprojlim \mathcal{I}_n \neq \emptyset$. The first case can be reduced easily to (MT $_{d-1}(q)$). In the second case, take an element of $\varprojlim \mathcal{I}_n \neq \emptyset$, and, for each $n \geq 0$, let $T_n \subset S'_n \subset S_n$ be the corresponding irreducible component of dimension d , which is regarded as a reduced closed subscheme. (Observe that $T_n(k)$ is dense in T_n .) Let F_n denote the function field of the integral scheme T_n and K_n the perfect closure of F_n . (That is, $K_n = F_n$ for $q = 0$ and $K_n = \cup_{s \geq 0} F_n^{q^{-s}}$ for $q > 0$.) Set $F := F_0$.

First, we claim that the natural map $\Gamma_{K_n} \rightarrow \Gamma_k$ is surjective, which is equivalent to saying that the natural map $\Gamma_{F_n} \rightarrow \Gamma_k$ is surjective. Indeed, for the latter surjectivity, take a nonempty open subscheme U_n of T_n which is normal. Then $\Gamma_{F_n} \rightarrow \Gamma_k$ factors through $\Gamma_{F_n} \rightarrow \pi_1(U_n)$. So, it suffices to prove that $\pi_1(U_n) \rightarrow \Gamma_k$ is surjective. But this is clear, as $U_n(k) \neq \emptyset$. In particular, since the cyclotomic character $\chi : \Gamma_{F_n} \rightarrow \mathbb{Z}_p^*$ is the composite of $\Gamma_{F_n} \rightarrow \Gamma_k$ and the cyclotomic character $\Gamma_k \rightarrow \mathbb{Z}_p^*$, the index c of $\chi(\Gamma_{F_n})$ in \mathbb{Z}_p^* is independent of n .

Next, as $T_{n+1} \rightarrow T_n$ is a finite morphism between d -dimensional schemes, we may regard F_{n+1} as a finite extension of F_n . So, we may take an algebraic closure $\Omega = \overline{F}$ of F , so that $F_n \subset \Omega$. Observe that $\Omega = \overline{K_n} = K_n^{sep}$ for each $n \geq 0$. Thus, S_n admits a natural Ω -rational point, which induces $(x_n)_{n \geq 0} \in \varprojlim H_{g, G_n, r}(\Omega)$. Let $f_n : Y_n \rightarrow X_n$ be a G -cover with group G_n over Ω corresponding to x_n . (Note that Ω is not only separably closed but also algebraically closed. Thus, by the very definition

of coarse space, such f_n exists.) Since $(x_n)_{n \geq 0}$ is an element of the projective limit, we may assume that $X_n = X_0 =: X$ and that $f_n : Y_n \rightarrow \bar{X}$ is a subcover of $f_{n+1} : Y_{n+1} \rightarrow X$ (compatibly with $G_{n+1} \rightarrow G_n$). Let $E_{f_n} \subset \text{Aut}(X)$ be the base group for f_n . As $E_{f_0} \supset E_{f_1} \supset \cdots$ are inclusions of finite groups, there exists $n_1 \geq 0$ such that $E_{f_n} = E_{f_{n_1}}$ for $n \geq n_1$. So, by renumbering if necessary, we may assume that $E_{f_n} = E_{f_0} =: E$. Let B_{n, K_n} be the canonical model over K_n of $B = X/E$ with respect to f_n , as in lemma 5.4. Then, by lemma 5.5, we may identify B_{n, K_n} with $B_{0, K_0} \times_{K_0} K_n$. So, we shall write $B_{n, K_n} = B_{K_n}$.

Let $1 \rightarrow G_n \rightarrow \tilde{E}_{f_n} \rightarrow E \rightarrow 1$ be the exact sequence as in 5.2. In the G-cover $\tilde{f}_n : Y_n \rightarrow B$ with group \tilde{E}_{f_n} , consider the subcover $Y_n^{ab} \rightarrow B$ corresponding to the quotient $\tilde{E}_{f_n} \rightarrow (\tilde{E}_{f_n})^{ab}$. Moreover, for each positive integer b , consider the subcover $Z_{n,b} \rightarrow B$ of $Y_n^{ab} \rightarrow B$ corresponding to the quotient $(\tilde{E}_{f_n})^{ab} \rightarrow (\tilde{E}_{f_n})^{ab}/(\tilde{E}_{f_n})^{ab}[b] =: N_{n,b}$.

Now, take $\beta = \beta(p, c, r, e) \geq 1$ and $\delta = \delta(p, c, r, e) \geq 0$ as in corollary 5.7. Then, since G_n is a cyclic group of order $p^{m(n)}$, $N_{n,\beta}$ is cyclic of order $p^{\mu(n)}$ with $\mu(n) \geq m(n) - \delta$; $Z_{n,\beta} \rightarrow B$ is a G-cover with group $N_{n,\beta}$ and unramified everywhere; $Z_{n,\beta}$ is Galois over B_{K_n} ; and the natural action of Γ_{K_n} on $N_{n,\beta}$ is trivial. The natural surjection $\tilde{E}_{f_{n+1}} \twoheadrightarrow \tilde{E}_{f_n}$ induces a surjection $N_{n+1,\beta} \twoheadrightarrow N_{n,\beta}$. Moreover, we may consider $Z_{n,\beta} \rightarrow B$ as a subcover of $Z_{n+1,\beta} \rightarrow B$, compatibly with the latter surjection.

There exist a finite extension F' of F included in K_0 and a proper, smooth, geometrically connected curve $B_{F'}$ over F' , such that $B_{F'} \times_{F'} K_0$ is K_0 -isomorphic to B_{K_0} . When $q = 0$, we have $K_0 = F' = F$. when $q > 0$, we may assume that $F' = F^{q^{-s}}$ for some $s \geq 0$, up to replacing F' by an extension. Now, set $k' := k$ for $q = 0$ and $k' := k^{q^{-s}}$ for $q > 0$. Also, set $F'_n := F_n$ for $q = 0$ and $F'_n := F_n^{q^{-s}}$ for $q > 0$.

When $q = 0$, set $T'_n := T_n$, and, when $q > 0$, set $T'_n := T_n^{q^{-s}}$. Note that T'_n is an integral scheme of finite type over k' with function field F'_n . There exists a nonempty open subscheme U of T'_0 such that $B_{F'} \rightarrow F'$ extends to a proper, smooth, geometrically connected scheme $B_U \rightarrow U$. Let $J := \text{Pic}_{B_U/U}^0$ be the jacobian. Now, applying (UB_d(q)) to the abelian scheme J over U and the cyclotomic character $\chi : \Gamma_{k'} \rightarrow \mathbb{Z}_p^*$, we see that there exists an integer $N := N(J, U, k', p, \chi)$, such that $J_s[p^\infty](\chi) \subset J_s[p^N]$ holds for any $s \in U(k')$.

We have a surjection $T_p(J_{F'}) \twoheadrightarrow N_{n,\beta} \simeq \mathbb{Z}/p^{\mu(n)}$ as modules over $\Gamma_{F'_n} = \Gamma_{K_n}$, hence, by duality, an injection $\mathbb{Z}/p^{\mu(n)}(1) \simeq (N_{n,\beta})^\vee \hookrightarrow J_{F'}[p^\infty]$ as $\Gamma_{F'_n}$ -modules. Since $m(n) \rightarrow \infty$ as $n \rightarrow \infty$, there exists an n_0 such that $\mu(n_0) > N$. Take a nonempty, open, normal subscheme V_{n_0} of $\pi^{-1}(U)$, where $\pi : T'_{n_0} \rightarrow T'_0$ denotes the natural morphism induced by $T_{n_0} \rightarrow T_0$. Since $J_{V_{n_0}} := J \times_U V_{n_0}$ is an abelian scheme over V_{n_0} , the action of $\Gamma_{F'_{n_0}}$ on $J_{F'}[p^\infty]$ factors through $\Gamma_{F'_n} \twoheadrightarrow \pi_1(V_{n_0})$. Thus, the injection $(N_{n_0,\beta})^\vee \hookrightarrow J_{F'}[p^\infty]$ as $\Gamma_{F'_{n_0}}$ -modules extends to an injection $(N_{n_0,\beta})_{V_{n_0}}^\vee \hookrightarrow J_{V_{n_0}}[p^\infty]$ as (ind-)finite-etale group schemes over V_{n_0} .

Recall that $T_{n_0}(k)$ is dense in T_{n_0} , hence $T'_{n_0}(k')$ is dense in T'_{n_0} . In particular, $V_{n_0}(k') \neq \emptyset$. Take $s_{n_0} \in V_{n_0}(k')$ and let s denote the image of s_{n_0} in U . Considering the fiber at s_{n_0} of the injection $(N_{n_0,\beta})_{V_{n_0}}^\vee \hookrightarrow J_{V_{n_0}}[p^\infty]$, one gets the injection $(N_{n_0,\beta})_{k'}^\vee \hookrightarrow J_s[p^\infty]$. Since $(N_{n_0,\beta})_{k'}^\vee \simeq \mathbb{Z}/p^{\mu(n)}(1)$ and $\mu(n) > N$. This contradicts the choice of N .

Thus, the proof of proposition 5.13 is completed. \square

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