Abstract. Many applications start by describing those curves of form
\[ C_{f,g} \overset{\text{def}}{=} \{(x,y)|f(x) - g(y) = 0\} \]
with infinitely many solutions over localizations of the ring of integers of a number field. Immediately you must address finding \((f,g)\) for which the projective normalization of \(C_{f,g}\) has a genus 0 (or 1) component. The case when \(f\) and \(g\) are polynomials and \(f\) is indecomposable was essentially solved by distinguishing between when the number, \(u\), of components was 1 versus > 1. For \(u = 1\), a direct formula for the genus worked: [Fr73b, (1.6) of Prop. 1]. For \(u > 1\), the result came from solving Schinzel’s problem [Fr73a]: Describe all such components. Here, though, the computation of component genuses was ad hoc.

Recently [Fr12] revisited work that related to [Fr73a] during the intervening years. Pakovich [Pak15] – who tried dropping both the indecomposable and polynomial restrictions – is a further example of that. Yet, he added a telling assumption:

\(^{(*)}{IC}\) That \(C_{f,g}\) is irreducible \((u = 1)\).

Assuming \(^{(*)}{IC}\) he showed – for fixed \(f\) – unless the Galois closure of \(f\) has genus 0 (or 1), the resulting genus grows linearly in the degree of \(g\).

We extend [Fr73b, Prop. 1] in Cor. 3.9 and Cor. 3.10, to when \(u > 1\). Thm. 4.11 extending Pakovich’s result must tend to composition factors of \(g\). We precisely describe for pairs \((f,g)\), as \(g\) varies, when infinitely many \(\tilde{C}_{f,g}\) have a genus 0 component, describing cases when that – in not obvious ways – does happen.

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2010 Mathematics Subject Classification. Primary 141130, 20B15, 20C15, 30F10; Secondary 12D05, 12E30, 12F10, 20E22.

Key words and phrases. Davenport’s Problem, Schinzel’s Problem, factorization of variables separated polynomials, Riemann’s Existence Theorem, imprimitive groups, Pakovich’s Theorem, Nielsen classes.
1. Fiber products of covers by rational functions

We use notation from [Fr12], quoting from it to relax the hypotheses (1.1a) and (1.1b) below. We often use the same language as in [Pak15] as if we are referencing one rational function $f$ at a time. Yet, almost always, results depend only on the Nielsen class (§3.2) of the cover given by $f$.

1.1. Context for our main genus computation. For $z$ a complex variable we denote the projective line uniformized by the $z$ – including $z = \infty$ – by $\mathbb{P}^1_z$. We will consider the set of variables separated equations

$$\mathcal{C}_{f,g} \overset{\text{def}}{=} \{(x,y)|f(x) - g(y) = 0\} \text{ with } f, g \in \mathbb{C}(x), \text{ and } f \text{ fixed.}$$

Regard $\mathcal{C}_{f,g}$ as an affine piece of the fiber product $\mathbb{P}^1_x \times_{\mathbb{P}^1_z} \mathbb{P}^1_y$ of the rational function pair $(f, g)$ of respective degrees $m$ and $n$. Denote by $\tilde{\mathcal{C}}_{f,g}$ the projective normalization of $\mathcal{C}_{f,g}$.

It is a disjoint union of (say, $u$) compact Riemann surfaces. Each has a genus, given by notation like $(g_{f,g})$, $i = 1, \ldots, u$. Each surface inherits canonical maps $\text{pr}_x$ and $\text{pr}_y$, resp., to $\mathbb{P}^1_x$ and $\mathbb{P}^1_y$. Excluding a finite set of points, $\tilde{\mathcal{C}}_{f,g}$ is the fiber product without normalization. With $f \in \mathbb{C}(x)$ fixed, consider

$$\mathcal{R}_f = \{g \in \mathbb{C}(x)|f(x) - g(y) \text{ is reducible}\}.$$

**Theorem 1.1** (Pakovich [Pak15]). Assume also:
(1.1a) the Galois closure of the cover \( \mathbb{P}^1_x \rightarrow \mathbb{P}^1_z \) is not of genus 0 or 1.
(1.1b) we run over all \( g \) not in \( R_f \).

Then, the genus of \( \tilde{C}_{f,g} \) goes to infinity with the degree of \( g \).

Cors. 3.9 and 3.10 (Methods I and II) computes the genus of \( \tilde{C}_{f,g} \) components. Cor. 3.9 is applicable whenever components arise from a fiber product of nonsingular covers. Cor. 3.10 is special for situations for allowing \( g \in R_f \) in (1.1b). [Fr73b, (1.6) of Prop. 1] was the case the fiber product is irreducible. So, this precisely extends adhoc discussions in [Fr73b] and [Fr74].

§A.1 reminds that the Galois closure of an irreducible cover has a natural description as a component of (projective normalization of) fiber products of a cover with itself. You need at most \( \text{deg}(f) \) fiber products. Princ. A.1 produces the minimal number of fiber products required.

Most \( f \)'s failing hypothesis (1.1a) in [Pak15] have Galois closures a component of the 2-fold fiber product of \( f : \mathbb{P}^1_x \rightarrow \mathbb{P}^1_z \) (see §A.2).

Pakovich notes that cases where (1.1a) fails go back to the late 1800s. True, but the condition needs more discussion than [Pak15, p. 2] gave it. For example, §A.2 constructs families of towers of genus 0 components that failure of (1.1a) using Nielsen classes. Those, for a given \( f \), give irreducible, genus 0, \( \tilde{C}_{f,g} \) with \( g \) of arbitrary large degree.

Our examples show what happens to components of \( \tilde{C}_{f,g(g_2)} \) when \( \tilde{C}_{f,g} \) is reducible – (1.1b) fails; especially when it has a genus 0 component – as \( g_2 \in \mathbb{C}(y) \) varies. Thm. 4.11 uses this to extend Thm. 1.1. Thm. 2.9 is a model for the best possible explicitness, relating §2.1 to the genus 0 problem. Here \( f, g \) are in \( \mathbb{C}[x] \) – polynomials – and \( f \) is indecomposable.

App. B considers how often we can expect \( \tilde{C}_{f,g} \) to be reducible and have a genus 0 component. As in all examples, collections of such pairs \( (f, g) \) fall in natural Nielsen classes. App.B.1 produces infinitely many, again shown by applying Cor. 3.10. This example shows the, now proven, (Primitive) Genus 0 Monodromy Conjecture doesn’t suffice for all issues.

1.2. When irreducibility hypothesis (1.1b) fails. We start by characterizing the failure of (1.1b) using Prop. 1.3.

**Definition 1.2.** A cover \( f : X \rightarrow Z \) is indecomposable if it is not composed of two covers, \( f_1 : X \rightarrow W \) and \( f_2 : W \rightarrow Z \), with both degrees exceeding 1. Equivalently, the monodromy group of \( f \) in its natural \( \text{deg}(f) \) permutation representation is primitive [Fr70, Lem. 2]. [Fr12, Thm. 4.5] reviews characterizations of this when

\[
(1.2) \quad f, g \in \mathbb{C}[x] \text{ and } f \text{ is indecomposable.}
\]
§2.1 includes reminders on how to use branch cycles. This is the effective computational tool for producing and computing with Riemann surface covers. The genus, \( g \), of the cover appears immediately from branch cycles a la Riemann-Hurwitz (RH, (3.3)), when the cover is irreducible.

That is why the space of absolute covers in a given Nielsen class (Princ. 3.2) is a generalization of the moduli space of covers of genus \( g \), if the natural permutation representation attached to a cover is transitive.

We must consider reducible fiber products of covers. Say, just to figure when Thm. 1.1, if not extended, would apply (§1.3). §3.3 extends Nielsen classes to drop the often used transitive representation assumption. Our fiber product situation is illustrative.

1.2.1. Characterizing reducibility of \( \tilde{\mathcal{C}}_{f,g} \). For the monodromy group, \( G_f \), of a cover with its (faithful) permutation representation \( T_f : G \to S_{\deg(f)} \), Galois theory associates the cover to the subgroup \( G(T_f,i) \), the stabilizer of any integer, \( 1 \leq i \leq \deg(f) = m \), in the representation.

From Prop. 1.3, [Fr73a, Prop. 2] or [Fr12, Lem. 4.2], the main device to consider reducibility of \( \tilde{\mathcal{C}}_{f,g} \) is the appearance of a group with two (faithful) permutation representations, \( T_1, T_2 \) of \( G \).

Consider (up to equivalence of covers), the collection of pairs,

\[
\mathcal{F}_{f,g} \overset{\text{def}}{=} \left\{ (f^*, g^*) \mid \text{of covers } f^* : X^* \to \mathbb{P}^1_{\mathbb{R}} \text{ (resp. } g^* : Y^* \to \mathbb{P}^1_{\mathbb{R}}) \right\}
\]

through which \( f \) (resp. \( g \)) factors.

**Proposition 1.3.** There exists \( (f^*, g^*) \in \mathcal{F}_{f,g} \) satisfying the following.

(1.3a) Their Galois closures, \( \hat{f}^* : \hat{X}^* \to \mathbb{P}^1_{\mathbb{R}} \) and \( \hat{g}^* : \hat{Y}^* \to \mathbb{P}^1_{\mathbb{R}} \), are equivalent (can be taken to be the same) as Galois covers.

(1.3b) Components of \( X^* \times_{\mathbb{P}^1} Y \) and \( X^* \times_{\mathbb{P}^1} Y^* \) correspond one-one.

Denote the \( (f^*, g^*) \in \mathcal{F}_{f,g} \) satisfying (1.3a) by \( \mathcal{G}_{f,g} \). For \( (f^*, g^*) \in \mathcal{G}_{f,g} \):

(1.4a) \( f^* \) and \( g^* \) have exactly the same branch points, with the respective branch cycles of \( f^* \) and \( g^* \) of the same order.

(1.4b) Components on \( X^* \times_{\mathbb{P}^1} Y^* \leftrightarrow \text{orbits of } G_f(T_{f*}, 1) \text{ under } G_g \leftrightarrow \text{orbits of } G_f(T_{g^*}, 1) \text{ under } T_{f^*} \text{ (switch } T_{f^*} \text{ and } T_{g^*}).

(1.4c) If \( C \) is a component corresponding to an orbit \( I \) in (1.4b), then the degree of \( C \) over \( X^* \) is \( |I| \).

From (1.3a), the complete collection of \( g \in \mathcal{R}_f \) has the form \( g = g^*(g_2) \) where \( g^* \), up to equivalence of covers, runs over a finite set, satisfying the conditions of (1.3) for some \( f^* \) decomposition factor of \( f \).

Equivalence of the Galois closures \( \hat{X}^*_f = \hat{Y}^*_g \) in (1.3a) implies that the \( \tilde{\mathcal{C}}_{f^*, g^*} \) components are quotients of \( \hat{X}^*_f \) by groups listed in Lem. 4.1.
Assume \( f = f_1(f_2) \), \( \deg(f_1) > 1 \), with \( f_1 \) and \( g_1 \) equivalent covers of \( \mathbb{P}^1_z \).

(1.5) Then \( \tilde{C}_{f_1, g_1} \) is reducible; one component is isomorphic to the diagonal in the fiber product of \( f_1 \) with itself.

At the outset it would appear we could just assume from here on that (1.5) does not hold. In Thm. 4.11, however, we find that its reappearance is the main point. Therefore, given \( f \), the first consideration for finding cases where Thm. 1.1 does not hold, is to consider these steps.

(1.6a) List nontrivial composition factors \( f^* \) of \( f \).
(1.6b) For each \( f^* \) in (1.6a) list (up to equivalence of covers) \( g^* \) s (including \( f^* \)) where \( \tilde{C}_{f^*, g^*} \) has more than one component. (1.3).
(1.6c) Compute the genuses of the components of \( \tilde{C}_{f^*, g^*} \).

Remark 1.4 (Non-uniserial situation). If \( \tilde{C}_{f^*, g^*} \) has a genus 0 component, and (1.5) does not hold, then there is an \( h^* \in \mathbb{C}(x) \) with two distinct decompositions: as \( h^* = f^*(f_2) = g^*(g_2) \).

Definition 1.5. Two such decompositions for \( h^* \) mean decomposition of \( h^* \) is not uniserial. Indecomposable, so uniserial, is typical for a “random” rational function. Still, random covers are not the interesting cases.

Remark 1.6 (The decomposition chain). Maximal chains of decompositions of \( f \in \mathbb{C}(x) \) (\( \deg(f) = m \)) correspond to maximal chains of subsets

\[
\{1\} < I_2 < \ldots < I_{u-1} < \{1, \ldots, m\}, \text{ with } 1 \in I_j; \text{ and for } \sigma \in G_f, \\
\text{and } 1 \leq j \leq u, \text{ if } 1 \in (I_j)T_f(\sigma), \text{ then } (I_j)T_f(\sigma) = I_j.
\]

This is equivalent to a maximal chain (by inclusion) of groups between \( G_f \) and \( G_f(T_f, 1) \): overgroups of \( G_f(T_f, 1) \). Fix \( f \). Locating proper subgroups \( H < G_f(T_f, 1) \), with the Nielsen class of \( T_H \) of genus 0, is the 2nd key to finding genus 0 components of reducible \( \tilde{C}_{f, g} \) as \( g \) varies (Lem. 4.1).

[Fr73b, Thm. 2] viewed searching for genus 0 components of \( \tilde{C}_{f^*, g^*} \) with \( f^* \), \( g^* \in \mathbb{C}[x] \) (polynomials) as generalizing Ritt’s Theorem: Listing all cases of \( h^* \in \mathbb{C}[x] \) (polynomial) in Rem. 1.4. Ritt reverts to \( (\deg(f^*), \deg(g^*)) = 1 \) – automatically \( \tilde{C}_{f^*, g^*} \) is irreducible – through a device called a

Cyclic factor reduction Diagram [Fr73b, p. 46].

Remark 1.7 (non-unique \( (f^*, g^*) \) in Prop. 1.3). If \( g \not\in \mathcal{R}_f \) (\( \tilde{C}_{f, g} \) is irreducible), then \( \hat{X}_f \to \mathbb{P}^1_z \) and \( \hat{Y}_g \to \mathbb{P}^1_z \) could still be equivalent Galois covers (as in (4.9b)). Then, \( (f^*(x) = x, g^*(y) = y) \) is a possible choice.

1.2.2. An archetype when (1.1b) fails. Prop. 1.3 is the fundamental lemma that describes exceptions to irreducibility assumption (1.1b) in Thm. 1.1. That says, for given \( f \), that \( g \in \mathcal{R}_f \) has the form \( g_1(g_2(x)) \), with \( g_2 \) arbitrary.
and with the \( g_1 \)s, up to equivalence of covers, running over a precise finite set \( U_f \) (depending only on the Nielsen class, \( \text{Ni}(G, C)^{\text{abs}} \), of \( f \)). §3.3 shows how to treat Nielsen classes of pairs \((f, g_1)\). With a slight abuse of notation we also use \( U_f \) (or \( U_{\text{Ni}(G, C)^{\text{abs}}} \)) to refer to those Nielsen classes.

§4 extends Thm. 1.1 to not require (\( \ast_{1C} \)). We run separately over the \( f^* \) in (1.6b) (systems of imprimitivity of \( T_f \)) to consider \((f, g_1) \in U_f \) satisfying the conclusion of Prop. 1.3: \( f \) and \( g_1 \) have the same Galois closures, and \( \tilde{C}_{f,g_1} \) has at least two components. Also, for each component \( C \) of \( \tilde{C}_{f,g_1} \), it decides between two alternatives for the genus of the (unique) component of \( U_{f,g_1,C} \) of \( \tilde{C}_{f,g_1(g_2)} \) lying over \( C \), as \( g_2 \) varies.

(1.7a) It goes to \( \infty \) with \( \deg(g_2) \); or
(1.7b) each has a genus 0 (resp. for \( \deg(g_2) \) large, a genus 1) component.

§A.2 refers to the family of genus 0 or 1 covers of \( \tilde{C}_{f,g_1} \) that arise from negating condition (1.1a) as an o-char-fan. They fan into a precise web of \( g_2 \) values for which \( \tilde{C}_{f,g_1(g_2)} \) remains irreducible, but still has bounded genus among all the other possible values of \( g_2 \) for which the genus rises just as in (1.7a). By contrast, we refer to (1.7b) as a Nielsen-component bound. A genus 0 component of \( \tilde{C}_{f,g_1(g_2)} \) that factors through one of those components automatically has a genus bound independent of \( \deg(g_2) \).

Results, especially the nature of the sets \( U_{f,g_1,C} \), depend on just the Nielsen class of \( f \). Given \( f \) (rather than its Nielsen class), explicitly writing coordinates for, say \( U_{f,g_1,C} \) is nontrivial. Still, explicitly describing \( g \) for which Pakovich's criterion (1.1b) fails – for a given \( f \) – is harder yet.

§B.1 shows why more \( f \)s than you might think have (nontrivial) \( g \) for which \( \tilde{C}_{f,g} \) is reducible, further motivating our generalization of Thm. 1.1.

§2.1 gives the lowest degree (\( \deg(f) = 7 \)) example with a corresponding polynomial pair \((f, g_1)\) with Galois condition (1.1a), though neither (1.5) nor (1.1b) holds. Here a component of \( \tilde{C}_{f,g_1} \) has genus 0, and so (1.7b) holds.

These arise from the solution of Schinzel's problem describing indecomposable polynomials \( f \) from [Fr73a] with a nontrivial polynomial \( g_1 \) for which the irreducibility hypothesis does not hold; (2.19b) gives the finite list of degrees for indecomposable \( f \).

The anomalous (decomposable) example \((f, g) = (T_4, -T_4)\) for which \( \tilde{C}_{f,g} \) has two degree 2 components has been discovered numerous times (Rem. A.4). For it, though, (1.1a) fails (the Galois closure has genus 0). Also, if we keep \( f \) a polynomial, but allow for \( g \) any rational function, §B.2 has the significant case where \( \deg(f) = 5 \), \( \deg(g) = 10 \).
Consider a cover $f : W \rightarrow V$ of algebraic varieties (finite flat map, we usually have ramification). Recall, $f$ has a monodromy group $G_f$, the group of the smallest Galois cover, $\hat{f} : \hat{W} \rightarrow V$, of $f$. If $V = \mathbb{P}^1_z$, then the cover has branch cycles for $f$ (generators of $G_f$; see §3). Further, branch cycles for $f$ and $g$ produce branch cycles (3.5) for $\tilde{C}_{f,g}$ as a cover of $\mathbb{P}^1_z$.

Two appendix sections §B.1 and §B.2 come up against known exceptions — here, alternating group Nielsen classes — to the Genus 0 Problem. The former section produces infinitely many Nielsen classes of indecomposable rational function pairs $f, g$, with $\tilde{C}_{f,g}$ reducible and having a genus 0 component. In the latter section, a computation long before the resolution of the genus 0 problem (of concern to arithmetic geometry) produced only one Nielsen class with the genus 0 component property.

1.3. Literature: Over $\mathbb{Q}$; not over $\mathbb{Q}$. §1.3.2 depends on the simple group classification. §1.3.1 does not.

1.3.1. Groups but no simple group classification. In the overlapping topic of [Fr73a] and [Fr12, §1-4], both used the B(ranch)C(ycle)L(emma), for gleaning the definition field of the natural (Hurwitz) families into which such covers fall. Still, by the time of [Fr12], there were many more applications of Hurwitz spaces, and the BCL and its variants.

Half the title of [Fr12] reminds that the BCL showed Davenport’s problem over $\mathbb{Q}$ for indecomposable polynomials without the simple group classification. It hinged on technical discoveries [Fr73a, Lems. 1-5].

(1.8a) The monodromy group, $G_f = G$, supports a pair of permutation representations $(T_1, T_2)$ attached to a doubly transitive design.

(1.8b) The $n$ cycles in these groups consist of more than one conjugacy class, even modulo the normalizer in $S_n$ of $(G, T_i), i = 1, 2$.

(1.8c) $-1$ is not a multiplier in a doubly transitive design [Fr73a, Lem. 5].

(1.8d) The Davenport covers could have no more than three finite (not including $\infty$) branch points [Fr73a, Thm. 1].

Indeed, from a special case of the BCL, the absolute Galois group, $G_{\mathbb{Q}}$, acts nontrivially on the Nielsen class of the covers through through all the non-multipliers appearing in (1.8c), not just the ubiquitous non-multiplier -1. The proof of Cor. 4.4 shows how to use those (non-)multipliers. This is a good place to state the result encoded in Cor. 4.4, Cor. 4.7 and Cor. 4.8, a collective we refer to as the Degree 7 Corollaries.

On one hand they are a model for very precise results; on the other, we use their ingredients for Main Thm. 4.11, though we cannot be as precise
in general. What the collective gives is the case for (genus 0 cover) Nielsen classes of degree \( n = 7 \) whose conjugacy class set contains an \( n = 7 \)-cycle. Then, §4.3.2 outlines the cases – as listed in Thm. 2.9 – for general \( n \), with an \( n \)-cycle where \( T_f \) is a primitive representation of the monodromy group. It does this by referencing [Fr73a] and its aftermath, while noting the cases \( n = 13 \) and \( 15 \) are challenging. The case \( n = 7 \) has all the ingredients for the generalization. Still, the group theory is challenging, and the group theory beyond when \( T_f \) is primitive is more than just challenging.

[Fr12, Prop. 4.4] gives a modern proof of (1.8c); [Fr73a, Lem. 5] used a classical idea hinted to the author by Tom Storer. Still, the upshot was the same: Davenport pairs weren’t defined over \( \mathbb{Q} \). Those genus 0 components, say in §2.1, don’t have definition field \( \mathbb{Q} \). Even without the classification this would be true of Davenport pairs of very general type. We discover this without explicit equations. Yet, for the list generating (1.8b) [CoCa99] used Pari (and [Fr73a]) to write such equations.

1.3.2. Enters the classification. [O15, p. 72] has a cocktail party definition of the simple group classification. Math could use a a guide to connecting pieces of the proof, aimed at non-group theorists. [O15, p. 70] motivates the classification by reference is to the Higgs boson. This has problems.

(1.9a) Awe for the Higgs boson is far more common than ability to define a boson (hint: foremost it includes photons).

(1.9b) Practical molecular chemistry, unmentioned, as much as elementary particles seriously uses (simple and other) groups.

(1.9c) The genus 0 problem (§2.2.3) via [AOS85] – now a common use of the classification to rational functions (from 9th grade) – is closer to expertise of the four mathematicians at the center of [O15].

There was no classification of simple groups in 1969 when [Fr73a] was written. The conjecture was that the core of those monodromy groups were – with one exception – of the form \( \text{PSL}_m(\mathbb{Z}_q) \). If so, then – based on genus 0 monodromy thinking – all degrees for Davenport pairs were in the list (2.19b). As [Fr99, §8 and §9] documents, using those Hurwitz space properties, especially that they had parametrizing transcendental parameters, was done slowly over the years so as to combine with applications.

Also, starting from [Fr73b], there were many applications to problems involving covers with wild ramification. To contrast with characteristic 0 phenomena, [Fr99, §7] shows the genus 0 conclusion limiting the monodromy groups won’t hold. By solving Davenport’s problem in positive characteristic [Fr99, Thm. 5.7] shows that not only isn’t the problem hopeless, but it
comes out quite pretty. Again, however, the author eschewed writing their equations explicitly. Yet, [Bl04] did just that.

[Fr05, §3.2] generalizes the Davenport definition from [Fr73a] to apply over a given finite field and including wild ramification. The key was the property of Monodromy Precision: that the application of Chebotarev Density – usually holding approximately – was here a precise equality for Davenport pairs \((f, g)\). Result: Over a given finite field, infinitely many genus 0 groups give Davenport pairs and the reducibility of \(C_{f,g}\).

Fried conjectured the degrees (2.19b) of Davenport pairs in characteristic 0 based on what would be the part of the simple group classification relevant to diophantine problems in the extant literature. Since this was right, a reader can use [Fr99, §9] to test Cor. 3.10 on the rest of those examples.

1.3.3. Groups versus equations. Hurwitz spaces do more than avoid writing specific equations for families of covers. They are moduli spaces on which group theory can divine the properties of the spaces of covers. In 1969 there was only a smattering of literature on Hurwitz spaces (time of the first version of [Fr73a]); almost all on simple-branched covers. None distinguished between conjugacy classes and ramification (or cycle) type.

Thm. 1.1 does not assume equations over \(\mathbb{Q}\). We don’t either. Yet, the literature that motivated [Fr73a] to generalize Ritt’s theorem considered explicit separated variables equations over \(\mathbb{Q}\): [DLSc61], [Le64] and [Sc82]. These showed their equations did not pass muster for having infinitely many integral solutions. Solving Davenport’s problem – using RET – produced separated variable equations, and a complete list of all number fields over which they had infinitely many quasi-integral solutions.

[AZ01], [AZ03], [B99], [BT00], [Haj97], [Haj98] and [BeShTi99] continued in that vein, over \(\mathbb{Q}\), but they added finding genus 1 curves with infinitely many \(\mathbb{Q}\) points. The last three papers took on specific equations with complicated coefficients, [Fr99, §11]. Our general results, however, reduced their problems to showing such particular expressions as

\[x(x + a)...(x + (k-1)a) = y(y + b)...(y + (m-1)b).\]

were not compositions of polynomials linearly equivalent to Chebyshev polynomials. They used Mazur’s famous result on modular curves – delineating precisely the small torsion groups possible for elliptic curves over \(\mathbb{Q}\) – by showing their equations had too many solutions to be given by torsion. Again, Mazur’s theorem was only applicable if equations were over \(\mathbb{Q}\).
[Fr12, §7.2.3] revisits [AZ03] and [BeShTi99]. Both run into showing irreducibility of specific fiber products. They didn’t use much of [Fr73a] which handles their examples quite well. [AZ03] mistakenly thought [Fr73a] used the classification (see above) possibly because they based their connection to the problem through [Fr73b].

**Problem 1.8.** [AZ03, Prop. 2.6] produced six separated variables polynomial equations with $\infty$-ly many $\mathbb{Q}$ solutions with $f \in \mathbb{Q}[x]$ and $g(x) = cf(x)$, $c \neq 0, 1$. Relate such examples to the results of this paper.

### 2. Examples of genus 0 components of $\tilde{C}_{f,g}$

§3 produces our genus computing results, Cors. 3.9 and 3.10, for fiber products of components of nonsingular covers. It also reviews assigning branch cycles to a cover. Even those familiar with branch cycles will benefit from the §2.1 examples, as previous papers we quote had no detailed method when the fiber product is reducible. It shows how one Nielsen class can provide several challenges to extending Pakovich’s Theorem.

#### 2.1. Displaying examples of $g \in \mathcal{R}_f$

Suppose $\tilde{C}_{f,g} = \bigcup_{j=1}^{u} C_j$ has $u > 1$ components. On $\mathcal{C}_{f,g}$ the union of the image of these components is connected, tied together at singular points. On $\tilde{C}_{f,g}$ the components will be disjoint. In our illustrative examples, $f$ is indecomposable. At the outset we assume (1.5) doesn’t hold: $f, g$ give inequivalent covers. We put a lot into the examples of this section because many of the harder points of them show us how to generalize Thm. 1.1.

##### 2.1.1. What if $C_1$ has genus 0?

As in Rem. 1.4, assume $C_1$ has genus 0. As a cover of $\mathbb{P}^1_y$ (with $g : \mathbb{P}^1_y \to \mathbb{P}^1_z$), some rational function

$$h_1 : C_1 \cong \mathbb{P}^1_w \to \mathbb{P}^1_y$$

represents it.

Suppose we could restrict to cases $g_1 : \mathbb{P}^1_{y_1} \to \mathbb{P}^1_y$ where $\mathbb{P}^1_w \times_{\mathbb{P}^1_{y_1}} \mathbb{P}^1_{y_1}$ satisfies both conditions of (1.1): it is irreducible and the Galois closure of $h_1$, as a cover of $\mathbb{P}^1_{y_1}$, has genus exceeding 1.

Then, the conclusion of Thm. 1.1 applies to $\tilde{C}_{f,g(g_1)}$ as $g_1$ varies. Alas, in a critical case we cannot assume that.

**Lemma 2.1.** If $g_1 = h_1$, then $\mathbb{P}^1_w \times_{\mathbb{P}^1_{y_1}} \mathbb{P}^1_{y_1}$ has the diagonal as a component. Further, for any $h_2 \in \mathbb{C}(y_2)$, with $g_2(y_2) = h_1(h_2(y_2))$, then $\mathbb{P}^1_x \times_{\mathbb{P}^1_{y_2}} \mathbb{P}^1_{y_2}$ has a genus 0 component. In particular, a component of $\tilde{C}_{f,g(h_1(h_2))}$ over $C_1$ has genus 0 for each (nonconstant) $h_2 \in \mathbb{C}(y_2)$. 
By contrast, suppose \( \tilde{\mathcal{C}}_{h_1,g_1} \) is irreducible and its Galois closure as a cover of \( \mathbb{P}^1_y \) is not genus 0 or 1. Then, Thm. 1.1 says the genus of the (unique) component of \( \tilde{\mathcal{C}}_{f,g(h_1)} \) over \( C_1 \) rises with the degree of \( h_2 \).

We comment on Lem. 2.1 as we expand its notation—especially in considering (2.1c)–greatly in Thm. 4.11.

(2.1a) We cannot assume here (1.5), starting from \( \mathbb{P}^1_y \), since we are not at the bottom of the chain.

(2.1b) There is much territory between \( \mathbb{P}^1_w \times_{\mathbb{P}^1_y} \mathbb{P}^1_{1/2} \) having the diagonal as a component and it being irreducible.

(2.1c) (2.1b) suits induction: when fiber products are reducible, crucial components are subcovers of the Galois closure of \( f : \mathbb{P}^1_x \to \mathbb{P}^1_z \).

(2.1d) The cover \( g^* \overset{\text{def}}{=} g(g_1) : \mathbb{P}^1_w \to \mathbb{P}^1_{1/2} \) factors through \( f : \mathbb{P}^1_x \to \mathbb{P}^1_z \).

**Lemma 2.2.** Condition (2.1d) implies \( g \circ g_1 = f \circ f_1 \). So, \( h \) factors through a genus 0 component of \( \tilde{\mathcal{C}}_{f,g_1} \), a situation handled, say, by Lem. 4.1.

To extend Thm. 1.1, generally than (2.1d) we must consider the possibility that \( g \circ h_1 = g^* \), \( h_1 \in \mathbb{C}(w) \), and \( f \) have equivalent left composition factors (of degree \( > 1 \)). Denote the set of such \( g^* \) by \( \mathcal{D}_f \). The notation reminds that \( \tilde{\mathcal{C}}_{f,g^*} \) has a proper component, as in Lem. 2.3.

**Lemma 2.3.** Not only is \( \mathcal{D}_f \subset \mathcal{R}_f \), but there is an effective check if \( g \in \mathcal{D}_f \). This is equivalent to \( f(x) \)–\( g(y) \) has a variables separated factor of form \( u(x) - v(y) \) with \( f = f^* \circ u \) and \( g(y) = f^* \circ v(y) \) (deg(\( f^* \)) > 1).

A Nielsen class equivalent is that the representation \( T_f \otimes T_g \) on the group \( G_{f,g} \) has a component isomorphic to the pullback of \( T_{f^*} \) from \( G_{f^*} \).

**Proof.** If \( g \in \mathcal{D}_f \), then write \( g = f_1(v) \) and \( f = f_1(u) \), \( u, v \in \mathbb{C}(x) \). Then, \( \tilde{\mathcal{C}}_{f,g} \) has a component isomorphic to \( \tilde{\mathcal{C}}_{u,v} \) with \( u : \mathbb{P}^1_x \to \mathbb{P}^1_w \) and \( v : \mathbb{P}^1_y \to \mathbb{P}^1_w \) where \( w \) uniformizes the diagonal component in \( \tilde{\mathcal{C}}_{f_1,f_1} \).

Now we want to turn that around, by supposing that \( f = f_1(u) \) and \( g = g_1(v) \), and \( \tilde{\mathcal{C}}_{f,g} \) has a component isomorphic to \( \tilde{\mathcal{C}}_{u,v} \) regarding both \( u \) and \( v \) giving covering maps to \( \mathbb{P}^1_w \). For each \( w' \in \mathbb{P}^1_w \), consider \( (x', y') \in \tilde{\mathcal{C}}_{u,v} \) lying over \( w' \) (extend to singular points using the removable singularities theorem). Then, \( f(x') = g(y') = z' \in \mathbb{P}^1_z \). Define the map \( f^* : \mathbb{P}^1_w \to \mathbb{P}^1_z \) by \( w' \mapsto z' \). Since \( f \) factors through \( u \) and \( g \) factors through \( v \), the result \( z' \) doesn’t depend on the choice of \( (x', y') \). Therefore \( f^* \) provides the common left composition factor to \( f \) and \( g \).

Further, supposing Prop. 1.3, notation of §1.2.1, consider \( \mathcal{F}_f \), the set of inequivalent left composition factors, \( f^* \) of \( f \). This has a partial ordering,
and we may speak of $D_{f,f^*}$, those $g^* \in D_f$ for which $f^*$ is a maximal element of $F_f$ through which $g^*$ factors.

Given $f$ (as with other formulations we use its Nielsen class, $\text{Ni}(G, C)^{\text{abs}}$), (2.1d) points to a 2-step approach in §4.3 to generalizing Thm. 1.1.

(2.2a) First replace $R_f$ by $D_f$ in the statement.
(2.2b) Then treat inductively the possibility that $g^* \in D_{f,f^*}$.

We have regarded (2.2a) as already a substantial generalization. Then, we have relied on the example(s) of the rest of this section, continued in §4.2 to aid the reader with the intricacies of handling (2.2b).

2.1.2. Branch cycles for Lem. 2.1. We now complement [Fr74, Ex. 5, p. 246] to produce examples of the situation in Lem. 2.1.

§2.2 explains they are the first of a set of six examples in (2.19b), all from groups with their core a projective linear group. Expository sections [Fr12, §1-4], stemming from [Fr73a], document the literature.

They arise from coalescing branch cycles from one Nielsen class (below) using a group $G = \text{GL}_3(\mathbb{Z}/2)$ and explicit conjugacy classes $C$ in $G$. Covers in this $\text{Ni}(G, C)$ give pairs $(f, g)$ for which $\tilde{C}_{f,g}$ is reducible. Here the components have genus 1 (Rem. 2.6). Yet, after two different coalescings of branch cycles, the pairs of $(f, g)$ give $\tilde{C}_{f,g}$ with genus 0 components. The coalescings amount to leaving $G$ the same, but changing $C$.

We use the notation $(1f, 1g)$ and $(2f, 2g)$ for respective pairs of degree 7 polynomials in these two Nielsen classes. Each polynomial has three branch points (including $\infty$) which we can – if we desire – take to be 0, 1, $\infty$ up to linear transformation (fixing $\infty$) on $\mathbb{P}^1$.

The monodromy groups are $\text{GL}_3(\mathbb{Z}/2) = G_{1f} = G_{1g}$, $j = 1, 2$, corresponding to two natural, but inequivalent, permutation representations on $\text{GL}_3(\mathbb{Z}/2) = \text{PGL}_2(\mathbb{Z}/2)$: On lines ($T_L$; for $jf$) and on hyperplanes ($T_H$; for $jg$) in the 2-dimensional projective plane $\mathbb{P}^2(\mathbb{Z}/2)$ over $\mathbb{Z}/2$.

Designate the permutation representations by $T_{jf}$ (resp. $T_{jg}$) on the letters $x_1, \ldots, x_7$ with the branch cycles given by permutations $j\sigma_i$ (resp. using letters $y_1, \ldots, y_7$ in permutations $j\tau_i$).

Example branch cycles in the Nielsen classes correspond to subscripts 1 and 2 on the $\sigma$ and $\tau$ s, as in [Fr74, (2.42)] and [Fr74, (2.41)].

\begin{align*}
T_{1f} : 1\sigma_1 &= (x_1 x_3)(x_4 x_5) & 1\sigma_2 &= (x_1 x_4 x_6 x_7)(x_2 x_3) \\
T_{1g} : 1\tau_1 &= (y_1 y_2)(y_3 y_5) & 1\tau_2 &= (y_1 y_3 y_6 y_7)(y_4 y_5) \\
T_{2f} : 2\sigma_1 &= (x_1 x_2 x_3)(x_4 x_5 x_7) & 2\sigma_2 &= (x_1 x_4)(x_6 x_7) \\
T_{2g} : 2\tau_1 &= (y_1 y_2 y_7)(y_3 y_5 y_6) & 2\tau_2 &= (y_3 y_7)(y_4 y_5)
\end{align*}
When it is clear whether it is $x$ or $y$ involved, we indicate the cycle notation with integers. For example, we have normalized each representation up to one conjugation in $S_7$ so that the branch cycle at $\infty$ is always the 7-cycle $(1\,2\,3\,4\,5\,6\,7)^{-1} = \sigma_\infty = \tau_\infty$. §2.1.3 explains a potential confusion from doing this, despite its computational advantage. Just remember the $\sigma$ entries are acting on different letters than the $\tau$ entries. The product, in order, of the branch cycles – as always – is 1, where the 3rd cycle is $(1\,2\,3\,4\,5\,6\,7)^{-1}$. Permutations act on the right of the letters: The result of $\sigma$ on $i \in \{1,\ldots,7\}$ is $(i)\,\sigma$, so

$$T_f(j\sigma_1)T_f(j\sigma_2) = (1\,2\ldots\,7), \ j = 1,2,\text{ etc}..$$

Given these choices the only leeway is conjugation by the branch cycle at $\infty$. That amounts to a cycling, $(1\,2\,3\,4\,5\,6\,7)$ of $\{1,2,3,4,5,6,7\}$.

2.1.3. **Essential differences between the $\sigma$ s and $\tau$ s.** Consider a pair $(f,g)$ of degree 7 rational functions that give examples of simultaneous branch cycles for $\text{GL}_3(\mathbb{Z}/2)$ in the respective representations. In the identification of $G_f$ and $G_g$ as subgroups of $S_7$, we have simplified in two ways: Dropping the $x$s and $y$s in the notation, and then identifying the 3rd terms of the $\sigma$s and $\tau$s in (2.3) with $(1\,2\,3\,4\,5\,6\,7)^{-1}$.

The actual relation between the two Galois closures is preserved by continuing the embedding of $G_f \to S_7$ to $\text{GL}_7(\mathbb{Q})$, and then conjugating by an incidence matrix $I_{f,g}$ ([Fr73a, Proof of Thm. 1] or [Fr12, §4.2]) that produces the representation $T_g$ in $\text{GL}_7(\mathbb{Q})$. The latter, in this case, happens to conjugate $S_7$ into itself, though $I_{f,g}$ is not a permutation matrix.

Indeed, apply this to $\sigma_\infty$. The result is in the conjugacy class of $\sigma_\infty^{-1}$. That is the statement that $-1$ is a non-multiplier of the design, or that $\sigma_\infty$ and $\sigma_\infty^{-1}$ (albeit with the same cycle type) are not conjugate in the image permutation group of $T_f$. Similar statements apply to the other degrees $m$ in (2.19b), though for those the quotient of $(\mathbb{Z}/m)^*$ by the multipliers is a larger group (as in (4.11); compare with deg = 13 in §4.3.2).

The most general Nielsen class that contains degree 7 polynomial covers, $f$, for which we get reducible $\tilde{C}_{f,g}$ has branch cycles of type

$$((2)(2),(2)(2),(2)(2),(7));$$

the integers here between ()’s are cycle lengths. Most general means largest number of branch points (elements in $\mathbb{C}$).

As above, there are actually two Nielsen classes, differing between the two conjugacy classes of 7-cycles in this group. They come together in the tensor product of the representation on points and lines as in §3.3. The
design referred to above clarifies that the most possible fixed points for an element $M$ in this group is 3. The fixed points in that extreme case correspond to 3 points on the projective plane lying on a line $L$. RH shows that to get such branch cycles for a genus 0 cover, requires the 3 elements that aren’t 7-cycles have the cycle type $(2)(2)$ (of index 2) above.

Such an element $M \in \text{GL}_3(\mathbb{Z}/2)$ is a transvection: having the form

$$M : \mathbf{v} \mapsto \mathbf{v} + \varphi(\mathbf{v})\mathbf{h}$$

where $\varphi$ is the linear functional with the points of $L$ in its kernel, and $\mathbf{h}$ is also in $L$. In, however, Rem. 2.6 we find that for elements in these Nielsen classes the components of $\tilde{C}_{f,g}$ have genus exceeding 0.

The cycle-types of $\sigma_1$ and $\sigma_2$ are different. So it is no surprise in §2.1.4 their contributions generalizing Thm. 1.1 have a significant difference, as in Cor. 4.4 and 4.8. §2.2.1 explains the following statements.

Reordering the conjugacy classes in $S_7$ of the entries of these branch cycles is a minor change. Still, have we left out some significant 3-entry branch cycles that come from coalescings of the main Nielsen class consisting of 4-tuples? The proof of Cor. 4.7 shows we have not. §4.2.4 explains – under the rubric of all such coalescings giving Nielsen class satellites to one fixed Nielsen class – the value of all these degree 7 examples cohering.

Remark 2.4 (Other degrees in (2.19b)). The case $n = 7$ has several regularities that might be misleading. Example: In Cor. 4.8 we take advantage that three branch cycles for covers in the main Nielsen class are transvections. That isn’t the case for $n = 13$ and 15 [Fr05b, §3.4]. [Fr12, §4.3, Thm. 4.5] notes the missing difference set for $n = 15$ in [Fr73a, p. 134]. It also shows precisely how Fried (in 1969) could write branch cycles for the Nielsen classes of the other examples. Thereby, modulo the conjecture, verified later, that we knew all the groups that had such doubly transitive designs, display the precise list of degrees for Davenport pairs.

2.1.4. Degree 7 example components using Cors. 3.9 and 3.10. The distinction between Cor. 3.9 and Cor. 3.10 is that the former computes the genus from an orbit of $x_i \otimes y_1$ under the group $G(\sigma \cdot \tau)$ (as in (3.5)) directly. The latter computes it from the orbit, under the smaller group $G(\sigma \cdot \tau, y_1)$ stabilizing $y_1$. While the genus is of the same component (so, the same), the two methods differ in what they reveal.

The orbits of $G(\sigma \cdot \tau, y_1)$ for $(jf, jg)$, $j = 1, 2$, are

$$(2.4) \quad O_1 = \{x_1, x_2, x_4\} \otimes y_1 \quad \text{and} \quad O_2 = \{x_7, x_3, x_5, x_6\} \otimes y_1.$$
Denote the genus in each computation \( jg_k \), \( k = 1, 2 \). Example: \( 2g_1 \) for the component associated with \( O_1 \) and polynomials \((2f, 2g)\).

To use Cor. 3.9, compute the complete orbit of \( x_1 \otimes y_1 \) under the action of \( G(\sigma \cdot \tau) \). We start with \( 1\sigma = \sigma \) and \( 1\tau = \tau \) in (2.3), dropping the pre-subscript 1 to simplify notation.

Apply \( (\sigma_1, \tau_1) \) (resp. \( (\sigma_2, \tau_2) \)) to \( x_1 \otimes y_1 \) (resp. \( x_3 \otimes y_3 \)) to get

\[
(2.5) \quad x_3 \otimes y_2 \text{ (resp. } x_2 \otimes y_6). \]

That alone gives us the expected orbit of length 21 by applying \((\sigma_\infty, \tau_\infty)\) to the subscripts, equivalencing them \( \text{mod } 7 \):

\[
(2.6) \quad x_{1+i} \otimes y_{1+i}, x_{3+i} \otimes y_{2+i}, x_{2+i} \otimes y_{6+i}, i = 0, \ldots, 6. \]

In abbreviated notation the first two branch cycles give

\[
(\sigma_1, \tau_1) : (((1,1) (3,2))((2,2) (2,1)))*((3,3) (1,5))((4,4) (5,4))((5,5) (4,3))
((6,5) (6,3))((1,7) (3,7))((4,1) (5,2))
(\sigma_2, \tau_2) : ((5,5) (5,4))((3,2) (2,2))((1,1) (4,3) (6,6) (7,7))
((7,6) (1,7) (4,1) (6,3))((1,5) (4,4) (6,5) (7,4))((2,6) (3,7) (2,1) (3,3)).
\]

Indices of \( (\sigma_1, \tau_1), (\sigma_2, \tau_2), (\sigma_\infty, \tau_\infty) \) are the 3 numbers – in order – on the right of (2.7). RH applies to the degree 21 cover of \( \mathbb{P}_\mathbb{Z}^1 \) to compute \( 1g_1 \):

\[
(2.7) \quad 2(21 + 1g_1 - 1) = 8 + 14 + 18, \text{ or } 1g_1 = 0. \]

Now we use Method II, Cor. 3.10. to compute \( 1g_1 \). Conjugate any of the cycles by \( (\sigma_\infty, \tau_\infty) \) to – translate their subscripts – to find the orbits a given cycle falls into in our examples.

We deal with the branch cycle \((1\sigma_1, 1\tau_1)\) for the fiber product over \( z_1 = 0 \). Since \( 1\tau_1 \) fixes \( y_1, y_6, y_7 \), non-trivial contributions to ramification come only from the three points corresponding to these length 1 orbits in \( 1\tau_1 \). So from the following cycles – as in Lem. 3.8:

\[
(2.8) \quad \begin{align*}
(x_1 \otimes y_4 x_3 \otimes y_4) & \quad (x_4 \otimes y_4 x_5 \otimes y_4)^* \\
(x_1 \otimes y_6 x_3 \otimes y_6) & \quad (x_4 \otimes y_6 x_5 \otimes y_6) \\
(x_1 \otimes y_7 x_3 \otimes y_7)^* & \quad (x_4 \otimes y_7 x_5 \otimes y_7).
\end{align*}
\]

Use \( \sigma_\infty \) to translate the subscripts to change \( y_i \) to \( y_1 \). Then check which cycles end up with their support in \( O_1 \). Example:

\[
(x_4 \otimes y_4 x_5 \otimes y_4) \mapsto (x_1 \otimes y_1 x_2 \otimes y_1) \text{ while } (x_1 \otimes y_4 x_3 \otimes y_4) \mapsto (x_5 \otimes y_1 x_7 \otimes y_1).
\]

In (2.8), cycles with superscript \( * \) have support in \( O_1 \). The rest are in \( O_2 \).

Now do the computation over \( z_2 = 1 \) where the branch cycle is \((1\sigma_2, 1\tau_2)\). The 4-cycle in \( 1\tau_2 \) contributes nothing to ramification, but the fixed point
We choose $v = 4$. As above, list those contributions, with the * superscript from translating the subscripts to see which cycles end up in $O_1$.

\begin{equation}
(x_1 \otimes y_2 x_4 \otimes y_2 x_6 \otimes y_2 x_7 \otimes y_2) (x_2 \otimes y_2 x_3 \otimes y_2)^* \\
(x_1 \otimes y_4 x_6 \otimes y_4) (x_4 \times y_4 x_7 \otimes y_4)^*.
\end{equation}

(2.9)

As both branch cycles over $z_3 = \infty$ are 7-cycles, they contribute nothing to ramification. Denote cycles with * superscripts by $\mu^*$. Compute $g_1$:

\begin{equation}
2(3 + 1g_1 - 1) = \sum_{\mu^*} \text{ind}(\mu^*) = 4, \text{ or } 1g_1 = 0.
\end{equation}

Finally, use Method II for orbit $O_1$ (in (2.4)) of $G(\sigma \cdot \tau, y_1)$ for $(2f, 2g)$. From the branch cycles in (2.3) the analogs of (2.8) and (2.9) are

\begin{equation}
(x_1 \otimes y_4 x_2 \otimes y_4 x_3 \otimes y_4) (x_4 \otimes y_4 x_5 \otimes y_4 x_7 \otimes y_4)^* \\
(x_1 \otimes y_1 x_4 \otimes y_1)^* (x_6 \otimes y_1 x_7 \otimes y_1) \\
(x_1 \otimes y_2 x_4 \otimes y_2) (x_6 \otimes y_2 x_7 \otimes y_2) \\
(x_1 \otimes y_6 x_4 \otimes y_6) (x_6 \otimes y_6 x_7 \otimes y_6)^*.
\end{equation}

(2.10)

We have already put the * s on the translates in the right orbit when all the $y_i$ s are set back to $y_1$. As above we compute $2g_1$ using summing the indices of the $\mu^*$ s (which is 4 again), to get $2g_1 = 0$.

These examples (including those of Rem. 2.5 and Ex. 2.6), continue in §4.3 to illustrate Thm. 4.11.

\textbf{Remark 2.5.} For the complementary orbit of $G(\sigma \cdot \tau, y_1)$, $O_2$ in (2.4), use the analogous formula for the genus $g$ in either case. Then, the left side of RH is $2(4 + g - 1)$ and the right side is the sum of the indices of the $\mu$ s that don't have a * superscript.

In the resp. cases this sum is 4+ 4, and 6: genuses $1g_2$ and $2g_2$ of the complementary orbits are, resp. 1 and 0. That gives three genus 0 and one genus 1 component on fiber products of degree 7 polynomial covers.

\textbf{2.2. Context of §2.1.4.} The attachment of a space to elements in a Nielsen class, is done for absolute classes in detail in [Fr77, §5]. It and [BaFr02] emphasize that properties of these spaces come from the explicit Hurwitz monodromy (braid) action on Nielsen classes. (2.12) is a special case. The second author, in exposing any new aspect of this construction, often used spaces of Davenport pairs. The most transparent property: Components of the spaces interpret as orbits of the braid action on Nielsen classes.
Many exposition elements appear in [Fr12, §5] including using the branch cycle lemma for finding the definition field of the whole Hurwitz space. [Fr12, §6.4] uses deg 7 again ([Fr12, Thm. 6.7]) including to explicitly display the relation between inner and absolute spaces for a given Nielsen class. [BaFr02, §4] starts the more advanced topics on cusps, including effective computation of the genus of reduced Hurwitz space components for 4 branch point covers. Then, each component is an upper half-plane quotient and j-line cover, though it is rarely a modular curve.

2.2.1. **Coalescing gives §2.1.4.** We now explain how the §2.1.4 examples arise from degree 7 pairs \((f, g)\) (defined by absolute equivalence), each branched at four (not three) points. Such pairs are parametrized by a connected space \(H_7\). For \((f, g) \in H_7\), \(\tilde{C}_{f,g}\) (fiber product over \(\mathbb{P}^1\)) has two connected components. Recall, §2.1.3 explains the difference between the Nielsen classes (permutation representations) for \(f\) and \(g\).

We give two examples of these coalescings. Then, all examples naturally related to the §2.1.4 examples of reducible \(\tilde{C}_{f,g}\), with genus 0 components, come – in a precise sense – from coalescings in the Nielsen class of \((f, g)\) in Ex. 2.6. So, Cor. 4.8 finishes Rem. 2.1.3. Here are the coalescings.

\[
\begin{align*}
1\sigma\text{-coalesce: } & \quad ((1\,3)(4\,5), (1\,6)(2\,3), (6\,4)(1\,7), \sigma_\infty) \\
& \rightarrow \quad ((1\,3)(45), (1\,467)(2\,3), \sigma_\infty) \\
2\sigma\text{-coalesce: } & \quad ((1\,3)(2\,7), (2\,3)(5\,7), (1\,4)(6\,7), \sigma_\infty) \\
& \rightarrow \quad ((1\,2\,3)(4\,5\,7), (1\,4)(6\,7), \sigma_\infty) 
\end{align*}
\]

(2.11)

The coalescing procedure is just to multiply the 2nd and 3rd (resp. 1st and 2nd) entries in the line for \(1\sigma\)-coalesce (resp. \(2\sigma\)-coalesce). This would be a 1-step coalescing, but we can form coalescings of many steps.

**Example 2.6** (Before coalescing). Prior to coalescing the components of \(\tilde{C}_{f,g}\) have genus 1 for the following reason. A convenient notation for the Nielsen class is \(\text{Ni}(G(\sigma \cdot \tau), C)^{\text{abs}}\) where \(C\) consists of three copies of the conjugacy class of \((\sigma_1, \tau_1)\) in (2.3) and one copy of a class of a 7-cycle. The abs superscript refers to the tensor product representation that we have been using all along. We can take branch cycles for these covers to be

\[
((\sigma'_1, \tau'_1), (\sigma'_2, \tau'_2), (\sigma'_3, \tau'_3), (\sigma_\infty, \tau_\infty))
\]

with each \((\sigma'_i, \tau'_i)\), \(i = 1, 2, 3\), conjugate in \(G(\sigma \cdot \tau)\) to \((\sigma_1, \tau_1)\) in (2.3). So, the indices of each of these in the representation computed there will be 8. The analog computation for (2.7) of this genus \(g_*\) gives

\[
2(21 + g_* - 1) = 3 \cdot 8 + 18, \text{ or } g_* = 1.
\]
Example 2.7 (A bigger coalescing). We can also form Nielsen classes with 
\( G = \text{GL}_3(\mathbb{Z}/2) \) for rational functions \((f, g)\) by replacing the conjugacy class of \((\sigma_\infty, \tau_\infty)\) by a repetition of three copies of the conjugacy class of \((\sigma_1, \tau_1)\). Now the genus, \( g_{**} \), of the degree 3 component of \( \tilde{C}_{f,g} \) is given by

\[
2(21 + g_{**} - 1) = 6 \cdot 8, \text{ or } g_{**} = 4.
\]

2.2.2. Coalescing and braids. This section’s material allows us to finish the context of these degree 7 coalescings, describing their Nielsen classes as satellites cohering to one space attached to a fixed Nielsen class (Thm. 4.7).

The braid action on an \( r \)-tuple \( \sigma = (\sigma_1, \ldots, \sigma_r) \) satisfying the branch-cycle conditions (3.1), is generated by two elements:

(2.12a) \( q_1 : g \mapsto (\sigma_1\sigma_2\sigma_1^{-1}, \sigma_1, \sigma_3, \ldots, \sigma_r) \) the 1st (coordinate) twist, and
(2.12b) \( \text{sh} : g \mapsto (\sigma_2, \sigma_3, \ldots, \sigma_r, \sigma_1) \), the left shift.

Conjugating \( q_1 \) by \( \text{sh} \), gives \( q_2 \), the twist moved to the right. Repeating gives \( q_3, \ldots, q_{r-1} \). Denote the group generated by the braids by \( H_r \) (more accurately described as the Hurwitz monodromy quotient of the braid group).

Here are uses of these braids as applied to a given Nielsen class \( \text{Ni} \). Denote the absolute Galois group of the number field \( L \) by \( G_L \).

(2.13a) Each braid preserves generation, product-one and the conjugacy class conditions. So it preserves branch cycles in \( \text{Ni} \).
(2.13b) Given \( \sigma \in \text{Ni} \), applying braids to \( \sigma \) allows forming \( \sigma' \) whose entries represent the elements of \( C \) in any desired order.
(2.13c) There is a minimal cyclotomic field \( L_{\text{Ni}} \) for which \( G_L \) maps all covers over \( \bar{L} \) in \( \text{Ni} \) into covers in \( \text{Ni} \) if and only if \( L_{\text{Ni}} \subset L \).

The branch cycle argument ([Fr77, p. 62] or [Fr12, §5.1.3]) gives (2.13c). Deeper points follow from this assumption [Fr77, Thm. 5.1]:

(2.14) For \( \varphi : X \to \mathbb{P}^1 \) in \( \text{Ni} \), \( H_r \) transitive on \( \text{Ni} \) (equivalent to there being one connected component of covers in \( \text{Ni} \)).

(2.15a) Then, for any \( \sigma \in \text{Ni} \) (Def. 3.1), for some classical generators (§3.2), \( \sigma \) is the branch cycle description for \( \varphi \).
(2.15b) For absolute Nielsen classes with \( G(\varphi, 1) \) its own normalizer in \( G \), the intersection of all definition fields of covers in \( \text{Ni} \) is \( L_{\text{Ni}} \).

Indeed, (2.15a) is equivalent to (2.14), and even without it, \( L_{\text{Ni}} \) is the right definition field for the Hurwitz space. Here, though, there are three such spaces, \( \mathcal{H}_7, \mathcal{H}_{13} \) and \( \mathcal{H}_{15} \) corresponding to the most interesting of the degrees that appear in (b) and referred to in the examples, Cor. 4.4 and 4.8. Hypothesis (2.14) does hold for them.
Connectedness of $\mathcal{H}_7$ translates to transitive braid action for $r = 4$ on all elements of $\text{Ni}(\text{GL}_3(\mathbb{Z}/2), \mathbb{C}_2^{3,7})^{\text{abs}}$, as in [Fr05b, Prop. 4.1] (Princ. 3.2). This Nielsen class is of 4-branch point covers of $\mathbb{P}_1$. Lem. 2.8 effectively detects orbits of $H_3$ on Nielsen classes of 3-branch point covers. There is no such easy conclusion for $H_r$, $r \geq 4$ on Nielsen classes. Recall: The notation $\text{abs}$ means to mod our Nielsen classes by the normalizer, $N_{S_{\text{deg}(T)}}(G)$ of $G$ in $S_{\text{deg}(T)}$: $\text{Ni}(G, C)^{\text{abs}} = \text{Ni}(G, C)/N_{S_{\text{deg}(T)}}(G)$.

**Lemma 2.8.** The group $H_3 = \langle q_1, q_2 \rangle$ acts on a Nielsen class $\text{Ni}(G, C)^{\text{abs}}$, with $C$ consisting of 3 conjugacy classes, as a quotient of the dihedral group $D_3 = S_3$ of order 6, since $q_1^2$ and $q_2^2$ act trivially. If all three classes in $C$ are distinct modulo $N_{S_{\text{deg}(T)}}(G)$ then all $H_3$ orbits have length 6.

**Proof.** This follows fairly straightforwardly from [BaFr02, §2.4.1], which notes that for $\tau_1 = q_1q_2$, $\tau_1^3 = q_1^2 = q_2^2$ from standard braid group relations. Now consider the action of $q_2^2$ on a Nielsen class element:

$$(g_1, g_2, g_3)q_2^2 \mapsto (g_1, \alpha(g_2\alpha^{-1}, \alpha(g_3\alpha^{-1}), \text{ with } \alpha = (g_2g_3)^{-1}.$$  

From the product-one condition, $\alpha = g_1^{-1}$. So, $q_2^2$ fixes the absolute class of $(g_1, g_2, g_3)$. Similarly for $q_1^2$. Therefore, in its action on Nielsen classes, $H_3$ is generated by two involutions, and therefore it is a dihedral group. It suffices to decide which dihedral group it factors through.

Since $\tau_1$ has order 3, this is $D_3 = S_3$, acting as permutations of the conjugacy classes. As the classes are distinct modulo $N_{S_{\text{deg}(T)}}(G)$, the action on those will give all permutations since their orderings are also distinct. □

2.2.3. §2.1.4 and the genus 0 problem. Monodromy groups $(G, T)$ of indecomposable rational functions are called (primitive) genus 0 groups.

Guranick’s version of the genus 0 problem (over $\mathbb{C}$) formulates what pairs $(G, T)$ – group with deg $n$ permutation representation $T$ – could be monodromy groups of $f : X \to \mathbb{P}_1$ satisfying these conditions:

(2.16) $X$ has genus 0 and $f$ is indecomposable ($T$ is primitive).

He also extended this to genus 1 and any fixed genus > 1. Our qualitative results don’t require these higher genus generalizations.

The idea – akin to the classification, but not restricted to simple groups – divided possible $(G, T)$ into two sets by appeal to the classification of primitive groups – an offshoot primed by essentially indexing them using simple groups [AOS85]. Primitive genus 0 monodromy groups have two types:

(2.17a) Genus 0 series: Infinite series of $(G, T)$ with $G$ having core $A_n$ or $=(\mathbb{Z}/p)^a \times \mathbb{Z}/d$, $d \in \{1, 2, 3, 4, 6\}$ appearing as genus 0 monodromy groups (more precisely [Fr05b, p. 78] or [Fr12, §7.1]).
(2.17b) Genus 0 exceptional: excluding (2.17a) there are only finitely many (primitive) genus zero \((G, T)\) s.

In Thm. 2.9 abs\(_{1,2}\), refers to the pairs, \(T_1 = T_f\) and \(T_2 = T_{g_1}\) of representations of \(G\). Further:

(2.18a) They produce polynomial pairs. So \(C\) has a conjugacy class \(C = C_{\infty}\) whose elements have order \(\deg(T_1) = \deg(T_2)\) and;

(2.18b) both \(T_1\) and \(T_2\) are primitive \((f\) and \(g_1\) are indecomposable).

Thm. 2.9 is a polynomial version of the main result, Thm. 4.11, with the extra assumption \(f\) is indecomposable. The result seems to suggest that only in trivial ways is the Pakovich conclusion violated. That is misleading.

Cors. 4.4 and 4.8 consider just the case \(\deg(f) = 7\), allowing \(g\) to be rational. Indeed, even with \(g = g_1(g_2)\) in the Thm., with \(g_1 \in \mathbb{C}[y]\), but \(g_2 \in \mathbb{C}(y)\) (allowed to be rational), all the nontrivial phenomena of Thm. 4.11 appear, except the induction. That is because condition (2.20a) (without (2.20b)) reappears in that result with a very different conclusion.

**Theorem 2.9.** Assume \(f, g \in \mathbb{C}[x]\) and \(f\) is indecomposable. Then, (2.19) lists all degrees of \(T_1\) for the Nielsen classes \(\text{Ni}(G, C)^{ab_{1,2}}\) for which \((f, g)\) in this class has \(\mathcal{C}_{f,g}\) reducible. For some decomposition \(g = g_1(g_2)\):

(2.19a) either the cover for \(g_1\) is equivalent to the cover for \(f\) (as in (1.5)),

(2.19b) or \(\deg(f) = \deg(g_1)\) is in \(\{7, 11, 13, 15, 21, 31\}\).

For each degree in (2.19b), there is one main Nielsen class for which \(\mathcal{C}_{f,g_1}\) is reducible. Then, that class is either unique, or in degrees 7, 13 and 15, it is one of a finite number of Nielsen classes obtained from coalescing the main class (as with the degree 7 case of §2.1.4).

All component genuses of \(\mathcal{C}_{f,g}\) rise with \(\deg(g)\), unless: (2.19a) holds, so \(\mathcal{C}_{f,g_1(g_2)}\) has a genus 0 component for each \(g_2 \in \mathbb{C}[x]\).

**Proof.** §4.3.2 describes explicitly all the Nielsen classes of Thm. 2.9 (corresponding to (2.19b)) that give exceptions to the conclusion of Thm. 1.1. Especially note that \(T_1\) and \(T_2\) are distinct permutation reps, equivalent as group representations. Applying Thm. 1.1, we have only to consider those cases where \(\mathcal{C}_{f,g}\) is reducible with a component \(C\) of genus 0. Prop. 4.12 handles the case the component has genus 1 (or higher), but restricting to polynomials this is even simpler.

Further, Lem. 2.1 (or (4.3)) reduces us to realizing two conditions:

(2.20a) for one of the degrees in (2.19b), a component \(C\) of \(\mathcal{C}_{f,g_1}\) has genus 0 and a totally ramified place over some \(y' \in \mathbb{P}^1_{y'}\), and
In the expression below (2.9), in applying Method II, there is the statement: “As both branch cycles over \( z_3 = \infty \) are 7-cycles, they contribute nothing to ramification.” The same statement applies here, replacing 7 by \( \deg(f) = \deg(g_1) \). Therefore, even if (2.20a) applies, you can’t take \( y' \) to be the unique value over \( z = \infty \), a necessity for (2.20b) to apply.

This concludes the proof. \( \square \)

3. Branch cycles and genus computation

There are many expositions on this 1st part of Riemann’s Existence Theorem (RET) including [Fr12, §1-4], [Vo96, Chap. 5] and [FrRET, Chap. 4].

3.1. Covers/branch cycles in a Nielsen class. Consider a degree \( n \) cover \( \varphi : X \to \mathbb{P}^1_z \), ramified over \( z = \{z_1, \ldots, z_r\} \). Associate to it classical generators, \( P_1, \ldots, P_r \), of the r-punctured sphere

\[
U_z \overset{\text{def}}{=} \mathbb{P}^1_z \setminus \{z_1, \ldots, z_r\}, \text{ based at a point } z_0.
\]

For this we have chosen an ordering of \( z_1, \ldots, z_r \) given by their subscripts. To avoid ambiguity we need a little extra notation on the branch points, so we use \( B_f \) for \( \{z\} \). These generators are disjoint (piecewise smooth) paths except for the base point, and they issue from that base point in clockwise order according to the ordering on \( z \).

To the cover associate branch cycles \( \sigma = (\sigma_1, \ldots, \sigma_r) \) relative to \( P_1, \ldots, P_r \): \( \sigma_i \in S_n \) corresponds to a closed path from \( z_0 \) going (clockwise) around \( z_i \).

Basic properties of \( \sigma \) (giving branch cycles for \( \varphi \));

(3.1a) Generation: Its entries generate \( G_\varphi \);
(3.1b) Product-one: \( \sigma_1 \cdots \sigma_r = 1 \); and
(3.1c) Conjugacy classes: Independent of the classical generators, \( \sigma \) defines \( r \) conjugacy classes (some possibly repeated), \( C_i \), in \( G_\varphi \).

Further, \( \varphi \) (for any ramified cover) produces a canonical permutation representation, \( T_\varphi : G_\varphi \to S_n \), up to inner automorphism by \( G_\varphi \) in 2 steps.

(3.2a) Produce the Galois closure \( \hat{\varphi} : \hat{X}_\varphi \to \mathbb{P}^1_z \) by the fiber product construction (§A.1).
(3.2b) Take \( G(T_\varphi, 1) \) to be the group giving \( X \to \mathbb{P}^1_z \) as the quotient of \( \hat{X}_\varphi \to \mathbb{P}^1_z \).

Riemann’s Existence Theorem (RET) uses that, given classical generators, \( P_1, \ldots, P_r \), elements \( \sigma \in G^v \) (replacing \( G_\varphi \) by \( G \)), satisfying (3.1), automatically define a cover \( \varphi : X \to \mathbb{P}^1_z \) with \( \sigma \) as branch cycles, unique.
up to equivalence of covers. We write \( \sigma \in G^r \cap C \) to indicate the conjugacy classes they define.

**Definition 3.1 (Nielsen class).** We say either a cover \( \varphi \) or any branch cycle description of it is in the Nielsen class \( \text{Ni}(G, C) \) defined by (3.1).

### 3.2. Nielsen class basics.

For any \( \sigma \in S_n \), \( \text{ord}(\sigma) \) is the least common multiple of its disjoint cycle orders. Fix the number \( r \) of branch points for the covers we take for our genus computations.

#### 3.2.1. Nielsen classes generalize curves of genus \( g \).
The genus stays the same if we move the branch points (keeping them separate) to any location we desire. Therefore, we can take all branch cycles relative to a fixed set of classical generators for any particular situation. Covers of \( \mathbb{P}^1_z \) ramified over \( z \) correspond to branch cycles computed from those classical generators.

**Principle 3.2.** Deforming branch points – keeping them distinct – canonically pulls an initial cover uniquely along a trail of covers over the branch point path. So, it defines (moduli) spaces of cover equivalence classes in a Nielsen class, \( \text{Ni}(G(\sigma), C) \): \( C \) classes in \( G(\sigma) \) defined by the initial cover.

Different equivalences between covers (of \( \mathbb{P}^1_z \)) in Princ. 3.2 defines different moduli spaces. In this paper it is usually absolute equivalence; covers are isomorphic by a continuous map commuting with the maps to \( \mathbb{P}^1 \).

The index, \( \text{ind}(\sigma) \), of a cycle \( \sigma \in S_n \) is \( \text{ord}(\sigma) - 1 \). Extend the definition of \( \text{ind} \) to a product of disjoint cycles additively. Then, the genus, \( g_f \), of an irreducible cover \( f : X \to \mathbb{P}^1_z \) is immediate from branch cycles defining the cover, as is the genus, \( \hat{g}_f \), of the Galois closure \( \hat{f} : \hat{X} \to \mathbb{P}^1_z \).

\[
2(\deg(f) + g_f - 1) = \sum_{i=1}^r \text{ind}(\sigma_i)
2(|G_f| + \hat{g}_f - 1) = \sum_{i=1}^r |G_f|/\text{ord}(\sigma_i)(\text{ord}(\sigma_i) - 1).
\]

The latter formula uses that \( \sigma_i \) in the regular representation of \( G_f \) is the product of \( |G_f|/\text{ord}(\sigma_i) \) disjoint cycles of length \( \text{ord}(\sigma_i) \).

#### 3.2.2. The o-char. Pakovich (Rem. 3.3) uses an orbifold characteristic for a cover. From branch cycles \( \sigma \), its expression from RH for \( g_f \), appears essentially by dividing by \( |G_f| \):

\[
\text{o-char}_f = \frac{2(1-g_f)}{|G_f|} = 2 + \sum_{i=1}^r (1/\text{ord}(\sigma_i) - 1).
\]

From our viewpoint, Riemann knew this. Still, Rem. 3.3 recounts its generalization and prestige tied to (real) 3-manifolds.
Remark 3.3. Clearly, only if the Galois closure cover of $f$ has genus 0 or 1 is $o$-char$_f$ nonnegative: condition (1.1a) does not hold. §A.2 presents, distinctly differently than does the short exposition of [Pak15, p. 2–3], on how that produces examples violating the conclusion of Thm. 1.1.

[Da, p. 5-6] says orbifolds first appeared (as V-manifolds; see right below (3.4)) in [Sa56]. [Th76, §13.3] changed the name V-manifold.

Remark 3.4 (One cover from many). Given branch cycles $g$ for one cover $\varphi : X \to \mathbb{P}^1_z$ with representation $T_\varphi$ as in (3.2), the elements $\sigma_i$ in the formula (3.3) should be understood to be $T_\varphi(\sigma_i)$. Now take any other (faithful, transitive) permutation representation $T_{\varphi'}$ of $G_\varphi$. Canonically produce a new cover $\varphi' : X' \to \mathbb{P}^1_z$ with branch cycles $T_{\varphi'}(\sigma)$. Get the genus of this cover from (3.3) by replacing $n_\varphi$ by $n_{\varphi'}$ and each $T_\varphi(\sigma_i)$ by $T_{\varphi'}(\sigma_i)$.

We use the following remarks inCors. 4.4. and 4.8. In each of the Nielsen classes pointed to by (2.19b), $o$-char $< 0$. Yet, the condition $o$-char $\geq 0$ arises in the complete description of Thm. 4.11, even in the rational function version starting with the degree 7 case.

Remark 3.5 (Appearance of $o$-char $\geq 0$ for degree 7). Suppose in any of the degrees in (2.19b) there is a (at least one) Nielsen class Ni$(G_f, C)_{\text{abs},2}$ so that $C_{f,g_1}$ has a genus 0 component $C$. Then, take $g = g_1(h_2)$ with $h_2 : \mathbb{P}^1_w \to \mathbb{P}^1_y$ representing the cover $C \to \mathbb{P}^1_y$.

For the Nielsen class represented by $(1_1\sigma_1, 1\tau_1)$ and $(1_1\sigma_2, 1\tau_2)$ (and the 7-cycle) in (2.3) we found the degree 3 component genus to be $1g_1 = 0$ using the four cycles in (2.8) and (2.9) with *s. Here $o$-char $= 0$. Denote this $h_2$ by $h_{1,\text{deg} - 3}$. This is in the Nielsen class of degree 3 covers in Ex. A.2.

Similarly, for the the degree 3 component $C_{2,\text{deg} - 3}$ with Nielsen class represented by $(2_1\sigma_1, 2\tau_1)$ and $(2_2\sigma_2, 2\tau_2)$ in (2.10). Here:

$$o$$-char $= 2 + (1/3 - 1) - 2(1/2) = 1/3$.

Branch cycles for $h_{2,\text{deg} - 3} : C_{2,\text{deg} - 3} = \mathbb{P}^1_w \to \mathbb{P}^1_y$ have a 3-cycle, and its Nielsen class is of the degree 3 Chebychev polynomial (§A.2.2).

From Rem. 2.5, the degree 4 complementary component $C_{2,\text{deg} - 4}$ also has genus 0. Directly compute $o$-char $= 2 + (1/3 - 1) + 4(1/2 - 1) < 0$. By inspection of the branch cycles of the degree 4

$$h_{2,\text{deg} - 4} : C_{2,\text{deg} - 4} = \mathbb{P}^1_w \to \mathbb{P}^1_y$$

its monodromy is $S_4$ as its branch cycles contain both a 2 and a 3-cycle.
3.3. Extension to nontransitive representations. As in §2.2, we need the notion of Nielsen class, but it has been standard to assume the natural permutation representation on the group $G$ is transitive. In this paper, we care about when it is not, and the representation breaks up into a direct sum of transitive representations on the orbits.

Denote the projective normalization of the fiber product of

$$\varphi_X: X \to \mathbb{P}^1_z \text{ and } \varphi_Y: Y \to \mathbb{P}^1_z \text{ by } X \times_{\mathbb{P}^1_z} Y = \tilde{C}_{\varphi_X, \varphi_Y}.$$ 

Suppose $\varphi_X$ (resp. $\varphi_Y$) has branch cycles $\sigma_1, \ldots, \sigma_r$ (resp. $\tau_1, \ldots, \tau_r$). Denote the subgroup of $G_{\varphi_X} \times G_{\varphi_Y}$ generated by

$$\sigma \cdot \tau \overset{\text{def}}{=} (\sigma_1, \tau_1), \ldots, (\sigma_r, \tau_r) \text{ by } G(\sigma \cdot \tau).$$

Then, denote the subgroup of $G(\sigma \cdot \tau)$ that stabilizes $y_i$ by $G(\sigma \cdot \tau, y_i)$.

(3.6a) Points on $X$ over $z_i$ correspond one-one with disjoint cycles of $\sigma_i$ whose lengths are the ramification indices of those points.

(3.6b) $G(\sigma \cdot \tau)$ acts naturally on the tensor product of the permutation representations $T_{\varphi_X}$ and $T_{\varphi_Y}$.

(3.6c) Components of $\tilde{C}_{\varphi_X, \varphi_Y}$ identify with orbits of $G(\sigma \cdot \tau)$ on the symbols $x_i \otimes y_j, i = 1, \ldots, m; j = 1, \ldots n$.

(3.6d) For $O$, a $G(\sigma \cdot \tau)$ orbit in (3.6c), branch cycles for the corresponding component $W_O$ are the restriction of $\sigma \cdot \tau$ to $O$.

(3.6e) $G_{\varphi_Y}$ is transitive on $\{y_1, \ldots, y_n\}$. So orbits of (3.6c) correspond 1-1 to orbits of $G(\sigma \cdot \tau, y_j)$ on $\{x_1, \ldots, x_m\} \otimes y_j$.

Our examples usually have $X$ and $Y$ irreducible – as when those covers are given by a pair $(f, g)$ of rational functions in one variable – unless otherwise said. Denote points on $\tilde{C}_{f,g}$ simultaneously over both $x' \in X$ and $y' \in Y$ by $P_{x',y'}$. Then, $P_{x',y'}$ is nonempty if and only if $\varphi_X(x') = \varphi_Y(y')$.

Consider $p$ on $\tilde{C}_{f,g}$ with image $x' = x'_p \in \mathbb{P}^1_x$ (resp. $y' = y'_p \in \mathbb{P}^1_y$, $z' = z'_p \in \mathbb{P}^1_z$). We compute the precise ramification, $e_{p/y'}$, of $p$ over $y'$.

Write $s_{x'} \overset{\text{def}}{=} e_{x'/z'}$ and $t_{y'} \overset{\text{def}}{=} e_{y'/z'}$. For $u, v \in \mathbb{Z}$, denote the greatest common divisor (resp. least common multiple) of $u$ and $v$ by $(u, v)$ (resp. $[u, v]$). [Fr74, Proof of Prop. 2]: Prop. 3.6 results from computing points and their ramification over $y = 0$ on the normalization of $\{(x, y) \mid x^u = y^v\}$.

**Proposition 3.6.** Let $\sigma_i, x'$ (resp. $\sigma_i, y'$) be the disjoint cycle in $\sigma_i$ (resp. $\tau_i$) corresponding to $x'$ (resp. $y'$). The following are equivalent.

(3.7a) $p \in P_{x',y'}$, $e_{p/y'} = \frac{|s_{x'}, t_{y'}|}{t_{y'}}$, and $|P_{x',y'}| = (s_{x'}, t_{y'})$.

(3.7b) $p \in P_{x',y'}$ correspond one-one to disjoint cycles in $\sigma_i^{\text{ord}(\tau_i, y')}$.
Remark 3.7. Assume $X$ and $Y$ (as usual) are irreducible, but $\tilde{C}_{f,g}$ has more than one component. Then, points in $P_{x',y'}$ corresponding to the cycles in $\sigma_{i,x'}^{\text{ord}(\tau_{i,y'})}$ may fall in different components. This is precisely what happens to the two 2-cycles in the 2nd line of (2.9).

3.4. Prop. 3.6 Corollaries. We use the Riemann-Hurwitz formula (RH). Equation (3.8) is immediate from Prop. 3.6 and RH applied to $\text{pr}_{y}: \tilde{C}_{f,g} \to \mathbb{P}^1_y$. For $i = 1, \ldots, r$, consider cycles, $\beta^*$, that appear in one of the permutations $\sigma_{t_y'}$ where $t_{y'}$ is the length of the cycle $\tau_{y'}$ corresponding to $y'$ in $\tau_i$. Denote this collection by $\text{Cyc}_i$, and the number of orbits in (3.6e) by $u$.

Lemma 3.8. Each cycle $\beta^* \in \text{Cyc}_i$ has support $x_w \otimes y_v$ where $v$ is a fixed integer in the support of $\tau_{y'}$. The collection of $x_w$ runs over a subset of one orbit $O_{v,s}$, $s = 1, \ldots, u$. So, if $\mu \in G(\sigma \cdot \tau)$, then $\mu \beta^* \mu^{-1}$ has the same type of support as $\beta^*$ by substituting $y_v'$ for $y_v$, and $O_{v',s'}$ for $O_{v,s}$.

From (3.6e), there is a choice of $\mu$ for which $y_v' = y_1$. The orbit $O_{1,s'}$ in which the support of $\mu \beta^* \mu^{-1}$ ends up does not depend on this choice of $\mu$, nor on the initial choice of $v$ in the support of $\tau_{y'}$.

Denote the subset of $\text{Cyc}_i$ for which $s' = k$ by $\text{Cyc}_{i,k}$:

Those are the $\beta^*$s whose support is in $\{x_i \otimes y_1 \mid x_i \in O_{1,k}\}$.

Label the components corresponding to these orbits by $W_k$, $k = 1, \ldots, u$.

Corollary 3.9 (Method I). Assume, as in (3.6d) that $W_O$ is an orbit of $G(\sigma \cdot \tau)$. Compute the permutations $T_O(\sigma_1, \tau_1), \ldots, T_O(\sigma_r, \tau_r)$ on $W_O$. Then, the genus $g_O$ is computed from RH:

$$2(|W_O| + g_O - 1) = \sum_{i=1}^{r} \text{ind}(T_O(\sigma_i, \tau_i)).$$

The first case of Cor. 3.10 is included in the general case $u \geq 1$; both cases count ramification over a particular component of the fiber product running over each point on $Y \to \mathbb{P}^1_z$ that contributes nontrivially to the ramification. If $u = 1$, then running over all $\beta^*$s allowed by Lem. 3.8 is equivalent to running over all points in the unique component.

Corollary 3.10 (Method II; See Rem. 3.12). If $\tilde{C}_{f,g}$ is irreducible, then the genus, $g_{f,g}$, of its unique component is given is in [Fr73b, (1.6) of Prop. 1]:

$$2(\deg(f) + g_{f,g} - 1) = \sum_{p \in \mathcal{C}_{f,g}} e_{p/y' - 1}$$

$$= \sum_{\text{branch points } z' \text{ of } f} \sum_{(x',y') \to z'} (s_{x'}, t_{y'})(t_{y'}^{s_{x'}} - 1).$$

When $u \geq 1$, for $1 \leq k \leq u$, use the rules of Lem. 3.8 to give the analog of the left and right sides of (3.8) to compute the genus $g_{f,g,k}$ of $W_k$. 
(3.9a) Left side: $2(|O_{1,k}| + g_{f,g,k} - 1)$.
(3.9b) Right side: Include only $\text{ind}(\beta^*)$ for $\beta^* \in \text{Cyc}_{i,k}$, $i = 1, \ldots, r$.

Consider a Nielsen class $\text{Ni}(G, \mathbb{C})_{\text{abs}}^{1,2}$, with

$T = T_{1,2} = T_1 \otimes T_2, T_1$ and $T_2$ transitive faithful representations of $G$.

Assume $\text{Ni}(G, \mathbb{C})_{\text{abs}}^i, i = 1, 2$, are both Nielsen classes of genus 0 covers. 

Then, the genuses of the components in (3.8) depend only on $\text{Ni}(G, \mathbb{C})_{\text{abs}}^{1,2}$; not on branch cycles for the particular $(f, g)$.

Cor. 3.11 is just a special (but important) case of Cor. 3.10: The hypothesis implies that all ramification must appear in $C_2$, and the contribution to the sum where $s_{x''} = s_{x'}$ is automatically trivial.

**Corollary 3.11.** For, $f \in \mathbb{C}(x)$, $T_f : G_f \to S_m$ is doubly transitive if and only if $\tilde{C}_{f,g}$ has the diagonal and one other component $C_2$ of degree $m-1$ over $\mathbb{P}^1_y$. Assume this. Then the genus, $g_{O_2}$ of $C_2$, satisfies:

$$2(m-1 + g_{O_2} - 1) = \sum_{\text{branch points } z'} \sum_{f \sum_{(x',x'',x'\neq x'\prime) \to z'} (s_{x'}, s_{x''})([s_{x'}, s_{x''}]_{s_{x'}} - 1).$$

**Remark 3.12.** The last statement in Cor. 3.10 (dependency on only the Nielsen class) follows because inclusion of particular $\beta^*$s for counting a particular component depends only on conjugacy classes in $\mathbb{C}$.

**Remark 3.13 (T_f double transitive in Cor. 3.11).** Note that $T_f$ is automatically primitive if it is doubly transitive. Examples of this include the case when $f \in \mathbb{C}[x]$ (polynomial) is indecomposable and neither a cyclic nor Chebychev polynomial [Fr70, Thm. 1]. The genus zero problem is easier, but not trivial, if we assume $f$ is doubly transitive. An easy characterization as in the polynomial case for the doubly transitive hypothesis is unlikely.

4. DROPPING IRRREDUCIBILITY HYPOTHESIS (1.1b)

We drop conditions (1.1) in Thm. 1.1 for the Nielsen class of $f : \mathbb{P}^1_x \to \mathbb{P}^1_z$. We decided not to precisely consider growth of component genuses of $\tilde{C}_{f,g}$ with $\deg(g)$, when such growth happens. This is possible, as seen in Cors 4.4 and 4.8. Still, given the intertwining of the $o$-char $> 0$ Nielsen classes with the others, that would complicate the statements.

In §1.3.1 we referred to the collective Cor. 4.4, Cor. 4.7 and Cor. 4.8, as the Degree 7 Corollaries. We stated their ingredients are sufficient to produce Main Thm. 4.11. Though it is not possible to be generally so precise, they are a model for distinguishing how well Thm. 4.11 applies to various Nielsen classes of rational functions $f$ of a given degree.
4.1. \( \tilde{C}_{f,g} \) properties from \( T_1, T_2 \). If \( \tilde{C}_{f,g} \) is reducible, then so is \( \tilde{C}_{f(f_2),g(g_2)} \) for arbitrary (nonconstant) \( f_2, g_2 \in \mathbb{C}(x) \). Prop. 1.3 points to where we would find the heart of reducible \( \tilde{C}_{f,g} \). View Thm. 1.1 as starting from a Nielsen class, \( \text{Ni}(G, \mathbb{C})^{\text{abs}} \), of genus zero covers. Then, \( f : \mathbb{P}^1_x \rightarrow \mathbb{P}^1_z \) is in the Nielsen class if \( T_1 = T_f \) is the rep. of \( G \) corresponding to \( \text{abs}_1 \).

4.1.1. Using Prop. 1.3. Assume \( \sigma \in G' \cap \mathbb{C} \) generates \( G \) and its entries have product 1. Apply \( T_1 \) to \( \sigma \), giving \( T_1(\sigma) \) and, by RET, a representative \( f \) of \( \text{Ni}(G, \mathbb{C})^{\text{abs}} \). Another faithful transitive representation \( T_2 \) of \( G \) produces \( \text{Ni}(G, \mathbb{C})^{\text{abs}_2} \) (with \( \text{abs}_2 \) corresponding to \( T_2 \)). Then, \( T_2(\sigma) \) produces a cover \( X_{T_2(\sigma)} \rightarrow \mathbb{P}^1_z \). A rational function \( g : \mathbb{P}^1_y \rightarrow \mathbb{P}^1_z \) represents this if covers in \( \text{Ni}(G, \mathbb{C})^{\text{abs}_2} \) have genus zero.

We digress on fiber products (using the \((f, g)\) notation), as the relation with the Galois closure of covers and components of \( \tilde{C}_{f,g} \) has a non-group theory aspect. View this as coming either from Dedekind Domain theory, or as from Grothendieck’s approach to fiber products [Mum66, §6].

Consider the tensor product \( \mathbb{C}(x, z) \otimes_{\mathbb{C}(z)} \mathbb{C}(y, z) \). Direct summands of this ring correspond to the prime ideals that lie over the unique prime ideal \( (0) \) of \( \mathbb{C}(z) \). Those, in turn, correspond to the isomorphism classes of function field extensions of the components \( C \) of the fiber product \( \tilde{C}_{f,g} \). Given that function fields of the Galois closures of \( \mathbb{C}(z, x)/\mathbb{C}(z) \) and \( \mathbb{C}(z, y)/\mathbb{C}(z) \) are the same, we can identify the common Galois closure with

\[
\mathbb{C}(z, x_1, \ldots, x_m) = \mathbb{C}(z, y_1, \ldots, y_n) = \Omega_{f,g}.
\]

The function field of \( C \), up to isomorphism as an extension of \( \mathbb{C}(z) \), identifies with a composite \( \mathbb{C}(z) \subset \mathbb{C}(z, x_i, y_j) \subset \Omega_{f,g} \), for some \( i \) and \( j \). The extension \( \Omega_{f,g}/\mathbb{C}(z, x_i, y_j) \) is then dual to the cover \( \tilde{X}_f = \tilde{Y}_g \rightarrow C \).

Lem. 4.1 uses the Galois correspondence. Each cover \( f_W : W \rightarrow \mathbb{P}^1_z \) through which \( \tilde{X}_f \) factors has the form \( \tilde{X}_f/G(T_{f_W}, 1) \). Its conclusion lists the conjugacy classes of subgroups of \( G \) defining components of \( \tilde{C}_{f,g} \).

**Lemma 4.1.** A representation \( T \) of \( G \) is faithful if and only if the intersection of all conjugates of \( G(T, 1) \) in \( G \) is trivial. In particular, if \( G \) is a simple group, and \( T \) is a nontrivial permutation representation of \( G \), then \( T \) is faithful. Assume \((f, g)\) are above.

Assume a component \( C \) of \( \tilde{C}_{f,g} \) corresponds to an orbit, \( O \), of \( G(T_1, 1) \) on \( \{1, \ldots, n\} \). Then \( C \rightarrow \mathbb{P}^1_z \) is equivalent (as a cover) to \( \tilde{X}_f/H \rightarrow \mathbb{P}^1_z \) with \( H = G(T_1, 1) \cap G(T_2, j) \) for any \( j \in O \). The genus of \( C \) is the genus of the covers in the Nielsen class \( \text{Ni}(G, \mathbb{C})^{\text{abs}} \) where \( T = T_C \).
Corresponds to an orbit, orbit of $G$ subgroup of $G$. 4.1.2. from the Nielsen class of the cover by RH. This concludes the proof. □

Since $H \leq G(T_1,1)$ and $\leq G(T_2, j)$, then $C \to \mathbb{P}_z^1$ factors through $\hat{X}_f/G(T_1,1)$ (resp. $\hat{X}_g/G(T_2, j)$) a cover of $\mathbb{P}_z^1$ equivalent to $\mathbb{P}_x^1$ (resp. $\mathbb{P}_y^1$). From the universal property of fiber products, this gives a map from $C$ to $\hat{C}_{f,g}$. The image, $C^*$, of $C$ is a component of $\hat{C}_{f,g}$, corresponding to a subgroup of $G(T_1,1) \cap G(T_2, j)$. So $C^* = C$.

Now consider the converse: $C$ is a component of $\hat{C}_{f,g}$. Prop. 1.3 says $C$ corresponds to an orbit, $O$, of $G(T_f, 1)$ in the representation $T_g$ (or an orbit of $G(T_g, 1)$ in the representation of $T_f$). Suppose $j \in O$. Conclude, in the Galois correspondence, that $C \to \mathbb{P}_z^1$ is equivalent to the cover that corresponds to the subgroup $G(T_f,1) \cap G(T_g, j)$ of $G(T_f,1)$ leaving $j$ fixed.

The genus comment is a restatement of a cover’s genus being computed from the Nielsen class of the cover by RH. This concludes the proof. □

4.1.2. Collecting pieces to extend Thm. 1.1. Consider all representations $T_2$ of $G$ for which, in Lem. 4.1, the following hold:

(4.1a) $G(T_1,1)$ is intransitive in $T_2$; and
(4.1b) the orbit corresponding to $G(T_1,1) \cap G(T_2, j) = H$ gives cover in $\text{Ni}(G, C)^{\text{abs}_{T^0}}$ of genus 0.

Denote those $T_2$ satisfying (4.1) by $T^0(G, C, T_1)$. A genus 0 cover in $\text{Ni}(G, C)^{\text{abs}_{T^0}}$ surjects to a genus 0 cover in $\text{Ni}(G, C)^{\text{abs}_{T^1}}$. Denote systems of imprimitivity of $T_1$ by $D_{T_1}$. They correspond to (left) decomposition factors, $f_1 \in \mathbb{C}(x)$ of $f$ (up to equivalence of covers): There exists $f_2$ with $f_1(f_2) = f$. These also correspond to systems of imprimitivity, but denoting them $D_f$ is also appropriate: $D_f = D_{T_f} = D_{T_1}$ (Rem. 1.6).

Assume $m_1 \in D_{T_1}$ is nontrivial (its sets, permuted by $G$, have cardinality > 1). Vectors in the representation $T_1$ are formal sums of the “letters” for the representation. The $G$ action on the sets of $m_1$ gives an image group $G \to m_1 G$, representation $m_1 T$, and image classes $m_1 C$ (delete classes trivial in the image) to produce $T^0(m_1 G, m_1 C, m_1 T)$ with $m_1 T$ a faithful transitive rep. of $m_1 G$. Generating vectors in $m_1 T$ are the sum of letters in each of the $m_1$ sets. Rem. 4.9 gives example $m_1 G$’s the same, and different, from $G$.

Given branch cycles $\sigma$ for $f$, then $m_1 T(\sigma)$ are branch cycles for a composition factor of $f$, reasonably denoted $m_1 f$. Use similar notation for (non-constant) $g \in \mathbb{C}(y)$, with $T_g = T_2$. For any component $C$ on $\hat{C}_{f,g}$, there is $m_1 \in D_{T_1}$ and $m_2 \in D_{T_2}$ for which the following hold (Prop. 1.3).
(4.2a) $m_1G = m_2G$, and $m_2T(\sigma)$ gives branch cycles for $m_2g$.

(4.2b) There is a unique component $C^*$ on $\tilde{C}_{m_1f,m_2g}$ for which $C$ is the only component of $\tilde{C}_f$ above $C^*$.

Lem. 2.1 gives a quintessential situation in which a conclusion like that of Thm. 1.1 fails. First: There are at least two components of $\tilde{C}_f$. Then:

(4.3a) one, say $C_1$, has genus 0, so is isomorphic to $\mathbb{P}^1_{y_1}$;
(4.3b) thus, some $h_1 \in \mathbb{C}(y_1)$ gives $h_1 : C_1 \cong \mathbb{P}^1_{y_1} \rightarrow \mathbb{P}^1_{y_1}$; and
(4.3c) for any $h_2 \in \mathbb{C}(y_2) \setminus \mathbb{C}$, $\tilde{C}_{f,g(h_1(h_2))}$ has a genus 0 component.

Notice this about finding a genus 0 (resp. 1) component of $\tilde{C}_f$. That gives $G$ and three (faithful) permutation representations $(T_1, T_2, T)$ whose corresponding point stabilizers satisfy $G(T, 1) = G(T_1, 1) \cap G(T_2, 1)$ (as in Lem. 4.1), where the Galois correspondence gives a genus 0 cover of $\mathbb{P}^1_z$ corresponding to $\tilde{X}_f/H$ (resp. 1) with $H$ any one of these three groups. In §2.1.1 we suggested a 2-step approach to generalizing Thm. 1.1 whereby in applying (2.2a), we would exclude (4.3c) because $g(h_1) \in D_d$.

Yet, this won’t dispence with other complications. In two other situations that replace (4.3c), (4.3a) and (4.3b) hold, but Thm. 1.1 fails.

(4.4a) If there is a $k_1 \in \mathbb{C}(y_2)$ giving a cover inequivalent to $h_1$ for which $\tilde{C}_{h_1,k_1}$ is reducible and has a genus zero component.
(4.4b) if the cover $\mathbb{P}^1_{y_1} \rightarrow \mathbb{P}^1_{y_1}$ has orbifold characteristic $\geq 0$ (fails (1.1a)); akin to the examples of §A.2.

Comment on (4.4a): As with case (4.3a), the genus 0 component gives a subcovers (covering $\mathbb{P}^1_y$) of the full Galois cover $\tilde{X}_f \rightarrow \mathbb{P}^1_y$. Therefore, with a subgroup of $G_f$, we have the framework for an inductive argument, to produce another situation like (4.3c).

Comment on (4.4b): The precise paradigm that starts §4.3 requires we know precisely when we get the conclusion of situations §A.2. Then, we can assert that there are infinitely many choices of $k_1$ for which $\tilde{C}_{h_1,k_1}$ is irreducible, and yet has genus 0 or genus 1. That produces infinitely many examples of $\tilde{C}_{f,g(a)}$ sharing properties with both (4.3c) and §A.2.

4.1.3. Handling pairs of reps. $(T_1, T_2)$ of $G$. We fix below either $f$ (or its Nielsen class).

If $T_1$ is primitive, then the genus 0 problem (§1.3.2) puts serious limitations on $G$ and the permutation representations that give Nielsen classes from even primitive permutation representations. Still, that doesn’t imply $T_2$ is primitive (it could arise from decomposable rational functions). Certainly $T$ is not. We therefore lay out an approach starting from a particular
Nielsen class $\text{Ni}(G, C)^{\text{abs}}$ (say, for $f$), with $G_f = G$ a particular group, for locating all possible corresponding $T_2$ for which one must deal in generalizing Pakovic, without conditions (1.1).

This shows the description of the $g$s with $\tilde{C}_{f,g}$ having genus 0 (or 1) components can be stated entirely from the Nielsen class of $f$ with group $G$. Even that actual computations are feasible using Nielsen classes.

(4.5a) Find $T_2$ for which $T_2$ is intransitive on $G(T_1, 1)$ (as in (1.3b)).

(4.5b) Find those $T_2$ in (4.5a) for which the Nielsen class $\text{Ni}(G, C)^{\text{abs}}$ has genus 0 covers. From (4.5a), a $g : \mathbb{P}^1_y \to \mathbb{P}^1_z$ in this Nielsen class fails hypothesis (1.1b): $\tilde{C}_{f,g}$ is reducible.

(4.5c) Apply Cor. 3.10 to identify the genuses of the components of $\tilde{C}_{f,g}$.

§2.1 examples give Nielsen classes that label natural collections of pairs $(f, g)$ that produce the reducibility phenomena. The Nielsen class of $g$ is one of a finite number associated with the Nielsen class of $f$. If, however, you change the Nielsen class of $f$, you start over again.

Comments on (4.5b): Branch cycles for $f$ consist of an $r$-tuple $\sigma = (\sigma_1, \ldots, \sigma_r) \in G^r$ satisfying (3.1) with $T_1$ applied to the entries. Automatically create branch cycles for $g$ as $T_2(\sigma)$ by applying $T_2$ to $\sigma$.

The Prop. §B.1 example follows the steps in (4.5a) and (4.5b). Then, Cor. B.2 does step (4.5c), finding genus 0 components of $\tilde{C}_{f,g}$.

We comment on what was essential about using the genus 0 condition and what was not. Then, we add additional comments on $T_1, T_2$ used in the proof below. Start from Rem. 1.6.

(4.6a) Neither $f$ nor $g$ need be covers of genus 0 curves: the components are still determined by orbits of $G(T_1, 1)$ under $T_2$ in Prop. 1.3.

(4.6b) Orbits and component degrees apply even if $T_1 = T_2$; one length 1 orbit $\leftrightarrow$ a $\tilde{C}_{f,g}$ component isomorphic to the diagonal.

(4.6c) It is convenient, but not necessary, in Cor. 3.10 for the cover $f$ to have genus 0 to find the genus of a component of $\tilde{C}_{f,g}$.

In (4.6c), covers need only be of nonsingular curves. Then, as at the start of §2.1, singular points of $C_{f,g}$ arise from coinciding images $z'$ of values of $x'$ (resp. $y'$) ramified of order $e$ (resp. $f$) over $z'$ with $(e-1)(f-1) > 0$. Then, the gcd($e, f$) points (nonsingular) of $\tilde{C}_{f,g}$ in a neighborhood over of $z'$ are locally from normalizing $C_{x', y'}$ over $z' = 0$. Using lengths of branch cycles in, say, Prop. 3.6 doesn’t work as well unless $f$ is a genus 0 cover.

Remark 4.2 (Subdegrees). In (4.6b), orbit lengths of $G(T_1, 1)$ on $\{2, \ldots, m\}$ are so-called subdegrees. For $f$ equivalent to $x^n$, or resp. to Ex. A.3; (called
Chebychev), then subdegrees are 1, resp. 2. Other than these two examples, if \( f \in \mathbb{C}[x] \) (a polynomial) and \( T_1 \) is primitive, then the only subdegree is \( m-1 \) [Fr70, Thm. 1]: \( T_1 \) is doubly transitive.

**Remark 4.3** (Comment on (4.6b)). We need not limit \( T_1 \) and \( T_2 \) to faithful representations. For example, suppose \( T_1 \) is imprimitive, corresponding to \( f = f_1(f_2) \), (or to a system of imprimitivity \( m_1 \) as in §4.1.2). If \( G_{f_1} \) is a proper quotient of \( G_f \) (say, if \( f \in \mathbb{C}[x] \), as in Rem. 4.9). Then, the representation \( m_1T = T_{f_1} \) extends to a non-faithful representation of \( G_f \) by composing it with the natural cover \( G_f \to m_1G \). There is a copy of the identity in the kernel of \( T_1 \otimes T_1 \to m_1T \otimes m_1T \) corresponding to the diagonal on the projective normalization, \( \tilde{V} \), of

\[
V = \{ (x, y) \mid f_2(x) - f_2(y) = 0 \}, \quad \tilde{V} \subset \tilde{C}_{f,f}.
\]

4.2. **Treating o-char > 0 when \( f \in \mathbb{C}[x] \) has degree 7.** Cors. 4.4 and 4.8 extend Thm. 1.1 only when \( f, g_1 \in \mathbb{C}[x] \) with \( \text{deg}(f) = 7 \) and \( g = g_1(g_2) \) with \( g_2 \in \mathbb{C}(x) \) (rational, not polynomial). These lay out – for either §2.1 Nielsen class – those Nielsen classes of rational functions \( g \) we must avoid so that \( \tilde{C}_{f,g} \) has no genus 0 components for \( \text{deg}(g) \) large. Each corollary includes a case where o-char \( \geq 0 \). Example: (A.5) describes covers with

\[
o\text{-char} = 2 + (1/3 - 1) + 2(1/2 - 1) > 0.
\]

These two related degree 7 Nielsen classes illustrate this difficult aspect, with slight variant, with \( f \) (or its Nielsen class) replaced by any rational function (or genus 0 Nielsen class).

4.2.1. **Handling 2σ in (2.11).** Apply Rem. 3.5. Consider Nielsen classes of the degree \( k \) components,

\[
h_{2,\text{deg}-k} : \mathbb{P}^1_w = C_{2,\text{deg}-k} \to \mathbb{P}^1_y \to \mathbb{P}^1_z, k = 3, 4,
\]
as covers of \( \mathbb{P}^1_z \), from orbits of branch cycles from (2.3):

\[
(4.7) \quad ((2\sigma_1, 2\tau_1), (2\sigma_2, 2\tau_2), ((2\sigma_1 \cdot 2\sigma_2)^{-1}, (2\tau_1 \cdot 2\tau_2)^{-1})).
\]

Denote these by \( \text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{2,3,7})^{\text{abs}, 2, \text{deg}-k} \). Then, we have the projections of these Nielsen classes onto the two distinct degree 7 Nielsen class covers denoted previously \( \text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{2,3,7})^{\text{abs}, j}, j = 1, 2 \) (for resp. \( f, g \), or the representations \( T_1, T_2 \)).

The start of Cor. 4.4 produces the ingredients of an o-char-fan as in §1.2.2. This is the only serious contribution to extending Thm. 1.1 in this case if we restrict to \( g \not\in \mathcal{D}_f \) (according to (2.2)). Allowing \( g \in \mathcal{D}_f \) requires
adding the Nielsen classes for which $\tilde{C}_{f,g_1}$ has components that give Nielsen-component bounds for all $g_2$. §4.2.2 comprises the proof of Cor. 4.4.

**Corollary 4.4.** The representation indicated by $\text{abs}_{2,\text{deg} - k}$ for $k = 3$ (resp. 4) is on index 21 (resp. 28) cosets of a 2-Sylow (resp. $D_3$) as in Lem. 4.1.

For each, $(m,6) = 1$, there are well formed Nielsen classes, $\text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{2,3,7})_{\text{abs}}$, $\text{Ni}(D_m, C_{2^2,m})_{\text{abs}}$ \textit{def} $\text{Ni}_{2,\text{deg} - 3,m}$, of genus 0 covers whose representatives $g = g_1(g_2)$ have these properties.

(4.8a) $g_1 \in \text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{2,3,7})_{\text{abs}}$ and $g_2 \in \text{Ni}(D_m, C_{2^2,m})_{\text{abs}}$;

(4.8b) $\tilde{C}_{f,g}$ has a single irreducible component over $C_{2,\text{deg} - 3}$; and

(4.8c) for any (nonconstant) $g_3 \in \mathbb{C}(x)$, the (unique) component of $\tilde{C}_{f,g}(g_3)$ over $C_{2,\text{deg} - 3}$ has genus 0.

There is a unique transitive representation $T^*: \text{GL}_3(\mathbb{Z}/2) \rightarrow S_8$. Further,

(4.9a) Covers in the nonempty $\text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{2,3,7})_{\text{abs}^*}$ have genus 0;

(4.9b) Yet, for $g_2 \in \text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{2,3,7})_{\text{abs}^*}$, $\tilde{C}_{f,g_2}$ is irreducible:

$$G(T^*, 1)$$

is transitive in $T_1$.

For $g \notin D$, let $m$ be maximal among integers prime to 3, so $g$ factors through a cover in the Nielsen class $\text{Ni}_{2,\text{deg} - 3,m}$ at the branch points given by $h_{2,\text{deg} - 3}$ on $\mathbb{P}_y^1$. Denote the set of such $g$ corresponding to $m$ by $\mathcal{F}_m$. For $g \in \mathcal{F}_m$, the genus of $\tilde{C}_{f,g}$ rises linearly in $\text{deg}(g)/m$.

Now, drop the assumption $g \notin D$. The following, have genus 0 components that give Neilsen-component bounds on those components of $\tilde{C}_{f,g}$ that factor through them:

(4.10a) $g = f(g_2)$, $g_2 \in \mathbb{C}(x)$; and

(4.10b) with any $g_3 \in \mathbb{C}(x)$, $g = g_1(h_{2,\text{deg} - 4}(g_3))$.

Consider any $g \in \mathbb{C}(x)$ whose Nielsen class does not factor through one of the components defined by the Nielsen classes implicit in (4.10). Also, consider $g \in \mathcal{F}_m$ as above. Then, the genus of all components of $\tilde{C}_{f,g}$ rises linearly in $\text{deg}(g)/m$. Components of $\text{C}_{f,g}$ that factor through a fixed one of the components given in (4.10) have uniformly bounded genus.

**Remark 4.5 (Finding separated variable factors).** Factoring 2-variable polynomials is easier than finding composition factors of a rational function. Then, checking if a two variable polynomial has a separated variables factor, as in Lem. 2.3 is easier still. Therefore, checking for a given $g$, if $f$ and $g$ have a common composition factor is intuitively not so hard. How will this actually play out should someone find a great problem requiring understanding Nielsen classes of genus 0 covers with imprimitive monodromy?
Handling such cases will require new ideas, going beyond the genus 0 problem (for primitive groups). [Fr12, §7.4] discusses some historical attempts to do this for \( f, g \in \mathbb{C}[x] \) (polynomials).

4.2.2. Proof of Cor. 4.4. As both \( G(T_1, 1) \) and \( G(T_2, 1) \) both contain a 2-cycle, by the Sylow Theorems, \( G(T_2, j) \) contains the same 2-Sylow as \( G(T_1, 1) \). Since the two groups define different permutation representations, they cannot be equal, so \( G(T_1, 1) \cap G(T_2, j) \) is exactly the 2-Sylow. From Rem. 3.5 that gives the cosets defining abs_{2,deg−k} for \( k = 3 \) (resp. 4).

Now we show how those constituent Nielsen classes in \( \text{Ni}_{2,\text{deg−3},m} \) fit together. From Rem. 3.5, the degree 3 component \( C_{2,\text{deg−3}} \to \mathbb{P}^1 \) has precisely 3 ramified points: one index 3, two of index 2, over 3 branch points. The (A.5) “Chebychev” construction gives a precise rational function mapping those 3 ramified points to the branch points, giving a representative – a Chebychev transport – of the desired Nielsen class. See Rem. 4.9.

A well-known computation gives the order of any group of form \( \text{GL}_m(R) \) where \( R \) is a finite ring. Here

\[
|\text{GL}_3(\mathbb{Z}/2)| = (2^3-1)(2^3-2)(2^3-2^2).
\]

Now consider what faithful transitive permutation representations \( T^* \), excluding those of degree divisible by 7 that we already know about, could possibly give a Nielsen class \( \text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{2,3,7})^\text{abs*} \) having genus 0. It would be defined by the cosets of a subgroup \( H_\ast \) with

\[
7 \cdot 6 \cdot 4 = |H_\ast| \cdot \deg(T^*).
\]

Then, \( T^*: \text{GL}_3(\mathbb{Z}/2) \leq S_{\deg(T^*)} \), and \( \deg(T^*) = d > 7 \), to assure a group of order divisible by 7. That leaves \( d = 8, 12, \) or 24. Suppose there is a degree 12 Nielsen class \( g_1 \in \text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{2,3,7})^\text{abs*} \). The maximal possible index contributions to the genus \( g_\ast \) of the cover give

\[
2(12+g_\ast-1) \leq 6+4 \cdot 2+6; \text{ or } g_\ast \leq -1,
\]

an impossibility. A similar computation excludes \( d = 24 \).

Now we explain the unique degree 8 representation, using references in (1.8). Choose \( \sigma_\infty \in \text{PGL}_m(\mathbb{F}_q) \), \( m \geq 3 \), written as \( (1 2 \ldots u) \) to have order \( u = \frac{2^m-1}{q-1} \). Regard those integers as the points of the space \( \mathbb{P}^m(\mathbb{F}_q) \). Consider the corresponding pairs of representations \( T_1 \) and \( T_2 \) on points and hyperplanes. The design is given by the orbit, \( D, \) of \( G(T_2, 1) \) on 1.

A multiplier \( m \) is a nonzero integer for which \( mD = D+v \mod u \) for some integer \( v \). Multipliers are related to conjugacy classes of \( u \)-cycles by
the following formula:

\[(4.11) \quad \sigma_m^\infty \text{ is conjugate in } G \text{ to } \sigma_\infty \text{ if and only if } m \text{ is a multiplier.}\]

For \( u = 7 \), with \( D = \{1, 2, 4\} \), the group of multipliers is cyclic of order 3. That gives an element of order 3 that conjugates \( \langle \sigma_\infty \rangle \) into itself.

A \( u \)-cycle, \( \sigma_\infty \), in \( S_u \) commutes only with \( \langle \sigma_\infty \rangle \). Consider the normalizer, \( N_G(\langle \sigma_\infty \rangle) \), of this group in \( G \). Then, \( |N_G(\langle \sigma_\infty \rangle)|/|\langle \sigma_\infty \rangle| \) counts the number of powers of \( \sigma_\infty \) conjugate to \( \sigma_\infty \). Conclude, for \( u = 7 \), the normalizer, \( H_* \), in \( GL_3(\mathbb{Z}/2) \) of \( \langle \sigma_\infty \rangle \) is a group of order 21. Since \( \sigma_\infty \) generates the 7-Sylow (and all 7-Sylows are conjugate) all normalizers of an element of order 7 are conjugate in \( GL_3(\mathbb{Z}/2) \).

This gives the uniqueness of the degree 8 representation of \( GL_3(\mathbb{Z}/2) \). From RET (3.1), that also guarantees the Nielsen class defined by this data is nonempty (represented by actual covers).

To see (4.9b), note that \( \sigma_\infty \) is in \( H_* \), but it is not in \( G(T_1, 1) \). Therefore its action is transitive, a cycle of degree 7, on the \( G(T_1, 1) \) cosets. Finally, compute the genus, \( g_* \), of a cover in \( Ni(GL_3(\mathbb{Z}/2), C)^{abs} \) by indicating the maximal possible indices on the right:

\[2(8+g_*-1) \leq 6+4+2 \cdot 2, \text{ or } g_* = 0.\]

To conclude the Corollary, consider any \( g \) that does not factor through a Nielsen class list given in (4.10). Then, \( \tilde{C}_{f,g} \) is irreducible, and according to Thm. 1.1 we are done unless \( g \) is composite with a cover equivalent to \( g_1(h) \) but \( h : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) does not factor through a cover equivalent to \( h_{2,deg-3} \). Now we look to show the statement in the last paragraph of the corollary.

Assume that \( m \) is the maximal integer such that \( h \) is equivalent to \( h_{2,deg-m} \circ h^* \), where \( h_{2,deg-m} \) is in the Nielsen class of \( Ni(D_m, C_{2^m})^{abs} \), with \( (m, 3) = 1 \) and has the appropriate branch points stated in the last paragraph of the Cor. statement. Since \( h_{2,deg-m} \) and \( h_{2,deg-3} \) compositionally commute, the conclusion derives from the genus of the irreducible \( \tilde{C}_{h_{2,deg-3}, h^*} \) rising with \( deg(h^*) \). This is a special case of Thm. 1.1.

4.2.3. Handling \( 1^\sigma \) in (2.11). Cor. 4.8 shows what we need to generalize Thm. 1.1 when \( f \) falls among a reasonably limited set of Nielsen classes. Here we can be explicit. Exs. 2.6 and 2.7 could have remained explicit even if we only limited \( f \in C(x) \) to have degree 7, rather than to assume an element in \( C \) is of a 7-cycle. Still, this eliminates arguments that wouldn’t have exposed anything new. Recall there are two conjugacy classes of 7-cycles in \( GL_3(\mathbb{Z}/2) \), \( C_7 \) and \( C'_7 \) (§2.1.3).
Here we now separate the Nielsen classes that contain these branch cycles by the notation for the conjugacy classes: $C_{2^3,7}$ and $C_{2^3,7'}$, with the former the branch cycles $\sigma^*$ appearing in Ex. 2.6 with 3 appearances of the conjugacy class of a transvection (as in §2.1.3; indicated by $C_2$) and $C_7$.

Similarly for the branch cycles $T_2(\sigma^*) \overset{\text{def}}{=} \sigma^*$ in the Nielsen class with conjugacy classes $C_{2^3,7'}$ (replace $C_7$ by $C_{7'}$). Then, denote the Nielsen classes for the two coalescings in (2.11), respectively, by reference to $C_{2^1,4 \cdot 7}$ and $C_{3 \cdot 2 \cdot 7}$. The ordering of the $C$ integer subscripts is irrelevant according to (2.13b). Ditto for analogous coalescings for $C_{2^3,7'}$ by $C_{2^1,4 \cdot 7'}$ and $C_{3 \cdot 2 \cdot 7'}$.

For the Nielsen class represented by $((1\sigma_1,1\tau_1), (1\sigma_2,1\tau_2), (\sigma_\infty, \tau_\infty))$ in (2.3) we found the degree 3 component genus to be $1g_1 = 0$ using the four cycles in (2.8) and (2.9) with $^*$s. Here $o$-char = 0. Denote this $h_2$ by $h_{1,\deg -3}$. This is in the Nielsen class of degree 3 covers in Ex. A.2.

Let $\text{Ni}(G, C_b)$ and $\text{Ni}(G, C_t)$ be two Nielsen classes with the same group $G$ ($b$ is for bottom, $t$ is for top).

**Definition 4.6 (Nielsen satellites).** We say that $\text{Ni}(G, C_b)$ is a satellite of $\text{Ni}(G, C_t)$, if each $\sigma \in \text{Ni}(G, C_b)$ is a coalescing from some $\sigma^* \in \text{Ni}(G, C_t)$.

In Cor. 4.7, abs$_{1,2}$ corresponds to the tensor product $T_1 \otimes T_2$ as in §2.1.3 where $T_1 = T_f$ (resp. $T_2 = T_g$) is the action on lines (resp. hyperplanes) of the vector space. Also, $T_2$ is $T_1$ conjugated by the incidence matrix $I_{f,g}$. As previously, $\sigma_\infty$ denotes $(12\ldots 7)^{-1}$.

**Corollary 4.7.** The branch cycles (2.3) – produced by the coalescings (2.11) – represent the only Nielsen classes, $\text{Ni}_{2^3,7} \overset{\text{def}}{=} \text{Ni}((\mathbb{Z}/2), C_{2^1,4 \cdot 7})_{\text{abs}_{1,2}}$ and $\text{Ni}_{3 \cdot 2 \cdot 7} \overset{\text{def}}{=} \text{Ni}((\mathbb{Z}/2), C_{3 \cdot 2 \cdot 7})_{\text{abs}_{1,2}}$ with a pair of degree 7 representations $(T_1, T_2)$ and conjugacy classes $C$ of a group $G$ satisfying these constraints:

(4.12a) $\sigma_\infty$ represents a class, $C_7$, in $C$; and
(4.12b) for $f \in \text{Ni}(G, C)_{\text{abs}_{1}}$ there is $g \in \text{Ni}(G, C)_{\text{abs}_{2}}$ with $\tilde{C}_{f,g}$ reducible, having a genus 0 component.

Further, $\text{Ni}_{2^3,7}$ and $\text{Ni}_{3 \cdot 2 \cdot 7}$ each consist of precisely six elements and

$\text{Ni}((\mathbb{Z}/2), C_{2^3,7'})_{\text{abs}_{1,2}}$ has both as satellites.

Analogous statements hold with $7'$ replacing 7 in the Nielsen class notation. So, all degree 7 Davenport pairs, given by pairs $(f, g)$, correspond to
points on a compactification of the irreducible space $\mathcal{H}_7$, with points corresponding to the Nielsen class $\text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{23,7})^{\text{abs}_{1,2}}$ off the boundary.

Proof. Expression (2.11) has the coalescings that give the two Nielsen classes above. On the other hand, from $\sigma_{\infty}$-coalesce we get the result of coalescing 4 other Nielsen class elements by the following device.

(4.13a) $\tau = (1467)(23)$ is the product of the two entries of

$$\mu = ((16)(23),(64)(17)).$$

(4.13b) Conjugate $\mu$ by $\tau$ to see that 4 elements in the Nielsen class (with $\sigma_{\infty}$ in the 4th position) give exactly the same coalescings.

(4.13c) Achieve (4.13b) by the powers of the braid $shq_2^2sh$.

The references above (2.3) show that all possible branch cycles for Nielsen classes of degree 7, containing $\sigma_{\infty}$, are contained in those listed in (4.13). We now know those are coalescings from the single braid orbit comprising $\text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{23,7})^{\text{abs}_{1,2}}$. There are only two conjugacy classes of degree 7 elements in $\text{GL}_2(\mathbb{Z}/2)$. So completing all the Nielsen classes that have Davenport pair representatives. Therefore the Nielsen classes we have listed are satellites of $\text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{23,7})^{\text{abs}_{1,2}}$.

See the beginning of §2.2, for initial discussion of $\mathcal{H}_7$, and then §4.2.4 for an explanation of the compactification we refer to. Finally, we document that both $\text{Ni}_{2,4,7}$ and $\text{Ni}_{3,2,7}$ consist of precisely six elements.

First, we have the natural map $\psi : H_3 \to S_3$ given by the effect of $g \in H_3$ on the order of the conjugacy classes. There are 3 distinct conjugacy classes in $C$ modulo $N_{S_3}(G)$ (because the classes have distinct order), and Lem. 2.8 implies $\psi$ is surjective. There are exactly 6 elements in each $H_3$ orbit in $\text{Ni}_{2,4,7}$ and $\text{Ni}_{3,2,7}$.

Suppose each involution in a 3-tuple in one of our Nielsen classes, say $\text{Ni}_{2,4,7}$, conjugates to any other by some power of $\sigma_{\infty}$. Assume further,

$$g = (g_1, g_2, g_3), \quad g' = (g'_1, g'_2, g_3) \in \text{Ni}_{2,4,7}$$

with say $g_1$ and $g'_1$ involutions and $g_3 = \sigma_{\infty}^{-1}$. Again from product one, if $\sigma_{\infty}^i$ conjugates $g$ to $(g'_1, \sigma_{\infty}^ig_2\sigma_{\infty}^{-i}, g_3)$, then $\sigma_{\infty}^ig_2\sigma_{\infty}^{-i}$ is $g'_2$. Conclude: $g$ and $g'$ are the same in $\text{Ni}_{2,4,7}^{\text{abs}}$. So there are six elements in this Nielsen class. Involutions, however, in this example (indeed all examples in the groups of (2.19b)), correspond to transvections (§2.1.3). So, they are conjugate.

You can check this directly using the 7 Nielsen class representatives in $\text{Ni}(\text{GL}_3(\mathbb{Z}/2), C_{23,7})^{\text{abs}_{1,2}}$ listed, say, in [Fr05b, §3.3]. The same applies to $\text{Ni}_{3,2,7}$. This easily completes the theorem. □
Corollary 4.8. The rest of the cases for degree 7.

Remark 4.9 (Wreath products). We have typically labeled a Nielsen class with a group and a collection of conjugacy classes, and that can be done for \( \text{Ni}_{2, \text{deg}-3, m} \), with \((m, 3) = 1\). Indeed, for any composite cover
\[
f_1 \circ f_2 : X_2 \to X_1 \to Z,
\]
we may describe \( G_{f_1 \circ f_2} \) as a subgroup of the wreath product of the two covers, with their natural permutation reps. entwined [Fr70, §2]. This is a fruitful way when, as in [BiFr86], \( G \) is close to, even if not quite equal, to the full wreath product.

Here it is not the full wreath product. The same \( \mathbb{Z}/2 \) quotient of \( D_m \) is mapped onto by \( \text{GL}_3(\mathbb{Z}/2) \) in its map to \( \mathbb{P}^1_y \), for all \( m \). We don’t make use of it here, so give no details. More extreme even is when \( \tilde{C}_{f,g} \) has a genus 0 component, producing the cover \( h_1 : C \to \mathbb{P}^1_y \) as in §2.1. Then, the Galois closure of \( g \circ h_1 \) is still just \( G \).

Also, we haven’t done any detail on the case when \((m, 2) = 2\), as in Ex. A.4. The time for doing that would be when it plays an important role in some example, as it doesn’t work the same as the case \( m \) is odd.

Remark 4.10 (Effectiveness in Lem. 2.2). In the results Cor. 4.4 and 4.8, Lem. 2.2 turns the condition \( g \in D_f \) into a check on the fiber product \( \tilde{C}_{f,g_1} \) where \( T_f \) and \( T_{g_1} \) satisfy special conditions (1.3) and (1.4) for representations on \( G_f \). There is considerable subtlety to this question, including that there is a difference between dealing with actual rational functions versus the same question about Nielsen classes, where there is little history. Some perspective comes from noting in the former case, given \( g \), that factoring through \( f \) is equivalent to finding a factor of form \( x - m(y) \), \( m \in \mathbb{C}(y) \), for \( f(x) - g(y) \).

4.2.4. Significance of \( \mathcal{H}_7 \). The Hurwitz space \( \mathcal{H}_7 \) is one case of the varieties constructed in [Fr77, §4]. For almost all Nielsen classes in this paper, such as \( \text{Ni}_{23,7} \), each of those spaces, \( \mathcal{H} \), is a fine moduli space, because the stabilizing group \( G(T, 1) \) self-normalizes in \( G \) [Fr77, §4, Prop. 3]. The remainder of this subsection explains more on how the construction of [Fr77, §4] combined with [Fr95b, Thm. 3.21] shows the cohering of those satellites.

First: \( \mathcal{H} \) is an affine variety with a total space structure, \( \mathcal{T} \), over it: \( \mathcal{T} \) is a cover of \( \mathcal{H} \times \mathbb{P}^1_z \) with this property. For \( p \in \mathcal{H} \), a fiber of \( \mathcal{T} \) over \( p \times \mathbb{P}^1_z \) represents a cover in the equivalence class of \( p \). This was shown by producing complex analytic coordinates [Fr77, §4.B, pgs. 49–53], and then applying a famous result of Grauert and Remmert [GraR57]: An analytic (unramified) cover \( W \) of a quasiprojective variety \( V \) is quasi projective. So,
you may complete $W$ to a projective variety $\\bar{W}$ by normalization of $\\mathcal{H}$ in the function field of $W$.

[Fr77, Thm. 5.1] gives, from the Branch Cycle Lemma, the definition field of $\\mathcal{H}$ as a moduli space. We can understood that nicely when fine moduli holds to be the well-defined minimal definition of $T$ with its structural maps. Particularly it says that the two connected families of Davenport polynomials with respective nielsen classes $Ni(GL_3(\mathbb{Z}/2), C_{2^3,7})^{abs}$ and $Ni(GL_3(\mathbb{Z}/2), C_{2^3,7'})^{abs}$, are conjugate over $\mathbb{Q}(\sqrt{-7})$.

Continuing in generality, use the compactifications $\\bar{\mathcal{H}}$ and $\\bar{T}$ as projective varieties with extending maps $\\bar{T} \rightarrow \bar{\mathcal{H}} \times \mathbb{P}_2^1$, through normalization of (components $\\bar{\mathcal{H}}'$ of) $\bar{\mathcal{H}} \times \mathbb{P}_2^1$ in the function field of (components $T'$ of) $T$. We recover the families of covers attached to satellite Nielsen classes by inductively coalescing, using this normalization stratification of $\\bar{\mathcal{H}}$:

(4.14a) Restrict $\\bar{T}'$ over the boundary $\\bar{\mathcal{H}}' \times \mathbb{P}_2^1 \setminus \bar{\mathcal{H}}' \times \mathbb{P}_2^1$.
(4.14b) Normalize (components of) that result and identify open unions of subsets of them as spaces of covers attached to Nielsen classes.
(4.14c) Continue inductively on the dimension from (4.14a) applied to the Nielsen classes in (4.14b).

[Fr95b, proof of Thm. 3.21 and Lem. 3.22] carried out these steps, under the names specialization sequences and coalescing operators.

This means there is a path from a point on $\mathcal{H}$ to any point on the space representing a satellite Nielsen class, a path that runs through a family of nonsingular covers. This is the case, say, for a cover starting at any element in the Nielsen class $Ni(GL_3(\mathbb{Z}/2), C_{2^3,7})^{abs}$ to a cover in the Nielsen class $Ni_{2,4,7}$, or the Nielsen class of the other satellite. A different, Deligne-Mumford style, compactification was constructed by a number of authors (in particular, [DeEm99] and [We99]), motivated (in part) by the application [?, Thm. 3.21]. It would be a good idea to “see” explicitly the path mentioned above within this latter compactification.

4.3. The Main Theorem. Together Cors. 4.4 and 4.8 explicitly exhibit the concepts for the complete generalization, Thm. 4.11, of Thm. 1.1, dropping conditions (1.1). The Nielsen classes in these corollaries contain covers of degree 7 given by a polynomial.

Rem. 4.5 discusses present lack of knowledge of imprimitive genus 0 covers. That hampers considering decomposable $f$ (imprimitive genus 0 Nielsen classes) in the explicit style of the previous corollaries even with Lem. 2.3. As hinted in §4.3.2, we hope someone will take up that challenge,
using specific cases as we have done. We start from the Nielsen class (or just branch cycles) for a rational function $f$.

**Theorem 4.11.** The scenarios in (4.3c), (and more generally) (4.4a) and (4.4b) describe exactly those (finitely many) Nielsen classes $\text{Ni}_1, \ldots, \text{Ni}_{t_f}$ for which the following holds.

For (nonconstant) $g \in \mathbb{C}(y)$:

(4.15a) either, up to equivalence of covers, there are only finitely many $g_1 \in \mathcal{C}(y_1)$ for which $\tilde{\mathcal{C}}_{f,g(g_1)}$ has a genus 0 component; or

(4.15b) for all $g_1$, $\tilde{\mathcal{C}}_{f,g(g_1)}$ has a genus 0 component and the Nielsen class of $g$ is an extension of one of the Nielsen classes $\text{Ni}_1, \ldots, \text{Ni}_{t_f}$.

(4.15c) Other than

The key part of the argument occurs when we find a permutation representation of $G = G_f, T$ which corresponds to $g : \mathbb{P}^1_y \rightarrow \mathbb{P}^1_z$ that produces an $h_1$ as in (4.3b). To construct the exceptional Nielsen classes we end up doing an induction on $\deg(h_1)$. §4.3.1 proves the theorem.

4.3.1. *Proof of Thm. 4.11.* First consider the case that $f$ is indecomposable. From Prop. 1.3, in light of Prop. 1.1, the conclusion follows unless we are considering the case of reducibility of $\tilde{\mathcal{C}}_{f,g(g_1)}$. Then indecomposability of $f$ implies he respective Galois closures of $f$ and $g$ are equal. At this point we have two cases for a component $C$ of $\tilde{\mathcal{C}}_{f,g}$, always under the reducibility assumption.

Case 1: $C$ has genus 0: Denote by $I_C$ the orbit of $G(T_g, 1)$ under $T_f$. Apply (1.4c) (switching $X^*$ and $Y^*$) to see that the degree of $C$ over $\mathbb{P}^1_y$, is less than $\deg(f)$.

**Proposition 4.12.** Case of $\tilde{\mathcal{C}}_{f,g}$ is reducible with a component of genus 1 (or higher).

**Definition 4.13** (Newly Reducible). Continue the notation of Rem. 4.3, and assume $g = g_1(g_2)$ with either $\deg(f_2) > 1$ or $\deg(g_2) > 1$. A component $C$ of $\tilde{\mathcal{C}}_{f_1,g_1}$ is said to be *newly reducible* in $\tilde{\mathcal{C}}_{f,g}$ if there is more than one component of the latter lying over $C$. Otherwise, $C$ is stable in $\tilde{\mathcal{C}}_{f,g}$.

4.3.2. *Examples pointed to by* (2.17) and (2.19b). (2.17) has divided the genus 0 primitive monodromy groups into two cases. In (2.17a) are those related to small semidirect products and alternating groups. The former make their appearance here as the special exclusion (1.1a) in Pakovich’s theorem, playing a big role in Cors. 4.4 and 4.8 using §A.2.
Now consider the primitive genus 0 monodromy related to alternating
groups. §B.1 uses them to produce many examples of reducible fiber pro-
ducts with genus 0 components. That seems to illustrate the alternating
groups as the \emph{untamable} case of the (primitive) genus 0 problem. Yet, §B.2
reminds of a (serious diophantine motivated, really!) alternating group series
(before [GSh07]) with a surprising conclusion. Compatible with the genus 0
result, from infinitely many possible examples, only one degree fulfilled the
constraints on the problem, and produced rich diophantine examples.

Among the exceptional genus 0 groups of (2.17b), there are only finitely
many such \((G, T)\), the number is huge. A myriad of different authors con-
tributed, applying in each case their expertise on pieces of the (simple group)
classification through [AOS85]. We doubt that any tractable classification
could from these for a full, explicit, Pakovich extension. That is, with the
present state of the classification, that list might as well be infinite even
without dropping the primitive monodromy condition.

Instead, for developing insight into such an extension, we suggest com-
pleting for the list (2.19b) what Cors. 4.4 and 4.8 have done for degree 7.
That would fulfill an indecomposable polynomial extension of Pakovich.

[Fr99, §9] wraps up the classification of the cases (2.19b), describing the
spaces of such pairs of polynomials. Only the degrees 7, 13 and 15 cases
have their main Nielsen class with 4 branch points (including \(z = \infty\)).
Likely these are the most interesting cases.

Like the degree 7 case, general covers in the degrees 13 and 15 Nielsen
classes have four branch points. [Fr99, §8.2] lists elements of the degree 13
Nielsen class. Here \(\{1, 2, 4, 10\}\) is what \(\{1, 2, 4\}\) was in degree 7 (as in (2.4)):
a difference set for the doubly-transitive design. Now apply Cor. 3.10 to find
the \(\tilde{C}_{f,g}\) components genuses, as for degree 7.

[Fr99, §8.3] finds the braid group generator \(q_1\) and \(q_2\) actions. That pro-
duces the genus of the reduced Hurwitz space \(\mathcal{H}(1, 2, 4, 10)\) of such covers.
[Fr99, §8.4] finds the defining field, \(K_{13}\), for \(\mathcal{H}(1, 2, 4, 10)\) as moduli of such
pairs. it is the fixed field of \(M(1, 2, 4, 10)\) in \(\mathbb{Q}(\zeta_{13})\): \(\mathbb{Q}(\zeta_{13} + \zeta_{13}^3 + \zeta_{13}^9)\). The
Hurwitz space for these Davenport pairs is a rational variety [Fr99, §8.5].

[Fr99, Thm. 9.1] bounds degrees of Davenport pairs, with detail for
\(n = 31\), the highest degree, in [Fr99, §9.2.2]. [Fr05b, §2.3.2 and §3] dis-
cuss difference sets, and branch cycles of the maximal families of Davenport
pairs for \(\deg(f) = 7, 13, 15\) in considerable detail. Two points stand out in
these cases.
(4.16a) Given a particular conjugacy class of an \( m \)-cycle (resp. there are 2, 4, 2 for \( m = 7, 13, 15 \) [Fr05b, p. 61]), then there is just one Nielsen class of covers with 4 branch points (counting \( \infty \)).

(4.16b) Each Nielsen class of polynomials having these degrees comes from coalescing from the Nielsen class in (4.16a).

(4.16c)

A pretty part is the complete description of all possible genus 0 monodromy groups of polynomials [Mu95] (or [Fr05, App. C1]). Restricting to polynomial maps (among rational functions) is substantial, but much easier.

APPENDIX A. COMMENTS ON FIBER PRODUCTS

To form the Galois closure, \( \hat{f} : \hat{X} \rightarrow Z \), of a cover \( f : X \rightarrow Z \), start with the fiber product of \( f \), \( m = \text{deg}(f) \) times. As usual, normalize, then remove the fat diagonal (loci where two coordinates are equal) to get \( \hat{X}_m \).

Finally, take a connected component \( \hat{X}(m) \), of \( \hat{X}_m \), with its natural projection (restriction of) \( \hat{f} : \hat{X}(m) \rightarrow Z \). This works over any field \( F \) (of characteristic 0; even, with some care about inseparability, in characteristic \( p \)), but here take \( F = \mathbb{C} \).

A.1. Presenting the Galois closure cover. Denote the decomposition group – subgroup of \( S_m \), in its natural action on \( \hat{X}_m \), whose elements map \( \hat{X}(m) \) into itself – by \( G(\hat{X}(m)/Z) \).

We can also construct \( \hat{X}(k) \), based on the fiber product \( k \leq m \) times. Given \( k \leq m \), consider faithfulness condition (A.1) on \( G(\hat{X}(m)/Z) \):

(A.1) \( F_K \) : if \( \sigma \in G_f \) fixes \( k \) integers, then it \( \sigma = 1 \).

Principle A.1. There is an isomorphism \( \hat{f}^* : G(\hat{X}(m)/Z) \rightarrow G_f \), uniquely defined up to inner isomorphism of \( G_f \). Given another connected component \( \hat{X}(m)' \), then \( \hat{X}(m) \rightarrow Z \) and \( \hat{X}(m)' \rightarrow Z \) are equivalent as covers by an isomorphism that induces an isomorphism of

\[
G(\hat{X}(m)/Z) \rightarrow G(\hat{X}(m)'/Z),
\]

unique up to inner isomorphism of \( G_f \).

Suppose \( \hat{f} : \hat{X}(m) \rightarrow Z \) factors through a cover \( \psi : V \rightarrow W \). Then, \( \hat{f} \) factors through \( \hat{\psi} : \hat{V} \rightarrow W \), the Galois closure cover of \( \psi \).

Suppose \( G_f \) satisfies \( F_k \). Then, the Galois closure cover of \( f \) is a connected component of \( \hat{X}(k) \).

Proof. Since \( |S_m| \) is the degree of \( \hat{X}(m) \rightarrow Z \), the degree of \( \hat{f} \) is the same as the order of the decomposition group. The basic Galois principle shows
this gives the Galois closure cover. The restriction of $S_m$ to any fiber $\overset{\to}{X}_{m,z}$ over $z \in Z$ is transitive. Therefore it is transitive on components of $\overset{\to}{X}(m)$, and some element $\sigma \in S_m$ takes $\overset{\to}{X}(m)$ to $\overset{\to}{X}(m)'. Conclude the result by tracing the effect on the isomorphisms with $G_f$.

Now consider the morphism $\psi$. The morphism $\overset{\to}{X}(m) \to W$, factoring through $\psi$ is a Galois cover factoring through $V$, while $\overset{\to}{\psi}$ is the minimal Galois cover of $W$ factoring through $V$ by the Galois correspondence.

Now assume $G_f$ satisfies $F_k$. Then, the action of $G_f$ on $\overset{\to}{X}(k)$ is faithful. Take $Y$ to be an orbit of this action. Consider the projection $\overset{\to}{X}_m \to \overset{\to}{X}_k$ onto the first $k$ coordinates, Then, the pullback of $Y$, is the image of some connected component $\overset{\to}{X}(m)$ of $\overset{\to}{X}_m$, and the action of $G(\overset{\to}{X}(m)/Z)$ commutes with the projection. This identifies $Y \to Z$ as the Galois closure of $f$. □

A.2. When the orbifold characteristic is nonnegative. We give Nielsen class formulations of two cases when the o-char is nonnegative, with each appearing in examples in this paper’s body.

A.2.1. Examples with o-char = 0. Let $E$ be a copy of the complexes $\mathbb{C}$. A 1-dimensional complex torus has the form $E/L$ with $L$ isomorphic to $\mathbb{Z}^2$ viewed as a rank 2 module of translations by complex numbers on $E$. For any integers $n > 1$ form $L \cdot \frac{1}{n} \overset{\text{def}}{=} L_n$. An isogeny (group homomorphism) between complex torii has the form

$$\beta : E/L \to E/L'; \; L \leq L' \leq L_n \text{ for some integer } n.$$ ($A.2$) For the moment assume $|L'/L|$ is odd. See Rem. A.4.

To find a genus 0 cover $\mathbb{P}^1_z \to \mathbb{P}^1_z$ whose Galois closure has genus 1, note that $\beta$ commutes with modding out by $\{\pm 1\}$ generated by multiplication by -1 on both $E/L$ and $E/L'$. From Weierstrass normal form, write

$$E/L' \mod \{\pm 1\} = \mathbb{P}^1_z \text{ and } E/L \mod \{\pm 1\} = \mathbb{P}^1_x.$$ ($A.3$) The induced cover $f : \mathbb{P}^1_x \to \mathbb{P}^1_x$ has $G_f$ isomorphic to the semidirect product $(L'/L) \times^s \{\pm 1\}$,

As $L'/L$ is an abelian group of rank (minimal number of generators) 1 or 2, this produces two families of covers given by rational functions: corresponding to $L'/L = \mathbb{Z}/n$ and $L'/L = (\mathbb{Z}/n)^2$. We refer to these as rank 1 and rank 2 (degree $n$) cases. All other covers from this method are cofinal in these [Fr74, §3].

Covers in the former case are in the Nielsen class $\text{Ni}(D_n, \mathbb{C}_{2^4})^{\text{abs}}$: $D_n$ the dihedral group of order $2n$, $\mathbb{C}$, the class of involutions, with $\mathbb{C}_{2^4}$ having $\mathbb{C}$
repeated 4 times. Absolute equivalence is from \( T : D_n \to S_n \) with the action on \( \langle \sigma \rangle \) cosets, \( \sigma \in C \).

(A.4) For \( (n, n') = 1 \), fiber product of covers, with the same branch points, in \( \text{Ni}(D_n, C_{2^4})^{\text{abs}} \) and \( \text{Ni}(D_{n'}, C_{2^4})^{\text{abs}} \) is in \( \text{Ni}(D_{n \cdot n'}, C_{2^4})^{\text{abs}} \).

These are cases where the Galois Closure has genus 1. Indeed, this follows immediately from (3.4) and it having four branch cycles of order 2. So, the orbifold characteristic is 0.

This applies to condition (1.1a) in Thm. 1.1. The rational function covers \( f \) there have branch cycles that are coalescings (as in \( \S \) 2.2) in the Nielsen class \( \text{Ni}(D_n, C_{2^4})^{\text{abs}} \), and their Galois closures (if they aren’t already Galois) are components \( \hat{f}_2 : \hat{\mathbb{P}}^1_{x, 2} : \) of the \( k = 2 \)-fold fiber product of the cover \( f \) minus the fat diagonal.

Especially the most important part is this. Over a given algebraic point \( j_0 \) of \( \mathbb{P}_j^1 \), what the action of the absolute Galois group \( G_{\mathbb{Q}(j_0)} \) does to the components of \( \hat{\mathbb{P}}^1_{x, 2} \). Suppose we add the rank 2 case in a similar style to this (with \( G_n = (\mathbb{Z}/n)^2 \times^s \{ \pm 1 \} \)). Then, this characterizes the covers cover that arise in three seemingly disparate diophantine problems: Serre’s Open Image Theorem, the theory of complex multiplication and deciding precisely when such \( f \) represent exceptional covers: Covers for which over their definition field \( K \), for infinitely many primes \( p \) of the ring of integers, \( \mathcal{O}_K \), provide one-one maps on \( \mathcal{O}_K/p \cup \{ \infty \} \). [Fr05, \S 6.1] puts that whole story together, referring back to the relevant points of [Fr78, \S 2] and [GMS03].

In particular, only a handful of covers in the whole collection of Nielsen classes with \( G = D_n \) have definition field \( \mathbb{Q} \), while the \( \mathbb{Q} \) covers in the case \( G = G_n \) are dense in the space of covers, because they correspond to elliptic curves over \( \mathbb{Q} \) with the degree \( n^2 \) isogenies given by multiplication by \( n \). As [Fr74, \S 3] notes, this is an enhancement of Ritt’s Second Theorem (which has no allusion to number theory), and the very motivation for the generalization of that to [Fr73b].

**Example A.2** (o-char-fans for these Nielsen classes). A natural Nielsen class, \( \text{Ni}(\mathbb{Z} \times^s \mathbb{Z}/2, C_{2^4}) \) covers every Nielsen class in this subsection. It is one Nielsen class, but it is special because to cover all these examples, the group \( G \) is infinite. We call this the \( 1C_{2^4} \) o-char-fan.

An even larger o-char-fan \( \text{Ni}((\mathbb{Z})^2 \times^s \mathbb{Z}/2, C_{2^4}) \) of genus 0, degree \( n^2 \), covers gives covers of all the degree \( n^2 \) isogenies of elliptic curves \( E \to E \) from multiplication by \( n \) followed by modding out on both sides by \( \langle \pm 1 \rangle \). Refer to this as the \( 2C_{2^4} \) o-char-fan.
A.2.2. *Examples with \( o \)-char > 0.* There is a natural polynomial case, by coalescing branch cycles just as was done according to §2.2 to arrive at the genus 0 fiber product components of §2.1.4. The corresponding Nielsen classes are \( \text{Ni}(D_n, C_{2^2,n})_{\text{abs}} \) with \( C_{2^2,n} \) indicating two repetitions of the class of involutions together with the class of an \( n \)-cycle in \( D_n \). Modulo absolute (but not inner) equivalence there is just one \( n \) cycle class, because the outer automorphism group of \( D_n \) is

\[
\{(a \ b \ c) \mid a \in (\mathbb{Z}/n)\,^*, b \in \mathbb{Z}/n\}
\]

These are cases where the Galois Closure has genus 0. This follows immediately from (3.4) and two branch cycles of order 2, one of order \( n \). So, the orbifold characteristic is > 0. The family of these is easy to understand as they are just the Chebychev polynomials of degree \( n \) modulo changes of variable (as noted in [Fr70]). These are cases where the Galois closure of the cover has genus 0.

**Example A.3** (*o*-char-fan \( \text{Ni}(\mathbb{Z} \times^s \mathbb{Z}/2, C_{2^2,\infty}) \)). Start with the *o*-char-fan that is the analog of the fans in Ex. A.2, for the Nielsen classes in this subsection given by the projective limit of the Nielsen classes \( \text{Ni}(\mathbb{Z} \times^s \mathbb{Z}/2, C_{2^2,m})_{\text{abs}} \). Here the representation is on cosets of a group \( G(T, 1) \) generated by an involution. For \( m \) odd, the involutions form a unique conjugacy class.

Covers in this Nielsen class form a connected family parametrized by

\[
\mathcal{H}(D_m, C_{2^2,m}) \overset{\text{def}}{=} (\mathbb{P}^1)^3 \setminus \Delta_3/S_2 \times \{1\}
\]

where: \( \Delta_3 \) is the fat diagonal; and \( S_2 \times \{1\} \) equivalences \((y_1, y_2, y_3)\) to \((y_2, y_1, y_3)\). The cover over the equivalence class of \((y_1, y_2, y_3)\) is in (A.5).

Define the polynomial \( T_m \) by

\[
T_m \left( \frac{t + 1/t}{2} \right) = \frac{t^{m+1}/t^m}{2} = z.
\]

As a covering map it is in \( \text{Ni}(D_m, C_{2^2,m}) \), with finite branch points ±1. For \((3, m) = 1\), the (normalized) fiber product \( \tilde{C}_{T_3,T_m} \), as a cover of \( \mathbb{P}^1 \), is \( T_{3m} : \mathbb{P}^1_{w} \rightarrow \mathbb{P}^1_{z}, \) with \( w = \frac{t+1/t}{2} \). Take \( L \) to be the linear fractional transformation on \( \mathbb{P}^1 \).

Then, the fiber over \( \ell \times \mathbb{P}^1_y \) of the map

(A.5) \( L \times \mathbb{P}^1_w \rightarrow L \times \mathbb{P}^1_y \) by \( \ell \times w \mapsto \ell \times l(T_m(\ell^{-1}(w))) \)

has branch points \( \ell(-1), \ell(+1) \); respective images of \( \ell(-1), \ell(+1) \).

This is, however, a case with a cover in \( \text{Ni}(\mathbb{Z} \times^s \mathbb{Z}/2, C_{2^2,m})_{\text{abs}} \) having its Galois closure of genus 0:

(A.6) \( 2(2m-1+g_{m,\text{in}}) = 2(m-1) + 2(2m/2) \Rightarrow g_{m,\text{in}} = 0. \)
Remark A.4. Take \( n = 4 \), where \( (A.2) \) does not hold. Here

\[
G = D_4 = (\mathbb{Z}/4) \times \{\pm 1\} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \{\pm 1\}, b \in \mathbb{Z}/4 \right\},
\]

has 2 classes of involutions \( C_1 \) and \( C_2 \), resp. represented by \((a, b) = (-1, 0)\) and \((-1, 1)\). The Nielsen class analogous to that for odd \( n \) is \( C_{1 \times 2} \) indicating we take each involution class twice. Then, we get a 2-component fiber product \( \tilde{C}_{f,g} \) by taking the respective pairs of permutation representations \( T_f \) and \( T_g \) as the resp. representations on the cosets of \( \langle (-1, 0) \rangle \) and \( \langle (-1, 1) \rangle \).

There are two anomalies. First, \( \tilde{C}_{f,g} \) here fails both of Pakovich’s hypotheses \((1.1)\). The genus of the Galois closure of \( f \) is 1, and we get reducible fiber product. Since the involutions fix two integers in the representation, the 2-fold fiber product of \( f \) will not give the Galois closure (as in Princ. A.1).

Now consider the polynomial version by coalescing in this case, where the Nielsen class is \( \text{Ni}(D_4, C_{2 \times 4})^{\text{abs}} \), with the repetition of an involution class twice, and a 4-cycle. The orbifold characteristic is \( 2 + 2(\frac{1}{2} - 1) + (\frac{1}{4} - 1) > 0 \).

Normalize the polynomials here to have branch points \( 1, -1, \infty \), and this is the case of reducible degree 4, \( T_4(x) + T_4(y) \), or \( g = -f \).

Example A.5 (Ubiquitous \( T_4 \) example). Rem. A.4 gives group data about this example. Despite the rareness of nontrivially reducible \( f(x) - g(y) \) with \( f, g \in \mathbb{C}[x] \), this example pops up in many papers: \( f(x) = T_4(x) \), \( g = -T_4(y) \). Affine pieces of the components of \( \tilde{C}_{f,g} \) appear in [Sc82, p. 57]:

\[
(x^2 + \sqrt{2}xy + y^2 - 2)(x^2 - \sqrt{2}xy + y^2 - 2).
\]

Each component is the Galois closure of \( X_f \) with corresponding groups both \( \{1\} \) and \( (x, y) \mapsto (-x, y) \) displays the two components are isomorphic.

Appendix B. Expectations for \( \tilde{C}_{f,g} \) genus 0 components

§B.1 constructs \( \infty \)-ly many cases – thanks to RET – where \( \tilde{C}_{f,g} \) has \( u = 2 \) components, with both \( f \) and \( g \) indecomposable. It also shows that one of those components has genus 0. Yet, §B.2 gives evidence that those are subtle exceptions to Genus zero problem (§2.2) expectations.

B.1. \( \infty \)-ly many \( \tilde{C}_{f,g} \) with \( u = 2 \) components. For each \( m \geq 4 \), we produce \( m \) and \((f, g)\), with \( f \in \mathbb{C}[x] \) of \( \deg(f) = m \), and (nontrivially) \( \tilde{C}_{f,g} \) has two components. Use the notation at the beginning of §1.2.1. Here is a branch cycle description for \( f : \mathbb{P}_x^1 \to \mathbb{P}_z^1 \), with \( G_f = S_m \) and \( T_f = T_1 \) the standard representation of \( S_m \):

\[
\sigma_f \overset{\text{def}}{=} (\sigma_1, \sigma_2, \sigma_3) = ((12), (1 3 4 \ldots m), (1 2 \ldots m)^{-1}) \in (S_m)^3).
\]
Take \( T_g = T_2 \) the degree \( \frac{m(m-1)}{2} = n \) rep. of \( S_m \) on unordered pairs 
\[
\{\{i, j\} \mid i \neq j, 1 \leq i, j \leq m\}.
\]

Given classical generators (§3), RET produces a polynomial \( f \) in the Nielsen class \( \text{Ni}(S_m, C)^{\text{abs}_1} \) where \( C \) are the conjugacy classes represented by \( \sigma \) in \( S_m \), and \( \text{abs}_1 \) indicates the rep. \( T_1 \) for absolute equivalence.

The cover for \( g \) arises (use the same classical generators as for \( f \)) from the branch cycles \( \tau = (T_2(\sigma_1), T_2(\sigma_2), T_2(\sigma_3)) \). A rational function represents the cover \( g \) if \( g^* \) in the following equation is 0.

\[
\begin{align*}
2\left(\frac{m(m-1)}{2} + g^* - 1\right) &= \sum_{i=1}^{3} \text{ind}(T_g(\sigma_i)) = \sum_{i=1}^{3} \text{ind}(\tau_i). \\
\end{align*}
\]

**Proposition B.1.** Computing indices in (B.1) shows \( g^* = 0 \), producing \( g \in \mathbb{C}(y) \). The fiber product \( \hat{C}_{f,g} \) has exactly two components \( (g \in R_f, (1.1b) \) fails). For \( m > 4 \), the orbifold characteristic is negative \((1.1a) \) holds).

**Proof.** The action of \((1 2)\) on the pairs \( \{1, j\}, j \neq 1 \) or 2, moves them to the pairs \( \{2, j\} \). Conclude: \( \text{ind}(\tau_1) = m - 2 \). Also, every \( \tau_3 \) orbit has the form

\[
\{\{j, k+j-1\} \mid 1 \leq j \leq m, \ 2 \leq k \leq m\}.
\]

This will be an orbit of length \( m \), unless for a given \( k \), there are two distinct values of \( j \) (say, \( j' \) and \( j'' \)) for which

\[
j' = j'' + k - 1 \quad \text{and} \quad j'' = j' + k - 1, \quad \text{or} \ 2(k-1) \equiv 0 \mod m.
\]

That is, \( m \) is even and \( k = \frac{m}{2} + 1 \). Also, the orbit under translation of this set of two distinct elements is determined by the minimal absolute difference between the elements in the set.

Therefore, \( \tau_3 \) is a product of \( m \)-cycles, \( \frac{m-1}{2} \) of them if \( m \) is odd, but it has, besides \( m \)-cycles, one \( \frac{m}{2} \)-cycle if \( m \) is even. Thus:

\[
\text{ind}(\tau_3) = \begin{cases} 
\frac{(m-1)(m-1)}{2} & \text{if } m \text{ is odd} \\
\frac{(m-1)(m-2)}{2} + \frac{m}{2} - 1 = \frac{m(m-2)}{2} & \text{if } m \text{ is even}.
\end{cases}
\]

Now we compute \( \text{ind}(\tau_2) \) based on the idea that \( g^* \geq 0 \).

Case \( m \) is even: From RH,

\[
\sum_{i=1}^{3} \text{ind}(\tau_i) \geq 2\left(\frac{m(m-1)}{2} - 1\right) = m(m-1) - 2.
\]

Therefore, \( \text{ind}(\tau_2) \geq m(m-1) - 2 - \frac{m(m-2)}{2} = \frac{m(m-2)}{2} \). Since \( \tau_2 \) has order \( m - 1 \), the maximal value of \( \text{ind}(\tau_2) \) occurs if \( \tau_2 \) is a product only of \( (m-1) \)-cycles, \( \frac{m}{2} \) of them. So, \( \text{ind}(\tau_2) \) is this maximal value and \( g^* = 0 \).
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Case \( m \) is odd: Same as in the Case \( m \) is even, except that the maximal possible value of \( \tau_2 \) if besides \((m-1)\)-cycles, \( \tau_2 \) has one \((\frac{m-1}{2})\)-cycle.

Now notice that \( \tilde{C}_{f,g} \) has two components corresponding to the two orbits of \( G_f(1) \) on the unordered pairs: One orbit on all pairs of the form \( \{1, k\} \), \( k \neq 1 \); the other on the rest of the pairs.

This concludes the proof except to show that for \( m > 4 \), the orbifold characteristic is negative. Since it is given by

\[
2 + \left(\frac{1}{2} - 1\right) + \left(\frac{1}{m} - 1\right) + \left(\frac{1}{(m-1)} - 1\right),
\]

a decreasing function of \( m \). Check that it is -.05 for \( m = 5 \). □

Corollary B.2. Let \( m \geq 5 \) in Prop. B.1 be odd. For \( m = 5 \), both components of \( \tilde{C}_{f,g} \) have genus 0.

For all \( m \geq 5 \), the component of degree \( m-1 \) over \( \mathbb{P}_x^1 \) has genus 0. The genus of the other component grows quadratically with \( m \).

Proof. We do the case where \( m \) is odd in detail. Denote representatives of the Nielsen class pairs \((f, g)\) corresponding to \( m \) by \((f_m, g_m)\). We now find the genuses of the two components of \( \tilde{C}_{f_m,g_m} \). In applying Cor. 3.10, switch \( f_m \) and \( g_m \), since the branch cycles for \( f \) are simpler.

That is, we are dividing the points corresponding to cycles in the branch cycles of \( \tilde{C}_{f_m,g_m} \) over \( \mathbb{P}_x^1 \) between the two components. Start with \( m = 5 \). The method is completely analogous to that of §2.1.4. Since \( \sigma_3 \) is an \( m \)-cycle, the point corresponding to it contributes nothing to RH in either component. As \( \sigma_2 \) fixes only one point – corresponding to the integer 2 – and the cycles of \( \tau_2 \) all have order dividing \( m-1 \), we can take

\[
((2, \{2, 1\}) (2, \{2, 3\}) (2, \{2, 4\}) (2, \{2, 5\}))^* \\
((2, \{1, 3\}) (2, \{3, 4\}) (2, \{4, 5\}) (2, \{5, 1\})) \\
((2, \{3, 5\}) (2, \{1, 4\}))
\]

as a rep. of the branch cycle conjugacy class in \( G_f = G_g \) of \( \tau_2 \).

Similarly, the branch cycle classes for the points of \( \mathbb{P}_x^1 \) corresponding to each of the three fixed points \( w = 3, 4, 5 \) of \( \sigma_1 \) are represented by

\[
(w, \{1, 3\}) (w, \{2, 3\}))(w, \{1, 4\}) (w, \{2, 4\}))((w, \{1, 5\}) (w, \{2, 5\})) .
\]

We want conjugacy classes all in \( G(T_1, 1) \) stabilizing 1 in the representation \( T_1 \). Conveniently \( \sigma_3 \) is an \( m \)-cycle. As in §2.1.4, translate subscripts uniformly to get in \( G(T_1, 1) \). Translate by -1 in (B.2), and respectively by -2, +2, +1 in the expressions (B.3) corresponding to \( w = 3, 4, 5 \). Now we can drop the notation indicating 1 is fixed, and thereby recognize the two
components correspond to the orbits of $G(T_1,1)$ on
\begin{equation}
S_1 = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}\} \\
S_2 = \{\{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}.
\end{equation}
We find one 4-cycle (denoted by * in (B.2)) and one 2-tuple for each $w$
whose support is in $S_1$ in, resp., (B.2) and (B.3). Therefore the genus $g_{5,1}$
for the component corresponding to $S_1$, is given by
\[2(4+g_{5,1}-1) = 3 + 3 \cdot 1 = 6, \text{ or } g_{5,1} = 0.\]
Similarly, for these cycles remaining from (B.2) and (B.3), the genus $g_{5,2}$
for the component corresponding to $S_2$, is given by
\[2(6+g_{5,2}-1) = 3 + 1 + 3 \cdot 2 = 10, \text{ or } g_{5,2} = 0.\]
Induct on odd $m$. Assume, for $m' = 2(\ell-1)+1$, we know the correct
result and notation. Then, tack on $\{1,2\ell\}, \{1,2\ell+1\}$ to $S_1$ in (B.4), and
\[\{w,k\}, k = 2\ell, 2\ell+1, w = 2, \ldots, 2\ell-1, \text{ and } \{2\ell, 2\ell+1\} \text{ to } S_2.\]
Then, the * term in (B.2) becomes the $(m-1)$-cycle by adding $(2,\{2,2\ell\})$
and $(2,\{2,2\ell+1\})$ to the end. Then, to (B.3) add
\[((w, \{1,2\ell\}) (w, \{2,2\ell\})) \text{ and } ((w, \{1,2\ell+1\}) (w, \{2,2\ell+1\}))\]
for $w = 3, \ldots, 2\ell+1$ and for $w = 2\ell, 2\ell+1$, add $((w, \{1,k\}) (w, \{2,k\}))$,
k = 3, \ldots, 2\ell-1.
Then, by writing $w = 1+k$, translate all terms with a given value of $w$
by $-k$, to conclude that only one 2-cycle – corresponding to the given value
of $w$ (again drop the $w$ slot) now occupied by $1$ – will end up with support
in $S_1$: $(\{1-k, u-k\} \{2-k, u-k\})$ where $u-k = 1$.
Now compute the genus of the degree $m-1$ component as for $m = 5$:
\[2((m-1)+g_{m,1}-1) = m-2 + (m-2) \cdot 1 = 2(m-2), \text{ or } g_{m,1} = 0.\]
Finally, do the same for the degree $\frac{(m-1)(m-2)}{2}$ component by adding indices
of all the cycles not covered by the degree $m-1$ component:
\[2\left(\frac{(m-1)(m-2)}{2} + g_{m,2}-1\right) = m(m-3)+2g_{m,2} = \]
\[(m-4)(m-2)+\left(\frac{(m-1)(m-2)}{2}\right)-(m-3)(m-2).\]
From the leading terms of the expression, $g_{m,2}/m^2$ has limit $\frac{1}{2}$.
That completes the case for odd $m$. For even $m$, the only serious adjustment
comes in the case of arranging $\tau_2$, which now has $m/2$ disjoint cycles
of length $m-1$. The cycles, however, that appear in the degree $m-1$
component are entirely analogous. Thus, the genus computation has exactly
the same terms in the right side of RH, as a function of $m$. \qed
B.2. Rarity of genus 0 components of $\tilde{C}_{f,g}$. By the rarity of such components, we mean there are series of examples as in §B.1 that require serious work to see that from potentially infinitely many genus 0 components, there are just a finite number. This section gives an historical example.

[Fr12, §7.1] goes through many of the applications that fostered classifying separated equations $f(x) - g(y) = 0$, with $f, g \in \mathbb{Q}[x]$ that have infinitely many quasi-integral points. As previously, a direct statement was to generalize what we refer to as Ritts Theorem 2 as in §1.3. That used Siegel’s famous result on quasi-integral points on affine curves over a number field. Expanding on that were the more general Hilbert-Siegel problems.

The 1st Hilbert-Siegel Problem had these hypotheses. First, $f \in \mathbb{Q}[x]$ is indecomposable. Second,

(B.5) **Quasi-integral reducibility**: there are infinitely many $y_0 \in \mathbb{Z}[1/a]$, $a \in \mathbb{Z}$, for which $f(x) - y_0$ is reducible, but it has no zero in $\mathbb{Q}$.

The conclusion was that all but finitely many such $y_0$ are in the values of $g \in \mathbb{Q}(x)$ where $\tilde{C}_{f,g}$ has $u \geq 2$ components. There are two cases.

(B.6) **Polynomial**: either $g \in \mathbb{Q}[x]$; or

- **Double-degree**: with $\text{deg}(f) = m$, $\text{deg}(g) = 2m$ and a branch cycle $\sigma_\infty$ for $g$ over $\infty$ has the shape $(m)(m)$.

That no $g$ works in the polynomial case of (B.6) is the consequence of the branch cycle argument alluded to in the title of [Fr12]. That was a serious part of the solution of Davenport’s problem in [Fr74]. If we didn’t restrict to $\mathbb{Q}$ we would have to include the other cases of (2.19b).

The double-degree case of (B.6) had its own deeper conclusions:

(B.7) Only $m = 5$ gives nontrivial examples.

There are several Nielsen classes, but – as in the examples of §2.1 – all are coalescings of the main Nielsen class: $G = S_5$, $r = 4, C_2 = C_3$ is the class of 2-cycles, $C_1$ is the class of products of two disjoint 2-cycles and $C_4$ that of 5-cycles. [DeFr99, Thm. 1.2] concludes: Using the natural moduli space, for any fractional ideal of $\mathbb{Z}$, the set of such $f$ produces solutions dense in the points of the Hurwitz space.

This one example surprised the authors of [DeFr99, Thm. 1.1 and 1.2], for two reasons: Only one value of $m$ produced these. Yet, that one did in great abundance. [Fr99, Exp. 6.3] has Guralnick’s conjecture ([Fr12, §7.1.4]; what monodromy groups can arise often and the precise version of (2.17) for what would be the exceptional genus 0 monodromy (over $\mathbb{C}$). Now it is a theorem. In these lists you see several related to $A_n$, including with the
permutation representation of the cover acting on distinct, unordered pairs of integers. That is the case above.

Thus, even Guralnick’s strong formulation of the genus 0 problem is insufficient. [Fr86, §3] has the extra group theory for showing (B.5) holds for the groups in Guralnick’s list for the Hilbert-Siegel problem, when the double-degree condition of (B.6) holds.

[Mu96] has results under various conditions on more general versions of the Hilbert Siegel Problems as in [Fr86, §4]. To the first author’s knowledge he still relied on [Fr86, §2 and §3].

References


[AZ03] , The Equation \( f(X) = f(Y) \) in Rational Functions \( X = X(t), Y = Y(t) \), Comp. Math., Kluwer Acad. 139 (2003), 263–295.


[DeFr99] and M.D. Fried, Integral Specialization of families of rational functions, PJM 190, 1999, 75–103.


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[Fr86] ______, The Hilbert Siegel Problems and Group Theory solving cases of them, a long preprint from 1986 contained this material at the end: Rigidity and applications of the classification of simples group to monodromy. The remainder of that material has been placed in other papers, leaving only this remnant, online at http://www.math.uci.edu/~mfried/paplist-cov/HilbSieg86.pdf.


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