June 2, 1989

Dear Mike:

Enclosed is a short write up which eliminates 3.6a and 3.6c. ~~This~~ should allows us to remove pages 4.6 - 4.10 (Cases 1 and 2). Some reorganization of §5 will also be necessary

On page 4.19, I don't quite ~~understand~~ $C_G(t)\,t$, I think you mean $C_T(t)$

~~Moreover in your fusion table~~

In any case in the write up, I list the values for the char of degree nine (the way I computed was that since $G = TP$ where P is of order nine

$$\Psi(x) = \sum_{g\in P} \Psi_0(gxg^{-1}) \qquad \text{for } x\in T \;).$$

Also on page 4.9, last paragraph - it is not quite right there are 2 other chars of degree obtained by taking a one dim char Ψ on $(\mathbb{Z}_3)^2$ extending it to the centralizer - but there are 2 ways of extending it either trivially or -1

This presumably changes the computation of the sizes of the Nielsen classes

Also, the ref to [GT] is "Finite groups of genus zero" J. Algebra, to appear.

Some other comments:

On page 5.6, you remark that G acts faithfully on $H_1(\hat{X}, \mathbb{Q})$ & then say $\mathbb{Q}G$ (or $\mathbb{Z}G$) acts faithfully as well. This is <u>not</u> true (Indeed we would be done if so) in fact it cannot act faithfully since $\mathbb{Q}G$ acts faithfully is equiv. to saying every irred. constituent occurs, but we know G has no nonzero fixed points (since $g(\mathbb{P}^1)=0$). This is irrelevant for our purposes.

Also, you claim it suffices to check 4.18e by the "coalescing principle". I agree except for 4.18(d) for if $C_{\alpha\gamma, \pi, \pi, \rho\sigma, \rho, \rho}$ is "coalesced" from $C_{\pi, \pi, \rho\sigma, \rho, \rho, \rho, \rho}$ then $\alpha\gamma$ could be written as a product $\rho_1\rho_2$ where ρ_i is conj to ρ. Since $\rho_1\rho_2 \in N \Rightarrow \rho_2 = \rho_1 s$ some $s \in N$. But $\rho_1 s$ conj. to $\rho_1 \Rightarrow s \in [\rho_1, N] \Rightarrow s$ is conj. to α not $\alpha\gamma$

Thus this cannot be done. However, let

ν' be the character of $H_1^a(\hat{X}, \mathbb{Q})$ in this case

Then $(\nu', \psi) = 2$ ($\deg \psi = 9$ our fav. char)

$\Rightarrow$ $\exists$ a nontrivial endo. in this case

Here is the char ν'

~~Also my no~~

Conj class # τ	$\nu'(\tau)$
1	1154
2	—
3	—
4	—
5	—
6	—
7	2
8	−62
9	−94
10	2
11	2
12	2
13	−14
14	2
15	2
16	—
17	−46
18	2
19	2
20	2

Nota bene: D_n here denotes the dihedral group of order $2n$. This is not the same as in the paper.

We will eliminate groups (3.6)(a) and (c) as possible groups which appear as the group of the Galois closure of some map of the generic curve of genus 2 to $\mathbb{P}^1_2$.

We use the following principle.

~~Case 3.6a $G = D_5$~~

If $\phi: X \to \mathbb{P}^1_2$ is a cover of the generic curve of genus 2 and G is the group of the Galois closure $\hat{X}$ of X, then no element of $\mathbb{Z}G$ acting on $V = H_1(\hat{X}, \mathbb{Q})$ induces a nonscalar linear transformation on $H_1(X, \mathbb{Q})$. Here $H_1(X, \mathbb{Q})$ is identified with V^H, the H fixed points (where H is the subgroup corresponding to X).

The following lemma will be useful.

Lemma Let G be a finite group, $H \leq G$, and W a KG-module where K is a field of characteristic zero. If U_1 and U_2 are nonisomorphic irreducible constituents of I_H^G and W, then there exists c in the center of $\mathbb{Z}G$ such that $c|_{W^H}$ is not a scalar.

Proof. Let φ_1 and φ_2 be the characters of U_1 and U_2, respectively. Let $n_i = \varphi_i(1)$. Since φ_1 and φ_2 and linearly independent as functions on the center of $\mathbb{C}G$ (and so $\mathbb{Q}G$), there exists c in the center of $\mathbb{Z}G$ such that $n_2\varphi_1(c) \neq n_1\varphi_2(c)$. Since $c|_{U_i} = \frac{\varphi_i(c)}{n_i} I$ we see c has distinct eigenvalues on U_1 and U_2. Since $U_i \cap W^H \neq 0$ for each i (as U_i is a constituent of I_H^G), we see that $c|_{W^H}$ is not a scalar.

<u>Remark</u> Conversely, if W^* and I_H^G have at most one common constituent U and U occurs only once in I_H^G, then no such elements of $\mathbb{Z}G$ can exist.

<u>Proposition</u> If G is as in 3.6a or 3.6c, then there exists $c \in$ center $\mathbb{Z}G$ which induces a nonscalar on $H_1^*(X, \mathbb{Q})$.

<u>Proof.</u> By the previous lemma, we need only show that two different constituents are common to I_H^G and $V = H_1(\hat{X}, \mathbb{Q})$. We note the following facts:

(i) V is defined over $\mathbb{Q}$ (in particular the character of V is rational valued)

(ii) If L is a sugroup of G, then $V^L \cong H_1(Y_L, \mathbb{Q})$ where Y_L is the surface corresponding to L. In particular, $\dim V^L = 2g(Y_L)$ ~~and~~ So $\dim V^G = 2g(\mathbb{P}_2^1) = 0$, and $\dim V^H = 4$.

By (ii) G has no trivial constituents.

First consider 3.6a. So $G = D_5$ and $|H| = 2$. Then $1_H^G = 1_G + \chi_1 + \chi_2$ where χ_1 and χ_2 are Galois conjugates of one another. Since V is defined over $\mathbb{Q}$, the multiplicity of χ_1 and χ_2 are the same in V. Since 1_G does not occur in V, χ_1 and χ_2 each occur twice in V, yielding the result

Next consider 3.6c. Here $G = \mathbb{Z}_3^2 \times_s D_4$ and $H = D_4$. In this case $1_H^G = 1_G + \chi_1 + \chi_2$ where ~~χ_1 and χ~~ $\chi_2 = \chi_1 \circ f$, where f is an outer automorphism of G (which switches the two conjugacy classes of reflections). Let χ be the character of $V = H_1(\hat{X}, \mathbb{Q})$, ~~Note that $\{\sigma_1, \dots, \sigma_6\}$ (see 3.6) is invariant under f.~~ Since V depends only on the conjugacy classes of $\{C_1, \dots, C_6\}$ where C_i is the conjugacy class of σ_i (see 3.6) and $\{C_1, C_2, \dots, C_6\}$ is invariant under f, $\chi = \chi \circ f$. Thus the multiplicity of χ_1 and χ_2 in V is the same, and we can argue as in the previous case.

<u>Remarks</u> In cases 3.6d, ~~and 3.6f (in~~ G is a doubly transitive group on the cosets of H. Thus $1_H^G = 1_G + \Psi$ where Ψ is irredecible. So no such central element of $\mathbb{Z}G$ can exist. (Similarly, whenever G is 2 transitive on H, no such element exists. In particular, this ~~implies~~ shows that one cannot eliminate A_n in this manner)

In case 3.6f, one must explicitly determine the character of $H_1(\hat{X}, \mathbb{Q}) = V$ to ~~determ~~ see what happens. By (ii) in the proof of the Proposition above, we can determine $\dim V^x$ for each x in G. Since V is defined over $\mathbb{Q}$, this ~~determines~~ is sufficient to compute the character of V. In particular let ~~...~~ $T \in Syl_2(G)$. Let μ_0 be a one dimensional character of N with $\ker \mu = [T, N]$. Extend μ to T by letting it be trivial on D.

Then $\Psi = \chi_{oT}^{G}$ is irreducible of degree nine. Let ν be the character of $H^1(\hat{X}, \mathbb{Q})$ in the case 4.18e. Here are the values

#	$C = Con(\tau)$	$C = Con(\tau^2)$	ord τ	$\|Con(\tau)\|$	$\Psi(\tau)$	$\nu(\tau)$
1	ι	1	1	1	9	1730
2	λ	λ	3	16	0	–
3	$\lambda\mu$	$\lambda\mu$	3	64	0	–
4	$\lambda\delta$	λ	6	48	0	–
5	$\lambda\sigma$	λ	6	96	0	–
6	$\lambda\sigma\delta$	$\lambda\delta$	12	96	0	–
7	α	1	2	6	-3	2
8	$\alpha\gamma$	1	2	9	1	2
9	ρ	1	2	12	3	-190
10	$\rho\beta$	α	4	12	-3	2
11	$\rho\gamma$	1	2	36	-1	2
12	$\rho\gamma\beta$	α	4	36	1	2
13	$\rho\sigma$	1	2	36	1	-14
14	$\rho\sigma\beta$	α	4	72	-1	2
15	$\rho\sigma\beta\delta$	$\alpha\gamma$	4	36	1	2
16	$\lambda\mu\pi$	$\lambda\mu$	6	192	0	—
17	π	1	2	24	3	-46
18	$\pi\alpha$	$\alpha\gamma$	4	72	-1	2
19	$\rho\pi$	$\rho\sigma$	4	144	1	2
20	$\rho\pi\beta$	$\rho\sigma\beta\delta$	8	144	-1	2

Thus $(\Psi, \nu) = \frac{1}{|G|} \sum_{x \in G} \Psi(x)\overline{\nu(x)} = 4 = (1_H^G, \nu)$.

Hence Ψ is the only common constituent of 1_H^G and ν, and we cannot eliminate this group as a possibility by this method

<u>Theorem A</u> ~~Let~~ ~~For~~ $\exists$ a positive integer $N'(g)$ so that if $X \in \mathcal{M}_g$ and $\varphi: X \to \mathbb{P}^1$ is a solvable cover, $\exists$ $\varphi': X \to \mathbb{P}^1$ a solvable cover with $\deg \varphi' \le N'(g)$.

<u>Proof.</u> Choose Y so that

$$\varphi: X \xrightarrow{\alpha} Y \xrightarrow{\beta} \mathbb{P}^1 \quad \text{factors where}$$

$\alpha: X \to Y$ is primitive (note this is also solvable)

<u>Case 1</u> If $g(Y) = g' = 0$, then $\deg \alpha \le N(g)$ by $(*)$.

<u>Case 2</u> If $g(Y) = 1$, then by Riemann-Hurwitz

$$2g-2 = \sum_{i=1}^{r} \operatorname{ind} h_i \quad \text{where } h_i \in \text{Galois group}$$

of the normal closure of X over Y. If $r=0$, then $g=1$ and we can clearly take $N'(1) = 2$.

Otherwise $r \ge 1$ and by [FrG] (as $X \to Y$ is primitive) $\operatorname{ind} h_i \ge n/4$. ~~Thus~~ (where $n = \deg \alpha$). Thus $n \le 8g-8$. Moreover, we can obtain a solvable cover of degree $2n \le 16g-16$ via $X \xrightarrow{\alpha} Y \xrightarrow{\gamma} \mathbb{P}^1$ where $\deg \gamma = 2$.

<u>Case 3</u> If $g(Y) = g' > 1$, then by Rieman-Hurwitz, $2g-2 \le n\,(2g'-2)$, whence $n \le \dfrac{g-1}{g'-1}$.

G to F2

Since $\beta : Y \to \mathbb{P}^1$ is solvable, by induction on g

$\exists \quad \beta' : Y \to \mathbb{P}^1$ solvable of deg $\leq N'(g')$.

Hence $X \xrightarrow{\alpha} Y \xrightarrow{\beta'} \mathbb{P}^1$ is solvable with deg $\leq \frac{g-1}{g'-1} \cdot N'(g')$.

So the theorem holds if

$N'(0) = 1, \; N'(1) = 1$ & for $g > 1$

$$N'(g) = \max \left\{ N(g), \; 16g - 16, \; \frac{g-1}{g'-1} N'(g'), \; g' = 1, 2, \ldots, \left[\frac{g}{2}\right] \right\}$$

<u>Cor. B</u> $\mathcal{M}_g(\mathrm{Sol}) = \mathcal{M}_g(G_1) \cup \cdots \cup \mathcal{M}_g(G_r)$ for some finite list of solvable groups $G_1, \ldots, G_r$

<u>Cor C</u> If $g > 6$, $\mathcal{M}_g(\mathrm{Sol})$ is a quasiprojective subvariety with $\dim \mathcal{M}_g(\mathrm{Sol}) < \dim \mathcal{M}_g$.

<u>Proof</u> this holds for $\mathcal{M}_g(G_i)$ for each i by [FrG] (or [Z]).

Let $\mathcal{M}_g$ denote the moduli space of curves of genus g. Let $\mathcal{M}_g(\mathrm{Sol}) = \{ X \in \mathcal{M}_g \mid \exists$ a solvable (branched) cover $X \to \mathbb{P}^1 \}$. Let $\mathcal{M}_g(\mathrm{Prim\text{-}Sol}) = \{ X \in \mathcal{M}_g \mid \exists$ a solvable primitive cover $X \to \mathbb{P}^1\}$. If G is a fixed group, set $\mathcal{M}_g(G) = \{ X \in \mathcal{M}_g \mid \exists$ a cover $X \to \mathbb{P}^1$ with Galois group of the normal closure $G \}$

We assume the "almost" proved Conjecture (by Neubauer)

(*) If $g > 0$, there are only finitely many primitive solvable groups arising from primitive covers $X \to \mathbb{P}^1$, or ~~equivalently~~ equivalently $\exists$ a positive integer $N(g)$, so that if $X \to \mathbb{P}^1$ is a primitive solvable cover, then $\deg X/\mathbb{P}^1 = n \le N(g)$

This yields: