THE GENERIC CURVE OF GENUS g>6 IS NOT UNIFORMIZED BY RADICALS

M. Fried UCI and U of Florida[1]

R. Guralnick USC[2]

ABSTRACT: The collection of nonsingular projective curves of genus g is parametrized by an irreducible algebraic variety $\mathcal{M}_g$ of dimension 3g-3 if g>1, and of respective dimensions 0 and 1 if g=0 or 1. For $\mathbf{m} \in \mathcal{M}_g$ let $X_{\mathbf{m}}$ be the corresponding member of this collection. Denote its field of meromorphic functions by $\mathbb{C}(X_{\mathbf{m}})$. Then $X_{\mathbf{m}}$ is said to be uniformized by radicals if there exists $z \in \mathbb{C}(X_{\mathbf{m}})$ such that the Galois closure of $\mathbb{C}(X_{\mathbf{m}})/\mathbb{C}(z)$ has a solvable Galois group. The Main Theorem: If $\mathbf{m}$ is a generic point of $\mathcal{M}_g$ and g>6 then $X_{\mathbf{m}}$ is not uniformized by radicals. Discussion of the exceptional cases, especially g=0,1 and 2, relates this paper to the massive program of [GTh].

1. Do mention complete uniformization by radicals - but we don't know how to do it. (g. 2. Do we know of any g for which general curve X of genus g which X → $\mathbb{P}^1$ has G ≠ S_n as group. We can answer this, even we don't answer now.

[1]Partially supported by NSF grant DMS-8508962.

[2]Partially supported by NSF grant DMS-8401008.

§1. INTRODUCTION: Let $\mathbb{P}^1_z$ denote the Riemann sphere with a uniformizing variable z. That is, $\mathbb{P}^1_z = \mathbb{C}\cup\{\infty\}$ where $\mathbb{C}$ is the z-plane. We give a brief overview of Riemann's existence theorem.

§1.1. BRANCH CYCLES AND MODULI: Select $\{z_1,z_2,\dots,z_r\}$, distinct points of $\mathbb{P}^1_z$, and consider $\varphi: X \to \mathbb{P}^1_z$, a cover of compact connected Riemann surfaces of degree n ramified only over $\{z_1,z_2,\dots,z_r\}$. Form the Galois closure $\hat{\varphi}:\hat{X} \to \mathbb{P}^1_z$, the smallest Galois cover of $\mathbb{P}^1_z$ that factors through φ. From Riemann's existence theorem, (X,φ) is determined (up to equivalence of covers) by an r-tuple $\boldsymbol{\sigma}=\{\sigma_1,\sigma_2,\dots,\sigma_r\}\in S_n^r$ with these properties:

(1.1) $\sigma_1\sigma_2\cdots\sigma_r=1$, and $G(\boldsymbol{\sigma})=G(\hat{X}/\mathbb{P}^1_z)$ (noncanonically), where $G(\boldsymbol{\sigma})$ is the subgroup of S_n generated by the entries of $\boldsymbol{\sigma}$. The r-tuple $\boldsymbol{\sigma}$ is called a description of the branch cycles of (X,φ), but it should be stressed that this branch cycle information depends on the choice of other data : a base point z_0; and special homotopy classes of paths that generate the fundamental group $\pi_1(\mathbb{P}^1_z-\mathbf{z},z_0)$ where we let $\mathbf{z}$ here denote $\{z_1,z_2,\ldots,z_r\}$, as in [Fr,1; §0.A].

From the Riemann-Hurwitz formula the genus, $g(X)=g$, of X is given by

$$(1.2)\quad 2(n+g(X)-1)=\sum_{i=1}^{r}\mathrm{ind}(\sigma_i)$$

where $\mathrm{ind}(\sigma)$ denotes n minus the number of disjoint cycles of σ.

The moduli space $\mathfrak{M}_g$ of (compact) Riemann surfaces of genus g is an irreducible algebraic variety with the following properties:

(1.3) a) it is of dimension $3g-3$ if $g>1$ (of dimension g if $g=0$ or 1);

b) each point $\mathbf{m}\in\mathfrak{M}_g$ corresponds to one and only one isomorphism class of surfaces of genus g; and

c) the labeling of b) induces a unique complex analytic morphism from any complex analytic family of surfaces of genus g.

We explain (1.3) c) more fully. Suppose that $\Phi:\mathcal{T}\to\mathcal{P}$ is a complex analytic map of irreducible nonsingular complex manifolds such that for each $\mathbf{p}\in\mathcal{P}$ the fiber $\mathcal{T}_\mathbf{p}$ of points of $\mathcal{T}$ lying over $\mathbf{p}$ is a (compact) surface of genus g. Then b) determines a map $\Psi(\mathcal{P},\mathfrak{M}_g):\mathcal{P}\to\mathfrak{M}_g$ by sending $\mathbf{p}$ to the point representing the isomorphism class of $\mathcal{T}_\mathbf{p}$;

$\mathcal{T}_{\mathbf{p}}$; and c) asserts that this map is complex analytic. The construction of $\mathcal{M}_g$ is a heavy duty matter, and all proofs could incorporate the local moduli argument that already appears in [Fr,2; p.26]. Coordination of the facts, however, about moduli parameters comes more easily from acceptance of (1.3) a), b) and c).

§1.2. STATEMENT OF RESULTS: Consider the collection of groups, as subgroups of S_n for some integer n, that arise from some map $\varphi: X_{\mathbf{m}} \to \mathbb{P}^1_z$ with $X_{\mathbf{m}}$ the generic curve of genus g ($n=\deg(\varphi)$) as the Galois group of the Galois closure $\hat{\varphi}: \hat{X}_{\mathbf{m}} \to \mathbb{P}^1_z$ of the map. Denote this collection by $\mathcal{G}_g$.

This paper is a contribution to the study of $\mathcal{G}_g$. The group corresponding to the cover $\varphi: X_{\mathbf{m}} \to \mathbb{P}^1_z$ is a primitive subgroup of S_n if and only if for any diagram of Riemann surfaces,

(1.4) $\varphi: X_{\mathbf{m}} \to Y \to \mathbb{P}^1_z$,

either $Y = X_{\mathbf{m}}$ or $Y = \mathbb{P}^1_z$. Let $\mathcal{G}_g(\text{prim})$ be the subset of primitive groups that appear in $\mathcal{G}_g$. Let $\mathcal{G}_g(\text{sol})$ be the solvable subgroups of $\mathcal{G}_g$.

Main Theorem: *For $g>6$, $\mathcal{G}_g(\text{sol})$ is empty.*

The proof of the Main Theorem appears in §3.1 as a special case of Proposition 3.1. This answers the question of [FrJ; top of p.137]. In turn this was ancillary to the still unsolved [FrJ; Problem (10.16) a)]: Does every absolutely irreducible variety over the solvable closure of $\mathbb{Q}$ have a rational point?

The "Geometric Principles" of §2 combined with relatively modest group theory give us the Main Theorem and all of the exceptional cases for g=3 to 6 (cf., the exclusion of the case of (g,n)=(3,8) in Lemma 3.2). In particular, Proposition 2.3 reduces the main consideration to the study of $\mathcal{G}_g(\mathrm{sol})\cap\mathcal{G}_g(\mathrm{prim})$. We give a finite list in Theorem 3.3 containing all of the possible permutation groups that appear in this collection when g=2.

Finally, Theorem 3.4 nearly completes the proof that $\mathcal{G}_g(\mathrm{sol})\cap\mathcal{G}_g(\mathrm{prim})$ is finite for all g>0 by showing that in the case g=1, the only exceptions are primitive solvable groups of degree 2^e, 3^e, 5^e or 7^e for some integer e. Probably there are just finitely many possibilities for e, but the effort in carrying this through is likely to duplicate much of [GTh]. Since this carries us away from the major theme still left from this paper, completion of the topic of Theorem 3.4 will await a later work.

We continue the discussion on the finite number of exceptional groups that appear in $\mathcal{G}_g(\mathrm{sol})\cap\mathcal{G}_g(\mathrm{prim})$ as Galois groups of the Galois closure of some cover of $\mathbb{P}^1_z$ by the generic curve of genus g=2,...,6. These are a subset of the lists of Proposition 3.1 and Theorems 3.3 and 3.4. The possible presentations of these groups through branch cycles raises a serious question. Applying Riemann's existence theorem we easily see that each such group arises as the Galois group of the Galois closure of some cover $\varphi:X\to\mathbb{P}^1_z$ with g(X)=g. But how do we know if the generic curve of genus g produces this group by some map to $\mathbb{P}^1_z$ (or, more pointedly, by some map with the specified branch cycle type)? This special case of the computation of the *moduli dimension of a Nielsen class* (§5.1) may be the most interesting topic in the whole paper.

We concentrate on the four toughest cases where $g>1$: § 4.3 gives branch cycle descriptions for the cases $(g,n)=(2,5)$ from (3.6) a), and (2,9) from (3.6) c) and d); and §4.4 does the same for $(g,n)=(2,16)$ from (3.6) f). We organize here with the tools called Nielsen classes and Hurwitz monodromy group action. In §5.1 and §5.2 we use principles based on "coelescing of branch points" and "the generic curve of genus at least 1 has no nontrivial endomorphisms" to decide finally (Proposition 5.6) which groups (more precisely, Nielsen classes-remaining from §4) are in $\mathcal{G}_2(\text{sol})$. Indeed, Ries [R] independently-with calculations of distinctly different flavor from ours-considered D_{10} (i.e., $(g,n)=(2,5)$), and the conclusion that $D_{10}\notin\mathcal{G}_2(\text{sol})$ is due to him.

It is well known that for $n>2g-1$, there is a simple branched cover $\varphi: X_{\mathbf{m}} \to \mathbb{P}^1_z$ of degree n (i.e., the branch cycles are all 2-cycles). In particular $\mathcal{G}_g(\text{prim})$ contains S_n for $n>2g-1$. Principle 2.5 shows that S_n appears in $\mathcal{G}_g(\text{prim})$ if and only if $n\ge[(g+3)/2]$ where [ ] denotes the greatest integer function. By extending the treatment of the examples of § 4.3 and 4.4, § 5.3 considers how one might test for the n's for which $A_n\in \mathcal{G}_g(\text{prim})$. In the process it makes comment on the cases $(g,n)=(2,3)$ and (2,4) left over from the list (3.6). Here there arises the more delicate question of the minimal value of r for which the generic curve of genus 2 does produce a cover with the groups S_3 and S_4.

The results of this paper (e.g., the Main Theorem) use the phrase generic curve to mean that the statement applies to a Zariski open subset of points on $\mathcal{M}_g$. For example: For $G\in\mathcal{G}_g$, points of $\mathcal{M}_g$ represented by surfaces X that can be presented as a cover of $\mathbb{P}^1_z$ with Galois closure having group G form a Zariski open subset of points on $\mathcal{M}_g$. Also, for a fixed group G, $G\notin\mathcal{G}_g$, the subset of points of $\mathcal{M}_g$ represented by surfaces X that can be presented as a cover of $\mathbb{P}^1_z$ with Galois closure having group G is algebraic of codimension at least 1. On the other hand, consider the set $\{\mathbf{m}\in\mathcal{M}_g \mid X_{\mathbf{m}} \to \mathbb{P}^1_z$ has solvable Galois closure$\}$, denoted $\mathcal{M}_g(\mathrm{sol})$. The Main Theorem does not quite proclude the possibility that $\mathcal{M}_g(\mathrm{sol})$ may be dense in $\mathcal{M}_g$ for some $g>6$; but we suspect that it is not.

ACKNOWLEDGEMENT: This paper seems to be but one corner of a project that has been envisioned by John Thompson. The goal is to divine some natural limits placed by group theory on the generic way we may expect to realize groups as Galois groups of regular extensions of $\mathbb{Q}(t)$. Essentially, for any fixed genus g , excluding the usual elementary (cyclic by cyclic) affine groups , S_n and A_n, it seems to be that there are

only finitely many primitive groups that arise as the Galois group of the Galois closure of $\varphi: X \to \mathbb{P}^1_{\mathbb{Z}}$ with $g(X)=g$.

The case $g=0$ of [GTh] is especially important and has special application to this paper in relating $\mathcal{G}_g(\text{prim})$ to $\mathcal{G}_g$ (resp., $\mathcal{G}_g(\text{sol}) \cap \mathcal{G}_g(\text{prim})$ to $\mathcal{G}_g(\text{sol})$). In the case $g>1$ the former is a subgroup of a series of wreath products formed from groups $G_1,\dots,G_v$ where $G_i \in \mathcal{G}_g(\text{prim})$ (resp., $\mathcal{G}_g(\text{sol}) \cap \mathcal{G}_g(\text{prim})$) and $G_i \in \mathcal{G}_0(\text{prim})$ (resp., $\mathcal{G}_0(\text{sol}) \cap \mathcal{G}_0(\text{prim})$), $i=2,\dots,v$. (The case $g=1$ has an extra slight, but not terribly damaging, complication explained in §3.3.) We expect this all to have solid implications about the nature of the families of curves having specific arithmetic properties for their specializations over $\mathbb{Q}$; not just for the problem of groups as Galois groups, but as in [DFr] for the problem of finding elliptic curves of high rank over $\mathbb{Q}$. Thompson worked out many of the details of §4.4.

Finally, in addition to pointing out the applicability of the "enodomorphism principle" in §5.1, John Ries apprised us of the relevance of the computation of the moduli dimension of a Nielsen class to Zariski's paper [Z].

§2. GEOMETRIC PRINCIPLES: The proof of the Main Theorem depends only on the Riemann-Hurwitz formula, Principles 2.1 and 2.2 and Proposition 2.3.

§2.1. PRINCIPLES ABOUT $\mathfrak{M}_g$: GENERIC CURVES OF GENUS g:

Principle 2.1: *If $\Psi(\mathbb{P},\mathfrak{M}_g)$: $\mathbb{P} \to \mathfrak{M}_g$ has image a Zariski open subset of $\mathfrak{M}_g$ and the general fiber of $\Psi(P,\mathfrak{M}_g)$ has dimension k, then the dimension of P is $\dim(\mathfrak{M}_g)+k$.*

Principle 2.2: *Assume that $\boldsymbol{m}\in\mathfrak{M}_g$ is generic and that $X_{\boldsymbol{m}}$ represents the isomorphism class of m. If there exists $\varphi:X_{\boldsymbol{m}} \to \mathbb{P}^1_z$ with r branch points, then $r\geq 3g$ if $g>1$ and $r\geq 4$ if $g=1$.*

Argument: We may assume that $X_{\mathbf{m}}$ and φ are defined over a field L of finite type over $\mathbb{Q}$. Add to L three of the branch points z_1,z_2,z_3 of the cover. Let $\psi:\mathbb{P}^1_z \to \mathbb{P}^1_z$ be a linear fractional transformation that takes z_1,z_2,z_3, in order to u_1,u_2,u_3, where the u's are algebraically independent indeterminants over L. Instead of φ, consider the morphism $\varphi'=\psi\circ\varphi:X_{\mathbf{m}} \to$

$\mathbb{P}^1_z$ defined over $L(\mathbf{u})$, and denote the algebraic closure of $\mathbb{Q}$ in L by K. Note that specializations of $L(\mathbf{u})$ that leave φ invariant constitute a three dimensional family.

A standard argument shows that there is a variety $\mathcal{V}$ with function field over K isomorphic to $L(\mathbf{u})$ and a variety $\mathfrak{T}'$ with a finite morphism Φ': $\mathfrak{T}' \to \mathcal{V}\times\mathbb{P}^1_z$ with the following properties:

(2.1) a) $\mathfrak{T}'$ and Φ' are defined over K;

b) for $\mathbf{v}\in\mathcal{V}$ generic, the fiber $\mathfrak{T}'_{\mathbf{v}}$ as a cover of $\mathbb{P}^1_z$ by projection on the second factor of $\mathcal{V}\times\mathbb{P}^1_z$ is equivalent to the cover $\varphi':X_{\mathbf{m}} \to \mathbb{P}^1_z$;

c) the natural map $\Psi(\mathcal{V},\mathfrak{M}_g)$: $\mathcal{V} \to \mathfrak{M}_g$ is onto a Zariski open subset of $\mathfrak{M}_g$ and its general fiber is of dimension at least 3; and

d) for each $v\in\mathcal{V}$, the cover $\mathfrak{T}'_{\mathbf{v}} \to \mathbb{P}^1_z$ has exactly r branch points.

Indeed, once we have a variety $\mathcal{V}_1$ over K with function field isomorphic to $L(\mathbf{u})$ we form $\mathcal{V}_1\times\mathbb{P}^1_z$ and take the integral closure of this variety in $L(\mathbf{u})(X_{\mathbf{m}})$, the field of functions of $X_{\mathbf{m}}$ over $L(\mathbf{u})$. The result is a finite morphism $\Phi'_1 : \mathfrak{T}'_1 \to \mathcal{V}_1\times\mathbb{P}^1_z$ [M; p.396-397]. Also, all of the properties a), b) and c) follow for this family automatically. By removal of a closed subset of codimension at least 1 from $\mathcal{V}_1$ we get $\mathcal{V}$ for which d) also holds. □

Proposition 2.3: *Suppose that* $X_{\mathbf{m}}$ *is a generic curve of genus g with g>1. Suppose that there is a diagram (1.4). Then either* $Y=X_{\mathbf{m}}$ *or Y is of genus 0. In particular, for g>1, all subgroups of elements of* $\mathcal{G}_g(sol)$ *arising as the stabilizer of a maximal system of imprimitivity are members of* $\mathcal{G}_g(sol) \cap \mathcal{G}_g(prim)$.

Proof: The first part is in [Fr,2; p.26], but the key points can be said quickly in terms of $\mathcal{M}_g$ and $\mathcal{M}_{g'}$ where $g'=g(Y)$. From (1.4) there is a rational map $\Gamma:\mathcal{M}_g \to \mathcal{M}_{g'}$. Let $\mathcal{N}$ be the image of this map. Also let r be the number of branch points (on Y) of the cover $\varphi':X_{\mathbf{m}}\to Y$ in (1.4). By replacing $\mathcal{N}$ by a nonempty Zariski open subset of $\mathcal{N}$ we may construct a new variety $\mathcal{T}_{\mathcal{N}}$ with these properties:

(2.2) a) there is a surjective map $\nabla:\mathcal{T}_{\mathcal{N}}\to \mathcal{N}$ and a map $\Lambda:\mathcal{T}_{\mathcal{N}} \to\mathcal{M}_g$ such that $\Gamma\circ\Lambda$ is ∇;

b) for $\mathbf{n}$ contained in a Zariski open subset of $\mathcal{N}$ the fiber $\mathcal{T}_{\mathcal{N},\mathbf{n}}$ over $\mathbf{n}$ is isomorphic to $(Y_{\mathbf{n}})^{(r)}$, the symmetric product of a representative of the class of $\mathbf{n}$;

c) Λ is defined by mapping the class of $X_{\mathbf{m}}$ to the point of the fiber over $\mathbf{n}$=[Y] that consists of the divisor of degree r on Y composed from the branch points of φ' ; and

d) Λ is birational.

Indeed, construct $\mathfrak{T}_{\mathfrak{N}}$ by taking the integral closure of $\mathfrak{N}$ in the function field of $(Y_{\mathbf{n}})^{(r)}$ with $Y_{\mathbf{n}}$ a representative of the generic point $\mathbf{n}$ of $\mathfrak{N}$. Verifying (2.2) is similar to verifying (2.1).

Since $\dim(\mathfrak{T}_{\mathfrak{N}})$ is at most $3g'-3+r$ (Principle 2.1)conclude easily from (2.2) d) that $3g-3 \le 3g'-3+r$. Also, from the Riemann-Hurwitz formula, $2g-2 \ge \deg(\varphi')(2g'-2)+r$. Thus $g'-1+r/3 \ge \deg(\varphi')(g'-1)+r/2$. If $g'>1$, then $\deg(\varphi')=1$. Note that if $g'=1$, then r=0.

Now suppose that $\mathcal{G}_g(\mathrm{sol})$ is nonempty . We are done if $\mathcal{G}_g(\mathrm{sol}) \setminus \mathcal{G}_g(\mathrm{sol}) \cap \mathcal{G}_g(\mathrm{prim})$ is empty. Otherwise there is a diagram (1.4) with $Y \ne X_{\mathbf{m}}$ or $\mathbb{P}^1_Z$. But from the first part, Y is of genus zero (i.e., isomorphic to $\mathbb{P}^1_Z$). Thus the Galois group arising from $X_{\mathbf{m}} \to Y$, the subgroup stabilizing the system of imprimitivity corresponding to Y, is in $\mathcal{G}_g(\mathrm{sol})$. Continue this process until Y corresponds to a maximal system of imprimitivity. □

§2.2. **DEGREES OF SIMPLE COVERS:** We investigate the exceptional cases for g=3,...,6 that occur in Proposition 3.1 with the aid of the following two principles:

Principle 2.4: *[KL] The smallest possible degree n(g) for which every surface X of genus g has a covering $X \to \mathbb{P}^1_z$ of degree n is $n(g)=[(g+3)/2]$.*

Principle 2.5: *If any surface X of genus g has a covering $X \to \mathbb{P}^1_z$ of degree d, then some surface X′ of genus g has a simple branched covering $X' \to \mathbb{P}^1_z$ of degree n. In particular, $S_n \in \mathcal{G}_g(\text{prim})$ if and only if $n \geq [(g+3)/2]$.*

Proof: Apply Riemann's existence theorem (§1) to $X \to \mathbb{P}^1_z$ to get a description $\boldsymbol{\sigma}$ of the branch cycles of the cover. Each σ_i can be written as a product of $\text{ind}(\sigma_i)$ 2-cycles, $\tau_{i,1},\dots, \tau_{i,u}$, $u=\text{ind}(\sigma_i)$. Now juxtapose the τ's to give a branch cycle description of a simple branched cover $X' \to \mathbb{P}^1_z$ of the same genus and degree as $X \to \mathbb{P}^1_z$. An argument using the branch points as parameters can be easily constructed (as in [Fr,2;§4]) to show that such a simple branched covering $X' \to \mathbb{P}^1_z$ can be put in an algebraic family which includes the original cover $X \to \mathbb{P}^1_z$ as a specialization.

If $S_n \in \mathcal{G}_g(\text{prim})$, then by Principle 2.4, $n \geq [(g+3)/2]$. Conversely suppose that $n \geq [(g+3)/2]$. Then the general surface $X_{\mathbf{m}}$ of genus g has a map of degree n to $\mathbb{P}^1_z$. But from the last sentence of the first paragraph, there exists a simple branched cover $X' \to \mathbb{P}^1_z$, with X' of genus g, that specializes to the cover $X_{\mathbf{m}} \to \mathbb{P}^1_z$. Thus X' is a generic surface of genus g, and $S_n \in \mathcal{G}_g(\text{prim})$. □