§3. MAIN THEOREM; GROUP THEORY AND EXCEPTIONAL CASES: From Proposition 2.3 we have only to show that $\mathcal{G}_g(\mathrm{sol})\cap\mathcal{G}_g(\mathrm{prim})$ is empty for $g>6$. Excluding Theorem 3.4, we assume in this section that $g>1$.

§3.1. PROOF OF THE MAIN THEOREM: Suppose that $G\in\mathcal{G}_g(\mathrm{prim})$. Riemann's existence theorem says that there exists an integer r and $\sigma\in S_n^r$ with properties (1.1).

Suppose that there is a constant α such that

(3.1) $\mathrm{ind}(\sigma_i)\geq \alpha n$, $i=1,\ldots,r$.

According to Principle 2.2, $r\geq 3g$. Therefore an application of (1.2) gives

(3.2) $2(n-1)/(3\alpha n-2)\geq g$ with equality if and only if $\mathrm{ind}(\sigma_i)=\alpha n$, $i=1,\ldots,r$.

Proposition 3.1: *In the above formula, if G is a primitive subgroup of S_n with a minimal normal subgroup N that is abelian, then $n=p^e$ for some prime p. If $p\neq 2$ we may take $\alpha=1/3$ in (3.1). If $p=2$ then we may take $\alpha=1/4$. In particular, this applies to the case that $G\in\mathcal{G}_g(\mathrm{sol})\cap\mathcal{G}_g(\mathrm{prim})$.*

The exceptional values for (g,n) in (3.2), with $g>1$, are (6,4),(5,4),(4,4),(4,3),(3,3),(3,4),(3,8) and (2,n) for any primepower n.

Proof: Let G be a transitive subgroup of S_n. For $\sigma\in G$ the following hold:

(3.3) a) if σ has no fixed points, then $\mathrm{ind}(\sigma)\geq n/2$; and

b) $\mathrm{ind}(\sigma)\geq\mathrm{ind}(\sigma^k)$ for any integer k.

Now assume that G is primitive, that H is the stabilizer of a point in the permutation representation and that N, a minimal normal subgroup, is abelian. It is well known (e.g., [Bu]) that $p^e=n$, that the permutation action of H is equivalent to the faithful and irreducible action of H by conjugation on the elements of N; and that $G=N\times^s H$ (semidirect product). Thus if $\sigma\in G$ has a fixed point, then σ is conjugate to an element of H, and the permutation action of H is equivalent to the conjugation action on N.

From (3.3) we prove the statements about α if we show that for $\sigma\in H$ of prime order, say q,

(3.4) $\mathrm{ind}(\sigma)\ge (q-1)(p-1)n/qp$.

But $\mathrm{ind}(\sigma)$ is n minus the number of orbits of σ acting by conjugation on N. By the class equation the cardinality of these orbits is

$$|C_N(\sigma)|+(n-|C_N(\sigma)|)/q\ge (p^e-p^{e-1})/q.$$

From this (3.4) follows.

Consider the exceptional cases for (g,p^e) from the inequality (3.2), $2(n-1)/(3\alpha n-2)\ge g$, where αn is the right side of (3.4). For n odd we may replace α by 1/3. Thus $g\le 2(n-1)/(n-2)$ gives (2,any odd primepower), (3,3) and (4,3).

For n even we may replace α by 1/4. Thus $g\le(2^e-1)/(3\cdot 2^{e-3}-1)$ gives (g,4), g=3,4,5,6, (3,8) and (2,any power of 2). □

Remark: Let Ω denote those permutation groups G that have a normal subgroup N (not necessarily abelian) such that G=NH and the restriction of the permutation action to N is equivalent to the regular representation of N. Define $\mathcal{G}_g(\Omega)$ in a manner analogous to the above. With slight modification, the proof of Proposition 3.1 can be improved to show that

$\mathcal{G}_g(\Omega)\cap\mathcal{G}_g(\text{prim})$ is empty for $g>6$. □

§3.2. FINITENESS OF $|\mathcal{G}_g(\text{SOL})\cap\mathcal{G}_g(\text{PRIM})|$ FOR $g>1$: Since S_3 and S_4 are solvable, Principle 2.5 tells us that the cases of form $(g,3)$ or $(g,4)$ in Proposition 3.1 actually are exceptions to the Main Theorem if and only if $n(g)\le 3,4$ respectively. For example $n(6)=[9/2]=4$. This leaves the cases

(3.5) a) (2,any prime power); and

b) (3,8).

The next lemma eliminates (3.5) b) as a possibility and the theorem following it cuts down to a finite number the possible exceptional cases that appear under (3.5) a). After this our only concern in describing $\mathcal{G}_g(\text{sol})\cap\mathcal{G}_g(\text{prim})$ explicitly for $g>1$ is with the finite number of cases left over when $g=2$. In §4 we delineate the possible branch cycles which arise from the portion of the list remaining from Theorem 3.3.

Lemma 3.2: *Let $G=(\mathbb{Z}/2)^e\times^s H$ with H a solvable subgroup of $GL(e,\mathbb{Z}/2)$ acting irreducibly on $(\mathbb{Z}/2)^e$ with $e>2$. Let T be the subgroup of H generated by transvections. If $T\neq 1$, then the irreducible T submodules of $(\mathbb{Z}/2)^e$ are 2-dimensional. In particular, if H contains a transvection, then e is even.*

Finally , if $e=3$, then 7 divides the order of H. This excludes the cases of $(g,n)=(2,8)$ or $(3,8)$ from being exceptions to the Main Theorem.

Proof: We divide the proof into 4 parts, the first 3 of which consider a minimal normal subgroup A of H contained in T.

Part 1. *Decomposition of N under a minimal normal subgroup of H.* Let A be as above. Then A is an abelian p-group for some prime p. Decompose $N=(\mathbb{Z}/2)^e$ as $V_1 \oplus V_2 \oplus \cdots \oplus V_t$ with V_i an irreducible A module, $i=1,\ldots,t$. Indeed, if V_1 is a minimal A submodule, since H acts irreducibly on N, the images of V_1 under H form such a decomposition with $h(V_i)=V_i$ or $h(V_i) \cap V_i$ empty for each i and $h \in H$. Suppose that $(\mathbb{Z}/2)^e$ has a 1-dimensional T submodule, <v>. Since T is normal in H, then $h^{-1}\tau h(v)=v$ for each $\tau \in T$ and $h \in H$. Conclude that H doesn't act irreducibly on N. This argument works with T replaced by any normal subgroup of H, in particular A. Thus all of the irreducible A and T submodules are of dimension at least 2.

Part 2. *Reduction to the case H=T and A is cyclic acting irreducibly on N.* If τ is a transvection, it fixes a hyperplane of N and therefore τ fixes a hyperplane of V_i. Thus $\tau(V_i)=V_i$, $i=1,\ldots,t$, and we may reduce to the case that H=T and $N=V_1=V$. Let $(\mathbb{Z}/2)[A]$ denote the image of the group ring of A in End(V). This is a field. Thus the elements of A represent a subgroup of the multiplicative group of this field. Such a subgroup must be cyclic, and therefore A is a cyclic group.

Part 3. *H is dihedral and dim(V)=2.* Denote the normalizer of $(\mathbb{Z}/2)[A]$ in GL(V) by K (it includes the action of T=H). Since this acts as automorphisms of the finite field $(\mathbb{Z}/2)[A]$, the induced map $K \to \mathrm{Aut}((\mathbb{Z}/2)[A])$ has image a cyclic group. Conclude that T/A is cyclic. Since it is generated by involutions it is of order 2 (or 1). Thus H is dihedral, and for any involution τ we check that $\dim([\tau,V])=\dim(V)/2$. If, however τ is a transvection, then $\dim([\tau,V])=1$. This gives dim(V)=2 and concludes the first two sentences of the lemma. If we now return to the general case, the argument of Part 1 shows that (if T is nontrivial) N is a direct sum of irreducible 2-dimensional T submodules. In particular, e is even.

Part 4. *The case e=3.* From the above H contains no transvections. The order of $GL(3,\mathbb{Z}/2)$ is $7\cdot 6\cdot 4$. Let A be a minimal normal subgroup of H. Of course, this contains a copy of S_4 induced from its permutation action If A stabilizes a line then $C_N(A)\neq 0$, which implies that $C_N(A)=N$. This contradiction to the faithful action of H on N shows that A acts irreducibly on N. By Part 2 above, A is cyclic, and therefore $|A|=7$ and $|H|=7$ or 21. In this case every involution is in N and it has no fixed points: its index is at least 4. An element of order 7 in H has index 6 and an element of order 3 has index 4. We exclude the case (3.5) b) by noting that $r\geq 9$. Thus the sum of the σ_i's, which should be 2(8+3-1)=20, must exceed 36. For the case (g,n)=(2,8) we get a similar contradiction with $r\geq 6$. □

Theorem 3.3: *Exclude the well known case n=2 and $G=S_2$. Then the only possible exceptional (solvable) groups that appear as the group of the Galois closure of some map of the generic curve of genus 2 to $\mathbb{P}^1_z$ occur with $n=p^e$ and $G\subset S_n$ a primitive subgroup as follows:*

(3.6) a) $p=5=n$, $G=D_{10}$;

b) $p=3=n$, $G=S_3$;

c) $p=3$, $n=9$, $G=(\mathbb{Z}/3)^2{\times}^s D_8$;

d) $p=3$, $n=9$, $G=(\mathbb{Z}/3)^2{\times}^s Gl(2,3)$;

e) $p=2$, $n=4$, $G=S_4$; and

f) $p=2$, $n=16$, $G=(\mathbb{Z}/2)^2{\times}(\mathbb{Z}/2)^2{\times}^s((S_3{\times}S_3){\times}^s(\mathbb{Z}/2)$.

Proof: From Principle 2.2, $r\geq 6$, and in our previous notation, the Riemann-Hurwitz formula gives, $\sum_{i=1}^{r}\text{ind}(\sigma_i)=2n+2$. Apply (3.4). For $p\geq 7$, $\text{ind}(\sigma_i)\geq(3/7)n$. Therefore, $\sum_{i=1}^{r}\text{ind}(\sigma_i)\geq (18/7)n \geq 2n+4$ and there are no examples. We break the proof into two parts according to n odd or even.

Part 1. *n is odd.* If p=5 a similar computation shows that

$\sum_{i=1}^{r} \mathrm{ind}(\sigma_i) \geq 2n+2n/5>2n+2$ if $n>5$. If $n=5$, equality implies $\mathrm{ind}(\sigma_i)=2$, $i=1,\ldots,6$. The only solvable subgroup of S_5 generated by such elements is D_{10} where σ_i is a product of two disjoint 2-cycles, $i=1,\ldots,6$.

For $p=3$, consider the possibilities for the action of σ on the vector space N as in the proof of Proposition 3.1. If σ fixes no points, then $\mathrm{ind}(\sigma)\geq n/2$. Otherwise, we may assume for the index calculation that $\sigma \in H \subset GL(e,\mathbb{Z}/3)$. If σ is an involution, then $v+\sigma(v)$ is fixed by σ for each $v \in N$. These fixed vectors form a subspace N_1. There are 2 possibilities:

(3.7) a) N_1 is a hyperplane (i.e., σ is a reflection in N_1), and

$\mathrm{ind}(\sigma)=(3^e-3^{e-1})/2 =n/3$; or

b) the fixed subspace of σ has order no more than 3^{e-2} elements, and $\mathrm{ind}(\sigma)\geq (3^e-3^{e-2})/2 =n/3+n/9=4n/9$.

If σ is not an involution, but σ is a transvection (i.e., σ fixes a hyperplane N_1 and $\sigma(v)-v \in N_1$ for each $v \in N$) then $\mathrm{ind}(\sigma)=2(3^e-3^{e-1})/3 =4n/9$.

Otherwise, $\mathrm{ind}(\sigma)\geq n/2$. Clearly, if $n>9$, either all σ_i's are reflections and the sum of the indices is $rn/3\neq 2(n+1)$ for all r; or at least one of the σ_i's is not a reflection and, since $r\geq 6$, the sum of the indices exceeds $2(n+1)$. Here are the actual branch cycle possibilities for $n=p=3$, $G=S_3$:

(3.8) a) $r=6$, four of the σ_i's are 2-cycles and two are 3-cycles;

b) r=7, six of the σ_i's are 2-cycles and one is a 3-cycle; and

c) r=8, and all of the σ_i's are 2-cycles.

For n=9 the σ_i's that are reflections have index 3, transvections have index 4 and multiplication by -1 on N has index 4. Suppose that σ and τ are involutions that generate D_8, with $\sigma\tau$ of order 4. We may assume that the action of σ on $(\mathbb{Z}/3)^2$ is given by mapping (α,β) to $(-\alpha,\beta)$, and similarly, that the action of τ is given by mapping (α,β) to (β,α). Here are the actual branch cycle possibilities for n=9, $G=(\mathbb{Z}/3)^2\times^s D_8$:

(3.9) r=6, four of the σ_i's are reflections and the other two are involutions of the form $(v;(\sigma\tau)^2)$ with $v\in N-\{(0,0)\}$.

In order to list the branch cycle possibilities for n=9, $G=(\mathbb{Z}/3)^2\times^s G_1$ with $G_1=GL(2,\mathbb{Z}/3)$ or $SL(2,\mathbb{Z}/3)$ note that $SL(2,\mathbb{Z}/3)$ contains no reflections. Thus it is ruled out since, as above, at least four of the σ_i's must be reflections. Consider the transvection

$A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$, a matrix that is the product $\sigma\tau$ from D_8 above.

Also, A and B generate $SL(2,\mathbb{Z}/3)$. Since σ and τ have determinant -1, σ,τ and α generate $GL(2,\mathbb{Z}/3)$. We therefore have the possibility for (3.6) d) that

(3.10) r=6, two of the σ_i's project in $GL(2,\mathbb{Z}/3)$ to σ, two of them to τ, and two of them to A.

We have said enough about the odd degree case for now.

Part 2. *p=2.* If n=4, then the sum of the indices of the σ_i's must be 10. Since r≥6 this rules out the possibility that each σ_i has index at least 2, and therefore it isn't possible that $G=A_4$. As in (3.8) there is a long, but obvious list of the possibilities. The extremes for r are of the most interest to us:

(3.11) a) r=10, all of the σ_i's are 2-cycles; and

b) r=6, two of the σ_i's are 2-cycles and the remaining four are some combination of 3-cycles and products of two 2-cycles.

Now assume that $n=2^e$ with e≥4 (Lemma 3.2 excludes the possibility of e=3). Use the notation of Lemma 3.2. The following list indicates possibilities for the indices of elements $\sigma\epsilon G$:

(3.12) a) if σ is a transvection, then σ fixes 2^{e-1} vectors and ord(σ) is 2 (over$\mathbb{Z}/2$), so ind(σ)=n/4;

b) if ord(σ)=3, then ind(σ)$\geq(2/3)(2^e-2^{e-2})$=n/2; and

c) if ord(σ)=2 and not a), then ind(σ)$\geq$3n/8= $(2^e-2^{e-2})/2$.

d) if ord(σ)$\geq$5, then ind(σ)$\geq$5n/8=(below);

e) if ord(σ)=4, ind(σ)$\geq$n/**2** and if σ^2 isn't a transvection, then ind(σ)$\geq$5n/8 (below); and

f) if ord(σ)=3 and $|[\sigma,N]|\geq16$, then ind(σ)$\geq2(2^e-2^{e-4})/3$=5n/8 (below).

The remainder of the proof is organized into five points of which one covers expansion on (3.12) d) and e), three cover a list of possibilities on the number of transvections among the σ_i's and the final gives the example in which e=4.

Point 1. *(3.12) d) and e).* First consider (3.12) d). If ord(σ) is an odd prime p then ind(σ)=(p-1)(2^e-2^f)/p where f is the dimension of the fixed point space of σ. Since GL(3,$\mathbb{Z}$/2) has order relatively prime to 5, if p=5, then e-f>4. Similarly, since GL(2,$\mathbb{Z}$/2) has order relatively prime to p for p>3, then e-f>3 if p>5. Thus in these cases ind(σ)$\geq$3n/4. Applying (3.3) b) we are reduced to the case that ord(σ) is divisible only by the primes 2 and 3, and if suffices to establish this in the case that ord(σ)=6,8 or 9.

In the case that ord(σ)=9 we may count the number of 9-cycles by similar thinking to the above: The number of 9 cycles is the number of 3-cycles of σ^3. So there must be an integer t such that σ^3 has (2^e-2^t)/3 3-cycles, which come together in groups of three from the 9-cycles of σ. Thus ind(σ)≥8(2^e-2^t)/9 where t is the minimal integer such that 2^e-2^t is divisible by 9. That is t=e-6, and (3.12) d) follows from (8/9)(63/64)> 3/4. For the other two cases we use the following formula. If orb(σ) is the number of orbits of σ and ord(σ)=d, then

(3.13) a) $\mathrm{orb}(\sigma)=\left(\sum_{i=1}^{d}|\mathrm{Fix}(\sigma^i)|\right)/d.$

This follows from Frobenius reciprocity in the inner product formula $(\chi,\mathbf{1})_{\langle\sigma\rangle}=(\mathbf{1},\mathbf{1})_M$, where χ is the character of the permutation representation of <σ> and M is the subgroup that stabilizes an integer in this representation.

If $d=6$, then $orb(\sigma)=(2|Fix(\sigma)|+2|Fix(\sigma^2)|+|Fix(\sigma^3)|+n)/6$. If we write σ in rational canonical form it is clear that $|Fix(\sigma)|\leq n/8$, $|Fix(\sigma^2)|\leq n/4$ and $|Fix(\sigma^3)|\leq n/2$. Therefore $orb(\sigma)\leq 3n/8$ and $ind(\sigma)=n-orb(\sigma)\geq 5n/8$. The exact same argument applied in the case that σ is of order 8 gives $ind(\sigma)=n-orb(\sigma)\geq(11)n/16$.

Now we consider e). Apply the above for σ of order 4 to get $orb(\sigma)=(2|Fix(\sigma)|+2|Fix(\sigma^2)|+n)/4\leq n/2$. But, if σ^2 isn't a transvection, then $|Fix(\sigma^2)|\geq n/4$ and $|Fix(\sigma)|\geq n/8$. Conclude e).

Point 2. *Existence of at least 1 transvection and 2 nontransvections among the σ_i's.* Since $r\geq 6$ and $e>3$, if there are no transvections among the σ_i's the sum of the indices of the σ_i's is at least $2n+(n/4)>2n+2$. Conclude that there is at least one transvection. From Lemma 3.2, e is even. We treat the case $e=4$, as in (3.6) f) in Point 5. Assume now that $e\geq 6$.

Let T be the (normal) subgroup of H generated by transvections. If all but one of the σ_i's are transvections, then H=T and Lemma 3.2 implies that $e=2$.

Point 3. *The impossibility of exactly 2 nontransvections among σ_i's.* If exactly two aren't transvections, then H/T is generated by a single element, and so is cyclic. Suppose that $|H/T|=2$. From Lemma 3.2 the irreducible submodules of $(\mathbb{Z}/2)^e$ would be of dimension at most 4, contrary to the irreducibility of the action of H and e>5. Therefore $|H/T|=m>2$. Thus these two nontransvections have order at least m.

If $\sigma=\sigma_i$, then $H\subseteq\langle T,\sigma\rangle$, so σ must transitively permute the irreducible T submodules of N. Also, no two T submodules of N are T-isomorphic (a given transvection can act nontrivially of only one irreducible submodule). Thus $|[\sigma,N]|\geq 16$. By (3.12) d), e) and f), $\mathrm{ind}(\sigma)\geq 5n/8$ and the sum of the indices of the σ_i's must be at least $4(n/4)+2(5/8)n>2n+2$.

Point 4. *Impossibility of 3 nontransvections among σ_i's.* Since the sum of the indices of the σ_i's is $\not\equiv 0$ mod 4, one of the nontransvections must have index $\not\equiv 0$ mod 4. We show that this nontransvection must have index at least (5)n/8.

If not, then (3.12) d) implies that $\sigma=\sigma_i$ has order at most 4. We list the cases. If $\mathrm{ord}(\sigma)=2$, the Jordan canonical form of σ shows that σ fixes at least $2^{e/2}$ elements so that, since $e\geq 6$, $\mathrm{ind}(\sigma)\equiv 0$ mod 4. If $\mathrm{ord}(\sigma)=3$, then $|\mathrm{Fix}(\sigma)|=1$ and $\mathrm{ind}(\sigma)=2(2^e-1)/3>5n/8$. Finally, if $\mathrm{ord}(\sigma)=4$, then $-\mathrm{ind}(\sigma)\equiv\mathrm{orb}(\sigma)=(2|\mathrm{Fix}(\sigma)|+2|\mathrm{Fix}(\sigma^2)|+n)/4$ mod 4 from Point 1. If σ^2 is a transvection, then σ has exactly one Jordan block of size at most 3, and so $|\mathrm{Fix}(\sigma)|$ is a multiple of 8 (and $|\mathrm{Fix}(\sigma^2)|=n/2$). Hence $\mathrm{ind}(\sigma)\equiv 0$ mod 4. By (3.12) e), $\mathrm{ind}(\sigma)\geq 5n/8$ as claimed. Therefore (again the formula of Point 1) the sum of the indices of the σ_i's is at least $(3n/4)+6n/8+5n/8=2n+n/8>2n+2$.

Point 5. *e=4.* From the first part of the proof of Lemma 3.2 conclude that the subgroup T of H generated by transvections acts as $H_1=S_3\times S_3$ acting on $N=(\mathbb{Z}/2)^2\times(\mathbb{Z}/2)^2$ through action of S_3 on $(\mathbb{Z}/2)^3/\langle(1,1,1)\rangle\equiv(\mathbb{Z}/2)^2$ in the standard degree 3 permutation representation. Note that in the argument of Point 2 the possibility of $|H/T|=2$ was left open when e=4. It is easy to conclude that H is $S_3\times S_3\times^s\mathbb{Z}/2$ where the action of $\mathbb{Z}/2$ is the switch on the two copies of S_3 (and on the two factors of N). The details on possible branch cycle descriptions σ appear in §4.4. □

§3.3. INSPECTION OF $|\mathcal{G}_g(\mathrm{SOL})\cap\mathcal{G}_g(\mathrm{PRIM})|$ FOR g=1: The last theorem of this section shows that in the case g=1, the elements of $\mathcal{G}_g(\mathrm{sol})\cap\mathcal{G}_g(\mathrm{prim})$ are groups whose degrees n are either of the form 2^e, 3^e, 5^e or 7^e. Of course, as explained in the introduction, we expect that $\mathcal{G}_1(\mathrm{sol})\cap\mathcal{G}_1(\mathrm{prim})$ is actually finite. We have one further duty in this case before we go to Theorem 3.4. That is to explain the relation between $\mathcal{G}_1(\mathrm{sol})\cap\mathcal{G}_1(\mathrm{prim})$ and $\mathcal{G}_1(\mathrm{sol})$ considering that Proposition 2.3 assumes that g>1 (cf. Acknowledgements).

Indeed, in diagram (1.4) we must allow one further possibility. If

(3.14) $\varphi: X_{\mathbf{m}} \to Y \to \mathbb{P}^1_Z$, with $X_{\mathbf{m}}$ the generic curve of genus 1,

then either Y is of genus 1 and $X_{\mathbf{m}} \to Y$ is an unramified Galois cover with abelian group of rank at most 2, or Y is of genus 0. In the former case Y is itself a generic curve of genus 1 (not necessarily isomorphic to $X_{\mathbf{m}}$). Recall that genus 1 curves (over an algebraically closed field) have the structure of an abelian group. With no loss we may assume that the origins of the group structures for $X_{\mathbf{m}}$ and Y have been chosen so that $X_{\mathbf{m}} \to$ Y is an isogeny of elliptic curves. In particular, this is a Galois cover with group a quotient of $(\mathbb{Z}/u)^2$ for some integer u [L; p.24]. We explain the implications for the relationship between $\mathcal{G}_1(\text{sol}) \cap \mathcal{G}_1(\text{prim}))$ and $\mathcal{G}_1(\text{sol})$ (or between $\mathcal{G}_1(\text{prim}))$ and $\mathcal{G}_1$).

Suppose that $G_1 \in \mathcal{G}_1$ is a subgroup of the wreath product of V, a quotient of $(\mathbb{Z}/u)^2$ for some integer u, and a group G (i.e., a subgroup of $V_k \times^s G$ via a permutation representation of G of degree k). We say that G and G_1 are elementary wreath equivalent. This generates an equivalence relation. From the above comments if $G \in \mathcal{G}_1$(resp., $\mathcal{G}_1(\text{sol})$), then it is a subgroup of a series of wreath products formed from groups $G_1, \dots, G_v$

where G_1, is elementary wreath equivalent to an element of $\mathcal{G}_1$(prim) (resp., $\mathcal{G}_1$(sol)$\cap\mathcal{G}_1$(prim)) and $G_i,\epsilon\mathcal{G}_0$(prim) (resp., $\mathcal{G}_0$(sol)$\cap\mathcal{G}_0$(prim)), $i=2,\ldots,v$. We are willing to use the elementary formula (3.15) from the still incomplete [GTh] on the principles that this will appear right up front in that paper and that this use will help clarify the relationship between this paper and that.

Theorem 3.4: *The only possible degrees of primitive solvable groups that appear as the group of the Galois closure of some map of the generic curve of genus 1 to $\mathbb{P}^1_z$ occur with $n=p^e$, with p equal to 2,3,5 or 7.*

Proof: From Principle 2.2, $r\geq4$, and in our previous notation, the Riemann-Hurwitz formula gives, $\sum_{i=1}^{r}\text{ind}(\sigma_i)=2n$. An application of (3.4) here when g=1 falls short of giving us the opening argument of Theorem 3.3. Instead we borrow a more precise statement from [GTh]. If $|\sigma|=d$, then

(3.15) a) $\quad\text{ind}(\sigma)\geq (d-1)(p-1)n/d\cdot p$ if σ has fixed points; and

b) $\quad\text{ind}(\sigma)\geq (p-1)n/p$ if σ has no fixed points.

We divide the remainder of the proof into two parts.

Part 1. *Reduction to the case that $r=4$ and $\text{ord}(\sigma_i)=2$, $i=1,2,3,4$.* First Assume that at least one σ_i, say σ_r, has order greater than 2 and, of course, that $p>7$. Therefore,

$$\sum_{i=1}^{r}\text{ind}(\sigma_i)\geq(3(p-1)/2p+2(p-1)/3p)n>2n+2$$

unless $p<13$; and in the case $p=13$, $|\sigma_i|=2$, $i=1,2,3$, and $|\sigma_4|=3$ and each σ_i leaves a hyperplane fixed. But in this case $\prod_{i=1}^{r}\det(\sigma_i)\neq 1$, contrary to $\prod_{i=1}^{r}\det(\sigma_i)=1$. Actually the same formula works for the case $p=11$, with the observation that since $11\not\equiv 1 \bmod 3$, (3.14) gives an improved bound for $|\sigma_4|$. Conclude that if p exceeds 7, then all of the σ_i's are of order 2.

Part 2. *Conclusion.* Use the above and that $\sigma_1\cdot\sigma_2\cdot\sigma_3\cdot\sigma_4=1$ to conclude that $\sigma_1\cdot\sigma_2$ generates a normal subgroup of G (which, because of primitivity, cannot fix an integer of the representation), and therefore that G is the dihedral group of order 2p. Finally, this implies that $\text{ind}(\sigma_i)=(p-1)/2$, contrary to the sum of the indices of the σ_i's equal to 2p. □