§4. DETAILS ON NIELSEN CLASSES IN THE EXCEPTIONAL CASES: In this section we give detailed description of the branch cycles σ, as in §1, that give rise to covers $X \to \mathbb{P}^1_z$ with X of genus 2 which correspond to the items of list (3.6) for which G is not S_n for some integer n. For most questions these are the toughest cases. Each of these consists of a finite number of subcases with each subcase collecting together a class of covers called a *Nielsen class*.

§4.1. NIELSEN CLASSES AND THE HURWITZ MONODROMY GROUP: Let G be a subgroup of S_n and let $C_1,\dots,C_r$ be nontrivial (not necessarily distinct) conjugacy classes of G.

Definition 4.1: The Nielsen Class of $(\mathbf{C},G)$ is the collection (assumed nonempty)

$$\mathrm{Ni}(\mathbf{C})=\{\boldsymbol{\sigma}\in G^r \mid G(\boldsymbol{\sigma})=G,\ \sigma_1\cdots\sigma_r=1 \text{ and for some } \alpha\in S_r,\ \sigma_{(i)\alpha}\in C_i, i=1,\dots,r\}.$$

We say that a cover $X \to \mathbb{P}^1$ is in $\mathrm{Ni}(\mathbf{C})$ if any (every) description of the branch cycles $\boldsymbol{\sigma}$ of the cover (as in §1) is in $\mathrm{Ni}(\mathbf{C})$. Note that it makes no difference in what order we list the conjugacy classes.

The normalizer, $N(\mathbf{C})$, of the Nielsen class is $\{\tau\in S_n \mid$ conjugation by τ permutes $C_1,\dots,C_r\}$. It acts on $\mathrm{Ni}(\mathbf{C})$ by conjugation: $\gamma\in N(\mathbf{C})$ maps $\sigma\in \mathrm{Ni}(\mathbf{C})$ to $\gamma^{-1}\sigma\gamma$. Denote the quotient of this action by $\mathrm{Ni}(\mathbf{C})^{ab}$, the absolute Nielsen classes.

The collection of covers $\mathcal{H}$ belonging to a given Nielsen class forms a natural moduli family. That is, $\mathcal{H}$ is a complex manifold with properties similar to those of $\mathcal{M}_g$ given in (1.3). If the Nielsen class is of covers of genus g, from (1.3) there is a natural complex analytic map

(4.1) $\Psi(\mathcal{H},\mathcal{M}_g)\colon \mathcal{H} \to \mathcal{M}_g.$

The main geometric principle merely restates the condition that the generic surface $X_{\mathbf{m}}$ of genus g can be presented as a cover $\mathbb{P}^1_z$ in the given Nielsen class:

Principle 4.2: *The map (4.1) is surjective onto a Zariski open subset of $\mathcal{M}_g$ if and only if the generic curve of genus g has a map to $\mathbb{P}^1_z$ with a description of its branch cycles in the Nielsen class described by $\mathcal{H}$.*

The reader does not need to learn the theory of the space $\mathcal{H}$ in detail since all of its known properties and applications (e.g., [BFr] and [Fr,1, 2]) are effected through explicit computations using the action of the Hurwitz monodromy group on explicit finite sets that arise from the Nielsen class data. For example, transitivity of H(r) on $\mathrm{Ni}(\mathbf{C})^{ab}$ is equivalent to the connectedness (or irreducibility) of $\mathcal{H}$.

The Hurwitz monodromy group of degree r is the free group on generators, Q_i , $i=1,\ldots,r-1$, subject to these relations:

(4.2) a) $Q_iQ_{i+1}Q_i = Q_{i+1}Q_iQ_{i+1}$, $i=1,\ldots,r-2$;

b) $Q_iQ_j = Q_jQ_i$ for $|i-j|>1$; and

c) $Q_1Q_2\cdots Q_{r-1}Q_{r-1}\cdots Q_1 = 1$.

We denote the group generated by the Q_i's by $H(r)$. Note that the Q_i's act on $\boldsymbol{\sigma}\in G^r$ by the formula $(\boldsymbol{\sigma})Q_i =$

(4.3) $(\sigma_1,\dots,\sigma_{i-1},\sigma_i\sigma_{i+1}\sigma_i^{-1},\sigma_i,\sigma_{i+2},\dots,\sigma_r)$, $i=1,\dots,r-1$.

Thus they induce a permutation representation of $H(r)$ on $\mathrm{Ni}(\mathbf{C})^{ab}$. Also there is a natural permutation representation $\Phi_r : H(r) \to S_r$ by $Q_i \to (i\ i+1)$, $i=1,\dots,r$.

In this section $H(r)$ appears as a tool for organizing the combinatorics of Nielsen classes. But in § 5, stretching the known theory a little, we propose $H(r)$ actions that may provide a combinatorial check for the surjectivity of the map of (4.1). Since the computations are difficult we do just one example as motivation for others to improve the method of computation. Thus there will still be Nielsen classes in the list of (3.6) for which we cannot state for certain whether or not the generic cover of genus 2 appears among them. This is also true (§5.3) for the analogue of Principle 2.5 for deciding, given g, for which integers n the alternating group $A_n \in \mathcal{G}_g(\mathrm{Prim})$.

§4.2. THE STRUCTURE CONSTANT LEMMA: The examples of §4.3 will baptize the reader in the combinatorics of these definitions. Suppose that we know the irreducible representations of a group G. Then there is a device from representation theory that makes short work out of counting the number of solutions $(\sigma_1,\ldots,\sigma_r)$ to

(4.4) $\sigma_1\cdots\sigma_r=1$ with $\sigma_i \in C_i$, $i=1,\ldots,r$.

In this next lemma only let C stand both for a conjugacy class in G and also for the sum of the elements in C as an element of the group ring of G over $\mathbb{Z}$.

The Structure Constant Lemma: *Let χ^a, $a=1,2,\ldots,s$, run over the characters of the irreducible representations of G, and let C_i, $i=1,\ldots,s$ run over the distinct conjugacy classes of G. Then the number of solutions to (4.4) is $a_G(C_1,\ldots,C_r)=$*

(4.5) $(|C_1|\cdots|C_r|/|G|)\sum_a \chi^a(\tau_1)\cdots\chi^a(\tau_r)/\chi^a(1)^{r-2}$,

where τ_i is a representative of C_i, $i=1,\ldots,r$.

Proof: Define c_{ijk} by the relation

$$C_i\cdot C_j=\sum_{k=1}^{s} c_{ijk}C_k \text{ (multiplication in } \mathbb{Z}[G]).$$

From it we get this relation [H;Theorem 16.6.10, p.277]:

(4.6) $(|C_i|\,|C_j|/\chi^a(1))(\chi^a(\tau_i)\chi^a(\tau_j))=\sum_\ell c_{ij\ell}|C_\ell|\,\chi^a(\tau_\ell)$.

where τ_i is one of the elements in the sum C_i. Multiply (4.6) by $\overline{\chi}^a(\tau_k)$ and sum over "a" using $\sum_a \chi^a(\tau_\ell)\overline{\chi}^a(\tau_k)=\delta_{\ell k}|G|/|C_k|$ [H;p.275], to get

(4.7) $$c_{ijk}= \sum_a (|C_i||C_j|/\chi^a(1))(\chi^a(\tau_i)\chi^a(\tau_j)\chi^a((\tau_k{}^{-1}))/|G|.$$

Now if we consider the equation, $\sigma_1\sigma_2\sigma_3=1$ with $\sigma_i \epsilon C_i$, $i=1,2,3$, by replacing τ_i by σ_1, τ_j by σ_2 and $(\tau_k)^{-1}$ by σ_3, the number of solutions is

$$a_G(C_1,C_2,C_3)= (|C_1||C_2||C_3|/|G|)\sum_a \chi^a(\sigma_1)\chi^a(\sigma_2)\chi^a(\sigma_3)/\chi^a(1).$$

For general r, count the number of solutions of (4.4) by an induction. We illustrate with the case r=4. The strategy is to compute the solutions to the equations $\sigma_1\sigma_2=\tau$ and $\sigma_3\sigma_4=\tau^{-1}$, and then let τ run over all of G. If k is the subscript of the conjugacy class of τ and t of τ^{-1}, then the number of simultaneous solutions to these equations with τ fixed is

(4.8) $$c_{12k}\cdot c_{34t}=(\sum_a (|C_1||C_2|/\chi^a(1))(\chi^a(\tau_1)\chi^a(\tau_2)\chi^a(\tau^{-1}))/|G|))\cdot$$

$$(\sum_b(|C_3||C_4|/\chi^b(1))(\chi^b(\tau_3)\chi^b(\tau_4)\chi^b(\tau))/|G|)).$$

Now apply the formula $\sum_\tau \chi^a(\tau^{-1})\chi^b(\tau)=|G|\delta_{ab}$, to see that the sum over τ of (4.8)) is (4.5) in the case r=4. □

§4.3. CASES (3.6) a), c) AND d) OF THEOREM 3.3: We treat the three cases separately using details from the proof of Theorem 3.3. There is a lesson here for the reader who may not be acquainted with group theory. As we move from case to case, an application of §4.2 requires increasing skill to construct the values of the characters of the group on conjugacy classes. To an untrained eye the groups look easy, but with the possible exception of (3.6) a) this is a deception. Use of character theory is a considerable shortcut to these computations. It is best to accept that the computation of the characters of these groups is a difficult task that takes up considerable space in literature known mostly to group theorists (e.g., [A]). In other words, it is better to have a group theory consultant available, rather than to regard this as a body of knowledge that must be learned, should you want to imitate these computations for any one of the many possible applications.

Note that after the computations of this section our main remaining concern (§5) is whether the *generic* surface of genus g=2 has a covering map in Ni(**C**), for each of the Nielsen classes of the cases below. That is, in which of these cases is (4.1) surjective?

Case 1: *(3.6) a) $p=5=n$, $G=D_{10}$*. We have already learned that r=6; that $\boldsymbol{\sigma}$ has entries from the conjugacy class of S_5 consisting of products of disjoint 2-cycles; and that $G(\boldsymbol{\sigma})=D_{10}$, the dihedral group of order 10.

Any two distinct elements σ and τ in S_5 that are both products of two disjoint 2-cycles will have $\sigma\tau$ equal to a 5-cycle. Thus $\langle\sigma,\tau\rangle=D_{10}$.

For our copy of D_{10} we choose $\sigma=(14)(23)$ and $\tau=(24)(15)$. In D_{10} there are five 2-Sylows conjugate under the action of $\sigma_\infty=(12345)$ $(=\sigma\tau)$, each characterized by the integer left fixed by its generator (e.g., σ corresponds to 5). In terms of Def. 4.1 there is only one possible Nielsen class to consider: $G=D_{10}$ and $C_1=C_2=\cdots=C_6$ and all are the conjugacy class of σ. Therefore absolute Nielsen classes , $\mathrm{Ni}(\mathbf{C})^{ab}$, are represented by elements $(\sigma,\sigma_2,\sigma_3,\ldots,\sigma_6)$ (i.e., $\sigma_1=\sigma$) as follows:

(4.9) a) $\sigma_i\neq\sigma$ for some $i=2,\ldots,6$; and

b) $\sigma\sigma_2\sigma_3\cdots\sigma_6=1$.

From (4.9) a) the element $(\sigma,\sigma_2,\sigma_3,\ldots,\sigma_6)$ and $\sigma(\sigma,\sigma_2,\sigma_3,\ldots,\sigma_6)\sigma$ represent the same element of $\mathrm{Ni}(\mathbf{C})^{ab}$. In the case at hand a special device works handily to efficiently list elements of $\mathrm{Ni}(\mathbf{C})^{ab}$: associate to $(\sigma,\sigma_2,\sigma_3,\ldots,\sigma_6)$ the **6**-tuple $(5,j_2,j_3,\ldots,j_6)$ of integers from $\{1,2,\ldots,5\}$ by σ_i leaves j_i fixed. A simple calculation shows that if $m=3\cdot(j_k-j_{k+1})$, then $\sigma_k\bullet\sigma_{k+1}=(\sigma_\infty)^m$. Thus (4.9) b) is equivalent to this:

(4.10) $\sum_{k=1,2,3} j_{2k-1}-j_{2k}\equiv 0 \bmod 5.$

From (4.10) conclude that given $\sigma_2,\sigma_3,\sigma_4,\sigma_5$ there is a unique σ_6 that satisfies (4.9) b), and that there are $(1+5+5^2+5^3)/2=13\cdot 24$ elements of $\mathrm{Ni}(\mathbf{C})^{ab}$. We, however, haven't yet dealt with whether the general surface of genus 2 has a map to $\mathbb{P}^1_z$ in this Nielsen class.

Case 2: *(3.6) c) $p=3$, $n=9$, $G=(\mathbb{Z}/3)^2\times^s D_8$*. Below we label elements of G using the notation $(v;\sigma')$ with $v\in(\mathbb{Z}/3)^2$ and $\sigma'\in D_8$. Following the format of Case 1, from expression (3.9) we have already deduced that we may assume with no loss the following: with σ and τ involutions where $\sigma\tau$ is of order 4, $D_8=\langle\sigma,\tau\rangle$; the action of σ on $(\mathbb{Z}/3)^2$ is given by $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$; and the action of τ is given by $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Furthermore, $r=6$ and actual branch cycle possibilities σ satisfy the following conditions:

(4.11) a) two of the σ_i's are conjugate to $(0;\sigma)$;

b) two of the σ_i's are conjugate to $(0;\tau)$;and

c) two of the σ_i's are conjugate to $(0;(\sigma\tau)^2)$.

We will denote the conjugacy class of $(v';\sigma')$ in G by $C(v';\sigma')$.

There are 5 conjugacy classes in D_8: $C(1)$; $C(\sigma)$; $C(\tau)$; $C((\sigma\tau)^2)$; and $C(\sigma\tau)$.

An easy computation gives the following list of conjugacy classes in G: $C(0;1)$ ord. 1, card. 1; $C((1,0);1)$ ord. 3, card. 4; $C((1,1);1)$ ord. 3, card. 4; $C((0,0);\sigma)$ ord. 2, card. 6; $C((0,1);\sigma)$ ord. 6, card. 12; $C((0,0);\tau)$ ord. 2, card. 6; $C((1,1);\tau)$ ord. 6, card. 12; $C((0,0);(\sigma\tau)^2)$ ord. 2, card. 9; $C((0,0);\sigma\tau)$ ord. 4, card. 18.

As in Case 1 there is but one Nielsen class to consider, Ni($\mathbf{C}$) with $C_1=C_2=C((0,0);\sigma)$, $C_3=C_4=C((0,0);\tau)$ and $C_5=C_6=C((0,0);(\sigma\tau)^2)$. We first compute the cardinality of the subset of Ni($\mathbf{C}$) consisting of τ such that $\tau_i \epsilon C_i$, $i=1,\dots,6$. Denote this SNi($\mathbf{C}$) (for straight Nielsen classes). There are 5 irreducible representations of D_8, 4 of degree 1 and one of degree 2 (the complex version of the representation used above on $(\mathbb{Z}/3)^2$). The sums of the squares of the degrees of the irreducible representations of G add up to $|G|=72$, and there are 9 of these (as many as there are conjugacy classes). From this data easily calculate that there are, in addition to the five from the canonical homomorphism to D_8 , four more, each of degree 4. In order to apply the Structure Constant Lemma, we need only consider those characters χ for which

(4.12) $\chi(0;\sigma)\chi(0;\tau)\chi(0;(\sigma\tau)^2)\neq 0.$

Formula (4.5) then becomes

(4.13) $2\cdot 3^6(\sum'(\chi(0;\sigma)\chi(0;\tau)\chi(0;(\sigma\tau)^2))^2/\deg(\chi)^4$, where the sum is over those χ for which (4.12) holds.

Each character of degree 4 is determined by taking a 1-dimensional non-trivial character ψ of $(\mathbb{Z}/3)^2$, extending it to be trivial on the centralizer in D_8 of the line L in the kernel of the homomorphism defining ψ, and then forming ψ^G, the character induced on G. The centralizer of a line will be a conjugate of either σ of τ; let's say the former for explicitness. Then the value of the character, $\sum_{i=1}^{4} \psi(\alpha_i\tau(\alpha_i)^{-1})$, with the α_i's coset representatives for the group generated by $L\times<\sigma>$ in G (with the understanding that $\psi(\alpha_i\tau(\alpha_i)^{-1})=0$ if $\alpha_i\tau(\alpha_i)^{-1}$ isn't in $L\times<\sigma>$) is 0.

Thus (4.13) is $8\cdot 3^6$. Of course, we must remove the solutions of (4.4) that give σ for which $G(\sigma)$ is a proper subgroup of G. Then we must quotient by the action of G by conjugation to deduce that the cardinality of $SNi(\mathbf{C})^{ab}$, the absolute Nielsen classes represented by SNi(C), is 3^4-1. Let H be the subgroup of S_6 that fixes each of the sets {1,2},{3,4} and {5,6}. Consider $(\Phi_r)^{-1}(H)$ in H(r) (expression (4.3)). Finally, $Ni(\mathbf{C})^{ab}$ is the union of the sets that we get by applying coset representatives of $(\Phi_r)^{-1}(H)$ in H(r) to $Ni(\mathbf{C})^{ab}$. Therefore the cardinality of $Ni(\mathbf{C})^{ab}$ is 6!/8 times 3^4-1. Again, however, we haven't yet dealt with whether the general surface of genus 2 has a map to $\mathbb{P}^1_z$ in this Nielsen class.

Case 3: *(3.6) d) p=3, n=9, $G=(\mathbb{Z}/3)^2\times^s GL(2,3)$* . Here G_1=GL(2,3), $\sigma=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$; and A is the transvection $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Of course, in GL(2,3) there is just one conjugacy class of reflections and one class of transvections. In the notation of Case 2, our Nielsen class $Ni(\mathbf{C})$ is given by this data: $C_1=C_2=C_3=C_4=C((0,0);\sigma)$; and $C_5=C_6=C((0,0);A)$.

From the Jordan normal form we deduce that there are 8 conjugacy classes in G_1 represented by I_2, $-I_2$, σ, A, $B_1=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$, $B_2=\begin{pmatrix}-1 & 1\\ 1 & 0\end{pmatrix}$,

$(B_2)^{-1}$ and $(B_2)^2$. The first task is to construct the character table of G_1.

We borrowed the character tables of S_4 and SL(2,3) from [I;p.287-288]. From this we got the table for GL(2,3) on the basis of these two simple principles: GL(2,3) contains SL(2,3) and S_4 is GL(2,3)/<$-I_2$> through the action on the lines of $(\mathbb{Z}/2)^2$; and each representation of the latter is a representation of GL(2,3) and the representations ρ of the former give induced representations $\rho_{SL(2,3)}^{GL(2,3)}$ of GL(2,3) which may or may not decompose into irreducible representations of GL(2,3). When we completed the representations of GL(2,3) that arose from these, we found that the sum of the squares of the degrees summed to 16·3; thus we were done.

Character Table of GL(2,3):

χ Class:	I_2	$-I_2$	σ	A	B_1	B_2	$(B_2)^{-1}$	$(B_2)^2$
χ^1	1	1	1	1	1	1	1	1
χ^2	1	1	-1	1	1	-1	-1	1
χ^3	2	2	0	-1	-1	0	0	2
χ^4	3	3	-1	0	0	-1	-1	1
χ^5	3	3	-1	0	0	1	1	-1
χ^6	2	-2	0	-1	1	$i\sqrt{2}$	$-i\sqrt{2}$	0
χ^7	2	-2	0	-1	1	$-i\sqrt{2}$	$i\sqrt{2}$	0
χ^8	4	-4	0	1	-1	0	0	0

The characters of S_4 are the first five in the table, χ^6 and χ^7 restrict to conjugate irreducible characters of SL(2,3) and χ^8 restricts to an irreducible character of SL(2,3).

Now compute the conjugacy classes of G. The conjugates of (v,τ) by elements of N consist of the elements $(w-w^{\tau}+v,\tau)$ as w runs over N. If τ has no eigenvalue equal to 1, then the linear map $w \to w-w^{\tau}$ is one-one and this conjugacy class equals $N\times\tau$. On the other hand, since GL(2,3) is transitive on N-{0}, $N\times 1$ consists of two conjugacy classes; and $N\times\sigma$ and $N\times A$ have, each, 3 conjugacy classes under N. The latter two instances, however, have two conjugacy classes under N fuse under the action of G_1 (e.g., $(0;-I)((0,b);\sigma)(0;-I)=(((0,-b);\sigma))$. Thus there 11 total conjugacy classes in G.

Each of the 8 characters of G_1 above gives an irreducible representation of G through the natural map $N\times^s G_1 \to G_1$. The sum of the squares of these representations add up to 48. Other characters of G arise by inducing the characters above from G_1 (embedded in G) to G. For each $\rho:G_1 \to GL(n,\mathbb{C})$ the induced representation applied to $(v;\tau)\in(\mathbb{Z}/3)^2\times^s GL(2,3)$ is given by a matrix (with matrix entries)

$$\rho^{G}_{GL(2,3)}(v;\tau)=\begin{pmatrix} \cdots & \cdots\cdots & \cdots \\ \cdots & \rho(w_1^{\tau}-w_2+v,\tau) & \cdots \\ \cdots & \cdots\cdots & \cdots \end{pmatrix} \text{ over all } w_1 \text{ and } w_2 \text{ in N,}$$

with the understanding that the matrix entries are 0 unless $w_1^{\tau}-w_2+v$ is 0. Thus $tr(\rho^G(v;\tau))=k_{v,\tau}tr(\rho(\tau))$ where $k_{v,\tau}$ is the number of w such that $w^{\tau}-w+v=0$. Note that there is a unique such w unless τ has 1 as an eigenvalue.

We need three more characters of G, and two of them are obvious: $\mathbf{1}_{GL(2,3)}^G$ breaks up into the identity character and a character ψ_1 of degree 8; and the sign character χ^2 induced to G breaks up into χ^2 and $\chi^2 \otimes \psi_1$, also of degree 8. Let ρ be any nontrivial character on N. For explicitness we take ρ to be given by $(a,b) \to \rho(a,b)=\zeta_3^a$. We will show that ρ_N^G (formula similar to that for $\rho_{GL(2,3)}^G$) breaks up into the sum of ψ^1, $\chi^2 \otimes \psi^1$ and two copies of ψ^2, an irreducible character of degree 16. This is all determined by using the following inner product computations:

$$\left(\rho_N^G, \mathbf{1}_{GL(2,3)}^G\right)=\sum_{(v;\sigma)} \rho_N^G(v;\sigma)\mathbf{1}_{GL(2,3)}^G(v;\sigma)/|G|=$$

$$\sum_v \sum_\sigma \rho(v^\sigma)\mathbf{1}_{GL(2,3)}^G(v;1)/|G|=1$$ (since only v=0 contributes); and

$$\left(\rho_N^G, \rho_N^G\right)=\sum_{(v;\sigma)} \rho_N^G(v;\sigma)\overline{\rho}_N^G(v;\sigma)/|G|=\sum_v \sum_\sigma \rho(v^\sigma)\sum_\sigma \overline{\rho}(v^\sigma)/|G|=$$

$$\left(|GL(2,3)|\,|GL(2,3)|+\sum_{v\neq 0} \sum_\sigma \rho(v^\sigma)\sum_\sigma \overline{\rho}(v^\sigma)\right)/|G|=\frac{2^4 3\cdot(3\cdot 2^4+6)}{3^3 2^4}=6,$$

where $\overline{\rho}$ indicates complex conjugate. That is, the first computation shows that the two representations of degree 8 appear in ρ_N^G with multiplicity 1. Since the other representation appears in $\mathbf{1}_1^G$ somewhere, and it appears in none of the representations of G_1, it must appear twice in ρ_N^G. The sums of the squares of the degrees of these representations add up to the order of G, so we have found them all.

In order to apply the structure constant formula we use only the characters that have nonzero values on both $(0;\sigma)$ and $(0;A)$. For the characters of G_1 it is just those of degree 1. Also, since $\mathbf{1}_{GL(2,3)}^G(0;\sigma)$ is the number of v's such that $v-v^\sigma=0$, $\psi^1(0;\sigma)$ is 2; similarly $\chi^2\otimes\psi^1(0;\sigma)$ is -2, $\psi^1(0;A)$ is 2 and $\chi^2\otimes\psi^1(0;A)$ is -2; and finally from the formula

$$2\psi^2+\psi^1+\chi^2\otimes\psi^1=\mathbf{1}_{GL(2,3)}^G,$$

$\psi^2(0;A)=0$. Thus $a_G(\mathbf{C})=2|C_{(0;\sigma)}|^4|C_{(0;A)}|^2\left(1+\frac{2^6}{8^4}\right)/|G|$. Again this suffices for the odd degree cases.

§4.4. CASE (3.6) F) OF THEOREM 3.3: Some readers may have thought at the beginning of this paper that solvable groups whose composition factors are small must be easy. This example will shred the vestiges of that naiveté. Recall that p=2, n=16, and that

$$G=(\mathbb{Z}/2)^2\times(\mathbb{Z}/2)^2\times^s((S_3\times S_3)\times^s\mathbb{Z}/2).$$

Following Thompson, present G as a subgroup of S_8 with these generators: $\alpha=(12)(34)$, $\beta=(13)(24)$, $\lambda=(123)$, $\rho=(12)$; $\gamma=(56)(78)$, $\delta=(57)(68)$, $\mu=(567)$, $\sigma=(56)$; $\pi=(15)(26)(37)(48)$. Identify $N=(\mathbb{Z}/2)^2\times(\mathbb{Z}/2)^2$ with $\langle\alpha,\beta,\gamma,\delta\rangle$, $P=\langle\lambda,\mu\rangle$ is of order 9, and $D=\langle\rho,\pi\rangle$ ($=\langle\rho,\pi,\sigma\rangle$) is dihedral of order 8. Furthermore, P is normal in G_1=PD which we identify with $(S_3\times S_3)\times^s\mathbb{Z}/2$. Finally, let G_0 be the normal subgroup of G of index 2 generated by the conjugates of ρ. Then G', the commutator subgroup of G, is $\langle\rho\sigma,\lambda,\mu,N\rangle$ and it is of index 4. Note here for use below that

(4.14) $1,\rho,\pi,\rho\pi$ are representatives for G/G'.

For $\sigma\in G$ denote the value of the permutation character on the cosets of G_1 applied to σ by $\chi(\sigma)$, and denote the index, 2^4-orb(σ) by ind(σ), orb(σ) is the number of orbits of σ in this representation. Rows 1-15 below give representatives for classes in G_0 and rows 16-20 for $G\backslash G_0$.

The Conjugacy Class Chart:

No.	C=Con(τ)	C=Con(τ^2)	ord(τ)	|Con(τ)|	$\chi(\tau)$	orb(τ)	ind(τ)
1	1	1	1	1	16	16	0
2	λ	λ	3	16	4	8	8
3	$\lambda\mu$	$\lambda\mu$	3	64	1	6	10
4	$\lambda\delta$	λ	6	48	0	4	12
5	$\lambda\sigma$	λ	6	96	2	6	10
6	$\lambda\sigma\delta$	$\lambda\delta$	12	96	0	2	14
7	α	1	2	6	0	8	8

8	$\alpha\gamma$	1	2	9	0	8	8
9	ρ	1	2	12	8	12	4
10	$\rho\beta$	α	4	12	0	4	12
11	$\rho\gamma$	12	36	36	0	8	8
12	$\rho\gamma\beta$	α	4	36	0	4	12
13	$\rho\sigma$	1	2	36	4	10	6
14	$\rho\sigma\beta$	α	4	72	0	4	12
15	$\rho\sigma\beta\delta$	$\alpha\gamma$	4	36	0	4	12
16	$\lambda\mu\pi$	$\lambda\mu$	6	192	1	4	12
17	π	1	2	24	4	10	6
18	$\pi\alpha$	$\alpha\gamma$	4	72	0	4	12
19	$\rho\pi$	$\rho\sigma$	4	144	2	6	10
20	$\rho\pi\beta$	$\rho\sigma\beta\delta$	8	144	0	2	14

Our goal is to find all integers $r \geq 6$ and all conjugacy classes, C_i, $i=1,\ldots,6$, such that for τ in the Nielsen class $Ni(\mathbf{C})$ of $(\mathbf{C},G)$, $\sum_{i=1}^{r} ind(\tau_i)=2(16+2-1)=34$. With no loss order conjugacy classes so that the indices of their elements are nonincreasing. Here are the possibilities for the values of the indices:

$I_1=\{12,6,4,4,4,4\}; I_2=\{10,6,6,4,4,4\}; I_3=\{8,6,6,6,4,4\}; I_4=\{6,6,6,6,6,4\};$

$I_5=\{6,6,6,6,4,4,4\}; I_6=\{10,8,4,4,4,4\}; I_7=\{10,4,4,4,4,4,4\}; I_8=\{8,8,6,4,4,4\};$

$I_9=\{8,6,4,4,4,4,4\}; I_{10}=\{14,4,4,4,4,4\}; I_{11}=\{6,4,4,4,4,4,4,4\}.$

Since $\mathrm{ind}(\tau)\epsilon\{4,8\}$ implies that $\tau\epsilon G_0$, in order for τ to be in Ni(**C**) at least two of the τ_i's must be in $G\backslash G_0$. This immediately eliminates $I_6,\ldots,I_{11}$.

We label the possible **C**'s by using the notation C together with the representative from the Conjugacy Class Chart:

for I_1 there is $C_{\lambda\mu\pi,\pi,\rho,\rho,\rho,\rho}$ and $C_{\pi\alpha,\pi,\rho,\rho,\rho,\rho}$; for I_2 there is $C_{\rho\pi,\pi,\rho\sigma,\rho,\rho,\rho}$, $C_{\lambda\sigma,\pi,\pi,\rho,\rho,\rho}$ and $C_{\lambda\mu,\pi,\pi,\rho,\rho,\rho}$#; for I_3 there is $C_{\lambda,\pi,\pi,\rho\sigma,\rho,\rho}$, $C_{\alpha,\pi,\pi,\rho\sigma,\rho,\rho}$, $C_{\alpha\gamma,\pi,\pi,\rho\sigma,\rho,\rho}$ and $C_{\rho\gamma,\pi,\pi,\rho\sigma,\rho,\rho}$#; for I_4 there is $C_{\pi,\pi,\pi,\pi,\rho\sigma,\rho}$# and $C_{\pi,\pi,\rho\sigma,\rho\sigma,\rho\sigma,\rho}$#; and for I_5 there is $C_{\pi,\pi,\rho\sigma,\rho,\rho,\rho,\rho}$.

Those with a # can be eliminated easily on the following principle. Since for $\tau\epsilon\mathrm{Ni}(\mathbf{C})$ the product $\tau_1\tau_2\cdots\tau_r$ $(=1)$ is congruent to the product of the subscripts of the entries of **C** modulo G', the commutator subgroup of G, the product of the subscripts of the **C** above must all be in G'. But for those with a # this product is in $G'\rho\neq G'$ from (4.14) (e.g., the product of the subscripts of $C_{\lambda\mu,\pi,\pi,\rho,\rho,\rho}$ is $\lambda\mu\rho$).

We are now prepared to apply the Structure Constant Lemma in order to count the number of elements of SNi(**C**) (as in the previous cases of this section) in each of the possibilities for above. As each **C** contains

C_π and C_ρ we need only use those characters χ such that $\chi(\pi)\chi(\rho)\neq 0$. There are precisely eight such characters (see the argument below (4.17) for their construction), four of degree 1 and four of degree 9. And the characters of degree 9 vanish on the elements of order divisible by 3.

Thus in the case when C is one of $C_{\lambda\mu\pi,\pi,\rho,\rho,\rho,\rho}$, $C_{\lambda\sigma,\pi,\pi,\rho,\rho,\rho}$ or $C_{\lambda,\pi,\pi,\rho\sigma,\rho,\rho}$ (i.e., when one of the classes represents elements of order divisible by 3) we have

(4.15) $a_G(\mathbf{C})=4|C_1|\cdots|C_6|/|G|$.

Check by explicit calculation in each of these cases that if $\boldsymbol{\tau}$ satisfies all of the criteria for being in Ni($\mathbf{C}$) except that $G(\boldsymbol{\tau})$ is not G, then $G(\boldsymbol{\tau})$ is contained in some conjugate of G_1=H. Similar to (4.15) we get $a_H(\mathbf{C})$. The calculations are not difficult from this point for each case:

$$a_G(C_{\lambda\mu\pi,\pi,\rho,\rho,\rho,\rho})=4\cdot 192\cdot 24\cdot 12^4/(2^7\cdot 3^2)=2^{12}\cdot 3^4; \text{ and}$$

$$a_H(C_{\lambda\mu\pi,\pi,\rho,\rho,\rho,\rho})=4\cdot 12\cdot 6\cdot 6^4/(2^3\cdot 3^2)=2^6\cdot 3^4.$$

Thus, precisely $2^{10}\cdot 3^4$ tuples $\boldsymbol{\tau}$ that are counted in $a_G(C_{\lambda\mu\pi,\pi,\rho,\rho,\rho,\rho})$ must be excluded since they generate proper subgroups of G. Thus

(4.16) a) $|SNi(C_{\lambda\mu\pi,\pi,\rho,\rho,\rho,\rho})^{ab}|=(2^{12}\cdot 3^4 - 2^{10}\cdot 3^4)/|G|=2^3\cdot 3^3$,

and from this $|Ni(C_{\lambda\mu\pi,\pi,\rho,\rho,\rho,\rho})^{ab}|$ is easily computed. Similarly, (4.16) b) $|SNi(C_{\lambda\sigma,\pi,\pi,\rho,\rho,\rho})_{ab}|=(2^{12}\cdot 3^4 - 2^{10}\cdot 3^4)/|G|=2^3\cdot 3^3$;and

c) $|SNi(C_{\lambda,\pi,\pi,\rho\sigma,\rho,\rho})_{ab}|=(2^{11}\cdot 3^4 - 2^{10}\cdot 3^4)/|G|=2^2\cdot 3^3$.

It remains to study the cases where **C** is one of

$C_{\pi\alpha,\pi,\rho,\rho,\rho,\rho}$, $C_{\rho\pi,\pi,\rho\sigma,\rho,\rho,\rho}$,

$C_{\alpha,\pi,\pi,\rho\sigma,\rho,\rho}$, $C_{\alpha\gamma,\pi,\pi,\rho\sigma,\rho,\rho}$ and $C_{\pi,\pi,\rho\sigma,\rho,\rho,\rho,\rho}$. In this case let ψ be some irreducible character of degree 9. Then in each of the five remaining cases

(4.17) $a_G(\mathbf{C})=4\cdot|C_1|\cdots|C_r|(1+\psi(C_1)\cdots\psi(C_r)/3^{2(r-2)})$.

If $\boldsymbol{\tau}$ satisfies all of the criteria for being in Ni(**C**) in one of these cases except that $G(\boldsymbol{\tau})$ is not G, then $G(\boldsymbol{\tau})\cdot N=T$ is the unique 2-Sylow of G containing $G(\boldsymbol{\tau})$. Therefore $(a_G(\mathbf{C})-3^2\cdot a_T(\mathbf{C}))/|G|$ is $|SNi(\mathbf{C})^{ab}|$. As our standard 2-Sylow take $T=D\cdot N$. Let ψ_0 be the unique linear character of T whose kernel is $D\cdot[D,N]$. Then the induced character $(\psi_0)^G$ is of degree 9 and can be taken to be ψ. Note that $[D,N]=\langle\alpha,\gamma,\beta\delta\rangle$. In order to conclude the computation of $|SNi(\mathbf{C})^{ab}|$ we need a little "fusion" data for use in the relation $(\psi_0)^G(\sigma)=\sum_t\psi_0(t)\cdot[C_G(t):C_G(t)(t)]$ where the sum is over conjugacy classes of T that fuse to the conjugacy class of σ in G.

Fusion Table:

Reps. σ for T^G	Reps. t for T^T	$[C_G(t):C_G(t)(t)]$	$\psi(\sigma)$
$\pi\alpha$	$\pi\alpha, \pi\beta \epsilon (\pi\alpha) G \cap T$	1,2	-1
π	π	3	3
ρ	ρ	3	3
$\rho\pi$	$\rho\pi$	1	1
$\rho\sigma$	$\rho\sigma$	1	1
α	α, β	3,6	-3
$\alpha\gamma$	$\alpha\gamma, \alpha\delta, \beta\delta$	1,4,4	1

Here are the final values of $|SNi(\mathbf{C})^{ab}|$:

(4.18) a) $|SNi(C_{\pi\alpha,\pi,\rho,\rho,\rho,\rho})^{ab}|=2\cdot 3^2$,

b) $|SNi(C_{\rho\pi,\pi,\rho\sigma,\rho,\rho,\rho})^{ab}|=2^7$,

c) $|SNi(C_{\alpha,\pi,\pi,\rho\sigma,\rho,\rho})^{ab}|=2^4\cdot 3$,

d) $|SNi(C_{\alpha\gamma,\pi,\pi,\rho\sigma,\rho,\rho})^{ab}|=2^3\cdot 3^2$ and

e) $|SNi(C_{\pi,\pi,\rho\sigma,\rho,\rho,\rho,\rho})^{ab}|=2^7\cdot 11$.

There is one last point to be made in this subsection. It might be called the "coalescing principle." Suppose that $\boldsymbol{\sigma} \epsilon (S_n)^{r+1}$ and that $\boldsymbol{\tau} \epsilon (S_n)^r$

Suppose further that there exists j such that $\sigma_i=\tau_i$ for $i<j$, $\sigma_j\sigma_{j+1}=\tau_j$, and $\sigma_{i+1}=\tau_i$ for $i>j$. We say that $\boldsymbol{\tau}$ is obtained by coelescing $\boldsymbol{\sigma}$.

Consider a transitive group $G \subset S_n$ and a fixed Nielsen class $\mathrm{Ni}(\mathbf{C})$ and another collection of Nielsen classes $\{\mathrm{Ni}(\mathbf{C}_1), \mathrm{Ni}(\mathbf{C}_2), \dots, \mathrm{Ni}(\mathbf{C}_u)\}$ (all representing G) that are related by these properties:

(4.19) a) for $\boldsymbol{\tau} \in \mathrm{Ni}(\mathbf{C}_i)$ or $\mathrm{Ni}(\mathbf{C})$, $i=1,\dots,r$, the indices of coordinates of $\boldsymbol{\tau}$ sum to $2(n+g-1)$ with g independent of $\boldsymbol{\tau}$; and

b) for each i and each $\boldsymbol{\tau} \in \mathrm{Ni}(\mathbf{C}_i)$ there exists $\boldsymbol{\sigma} \in \mathrm{Ni}(\mathbf{C})$ such that $\boldsymbol{\tau}$ is obtained by coalescing $\boldsymbol{\sigma}$.

The following is proved along the lines of Principle 2.5.

Principle 4.1: *Assume that the conditions of (4.19) hold. If for some i the generic curve of genus g has a cover to $\mathbb{P}^1_z$ in $\mathrm{Ni}(\mathbf{C}_i)$, then this is also true with $\mathrm{Ni}(\mathbf{C}_i)$ replaced by $\mathrm{Ni}(\mathbf{C})$.*

Our concluding point is that (4.19) holds with $r+1=7$, $\mathrm{Ni}(\mathbf{C})=\mathrm{Ni}(C_{\pi,\pi,\rho\sigma,\rho,\rho,\rho,\rho})$ and $\mathrm{Ni}(\mathbf{C}_i)$ running over the remaining Nielsen classes of (4.16) and (4.18). Thus, if the generic curve of genus 2 does not have a cover of $\mathbb{P}^1_z$ in the Nielsen class of (4.18) e), then the generic curve of genus 2 does not have a cover of $\mathbb{P}^1_z$ in any of the Nielsen classes in (4.16) or (4.18).