**§5. MODULI DIMENSION OF FAMILIES OF COVERS:** For each Nielsen class Ni(**C**), as in §4.1, there is a map $\Psi(\mathcal{H}(\mathbf{C}),\mathcal{M}_g)$: $\mathcal{H}(\mathbf{C}) \to \mathcal{M}_g$ as in (4.1). The goal of this section is to list ideas for using explicit group actions to compute the dimension of the image of $\Psi(\mathcal{H}(\mathbf{C}),\mathcal{M}_g)$ (in particular, to check if it is generically surjective). We call this the *moduli dimension* of Ni(**C**). This is the final check (§5.2) required for the many cases where g=2 that were produced in §4 to ascertain if they do, indeed, give covers of $\mathbb{P}^1_z$ by the generic curve of genus 2 that have solvable closure. That is, in these cases we check if the moduli dimension of the corresponding Nielsen class is 3. This will also be used (§5.3) to give preliminary ideas on how to check for which n the generic curve of genus g has a map to $\mathbb{P}^1_z$ with $A_n$ as monodromy group. Part of §5.1 gives an exposition on the results of [Fr,3].

**§5.1. GROUP ACTIONS ON COHOMOLOGY:** First, given a cover $\varphi: X \to \mathbb{P}^1_z$ we must be able to compute $\pi_1(X)$ (and $H_1(X,\mathbb{Z})$) in terms of a description of the branch cycles for the cover. Indeed, if the cover is Galois it is back to the well known Schreier construction for generators of a subgroup of a free group in an explicit way.

Denote by $S = \{\overline{\sigma}_1,\dots,\overline{\sigma}_r\}$ generators of the free group $F_r$ on r generators. Let $G=G(\boldsymbol{\sigma})$ with $\boldsymbol{\sigma}$ a description of the branch cycles of the cover. Let $G(1)= \{\gamma \in G(\boldsymbol{\sigma})$ that fix $1\}$ and $\delta: F_r \to G(\boldsymbol{\sigma})$ is the homomorphism induced by $\overline{\sigma}_i \to \sigma_i$, $i=1,\dots,r$. Finally , let $H(1)$ be $\delta^{-1}(G(1))$. In order to get free generators of $H(1)$ we need a function $\rho: F_r \to F_r$ representing right cosets of $H(1)$, with the following properties : $\rho(1)=1$, $\rho(\alpha)\in H(1)\alpha$, and $\rho(h\alpha)=\rho(\alpha)$ for each $h\in H(1)$ and $\alpha\in F_r$. Furthermore $\rho$ may be selected to have the following property :

(5.1) $\quad \mathrm{length}_{\overline{\boldsymbol{\sigma}}}(\rho(\alpha))=\min_{h\in H(1)}(\mathrm{length}_{\overline{\boldsymbol{\sigma}}}(h\alpha))$ for each $\alpha\in F_r$.

From this it is automatic that if $\rho(\alpha)=s_1^{\epsilon(1)}s_2^{\epsilon(2)}\cdots s_n^{\epsilon(n)}$ is a reduced presentation of $\rho(\alpha)$, $s_i\epsilon S$, $\epsilon(i)\epsilon\{+1 \text{ or } -1\}$, $i=1,\cdots,n$, then

$$(5.2)\qquad s_1^{\epsilon(1)}s_2^{\epsilon(2)}\cdots s_i^{\epsilon(i)}\epsilon\rho(F_r) \text{ for each } i=1,\ldots,n.$$

With these conditions, the collection $M=\{ rs\rho(rs)^{-1} \mid r\epsilon\rho(F_r),\ s\epsilon S \text{ and } rs\notin\rho(F_r)\}$ generates H(1) freely (e.g., [ FrJ; Lemma 15.23 ]).

The following is well known: *If $\varphi: X\to\mathbb{P}^1$ is a Galois cover, then the fundamental group of X is isomorphic to the image of H(1) in $F_r/N$ where N is the smallest normal subgroup of $F_r$ containing $\bar\sigma_1\cdots\bar\sigma_r$ and $\bar\sigma_i^{ord(\sigma_i)}$, $i=1,\ldots,r$.*

But this definitely doesn't hold if $\varphi: X\to\mathbb{P}^1$ isn't Galois. Return to the group H(1), a possibly nonnormal subgroup of $F_r$ as derived from the Schreier construction. It is easy to interpret its quotient, $H(1)'$, modulo the group N: Subgroups of $H(1)'$ are in one-one correspondence with covers $X'\to X$ with the property that the pullback over $\hat X$ (i.e., a connected component of the fiber product $\hat X\times_X X'$ ) is unramified over $X'$. A preliminary step reduces the situation of a general cover $X\to\mathbb{P}^1$ to the situation where each nontrivial cover $Y\to X$, fitting in a diagram $\hat X\to Y\to X$, is ramified. That is, replace X by the maximal unramified cover $X^{un}$ of X fitting between $\hat X$ and X [Fr,3; Lemma 2.3].

**Lemma 5.1:** *The cover $X^{un}$ corresponds to the minimal subgroup $H_{un}$ of G(1) with the property that the length of any orbit O of $\sigma_i$ on the right cosets of G(1) is the same as the lengths of each of the orbits of $\sigma_i$ on the right cosets of H that comprise O, $i=1,\ldots,r$. Finally, this condition is equivalent to the following :*

*(5.3) $\alpha\epsilon G(1)$ if and only if $\alpha\epsilon H$ as $\alpha$ runs over all elements of the form $g\sigma_i^k g^{-1}$ with $g\epsilon G$, k a divisor of $ord(\sigma_i)$, $i=1,\ldots,r$.*

Again recall the natural surjective map $\delta: H(1)'\to G(1)$ induced from $\delta: F_r\to G(\boldsymbol{\sigma})$. Whenever there can be no confusion we denote the normal subgroup of $H(1)'$ generated by torsion elements by tor. In the sense of presented groups it has generators $T=\{ \tau\alpha\tau^{-1} \mid$

$\tau\epsilon F_r$, $\alpha=\bar{\sigma}_i^k$ with k a divisor of $\mathrm{ord}(\sigma_i)$, $i=1,\ldots,r$, and $\delta(\tau\alpha\tau^{-1})\epsilon G(1)$ }. Denote the subgroup of G(1) generated by $\delta(\beta)$ as $\beta$ runs over T by $G_{tor}$. If $H_{un}=G(1)$, then [Fr,3; Theorem 2.5] gives $\pi_1(X)$ in terms of branch cycles.

**Theorem 5.2:** *Use the previous notation with $\hat{H}(1)$ the maximal normal subgroup of $F_r$ contained in H(1). We may compute $\pi_1(X)$ as the quotient $H(1)'/N'$ where N′ is the smallest normal subgroup of H(1)′ with the property that the induced map from $\hat{H}(1)$ is surjective. In particular the quotient H(1)′/tor of H(1)′ by tor maps surjectively to $\pi_1(X)$.*

*Furthermore, this is an isomorphism if and only if in the natural map $\delta: H(1)'\to G(1)$, the image of tor is surjective, and this holds if and only if $H_{un}=G(1)$.*

**Example 5.3:** *Ni(**C**) given by (4.18) a).* Recall that our concern is with the absolute Nielsen classes $\mathrm{Ni}(\mathbf{C})^{ab}$ with **C** given by $C_{\pi\alpha,\pi,\rho,\rho,\rho,\rho}$ where $G=(\mathbb{Z}/2)^2\times(\mathbb{Z}/2)^2\times^s((S_3\times S_3)\times^s\mathbb{Z}/2)$ presented as a subgroup of $S_8$ having these generators: $\alpha=(12)(34)$, $\beta=(13)(24)$, $\lambda=(123)$, $\rho=(12)$; $\gamma=(56)(78)$, $\delta=(57)(68)$, $\mu=(567)$, $\sigma=(56)$; $\pi=(15)(26)(37)(48)$. We choose $\boldsymbol{\sigma}=(\alpha\pi,\pi,(12),(34),(13),(13))$ as a representative of this Nielsen class in order to get going. To verify that $G(\boldsymbol{\sigma})$ is not a proper subgroup of G (and therefore, as previously observed, contained in a 2-Sylow) note that (13)(12)=(132) so that $G(\boldsymbol{\sigma})$ has an element of odd order.

Next we check if $G(1)=H_{un}$. Warning: The permutation representation in this problem is of degree 16, not 8, and as remarked around expression (4.18), $G(1)=(S_3\times S_3)\times^s\mathbb{Z}/2=PD$ (called $G_1$ in §4) with $D=\langle\rho,\pi\rangle$, $P=\langle\lambda,\mu\rangle$ and N is identified with $\langle\alpha,\beta,\gamma,\delta\rangle$. According to Lemma 5.1, $H_{un}$ is the minimal subgroup of G(1) containing all of the elements $G(1)\cap\{g\rho g^{-1}, g\pi g^{-1}, g\pi\alpha g^{-1}, g(\pi\alpha)^2g^{-1}$ with $g\epsilon G\}$. Since both (12) and (13) are in G(1), it is clear that this is all of G(1). □

Our next task is to understand the action of a subgroup of the Hurwitz monodromy group on $H_1(X,\mathbb{Z})$. Identify projective r-space, $\mathbb{P}^r$, with the space of nonzero polynomials in z of degree at most r modulo the action of multiplication by $\mathbb{C}^*$. For the sake of intuitive identification, consider the polynomials of degree less than r as having a zero at $\infty$ (y.m., those of degree at most r-2 as having at least two zeros at $\infty$). Denote the points representing polynomials with distinct roots by $\mathcal{U}^r$. Since H(r) is the fundamental group of $\mathcal{U}^r$ and the action of H(r) on $Ni(\mathbf{C})^{ab}$ corresponds to the cover $\Psi(\mathbf{C}):\mathcal{H}(\mathbf{C}) \to \mathcal{U}_r$ [Fr,1,2], the subgroup $H(r,\boldsymbol{\sigma})$ that stabilizes the absolute class of $\boldsymbol{\sigma}$ in $Ni(\mathbf{C})$ acts on $H_1(X,\mathbb{Z})$ (in fact on $\pi_1(X)$). Indeed, since the centralizer $Cen_{S_n}(G)$ of G in $S_n$ is trivial there is an algebraic family $\mathcal{F}(\mathbf{C})$,

$$\Phi(\mathbf{C}):\mathcal{T}(\mathbf{C}) \to \mathcal{H}(\mathbf{C})\times\mathbb{P}^1_z,$$

of complex analytic manifolds where for each $\mathbf{x}\in\mathcal{H}(\mathbf{C})$ the fiber $\mathcal{T}(\mathbf{C})_{\mathbf{x}} \to \mathbb{P}^1_z$ represents the equivalence class of covers corresponding to $\mathbf{x}$.

To see the $H(r,\boldsymbol{\sigma})$ action explicitly recall from Theorem 5.2 that $\pi_1(X)$ is identified with $H(1)N/N=H(1)'$ modulo $N'$, the minimal normal subgroup of $H(1)'$ such that the induced map from $\hat{H}(1)$ is surjective.

Denote the normal subgroup of $F_r$ generated by $\bar{\sigma}_1\cdots\bar{\sigma}_r$ by $N_0$. Then we identify $\pi_1(X-\varphi^{-1}(z))$ with $H(1)N_0/N_0$. Consider $Q\in H(r)$. It is in the group generated by $Q_1,\dots,Q_{r-1}$ as in (4.3). Let Q act on $\bar{\boldsymbol{\sigma}}=(\bar{\sigma}_1,\dots,\bar{\sigma}_r)$ through the same formula as in (4.3). Clearly Q maps $N_0$ into itself. Furthermore, consider the application of Q to one of the generators $rs\rho(rs)^{-1}\in M$ of H(1). If $Q\in H(r,\boldsymbol{\sigma})$ then the image of $(rs\rho(rs)^{-1})Q$ in $G(\boldsymbol{\sigma})$ is the same as the image of $rs\rho(rs)^{-1}$. Since H(1) consists of those elements whose image is in G(1), clearly Q maps H(1) into itself. Similarly, Q maps $\hat{H}(1)$ into itself.

Let $N^*$ be the smallest normal subgroup of $H(1)$ containing $N$ (discussion prior to Lemma 5.1) such that $\hat{H}(1)/N^*=H(1)/N^*$ . We have induced an action of $Q$ on $\pi_1(X)$ if we show that $Q$ maps $N^*$ into itself. But $Q(N^*)$ clearly has the same properties as does $N^*$, once it has been established that $Q(N)=N$. This reduces to showing that

$(\bar{\sigma}_i{}^{\mathrm{ord}(\sigma_i)})Q$ is in $N$, $i=1,\dots,r$.

Going back to the generating elements of $H(r)$, $(\bar{\sigma}_i)Q=\alpha\bar{\sigma}_j\alpha^{-1}$ for some $\alpha\in F_r$ and some $j$. Since the image of $(\bar{\sigma}_i)Q$ and $\bar{\sigma}_i$ in $G(1)$ are the same, $\mathrm{ord}(\sigma_j)= \mathrm{ord}(\sigma_i)$. As $N$ is a normal subgroup of $F_r$, $(\bar{\sigma}_i{}^{\mathrm{ord}(\sigma_i)})Q= \alpha\bar{\sigma}_j{}^{\mathrm{ord}(\sigma_j)}\alpha^{-1}$ is in $N$. The following includes a summary:

**Theorem 5.4:** *Suppose that $\mathcal{T}(\mathbf{C})_{\mathbf{x}}$ $=X$ is one of the fibers of the family $\mathcal{F}(\mathbf{C})$, and that this fiber , as a cover of $\mathbb{P}^1$ , has $\boldsymbol{\sigma}$ as a description of its branch cycles. The subgroup $H(r,\boldsymbol{\sigma})$ of $H(r)$ that leaves fixed the image of $\boldsymbol{\sigma}$ in $Ni(\mathbf{C})^{ab}$ induces an action on $\pi_1(X)$ through the action of $H(r)$ on $\bar{\boldsymbol{\sigma}}$ from (4.3). This action can be identified with the usual (Picard-Lefshetz) monodomy action of the fundamental group of a parameter space on the fibers of a smooth complex analytic family .*

*Furthermore, in the case that $g=1$ or $2$, the image of $H(r,\boldsymbol{\sigma})$ in $H_1(X,\mathbb{Z})$ is a finite group if and only if $\Psi(\mathcal{H}(\mathbf{C}),\mathcal{M}_g)$: $\mathcal{H}(\mathbf{C}) \to \mathcal{M}_g$ is constant.*

**Proof:** The action is described above. The identification with the usual monodromy action follows from [Fr,2;§4] which shows the effect of the $Q_i$'s on generating paths of $\pi_1(\mathbb{P}^1-\mathbf{z})$. If these are represented by $\bar{\boldsymbol{\sigma}}$ the action is given by (4.3). The induced action on paths representing $\pi_1(X-\varphi^{-1}(\mathbf{z}))$ is a part of the uniqueness up to homotopy of the natural fundamental group action.

The final statement comes from identifying generators of $H(r,\boldsymbol{\sigma})$ with the Picard-Lefshetz transformation around a branch at $\infty$, in the language of [Gr; §6, especially Theorem 6.4-The removable singularity theorem]. If each of these generators is of finite order on $H_1(X,\mathbb{Z})$, then $\Psi(\mathcal{H}(\mathbf{C}),\mathcal{M}_g)$: $\mathcal{H}(\mathbf{C}) \to \mathcal{M}_g$ extends to $\overline{\mathcal{H}}(\mathbf{C}) \to \mathcal{M}_g$ where $\overline{\mathcal{H}}(\mathbf{C})$ is a nonsingular projective compactification of $\mathcal{H}(\mathbf{C})$. But, in the case that g=1 or 2, $\mathcal{M}_g$ is an affine open subset of a projective variety $\overline{\mathcal{M}}_g$ [M,2; p.25]. By Chow's theorem the image of $\overline{\mathcal{H}}(\mathbf{C})$ is a projective subvariety of $\mathcal{M}_g$. Thus unless it is just a point it must meet one of the divisors in $\overline{\mathcal{M}}_g-\mathcal{M}_g$, contrary to our information. the converse of the last statement is much easier. □

**Remark:** For g≥3, $\mathcal{M}_g$ is not affine (e.g., it contains projective curves), but the "co lescing of branch points" argument in §5.2 (e.g., Claim 4) checks for the possibility of extending $\mathcal{H}(\mathbf{C}) \to \mathcal{M}_g$ to a map into $\mathcal{M}_g$ along a specific branch at ∞. □

Now consider the group $\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z)$ of automorphisms of the cover $\varphi: X \to \mathbb{P}^1_z$. In terms of branch cycles, this is naturally identified with the centralizer, $\mathrm{Cen}_{S_n}(G(\boldsymbol{\sigma}))$, of $G(\boldsymbol{\sigma})$ in $S_n$ [Fr,2; Lemma 2.1]. When we take X to be $\hat{X}$ (i.e., the cover is Galois) this is the regular representation of $G(\boldsymbol{\sigma})$ and $n=\hat{n}=|G(\boldsymbol{\sigma})|$ [Fr,2; Lemma 2.1]. This induces an action of $\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z)$ on $H_1(\hat{X},\mathbb{Z})$ (and $H_1(\hat{X},\mathbb{Q})$) which is known to be faithful [FaK; p.253]. Thus the group ring $A=\mathbb{Z}[\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z)]$ (resp., $A\otimes\mathbb{Q}$) acts faithfully on $H_1(\hat{X},\mathbb{Z})$ (resp., $H_1(\hat{X},\mathbb{Q})$). For each $\alpha\in\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z)$ choose a (homotopy class of) path $\overline{\sigma}_\alpha$ on $\hat{X}$ with initial point $x_1$ and end point $\alpha(x_1)$. (Note that such choices would have already been made in applying Schreier's construction to compute the fundamental group of $\hat{X}$ in terms of branch cycles.) The following uses a number of classical results, including the Lefshetz trace formula: The alternating sum of the traces of an automorphism of a Riemann surface on the integral homology spaces is the number of fixed points of the automorphism [FaK; p.265].

**Principle 5.5:** *In the notation above, $\alpha$ acts on $H_1(\hat{X},\mathbb{Z})$ by conjugation by $\bar{\sigma}_\alpha$. Denote the number of disjoint cycles of $\sigma_i$ (in the regular representation of G) that $\alpha$ centralizes by $t_i$. Then,*

*(5.4) the trace of the action of $\alpha$ on $H_1(\hat{X},\mathbb{Z})$ is $2-\sum_{i=1}^{r} t_i$.*

*We may effectively do the following: identify $H_1(X,\mathbb{Q})$ with a subspace of $H_1(\hat{X},\mathbb{Q})$; and given $\beta \epsilon A$, decide if $\beta$ maps $H_1(X,\mathbb{Q})$ into itself and in this case deduce whether the action of $\beta$ is nontrivial.*

*Denote the elements of $A\otimes\mathbb{Q}$ that leave $H_1(X,\mathbb{Q})$ stable by $A_X$. If the moduli dimension of Ni(**C**) is 3g-3, then $A_X=\mathbb{Q}$.*

**§5.2. Moduli Dimension of Exceptional Cases:** Continue the notation of §5.1. Denote the divisor classes (not necessarily positive) of degree k on X (resp., $\hat{X}$) by $\text{Pic}(X)^k$ (resp.,$\text{Pic}(\hat{X})^k$). Consider an element $c=\sum_{\tau\epsilon \text{Aut}(\hat{X}/\mathbb{P}^1_z)} a_\tau \tau \;\epsilon\; A$. Then c induces a map $c^*:\text{Pic}(\hat{X})^1 \rightarrow \text{Pic}(\hat{X})^k$ where $k=\sum_{\tau\epsilon \text{Aut}(\hat{X}/\mathbb{P}^1_z)} a_\tau$ by mapping $\hat{\mathbf{x}}\epsilon\hat{X}$ to $\sum_{\tau\epsilon \text{Aut}(\hat{X}/\mathbb{P}^1_z)} a_\tau \tau(\hat{\mathbf{x}})$.

Let $\psi:\hat{X} \to X$ be the natural map. Identify a point $\mathbf{x}\in X$ with $\sum_{\tau\in G(1)}\tau(\hat{\mathbf{x}})$ for any point $\hat{\mathbf{x}}\in\hat{X}$ lying above $\mathbf{x}$. Then we recognize $\mathrm{Pic}(X)^1$ as the image of $\mathrm{Pic}(\hat{X})^1$ under the map $c(\psi)^*:\mathrm{Pic}(\hat{X})^1 \to \mathrm{Pic}(\hat{X})^k$ with $c(\psi)=\sum_{\tau\in G(1)}\tau$.

Suppose $c\in A$ satisfies $c\cdot c(\psi)=c(\psi)\cdot c$. Then there exists a commutative diagram

$$\begin{array}{ccc} c(\hat{\psi}):\mathrm{Pic}(\hat{X})^1 & \to & \mathrm{Pic}(X)^1 \\ c\downarrow & & \downarrow c_0 \\ c(\psi):\mathrm{Pic}(\hat{X})^k & \to & \mathrm{Pic}(X)^k \end{array} \qquad (5.5)$$

where $c_0$ is defined by application to $c(\psi)(\hat{\mathbf{x}})$ where it yields $c(\psi)(c(\hat{\mathbf{x}}))$. Note that $c_0$ induces an endomorphism on $\mathrm{Pic}(X)^0$.

At any time, up to isogeny, this allows us to identify $\mathrm{Pic}(X)^0$ with the image of $c(\hat{\psi})$ in $\mathrm{Pic}(\hat{X})^0$, an observation that we use in our examples. We will check if there exists a c such that $c_0$ induces a nontrivial endomorphism (i.e., not in $\mathbb{Z}$) of $\mathrm{Pic}(X)^0$.

First consider Case 1 from §4.3: $p=5=n$, $G=G(\boldsymbol{\sigma})=D_{10}$; $r=6$; $\boldsymbol{\sigma}$ has entries from the conjugacy class of $S_5$ consisting of products of disjoint 2-cycles; and, with no loss, $G=\langle\sigma,\tau\rangle$ with $\sigma=(14)(23)$ and $\tau=(24)(15)$. Denote $\sigma\tau$ by $\sigma_\infty$. If $c\in A$ commutes with $c(\psi)$ and the corresponding endomorphism $c_0$: $\mathrm{Pic}(X)^0 \to \mathrm{Pic}(X)^0$ is not an element of $\mathbb{Z}$, then Principle 5.5 demonstrates that the moduli dimension of $\mathrm{Ni}(\mathbf{C})$ is no more than two, and therefore that $D_{10}\notin\mathcal{G}_2(\mathrm{sol})$.

It is natural to select c to have minimal support in the elements of G. Since $G(1)=\{1,(25)(34)\}$, the choice for c is down to $\sigma_\infty+\sigma_\infty^4$ (or $\sigma_\infty^2+\sigma_\infty^3$). From the faithfulness of the action of A on $H_1(\hat{X},\mathbb{Z})$ it is clear that c doesn't act through $\mathbb{Z}$ on this. In the next calculation denote $1+\sigma_\infty+\sigma_\infty^2+\sigma_\infty^3+\sigma_\infty^4$ by $c_\infty$.

The key to our computation is in this collection of homomorphisms:

(5.6) a) $1+\sigma=c_1:\mathrm{Pic}(\hat{X})^0 \to \mathrm{Pic}(X)^0$;

b) $-(1+\sigma)\sigma_\infty^k=c_2:\mathrm{Pic}(\hat{X})^0 \to \mathrm{Pic}(X)^0$; and

c) $c_\infty:\mathrm{Pic}(\hat{X})^0 \to \mathrm{Pic}(Y)^0$,

where Y is the qoutient of $\hat{X}$ by $<\sigma_\infty>$ and k is an integer to be selected below. The rest of the argument consists of four claims. Denote the origin of $\mathrm{Pic}(\hat{X})^0$ by **o**.

**Claim 1.** *$\mathrm{Pic}(\hat{X})^0$ is isogenous to $\mathrm{Pic}(X)^0\times\mathrm{Pic}(X)^0\times\mathrm{Pic}(Y)^0$.* Since $\hat{X}$ is of genus 6 (i.e., $2(10+g(\hat{X})+1)=6\cdot5$) and Y (and X) is of genus 2, the isogeny is given by $(c_1,c_2,c_\infty)$ if we show that for $\hat{\mathbf{x}}\epsilon\mathrm{Pic}(\hat{X})^0$, $(1+\sigma)(\hat{\mathbf{x}})=-(1+\sigma)\sigma_\infty^k(\hat{\mathbf{x}})=c_\infty(\hat{\mathbf{x}})=\mathbf{o}$, then $\hat{\mathbf{x}}$ is in a finite subgroup of $\mathrm{Pic}(\hat{X})^0$. From the first two of these conclude that for any integer j, $(1+\sigma-\sigma_\infty^j(1+\sigma)\sigma_\infty^k)(\hat{\mathbf{x}})=\mathbf{o}$. Choose j and k so that $j+k\equiv1 \bmod 5$ and so that $\sigma_\infty^j\sigma\sigma_\infty^k=\sigma$ (i.e., $j=k=3$ and $\sigma_\infty^3\sigma\sigma_\infty^3=((14253)(14)(23)(14253)=\sigma)$. In this case,

(5.7) $$(1-\sigma_\infty^j)(\hat{\mathbf{x}})=\mathbf{o}.$$

But (5.7) says that $c_\infty$ acts on $\hat{\mathbf{x}}$ as multiplication by the integer 5, $c_\infty(\hat{\mathbf{x}})=\mathbf{o}$ implies that $\hat{\mathbf{x}}$ is in the kernel of multiplication by 5.

**Claim 2.** *The trace of $\sigma_\infty^2+\sigma_\infty^3=c$ on $H_1(X,\mathbb{Q})$ is -2.* Indeed, from Claim 1 conclude that $H_1(\hat{X},\mathbb{Q})$ is naturally isomorphic to $H_1(X,\mathbb{Q})\oplus H_1(X,\mathbb{Q})\oplus H_1(Y,\mathbb{Q})$. As c commutes with $c_1$ and

$c_2$, the action of c on both copies of $H_1(X,\mathbb{Q})$ is the same; and the action of c on $H_1(Y,\mathbb{Q})$ is multiplication by 2. Thus, if u is the trace of c on $H_1(X,\mathbb{Q})$, then $2u+4\cdot2=2(2-t)$ where t is the total number of disjoint cycles that appear among the $\sigma_i$'s that are centralized by $\sigma_\infty^2$. The claim therefore follows if we show that t=0.

There is one subtlety for the reader here. While $\mathrm{Aut}(\hat{X}/\mathbb{P}^1_z)$ and $G(\boldsymbol{\sigma})$ are isomorphic groups, as subgroups of $S_{10}$, the former must be identified as the centralizer of the latter in ascertaining this calculation about disjoint cycles. Since, however, each $\sigma_i$ is a product of five disjoint 2-cycles and $\sigma_\infty^2$, a product of two disjoint 5-cycles, centralizes $\sigma_i$, it clearly permutes the five disjoint 2-cycles as a 5-cycle.

**Claim 3.** *The collection of covers in the Nielsen class Ni(**C**) consists of an irreducible family.* In extending the notation of the proof of Theorem 5.4, we consider again the algebraic family $\mathcal{F}(\mathbf{C})$,

$$\Phi(\mathbf{C}):\mathcal{T}(\mathbf{C}) \to \mathcal{H}(\mathbf{C})\times\mathbb{P}^1_z, \tag{5.8}$$

of complex analytic manifolds where for each $\mathbf{x}\in\mathcal{H}(\mathbf{C})$ the fiber $\mathcal{T}(\mathbf{C})_{\mathbf{x}} \to \mathbb{P}^1_z$ represents the equivalence class of covers corresponding to $\mathbf{x}$. Each cover (of degree 5 in our case) in the Nielsen class Ni(**C**) corresponds to a unique point of $\mathcal{H}(\mathbf{C})$. Again, (5.8) exists because the centralizer $\mathrm{Cen}_{S_n}(G)$ of G in $S_n$ is trivial, and the action of H(6) is transitive on $\mathrm{Ni}(\mathbf{C})^{ab}$ if and only if $\mathcal{H}(\mathbf{C})$ is irreducible (i.e., the family is irreducible). As a matter of fact, [R] has this calculation, based on (4.3), which he learned from [Arb,1] (and which starts the considerations of [Fr,2;§4]). For none of the other examples will this computation be so easy. Therefore we do it in terms of expressions following (4.9).

Suppose $(\sigma,\sigma_2,\sigma_3,\ldots,\sigma_6)$ represents an absolute Nielsen class and to it is associated the 6-tuple $(5,j_2,j_3,\ldots,j_6)$ of integers from $\{1,2,\ldots,5\}$ by $\sigma_i$ leaves $j_i$. Then for $k>1$, $Q_k$ moves this to the absolute Nielsen class that has the 6-tuple of integers associated to it in which $j_k$ has been switched with $j_{k+1}$. Also: conjugation by $\sigma$ switches the 1's and 4's, the 2's and 3's and leaves the 5's as they are: and application of $Q_1$ gives the 6-tuple associated to $(5,-j_2,j_3-j_2,\ldots,j_6-j_2)$ (where the entries are read appropriately mod 5). Denote the subset of elements of $(\mathbb{Z}/5)^5$ whose entries sum to 0 by V. Thus the irreducibility of the family follows from the transitivity of the subgroup of $GL(5,\mathbb{Z}/5)$ generated by $S_5$, multiplication by -1 and $\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$ on $V-\{\mathbf{0}\}$.

**Claim 4.** *The family has moduli dimension 2.* An element of $\mathbb{Z}$ acting on the 4-dimensional space $H_1(X,\mathbb{Q})$ has trace a multiple of 4. From Claim 2, c doesn't act like an element $\mathbb{Z}$ on $H_1(X,\mathbb{Q})$. Conclude that the moduli dimension isn't 3. We have only to show that it is 2. The combinatorial calculation for this uses the coelescing of branch cycles and Principle 4.1 from the end of §4.4. The theoretical foundation for the interpretation of this calculation arises from [Gr; §13-an unpublished result of Mayer and Mumford]. This raises a new issue: How do we test for the hypotheses of the boundary behavior criteria of [Gr; §13]? We now combine the remainder of the verification of Claim 4 with an exposition of this issue.

In addition to the nonsingular compactification $\overline{\mathcal{H}}(\mathbf{C})$ of $\mathcal{H}(\mathbf{C})$ used in the proof of Theorem 5.4, we require a smooth compactification of the family: a diagram

$$(5.9)\qquad \begin{array}{ccc} \mathfrak{T}(\mathbf{C}) & \rightarrow & \overline{\mathfrak{T}}(\mathbf{C}) \\ pr_1\circ\Phi(\mathbf{C})\downarrow & & \downarrow\overline{\Lambda} \\ \mathcal{H}(\mathbf{C}) & \rightarrow & \overline{\mathcal{H}}(\mathbf{C}) \end{array}$$

where $\overline{\mathfrak{T}}(\mathbf{C})$ is a compact nonsingular variety, $\overline{\Lambda}$ is a smooth morphism, and $\overline{\mathfrak{T}}(\mathbf{C})-\mathfrak{T}(\mathbf{C})$ and $\overline{\mathcal{H}}(\mathbf{C})-\mathcal{H}(\mathbf{C})$ are divisors with normal crossings. That is, local parameters $y_1,\dots,y_k$ for the variety locally describe a neighborhood of any given point on the divisor by the equation $y_1\cdots y_k=0$. Then the map $\Psi(\mathcal{H}(\mathbf{C}),\mathcal{M}_g)$: $\mathcal{H}(\mathbf{C}) \rightarrow \mathcal{M}_g$ extends to $\overline{\mathcal{H}}(\mathbf{C}) \rightarrow \overline{\mathcal{M}}_g$ where $\overline{\mathcal{M}}_g$ is the Sataki compactification of $\mathcal{M}_g$ [Gr; Theorem 13.1]. Here is how we intend to use this.

If we coalesce the elements of Ni($\mathbf{C}$), say, by replacing $\boldsymbol{\tau}\in$Ni($\mathbf{C}$) by $(\tau_1\tau_2,\tau_3,\dots,\tau_6)$ we find that the resulting collection, consists of two distinct Nielsen classes: Ni($\mathbf{C}_1$) with r=5-containing the class of $(\sigma_\infty^3,\sigma,\tau,\sigma,\tau)$-on which H(5) acts transitively; and Ni($\mathbf{C}_2$) with r=4-containing the class of $(\sigma,\sigma,\tau,\tau)$-on which H(4) acts transitively. The covers in the Nielsen class Ni($\mathbf{C}_1$) are of genus 2, and those of Ni($\mathbf{C}_2$) are of genus 0. Coordinates used by [Har] suggest the fibers of $\overline{\mathfrak{T}}(\mathbf{C})-\mathfrak{T}(\mathbf{C}) \rightarrow \overline{\mathcal{H}}(\mathbf{C})-\mathcal{H}(\mathbf{C})$, restricted to an irreducible component of $\overline{\mathcal{H}}(\mathbf{C})-\mathcal{H}(\mathbf{C})$, correspond to covers in one of these two Nielsen classes. But for the case of the Nielsen class Ni($\mathbf{C}_2$), the actual fibers would not be just genus 0 Riemann surfaces. Rather they would be represented by algebraic curves which would have an indication of the two singularities acquired as the two pairs of points on the family of Riemann surfaces degenerating over the coalescing locus came together.

## §5.3. GENERIC CURVES AND $A_n$, AND A FULL MODULI FORMULA:

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