Frattini towers and the shift-incidence cusp pairing: 
The Modular Tower view of modular curve towers

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Abstract. Modular curves give a special case of Frattini towers of spaces. Yet, they are only a special case of a general context, under the topic of MT (Modular Towers). In parallel, we treat two cases:

- modular curves, which in this view derive from the semi-direct product of \( Z/2 \) acting through multiplication by -1 on \( Z \); and
- an equally rich case which derives from \( Z/3 \) acting irreducibly on \( Z^2 \).

Modular curves are families of sphere covers attached to dihedral groups.

We compute the shift-incidence cusp pairing on reduced Hurwitz spaces for the curves \( X_j(p^{k+1}) \), \( j = 0 \) and 1, \( p \) an odd prime. From a MT view, modular curve cusps come in two types, \( g(roup)-p' \) (here the subtype, \( shift \) of Harbater-Mumford), and \( p \)-cusps. The 3rd MT type, \( o(ly)-p' \) is missing.

For both cases we also work out the reduced Hurwitz space monodromy group as a \( j \)-line cover, and the monodromy actions on 1st homology of a reduced Hurwitz space fiber. The shift-incidence pairing is new; a different slant on modular curve cusps. Monodromy for modular curves is not new.

Yet, our braid group approach extends the Hurwitz space description of modular curves to other systems of spaces. Extending Serre’s Open Image Theorem beyond modular curves, to general moduli of abelian varieties, has lacked a way to master the limiting effect of correspondences – read motives – of arithmetic monodromy on special tower fibers. Our \( Z/3 \) case shows how Frattini data in our Hurwitz space approach helps tame that structure. This is a step to generalizing the Open Image Theorem to new contexts.

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1. Abel and Dihedral functions

Let \( n \) be an odd (positive) integer. 1st year calculus uses \( T_n(\cos(\theta)) = \cos(n\theta) \), with \( T_n(w) = z \) the \( n \)th Chebychev polynomial. The goal is to express \( \cos(\theta)^n \) as a sum of \( \cos(k\theta) \) terms, \( 0 \leq k \leq n \). So, we can integrate any polynomial in \( \cos(\theta) \).

The trick: Induct on \( n \) to find \( T_n^*(w) = 2T_{n}(w/2) \) so \( T_n^*(u+1/u) = u^n+1/u^n \). Then substitute \( u \to e^{2\pi i \theta} \): The identity is just De Moivre’s formula

\[
(e^{i\theta})^n + (e^{-i\theta})^n/2 = ((\cos(\theta) + i \sin(\theta))^n + (\cos(\theta) - i \sin(\theta))^n)/2 = (e^{in\theta} + e^{-in\theta})/2 = \cos(n\theta).
\]

1.1. Dihedral polynomials. Consider any rational function \( f \) in \( w \) – whose degree we take as \( n \) – as a map between complex spheres:

\[
f: \mathbb{P}_w^1 = \mathbb{C}_w \cup \{\infty\} \to \mathbb{P}_z^1 = \mathbb{C}_z \cup \{\infty\}.
\]

Then, \( f \) has finitely many (branch) points, \( z' \), over which it ramifies: Instead of \( n \) distinct values of \( w \), there are fewer. We will designate the branch points by \( \{z_1, \ldots, z_r\} = \mathbf{z} \). Our key parameter will be this number \( r \geq 3 \) (after §1.1, \( r \geq 4 \)).

Select a point \( z_0 \in \mathbb{P}_z^1 \setminus \mathbf{z} \) \( \equiv U_z \). \( A \) explains the idea of a set of classical generators \( P_1, \ldots, P_r \) based at \( z_0 \) around \( \mathbf{z} \). That is, they are generators of the fundamental group \( \pi_1(U_z, z_0) \). By labeling the points of \( \mathbb{P}_w^1 \) over \( z_0 \) as \( \{1', \ldots, n'\} \), for each \( i, P_i \) is a loop around \( z_{\tau(i)} \) where \( \tau \) is a permutation of \( \{1, \ldots, r\} \).

Restrict the cover given by \( f \) over the pullback \( U_w \subset \mathbb{P}_w^1 \) of \( U_z \) in \( \mathbb{P}_w^1 \). Call this unramified cover \( f_\mathbf{z} \). On it use the unique path lifting property applied to \( P_i \): The unique path lifting \( P_i \) that starts at \( j' \in \{1', \ldots, n'\} \) ends at some \( j'' \). Denote the permutation from \( j'' \to j'' \) by \( g_i \).

DEF 1.1. The ordered collection \( \{g_1, \ldots, g_r\} \) is a branch cycle description of \( f \). This has nothing to do with the space \( \mathbb{P}_w^1 \) being a sphere; it applies equally with any compact Riemann surface \( W \) covering \( \mathbb{P}_z^1 \).
Here are the simplest facts about \((g_1, \ldots, g_r) = g\) - ordered by the condition that the classical generators emanate in order clockwise from \(z_0\).

(1.1a) Generation: The smallest Galois cover of \(\mathbb{P}^1\) over \(\mathbb{C}\) factoring through \(\mathbb{P}^1_w\) has group \((g_1, \ldots, g_r) = G_f\): \(f\) is a \(G_f\) cover.

(1.1b) Conjugacy classes: the \(g_i\)s represent \(r\) conjugacy classes \(C\) in \(G_f\) with well-defined multiplicity, independent of \(P_1, \ldots, P_r\).

(1.1c) Product-one: \(g_1 \cdots g_r = 1\).

The complete story of forming branch cycles is in many books. The most complete treatment is at [Fr08a, Chap. 4]. The set of \(r\)-tuples that you get from applying all possible classical generators to a given cover is a subset of all possible \(r\) tuples satisfying the conditions of (1.1). This is called a Nielsen class \(\text{Ni}(G, C)\).

Here is the main ramification fact that sometimes encourages researchers to consider only the cycle type, and not the more precise conjugacy classes.

(1.2) The lengths of the disjoint cycles in \(g\) are a listing of the orders of ramification of the points of \(\mathbb{P}^1_w\) lying over \(z_\tau(i)\).

Traditional algebraic geometry deals only with the case when each \(g_i\) is a 2-cycle in \(S_n\). We are not in that case, though computing our conjugacy classes will be easy. Still, our examples are all of the type where the cycle-type does not distinguish conjugacy classes used in a Nielsen class.

Since there are many sets of classical generators there could be (usually are) many possible branch cycle description of \(f\). Our first particular case when \(f\) is \(T_n\), leads us to refer to \(T_n\) (\(n\) odd) as a dihedral function.

**Lemma 1.2.** The polynomial cover \(T_n\) (\(n\) odd) is branched over 
\[\{z_1, z_2, z_3\} = \{-1, +1, \infty\}.\]

The Galois closure of that cover over \(\mathbb{C}\) has group \(D_n\), the dihedral group of degree \(2n\). Let \(\{P_{-1}, P_{+1}, P_\infty\}\) be classical generators based at an allowable \(z_0\), attached to the points indicated by the notation. Then, there is an \(h \in S_n\), so that simultaneous conjugation of a corresponding branch cycle description \((g_{-1}, g_1, g_\infty)\) by \(h\) gives this \((\text{b(ranch)} \text{ c(ycle)} \text{ d(escription)})\) of \(T_n\): 
\[
g_{-1} = (1n)(2n-1) \cdots (\frac{n-1}{2} \frac{n+3}{2}) \quad g_1 = (2n)(3n-1) \cdots (\frac{n+1}{2} \frac{n+3}{2}) \quad g_\infty = (n \cdot n-1 \cdots 1)\]

**Proof.** For the function \(H : u \mapsto (u^n + 1/u^n)/2\), compute the values of \(u\) – points that appear with multiplicity in the fibers of the map – by computing the zeros of the derivative \(\frac{dH}{du} = -n(u^{2n} - 1 - u^{-1})/u^{2n}\). In particular, all \(n\)-th roots of 1, and all their negatives are ramified values of \(u\).

Then, \(H\) is a composite in two different ways:
\[
(1.4) \quad u \mapsto u^n = t \mapsto (t + 1/t)/2 \quad \text{and as} \quad u \mapsto w = (u + 1/u)/2 \mapsto z = T_n(w).
\]

The \(u\)s multiple in the fiber of \(H(u)\) are \(0, \infty\) (with multiplicity \(n\)) and the zeros of \(\frac{dH}{du}\). The latter are the zeros of \(z^{2n} - 1\): \(n\)th roots of 1 and their negatives. Since these zeros have multiplicity 1, those \(u\)s have multiplicity 2 in the fiber of \(H(u)\).

The values of \(w\) appearing multiply in the fiber of \(T_n\) would be among the images of \(u\), under \(w(u)\), multiple in the fiber of \(H(u)\). Here \(0, \infty\) both go to \(\infty\) and the remainder are the real parts of the roots of 1 (over \(+1\)) and their negatives.
belong according to Lem. 1.2. These are represented by writing them as $C_\pm G$. The real parts of roots of $\pm 1$, the unique class of order 2 elements. The element $g_\pm$ is in the fiber of $S_n$. By conjugating by an element in $G$, we can indicate the conjugacy classes represented by the 3 elements of (1.3) and (1.4), especially in differential equations (as in §§1.2.1. and §1.2.2. branch point dihedral functions and our results). The example of §1.2 has a crucial invariant. It is the Nielsen class of 3-tuples satisfying the conditions of (1.1). Various equivalences attach to any Nielsen class. We will use several starting with that of absolute equivalence, already introduced: $g \equivabs g'$ if $g' = hgh^{-1}$ for some $h \in S_n$. We denote corresponding Nielsen (absolute) classes by $Ni(G, C)^{abs}$.}

1.2. Abel’s Nielsen class. In equivalencing elements that can appear in a 3-tuple indicated by (1.3), for some applications it is appropriate to equivalence them by conjugating by an element in $S_n$. Those are applications where we can – without loss – allow any change of variable that respects the map to $\mathbb{P}_1$. Given a copy of $G = Gf \subseteq S_n$, we can restrict to elements in the normalizer, $N_{S_n}(G)$, of $G$ in $S_n$. The elements $g_1$ and $g_{-1}$ in (1.3) are in the involution conjugacy class, $C_2$, of $D_n$: the unique class of order 2 elements. The element $g_{\infty}$ is in a conjugacy class of $n$-cycles. There are several such. Yet, in this equivalence – called absolute – we can choose an element of $N_{S_n}(D_n)$ that conjugates any such $n$ to $g_{\infty}$.

So, we can indicate the conjugacy classes represented by the 3 elements of (1.3) by writing them as $C = C_2^{2n_n}$. There are just three elements in $Ni(D_n, C)^{abs}$ where Chebyshev polynomials belong according to Lem. 1.2. These are represented by

$$(g_{-1}, g_1, g_{\infty}), (g_1, g_{\infty}, g_{-1}) \text{ and } (g_{\infty}, g_1, g_{-1}).$$
These are the representative from (1.3) and the application of the (left) shift applied repeatedly to it. Any $r$-tuple $g$ in any Nielsen class produces further representatives $(g)sh^k$ by applying iterations of the shift:

\[(1.5) \quad sh : (g_1, \ldots, g_r) \mapsto (g_2, \ldots, g_r, g_1).\]

Here is a review of the properties (1.1), applied to the Nielsen class of Chebychev polynomials, though we here write $D_n$ in matrix form.

\[(1.6a) \quad \text{ generation, (1.1a) } (g_{-1}, g_1) = D_n = \{( \begin{pmatrix} \pm 1 & 0 \\ b & 1 \end{pmatrix} \}_{b \in \mathbb{Z}/p}; \]

\text{ being in } C_2 \leftrightarrow 1 \text{ in the upper left matrix corner.}

\[(1.6b) \quad \text{ conjugacy classes, (1.1b) } g_{-1}, g_1 \text{ are in } C_2.\]

\[(1.6c) \quad \text{ product-one, (1.1c) } g_{-1}g_1 g_\infty = 1.\]

Finally, suppose there is another function which is also in this same Nielsen class. How is it related to $T_n$? Answer:

\[(1.7) \quad \text{ There exist M"obius transforms. } \alpha_1, \alpha_2 \in \text{PGL}_2(\mathbb{C}) \text{ with } f = \alpha_2 \circ T_p \circ \alpha_1^{-1}(w): f \sim_{\text{M"obius}} T_n.\]

Denote elements of $(\mathbb{P}^1)^d$ with no two entries the same by $U^4$. Then, $S_4$ (symmetries on \{1, 2, 3, 4\}) naturally permutes the coordinates of $U^4$. We denote its quotient $U^4/S_4$ by $U_4$. We may regard any unordered 4-tuple $z$ of distinct complex numbers as representing a PGL$_2(\mathbb{C})$ – M"obius – orbit:

\[z = \{z_1, \ldots, z_4\} \in \text{PGL}_2(\mathbb{C})/U^4 \text{ def } U_4 = \mathbb{P}_j^1.\]

Instead of the Nielsen class for Chebychev polynomials, we consider $\text{Ni}(D_p, C_{2^4})$. This replaces the 3-tuples of (1.3) by 4-tuples satisfying the properties of (1.1) with $G$ is still $D_p$ and with $C_{2^4}$, four repetitions of $C_2$. One representative of this Nielsen class is $(g_{-1}, g_1^{-1}, g_1, g_1^{-1}) \in C_{2^4}$. This is a H(arbater)-M(umford) representative. In a general Nielsen class such a rep. has the form $(g_1, g_1^{-1}, \ldots, g_s, g_s^{-1})$ whenever such a representing form in a Nielsen class is possible since it requires $r$ even and a special condition on conjugacy classes.

1.2.2. Modular Curves. The exposition will only outline Thm. 1.3, for it has served many purposes since its first display in [Fr78, §2] where it shows that for prime degree rational functions identifying those with the Schur covering property is equivalent to the theory of complex multiplication. The case $k = 0$ below:

Describing the prime-squared degree exceptional rational functions is equivalent to the GL$_2$-case of Serre’s Open Image Theorem [Fr05, §6.1–6.3], $k = 1$ below.

(This documents [GMS03] in showing all other degrees are sporadic.)

We use two other equivalences on Nielsen classes. One is inner – denoted by $\text{Ni}(G, \mathbb{C})/G \overset{\text{def}}{=} \text{Ni}(G, \mathbb{C})^{\text{in}}$, or between elements by $\equiv^\text{in}$. Inner equivalence of branch cycles assumes a corresponding cover is Galois, where we explicitly identify its covering group, up to inner conjugation, with $G$. The Inverse Galois Problems uses $\equiv^{\text{in}}$. §2.1.2 explains reduced $^*\text{-Nielsen classes, } \text{Ni}(G, \mathbb{C})^{*, \text{rd}}, ^* = \text{in or abs.}$

**Theorem 1.3.** Attached to $\text{Ni}(D_{p^{k+1}}, C_{2^4})^{*, \text{rd}}$ is the space of Möbius equivalence classes of degree $p^{k+1}$ functions, up to $\equiv^{\text{abs}}$, $\mathcal{H}(D^{p^{k+1}}, C_{2^4})^{\text{abs, rd}}$, that identifies with the modular curve $X_0(p^{k+1})$, $k \geq 0$, minus its cusps.

Similarly, the reduced inner space, $\mathcal{H}(D^{p^{k+1}}, C_{2^4})^{\text{in, rd}}$, identifies Möbius equivalence of the Galois closures of those functions with $X_1(p^{k+1})$, minus its cusps.
1.2.3. Explanation and use of Thm. 1.3. §1.3.1 explains the Hurwitz spaces attached to these Nielsen classes. Recall: Cusps refer to points over \( j = \infty \) on any cover of the complex \( j \)-line \( \mathbb{P}^1_j \). There is also a superscript notation rd indicating reduced equivalence. §2.1.2 defines reduced Nielsen classes. Reduced spaces arise as covering spaces from the group \( H_4 \) acting on Nielsen classes (modulo absolute, inner and reduced equivalence).

§2.1.1 defines the subgroup \( Q'' \) of \( H_4 \). When \( r = 4 \), all Hurwitz spaces are upper half-plane quotients covering classical \( j \)-line, with each component — like modular curves — an upper half-plane quotient by a subgroup of \( \text{PSL}_2(\mathbb{Z}) = H_4/Q'' \), ramified over \( \{0, 1, \infty\} \subset \mathbb{P}^1_j \) (when \( j \) is suitably normalized) [BFr02, Prop. 4.4].

§1.3.4 gives one MT setup that starts from the group \( \mathbb{Z}/n \) acting on the free group, \( F_{n-1} \) on \( n-1 \) generators. Serre’s O(pen) I(mage) T(heorem) is very precise about expectations — for modular curve covers, the case \( n = 2 \) of this setup — for the dependence on decomposition groups in such towers.

The geometric monodromy groups of the modular curve tower

\[
\cdots \rightarrow X_0(p^{k+1}) \rightarrow \cdots \rightarrow X_0(p) \rightarrow \mathbb{P}^1_j
\]

over the \( j \)-line form a sequence of groups \( \{\text{SL}_2(\mathbb{Z}/p^{k+1})/\{\pm 1\}\}_{k=0}^\infty \). For all primes \( p \geq 5 \), the \( k \)th term is a \( p \)-Frattini cover of the \( k = 0 \) term. Even for \( p = 2 \) and 3, there is a (small, explicit) \( k_0 \) so that for \( k \geq k_0 \), the \( k \)th term is a \( p \)-Frattini cover of the \( k_0 \) term. This is an example tower of \( j \)-line covers with the Frattini property.

We extend a step in the OIT for MTs, then show how it works with the non-modular curve case \( n = 3 \), above (§3.1.2). §4 explains why this implies, for most points of the \( j \)-line, any projective sequence of points over that point have decomposition group equal to the projective limit of the groups \( \{\text{GL}_2(\mathbb{Z}/p^{k+1})/\{\pm 1\}\}_{k=0}^\infty \). It also shows that the Frattini property is essential for a precise version of this.

1.3. Notation around extensions and spaces. Our main examples use split group extensions (§1.3.2). Still, somewhat mysterious essential properties of Hurwitz spaces come from Frattini covers, extensions far from split (§1.3.3).

So, though our use of groups is close to self-contained, we’ve added §1.3.1 to give some intuition in forming the spaces. Then, §1.3.4 clarifies precisely what it is we use from the group theory that forms appropriate sequences of spaces.

1.3.1. Spaces from dragging branch points. We construct reduced Hurwitz spaces, as in Thm. 1.3, so as to allow producing much more general towers.

Consider producing functions \( f : \mathbb{P}^1_w \rightarrow \mathbb{P}^1_z \) in the Nielsen class \( \text{Ni}(D_n, C_{24}) \) of §1.2.1 in analogy to how Chebychev polynomials are in the Nielsen class \( \text{Ni}(D_n, C_{22,n}) \).

Suppose we have a set of 4 distinct points \( z \) on \( \mathbb{P}^1_z \) and classical generators of \( \pi_1(\mathbb{P} \setminus z, z_0) \), as in App. A.

(1.8a) Then, App. A attaches a degree \( n \) compact surface cover \( f : X \rightarrow \mathbb{P}^1_z \) to each element of \( \text{Ni}(D_n, C_{24}) \).

(1.8b) Apply ramification fact (1.2) and R-H (App. B) to see \( X \) has genus 0.

(1.8c) R(iemann)-R(och) implies \( X \) is analytically isomorphic to \( \mathbb{P}^1_w \).

(1.8d) Each \( z_i \) has a unique unramified \( w_i \mapsto z_i \) with \( w_i \) corresponding to the length 1 disjoint cycle in \( g_i \).

From (1.8d), each function \( f \) produces \( f \leftrightarrow (w,z) \in U_{4,w} \times U_{4,z} \). Applying \( \text{PGL}_2(\mathbb{C}) \) action to both \( U_{4,w} \) and \( U_{4,z} \) embeds the Möbius equivalence classes of
such functions $f$ into

$$
PGL_2(\mathbb{C}) \backslash U_{4,r} \times U_{4,r}/PGL_2(\mathbb{C}) = (\mathbb{P}_1 \backslash \infty) \times (\mathbb{P}_1 \backslash \infty).
$$

There is a continuity on the space of such functions. That is, you can drag the classical generators on $U_{4,r} = \mathbb{P}_1 \backslash \mathbb{Z}^0$ along any path $P(t), t \in [0, 1]$ based at $\mathbb{Z}^0 \in U_r$ to classical generators on $U_{P(t)}$.

Upshot: You can drag $f_0$ to $f_1$ by its branch points. If $P$ is closed, representing $[P] \in \pi_1(U_r, \mathbb{Z}^0)$, then $f_1$ has the same set of branch points. Yet, if $G$ is nonabelian, it (usually) has a different branch cycle description, denoted $(g)_{[P]}$, relative to the original classical generators.

§2.1.2 continues gives references for forming the spaces and the precise identifications that allow computing with the corresponding $j$-line covers when $r = 4$.

### 1.3.2. Split extensions.

Consider a short exact sequence of groups:

$$
1 \rightarrow \text{Ker} \rightarrow G \rightarrow H \rightarrow 1.
$$

This section assumes (1.9) is split. That is, you can represent $G$ as 2x2 matrices:

$$
\left\{ \left( \begin{array}{cc} h & 0 \\ a & 1 \end{array} \right), \ | \ h \in H, a \in \text{Ker} \right\}.
$$

Then, matrix multiplication gives the multiplication in $G$. Call the matrix above $M(h, a)$. If we multiply $M(h_1, a_1)$ by $M(h_2, a_2)$, then the result is what you would expect from matrix multiplication:

$$
M(h_1 h_2, (a_1) h_2 \cdot a_2), \text{ as if } h_2 \text{ is a matrix acting on the right of } a_1.
$$

We write $G$ as $\text{Ker} \times H$ – the semi-direct product of Ker and $H$. The notation with $G$ on the right makes transparent our homomorphism $G \rightarrow H$. If instead you wanted a left action of $H$ on Ker, you could put $a$ in the upper right corner, and 0 in the lower left. Write this out and you see that to preserve the traditional notation of left to right multiplication, you must use the opposed multiplication on Ker in the matrix multiplication. So, we use the lower triangular matrix notation. Recall in any group $H$, the notation $(H, H)$ refers to the subgroup of $H$ generated by commutators, $h_1 h_2 h_1^{-1} h_2^{-1}, h_1, h_2 \in H$.

**Stray group notation:** $\mathbb{Z}/n$ is the integers mod $n$; for $p$ a prime, $\mathbb{Z}_p$ is the $p$-adic integers. We say sets of various kinds in a group are $p'$ if their elements have order prime to $p$. Example: A conjugacy class is $p'$ if . . . . We also say $p^n|n$ if $u$ is the highest power of $p$ that divides $n$.

A group $G$ is $p$-perfect if $p$ divides $|G|$, but $\mathbb{Z}/p$ is not a quotient of $G$. Denote the center of $G$ by $Z(G)$. Finally, some computations have permutations acting on a set of integers where we must write out the matrix multiplication. Compatible with our statements above, those permutations will act on the right of those integers (say, as in §3.1.2).

**Stray field notation:** For $K$ a field denote its absolute Galois group by $G_K$. For $V$ a quasi-projective algebraic variety defined over a field $K$, denote its set of $K$ points by $V(K)$.

### 1.3.3. Frattini extensions.

A covering (homomorphism) of groups, $\psi : G \rightarrow H$, is a Frattini cover if for each restriction of $\psi$ to a subgroup $G^*$ that is also a cover, then $G^* = G$. Any cover through which a Frattini cover factors is also a Frattini cover. It is a $p$-Frattini cover if the kernel, $\ker(\psi)$, is a $p$-group. We refer to the factor cover $\psi : G/(\ker(\psi), \ker(\psi)) \rightarrow H$ as the abelianization of the cover.
Finally, from [FrJ86, Chap. 22], for any (pro)finite group $H$ of order divisible by $p$, there is a Universal $p$-Frattini cover $\tilde{p}H \to H$. It is the minimal group that factors through all $p$-Frattini covers of $H$.

Its kernel is pro-free pro-$p$ of finite rank. It and its abelianization are projective limits of characteristic sequences of $p$-Frattini covers, respectively, $\{G_k(H)\}_{k=0}^{\infty}$ and $\{G_{k,\text{ab}}(H)\}_{k=0}^{\infty}$. Denote $\ker(G_{k+1} \to G_k)$ (resp. $\ker(G_{k+1,\text{ab}} \to G_{k,\text{ab}})$) by $M_k(H)$ (resp. $M_{k,\text{ab}}(H)$). Then:

(1.10a) $G_0(H) = H$, and $G_{1,\text{ab}}(H) = G_1(H)$; and
(1.10b) $M_k$ is an indecomposable $\mathbb{Z}/p[G_k]$ module [FrK97, Lem. 2.4], and $M_{k,\text{ab}} = M_0$ as an $H$ module, $k \geq 0$.

The following features in basic properties for our Hurwitz spaces towers.

**Def 1.4** ($p$-growth). A cover $\psi : G \to H$ with $p$-group kernel, is said to exhibit $p$-growth if all lifts of any order $p$ element of $H$ to $G$ have order $p^2$.

It may be hard to characterize covers that exhibit $p$-growth. So, we suffice with two observations related to Frattini covers. The centerless result is valuable for many applications of specific Hurwitz space: for example, its application to relating $p$-growth to cusp widths in Princ. 2.3.

**Proposition 1.5.** If $\psi : G \to H$ exhibits $p$-growth, then so does the restriction of $\psi$ to any subgroup $G^* \leq G$ that maps surjectively to $H$. In particular, any minimal such subgroup gives a $p$-Frattini cover of $H$ that exhibits $p$-growth.

Characteristic $p$-Frattini covers $G_{k+1} \to G_k$ and $G_{k+1,\text{ab}} \to G_{k,\text{ab}}$ exhibit $p$-growth. Also, if $G = G_0$ is $p$-perfect and centerless, then so are $G_k$ and $G_{k,\text{ab}}$.

**Proof.** Consider the hypotheses of the first sentence about $\psi : G \to H$ exhibiting $p$-growth. Since the set of lifts of any order $p$ element of $H$ to $G$ are a subset of its lifts to $G$, they all have order $p^2$. So, $G^* \leq G$ exhibits $p$-growth.

[FrK97, Lift Lem. 4.1] shows the 1st characteristic $p$-Frattini cover $G_1(H) \to H$ exhibits $p$-growth: It is the item (4.1b) $\Rightarrow$ item (4.1a) part of the proof. Since $G_{k+1}(H)$ is the 1st characteristic $p$-Frattini extension of $G_k(H)$, this inductively shows the result for $\ker(G_{k+1} \to G_k)$. For the abelianized case, starting with $k = 0$, all the order $p$ elements of $G_{1,\text{ab}}$ are in $M_0$. As the kernel of $\tilde{p}H \to H$ is profree of finite rank, its abelianization is just $(\mathbb{Z}/p)^u$ for some integer $u$. So, inductively the conclusion follows because the cover $(\mathbb{Z}/p^2)^u \to (\mathbb{Z}/p)^u$ exhibits $p$-growth.

The centerless result for $G_k$ is [BFr02, Prop. 3.21]. For $G_{k,\text{ab}}$ it reverts to the case $k = 1$ since it amounts to showing $\ker(G_{k+1,\text{ab}} \to G_{k,\text{ab}})$, isomorphic to $M_0$ as a $G_0$ or $G_{k,\text{ab}}$ module, has no fixed subspace under $G_0$. □

Examples show $G_1(H) \to H$ can have proper Frattini quotients that exhibit $p$-growth. The simplest – documented in [Fr95, Prop. 2.4] – is with $A_5$ and $p = 2$, where $G_1(A_5) \to A_5$ has kernel of order $2^3$ and all 2-Frattini covers factor through the spin cover $\text{SL}_2(\mathbb{Z}/5) \to A_5$ (with kernel $\mathbb{Z}/2$).

1.3.4. **Towers of spaces.** $\mathcal{M}$(oddular) $\mathcal{T}$(ower)s forms Hurwitz space towers from projective sequences of groups $G_p = \{G_{p,k}\}_{k=0}^{\infty}$. The initial insight for desirable properties of $G_p$ was that many properties of modular curve towers trace to the Frattini covering property of the sequence $\{D_{p,k+1}\}_{k=0}^{\infty}$. Also, for modular curve towers the ability to label and interpret cusp ramification, as in Princ. 2.3, derives from the $p$-Frattini property; it amounts to the group sequence exhibiting $p$-growth.
So, those are the keys: The Hurwitz spaces attached to a prime \( p \) are based on a projective sequence of Nielsen classes \( \mathcal{N}(G_{p,k}, C) \) where \( G_p \) has these properties.

(1.11a) Each \( G_{p,k} \) is a quotient of the \( k \)th characteristic \( p \)-Frattini cover of \( G_{p,0} \).

(1.11b) The homomorphism \( G_{p,k+1} \to G_{p,k} \) exhibits \( p \)-growth, and its kernel, \( M_k \), is a \( \mathbb{Z}/p[G_{p,k}] \) module.

(1.11c) The classes \( C \) lift uniquely from \( G_{p,0} \) to \( G_{p,k} \): “constant” with \( k \).

Item (1.11c) is based on Schur-Zassenhaus because the conjugacy classes are \( p' \) (as in §1.3.2). This independence of \( C \) with \( K \) is necessary for there to be any chance that any projective sequence of Hurwitz space components could possibly have a fixed number field as definition field. This meets the first requirement of any natural generalization of modular curve towers. [FrK97, Thm. 4.4] shows this for the full \( p \)-Frattini cover but it comes directly from from [FrK97, Lem. 4.3] which applies to any sequence exhibiting \( p \)-group.

To exhibit a full range of modular curve type properties, for almost all primes \( p \) you want corresponding sequences of groups. Yet, these conditions must hold.

(1.12) The classes, \( C \), must come from one group \( H \) that all the \( G_{p,k} \) cover.

That is, \( G_{p,0} \) covers \( H \), for each allowable \( p \). Indeed, the only primes one must avoid in this formulation are those dividing \( N_C \), the \( g(reatest) \) \( c(ommon) \) \( d(ivisor) \) of the order of elements in \( C \).

MTs aims to interpret properties of the objects – sequences of Hurwitz spaces – as comparable to those of modular curve towers. We can produce such systems by giving a finite group \( H \) (with chosen conjugacy classes \( C \)) acting faithfully on a lattice – torsion free \( \mathbb{Z} \) module – \( L_u \) of rank \( u \). This induces an action on \( L_u/pL_u = (\mathbb{Z}/p)^u \) for each prime \( p \) to which we refer below.

Let \( F_u \) be the free group on \( u \) generators. The following shows the information available from this action of \( H \). [Fr06, §4.1.3] calls \( u \) the rank of the MT system.

**Proposition 1.6.** For any \( p \), with \( (p,|H|) = 1 \), the \( p \)-pro completion, \( F_{p,u} \), of \( F_u \) gives the universal \( p \)-Frattini cover of \( (\mathbb{Z}/p)^u \times^s H \) (as above) as \( F_{p,u} \times^s H \).

**Proof.** We remind how [FrJ86, Prop. 22.12.2] gives this. First: The universal \( p \)-Frattini cover of any \( p \)-group is the pro-free pro-\( p \) group of the same rank [FrJ86, Lem. 22.7.10]. Since \( F_{p,u} \) is pro-free, you can extend an action of \( H \) – by extending it to generators of \( H \) – to give a group \( H' \) acting on \( F_{p,u} \). That gives a cover \( \psi: F_{p,u} \times^s H' \to (\mathbb{Z}/p)^u \times^s H \); so a homomorphism \( \psi_H \) from the former to \( H \).

Since \( (p,|H|) = 1 \), the profinite Schur-Zassenhaus says \( \psi_H \) splits, uniquely up to conjugation [FrJ86, Lem. 22.10.1]. This gives a conjugation action of \( H \) on \( F_{p,u} \) extending its action on \( (\mathbb{Z}/p)^u \). That makes \( F_{p,u} \times^s H \) the minimal cover of \( (\mathbb{Z}/p)^u \times^s H \) with a \( p \)-projective \( p \)-Sylow, the characterization of the universal \( p \)-Frattini cover of any finite group [FrJ86, Prop. 22.11.8]. \( \square \)

Applications, such as in [Fr10b], have \( H \) a transitive subgroup of \( S_n \) acting on the standard representation module modulo the identity module. Note: Despite all the actions of \( H \) on the completion of \( F_u \) in Prop. 1.6, that doesn’t imply an action of \( H \) on \( F_u \). The following example, including our two main examples, has an action of \( H \) on the free discrete group.

**Example 1.7.** Take \( H = \mathbb{Z}/n \) acting on \( F_n/\langle \sigma_1 \cdots \sigma_n \rangle \) (isomorphic to \( F_{n-1} \)) with the action given by having \( 1 \mod n \) act as the shift on \( \langle \sigma_1, \ldots, \sigma_n \rangle \).

The following is our standard goal generalizing a modular curve type property.
CONJ 1.8 (Main Conj.). Version 1: Given a prime $p$, and a system of groups satisfying (1.11), only finitely many of the spaces $\mathcal{H}(G_{p,k}, \mathbb{C})_{k \geq 0}$, have $K$ points over a given number field $K$.

Version 2: Add in (1.12). Then, the same conclusion as in Version 1, but running over all primes $p \nmid N_C$, and all $k \geq 0$.

Version 1 of Main Conj. 1.8 has been corroborated recently by results in [CaTa09] and [Fr10b] when $r = 4$. The set modular curves occurs in §2, where $u = 1$, $H = \langle \pm 1 \rangle$, with $-1$ acting by multiplication by $-1$. Mazur-Merrill is the affirmative conclusion of the modular curve case, over any number field. They make a stronger statement, also appropriate in Conj. 1.8, but not necessary to state here.

For $r > 4$, Version 1 of Conj. 1.8 is still unknown for any examples, even the dihedral case generalization of modular curves. We continue its notation. Assume $K$ is a definition field of the collection $\{\mathcal{H}(G_{p,k}, \mathbb{C})_{k \geq 0}\}$, including the maps between them and the maps to $J_r$.

Given a $K$ point $j_0 \in J_r$ (§2.3), suppose $\overline{p} = \{p_k \in \mathcal{H}(G_{p,k}, \mathbb{C})_{k \geq 0}\}$ is a projective sequence of points lying over $j_0$. We say $\overline{p} \in \mathcal{H}$. Then, there is a natural action of $\sigma \in G_K$ on $\overline{p}$ mapping it to another projective sequence, $\overline{p}^\sigma$, of points over $j_0$. This is a conjugate of $\overline{p}$.

From this we can define the decomposition group $D_{\overline{p}}$ to be the quotient of $G_K$ fixed on all conjugates of $\overline{p}$. It also makes sense to define the decomposition group $D_{\mathcal{H}}$ of $\mathcal{H}$. A weak version of Serre’s OIT would be this.

(1.13) Then, for a topologically dense set of points in $J_r(K)$, $D_{\overline{p}}$ is large compared to $D_{\mathcal{H}}$.

§4.2 gives a necessary and sufficient condition on $D_{\mathcal{H}}$ for (1.13) to hold, and compares it with conclusions of §3. Then, §4.3 discusses how to interpret stronger versions of the OIT as applying to $\mathcal{H}$. Version 1 of Conj. 1.8 and versions of a strong OIT are compatible, neither is a strengthening of the other. In practice, however, it is behavior over the boundary (think cusps) of the sequence $\mathcal{H}$ that guides the original OIT and which should do so here, too, using Nielsen classes.

2. MT view of modular curves

This section works out two aspects of modular curves:

(2.1a) The $sh$-incidence cusp pairing (§2.2, computed in §2.4).
(2.1b) Their monodromy groups as covers of the $j$-line (§2.5).

(2.1a) is new, giving a different slant on modular curve cusps. We do that in Prop. 2.4, where it illustrates several points about the $sh$-incidence matrix, especially the relation between cusp widths from one level to the next.

Not only is (2.1b) not new, it was known at level 0 by Abel, and then at all levels by Galois. His point was that the solutions of modular equations are not solvable functions of the $j$-invariant. This is in [Ri96], also describing the suicide, starting from p. 112 – counter to the stories passed for generations – of Galois.

We use this to show how the Hurwitz space description of modular curves applies to other systems of spaces. That includes generalizing monodromy action results for modular curves by seeing them through the braid group action.

2.1. Modular curves vs reduced Hurwitz spaces. Recall the classical group $\text{PSL}_2(\mathbb{Z}) = \langle \gamma_0, \gamma_1 \rangle$ generated by $\gamma_0$ and $\gamma_1$ of resp. orders 3 and 2. It acts on
the upper half-plane $\mathbb{H} \eqdef \{z = x + iy \in \mathbb{C} | y > 0\}$. The quotient of $\mathbb{H}$ by $\text{PSL}_2(\mathbb{Z})$ is a natural copy of the Riemann sphere, $\mathbb{P}^1$ missing the point at $\infty$.

2.1.1. $\text{PGL}_2(\mathbb{C})$ action; mapping class group $M_r$. \S 1.3.1 discussed “dragging a cover by its branch points” along $[P] \in \pi_1(U_r, z^0)$. We denote by $q_{[P]}$, for any $r \geq 4$, the operator giving the effect on the branch cycles for a cover relative to classical

we can from

This induces a Galois, but ramified, cover with group $\bar{U}$.

There is a natural subset $Z$ of $\mathbb{P}^1 \times U_r \to U_r$ for which restriction of the projection to $U_r$ to $Z$, is a fibration with fiber $U_z$ over $z \in U_r$. From covering space theory we can from $V_r \to U_r$, a fibration with fiber $\bar{U}_z$ over $z \in U_r$.

Now add $\alpha \in \text{PGL}_2(\mathbb{C})$ action, starting from

$$(z_1, \ldots, z_r) \in U^r \mapsto ((z_1)\alpha, \ldots, (z_r)\alpha).$$

This induces a Galois, but ramified, cover with group $\bar{M}_r$, the $r$-branch point mapping class group,

(2.3) \hspace{1cm} \frac{V_r/\text{PGL}_2(\mathbb{C})}{U_r/\text{PGL}_2(\mathbb{C})} \eqdef J_r \hspace{0.2cm} (J_4 = \mathbb{P}^1 \setminus \infty).

2.1.2. Branch cycles for reduced Hurwitz spaces, $r = 4$. [BFr02, Prop. 3.28] identifies $\text{PSL}_2(\mathbb{Z})$ with $M_4 \eqdef H_4/Q''$ by recognizing generators (of respective orders 3 and 2) $\gamma_0$ and $\gamma_1$ as the images of $q_1q_2$ and $\text{sh} = q_1q_2q_3$.

Then, $q_2 \mapsto \gamma_\infty$, acts as a clockwise path around $\infty \in \mathbb{P}^1$. Thus, $\{\gamma_0, \gamma_1, \gamma_\infty\}$ acting on reduced Nielsen classes $\text{Ni}(G, \mathbb{C})^* / Q'' \eqdef \text{Ni}(G, \mathbb{C})^*/\text{rd}$, are branch cycles for the nonsingular compactification of $\mathcal{H}(G, \mathbb{C})^*/\text{PGL}_2(\mathbb{C})$ as a j-line cover.

One standard normalization [Ah79, p. 282] (see \S 1.2.2 and \S 2.4) adjusts $j$ by an affine change so that only points over $j = 0, 1, \infty$ ramify with respective indices measured by the lengths of disjoint cycles, respectively, in $\gamma_0, \gamma_1, \gamma_\infty$. For any $r$, reduction theory shows $\mathcal{H}(G, \mathbb{C})^*/\text{PGL}_2(\mathbb{C})$ is an affine variety [Fr10a, Prop. A.8].

2.1.3. Classical cusp description for $\Gamma_0(p^{k+1})$. Let $\Gamma \leq \text{PSL}_2(\mathbb{Z})$ be a finite index subgroup. This produces a natural $j$-line cover.

- $\Gamma \to X^0_1 = \mathbb{H}/\Gamma$, ramified cover of $\bar{U}_j = \mathbb{P}^1 \setminus \{\infty\}$.
- Orbits of $\gamma_\infty = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ on the cosets of $\Gamma$ in $\text{PSL}_2(\mathbb{Z})$ correspond to the cusps over $\infty$. 


Classically, one counts the cusps when

\[ \Gamma = \Gamma_0(p^{k+1}) \overset{\text{def}}{=} \{(a/b \mod p^{k+1}): a \equiv b \equiv c \equiv d \mod p^{k+1}\}. \]

as \( \gamma_\infty \) cosets. You make this count by selecting orbit representatives \([Sh71, \S1.6]\).

There is no sh-incidence cusp pairing like that of \( \S 2.4 \).

2.1.4. Dihedral Nielsen classes. Assume \( p \) is odd. Let \( G_k = D_{p^{k+1}} \) (resp. \( N_k \)) be the order \( 2 \cdot p^{k+1} \) dihedral group (resp. its normalizer in \( S_{p^{k+1}} \); as in \( \S 1.3 \)):

\[ G_k \overset{\text{def}}{=} \{(\pm 1 0 \mod p^{k+1})\}_{b \in \mathbb{Z}/p^{k+1}} \text{(resp. } N_k \overset{\text{def}}{=} \{(a/b \mod p^{k+1})\}_{a \in \langle \mathbb{Z}/p^{k+1} \rangle^*, b \in \mathbb{Z}/p^{k+1}}\). \]

The linear algebra notation helps picture actions on \( \text{Ni}(G_k, C_{2^*}) \). Start with \( (a/b 0 1) \in N_k \) acting on \( \{(b', 1) | b' \in \mathbb{Z}/p^{k+1}\} \): \( (b', 1) \mapsto (a \cdot b' + b, 1) \): Matrix multiplication from the right. Our conjugacy classes \( C_{2^*} \) are four repetitions of the class \( C_2 = \{(\pm 1 0 \mod p^{k+1})\}_{b \in \mathbb{Z}/p^{k+1}} \).

Use the following \([\ ]\) notation for Nielsen class elements:

\[ g \in \text{Ni}(G_k, C_{2^*}) \mapsto [b_1, b_2, b_3, b_4] \in (\mathbb{Z}/p^{k+1})^4. \]

This automatically assures we have the right conjugacy classes. Now we assure respective generation and product-one – translating \( g_1g_2g_3g_4 = 1 \) – properties for absolute Nielsen classes \( \text{Ni}(G_k, C_{2^*})^{\text{abs}} \):

\[ \{g = (g_1, \ldots, g_4) \in C_{2^*} | \text{ condition (1.1a), for some } i \text{ and } j, b_i \neq b_j \mod p; \text{ and condition (1.1c), } b_1 - b_2 + b_3 - b_4 \equiv 0 \mod p^{k+1}\}/N_k. \]

For inner classes replace \( /N_k \) by \( /G_k \).

Generation only requires a condition \( \mod p \) because it is a Frattini property: If it holds in \( G_0 \), it automatically holds in any Frattini cover \( G_k \to G_0 \) for any \( k \).

2.2. Cusps as Nielsen class sub-orbits. We apply a number of principles from \([Fr06]\) to the dihedral Nielsen classes, and later to our other examples.

2.2.1. Interpreting cusps. The following points are from \([Fr06, \S2.4.2]\). As in \( \S 2.1.1 \), \( \langle q_1q_3^{-1}, \text{sh}^2 \rangle = Q'' \) acts as a quotient of the Klein 4-group on any (absolute or inner) Nielsen class defined by 4-tuples. We call \( \text{Cu}_4 \overset{\text{def}}{=} \langle Q'', q_2 \rangle < H_4 \), the cusp group \([Fr06, \text{Def. 2.3}]\).

Each reduced Hurwitz space \( \mathcal{H}' \) defined by the Nielsen class is a \( j \)-line cover whose physical cusps – points \( p \) over \( j = \infty \) on the projective completion \( \mathcal{H}' \) of the cover – correspond to \( \text{Cu}_4 \) orbits. Denote its ramification index by \( e(p/\infty) \).

The inner (resp. abs) reduced cusp orbit of a Nielsen class element \( g \) is the set of inner (resp. abs) equivalence classes in \( O_g \overset{\text{def}}{=} \text{Cu}_4(g)/Q'' \). The orbit length is the ramification index of the corresponding physical cusp – point \( p \) over \( j = \infty \) – on the nonsingular projective completion \( \mathcal{H}' \).

I list some general principles, followed by their application to the particular Nielsen class \( \text{Ni}(G_k, C_{2^*}) \). There will be analogues of these in \( \S 3.1 \). Denote the braid orbit of \( g \in \text{Ni}(G, \mathbb{C})^* \) (\( *=\) in or abs) by \( O_g \). Sometimes we also use the notation \( cO_g \) for the cusp orbit of \( g \).
PRINCIPLE 2.1 (\( Q'' \) stabilizing). If \( q_1q_3^{-1} \) and \( sh^2 \) fix the Nielsen class of \( g \), then \( Q'' \) fixes the class of any \( g' \in O_g \).

PROOF. Since \( q_1q_3^{-1} \) and \( sh \) generate \( Q'' \), the hypotheses imply \( Q'' \) fixes the Nielsen class of \( g \). Then, consider \( (g)q \) for any \( q \in H_4 \). Any \( \alpha \in Q'' \) fixes this if and only if \( qaq^{-1} \) fixes \( g \). Since \( Q'' \) is a normal subgroup of \( H_4 \) \cite{BFr02, §2.10}, \( qaq^{-1} \in Q'' \) and so it fixes \( g \).

COROLLARY 2.2. \( Q'' \) fixes each element of \( Ni(G_k, C_{2^4})^* \), \(* = in \) or abs.

PROOF. Among many accountings for one braid orbit on \( Ni(G_k, C_{2^4}) \), one goes back to \cite{Fr78, §2}. There is another in §2.4. Since there is only one orbit, Princ. 2.1 gives the result given an example \( g \) for which \( Q'' \) fixes its inner Nielsen class.

Take \( g = g_{H-M} \mapsto [0, 0, a, a] \), \( a \in (\mathbb{Z}/p^{k+1})^* \), a \( H(\text{arbater})-M(\text{umford}) \) rep. as in §1.2: \( g = (g_1, g_1^{-1}, g_2, g_2^{-1}) \). Clearly \( q_1q_3^{-1} \) fixes \( g_{H-M} \). Then, \( (g_{H-M})sh^2 \mapsto [a, a, 0, 0] \).

As \( p \) is odd, there is a solution, \( a' \), for \( x \) in \( a \equiv 2x \ mod \ p^{k+1} \). Conjugate \( g_{H-M} \) by \( \left( \begin{smallmatrix} -1 & 0 \\ a & 1 \end{smallmatrix} \right) \) to see \( sh^2 \) leaves invariant its inner class.

2.2.2. \( sh \)-incidence on cusps. Let \( eO_1 \) and \( eO_2 \) be two (inn. or abs.) cusp orbits. Then, we have the following pairing:

\[ (eO_1, eO_2) \mapsto eO_1 \cap (eO_2)sh. \]

This works for any value of \( r \), but when \( r > 4 \), cusp orbits are the same as \( q_2 \) orbits. In general, \( sh \)-incidence blocks \( \leftrightarrow \) components of the space \cite{BFr02, §2.10}.

We will here compute examples of this for \( r = 4 \). Then, since \( sh^2 \) is in \( Q'' \), the matrix is symmetric and we have the following definitions for \( g \in Ni(G, C)^{in} \).

\[ \begin{align*}
(2.4a) \quad & \langle g_2, g_3 \rangle = H_{2,3}(g) \quad \text{and} \quad \langle g_1, g_4 \rangle = H_{1,4}(g). \\
(2.4b) & \text{For } g \in Ni(G, C), \text{ its middle product is } mp_g = \text{ord}(g_2g_3). \\
\end{align*} \]

As we will see below, modular curves have two kinds of cusps:

\[ \begin{align*}
(2.5a) & \text{ } p \text{-cusps: } p \text{ divides } mp_g; \text{ and} \\
(2.5b) & \text{ } g \cdot p' \text{-cusps: both } H_{2,3} \text{ and } H_{1,4} \text{ are } p' \text{ groups.} \\
\end{align*} \]

If neither condition of (2.5) holds, we say the cusp is \( o(\text{only}) \)-\( p' \). §3.1.2 has such cusps.

Assume \( G_p = \{ G_{p,k} \}_{k=0}^\infty \) satisfies conditions (1.11). Then, if for \( g_k \in Ni(G_{p,k}, C), \) \( p^u|mp_g \) (as in §1.3.2), \( u > 0 \), then for any \( g' \in Ni(G_{p,k+1}, C) \) over \( g \), \( p^{u+1}|mp_{g'} \).

Notice: Princ. 2.3 is about inner cusps (points on inner, reduced Hurwitz spaces). The \( sh \)-incidence matrix for modular curves (Prop. 2.4) illustrates the subtlety in the absolute case (§2.4.1).

In all cases, finding inner (or absolute) cusp widths starts with finding the length of \( q_2 \) orbits on \( Ni(G, C) \). Denote \( q_2 \) by \( a \), and \( q_3 \) by \( b \). Consider the orbit of pairs generated by applying \( \gamma : (x, y) \mapsto (xyx^{-1}, x) \) starting with \( (a, b) \).

Then, \cite{BFr02, Prop. 2.17} computes the length of the orbits of \( \gamma^2 \) and \( \gamma \), denoting them, respectively, by \( o(a, b) = o \) and \( o'(a, b) = o' \). As in §1.3.2, \( Z_p(G) \) is the \( p \) part of \( G \)'s center. Then,

\[ o = \text{ord}(ab)/|(a, b) \cap Z((a, b)| \text{ and one of the following hold.} \]

\[ \begin{align*}
(2.6a) & \text{ Either } a = b \text{ and } o' = 1; \\
\end{align*} \]
(2.6b) or $o$ is odd, $b(ab) \overline{\infty}$ has order 2, and $o' = o$;
(2.6c) or $o' = 2o$.

To figure the inner (resp. absolute) cusp width represented by $g$, one takes the orbit of iterations of $q_2$ on $g$ using (2.6) and then decides if any elements in the orbit are conjugate under the action of $G$ (resp. $N_{S_n}(G)$) and $Q''$.

For large $k$, [Fr06, Princ. 3.5] says a $p$-cusp at level $k$ and width $p^u$ has above it only $p$-cusps of width $p^{u+1}$. For $p$ odd it has the following more precise statement. Princ. ?? extends this for $p = 2$, where the appearance of $Q''$ and the distinction between $o$ and $o'$ complicate matters.

**Principle 2.3 (p-width).** Assume $Z_p(G)$ is trivial, and $p$ is odd. Consider $p' \in \mathcal{H}(G_{p,k+1}, C)^{in,rd}$ over a $p$-cusp $p \in \mathcal{H}(G_{p,k}, C)^{in,rd}$. Then, $e(p'/\infty) = e(p/\infty)p$.

In the remainder of this section we don’t need $p = 2$. From Cor. 2.2, $Q''$ is trivial on Nielsen classes, so these are the same as reduced Nielsen classes.

For $g \in \text{Ni}(D_{p,k+1}, C) \leadsto [b_1, b_2, b_3, b_4]$, $mp_g$ is the order of $b_2 - b_3 = b_g$ in $\mathbb{Z}/p^{k+1}$. A product of an odd (resp. even) number of elements from $C_2$ is in $C_2$ (resp. the translate by some $b \in \mathbb{Z}/p^{k+1}$). So, (2.6b) gives these cusp orbit reps:

\begin{equation}
\{[b_1, b_2+mb_g, b_3+(m-1)b_g, b_4]\}_{m=0}^{mp_g^{-1}}.
\end{equation}

**2.3. Cusps in Ni(Gk, C2)abs.** We normalize reps. for absolute classes.

1st Normalization: Conjugate $[b_1, \ldots, b_4]$ by an element of $\langle \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \rangle$ to assume $b_1 = 0$ and $b_2 - b_3 + b_4 = 0$.

2nd Normalization: If $b_2 - b_3 = ap^u, a \in (\mathbb{Z}/p^{k+1-u})^*$, conjugate by $\left( \begin{array}{cc} a^{-1} & 0 \\ 0 & 1 \end{array} \right)$ to get an absolute rep. with $a = 1$. Take $c = b_2, b_3 = c - p^u$: $u$ is a parameter.

Third Normalization: Normalizations 1 and 2 still allow further conjugation by

$$H_u = \{ \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \mid a = 1 + bp^{k+1-u} \in \mathbb{Z}/p^{k+1} \mod p^u, b \in \mathbb{Z}/p^u \}.$$ 

(2.8a) $u = 0$: $(b_2, b_3) = (c, c - 1)$ has $q_2$ orbit of width $p^{k+1}$ containing

$$g_{H-M} \mapsto [0, 0, 1, 1].$$

(2.8b) $u > 0$: \langle $g \rangle = D_{p,k+1} \implies (c, p) = 1$. Conjugating by $H_u$ assures the cusp orbit of $g$ with $b_2 = c$ consists of Nielsen class reps. whose 2nd entries, $c + mp^u$, are distinct $\mod p^{k+1-u}$, running over integers $m$.

(2.8c) $u = k+1$, this is a $g$-$p'$ cusp: $(b_2, b_3) = (1, 1)$: \langle $g_{H-M}$sh (width 1),

$$H_{2,3} = \langle g_2, g_3 \rangle, H_{1,4} = \langle g_1, g_4 \rangle$$, both $p'$ groups (here cyclic of order 2).

**2.4. The sh-incidence pairing for modular curves.** \S 2.4.1 does the absolute case, modding out by $N_{S_{p,k+1}}(D_{p,k+1})$. The reduced Hurwitz space is the modular curve $X_0(p^{k+1})$ minus its cusps. \S 2.4.2 does the inner case where the reduced Hurwits space is $X_0(p^{k+1})$ minus its cusps.
2.4.1. *Abs. case.* Finding Hurwitz space components – corresponding to matrix blocks (§2.2.2) – is the first business. Taking advantage of exceptional long cusps quickly contributes to finding braid orbits. In our case there is one cusp of width \( p^{k+1} \) whose cusp orbit contains an H-M rep. There is also exactly one exceptional short cusp, the shift of the H-M rep. There is a dividing line in Prop. 2.4 at \( \frac{k+1}{2} \) in Prop. 2.4. At times we use the greatest integer, \([\frac{k+1}{2}]\), in it.

The \( \mathbf{sh} \)-incidence matrix has Row 1 entries from intersecting \( \mathbf{sh} \) applied to the length \( p^{k+1} \) cusp, \( c_{O^{p^{k+1}}} \), with each of the other cusp orbits. A virtual display would have a row for each collection of cusps of width \( p^{k+1-2u} \), in these two cases:

\[
\begin{align*}
(2.10a) \ & \ 0 \leq u \leq \frac{k-1}{2}, \text{ where the cusp have widths } k+1-2u; \text{ and} \\
(2.10b) \ & \ 0 \leq \frac{k+1}{2} \leq u \leq k+1, \text{ where all cusp widths are 1}.
\end{align*}
\]

Apply the shift, then translate by \( m \) and multiply by -1, to get

\[
\begin{equation}
(2.10) \quad (c_{O^{p^{k+1}}} \mathbf{sh})_{\equiv} = \{ [0, 1, 1 + a, a] \}_{a \in \mathbb{Z}/p^{k+1}}.
\end{equation}
\]

Denote \((\mathbb{Z}/p^{k+1-u})^* \mod p^u \) by \( L^{+,u}_{k+1,p} \). Then, cusps corresponding to \( u \) are \( c_{O^{p^{k+1-2u,a}}} \), \( a \in L^{+,u}_{k+1,p} \). If \( c' \in (\mathbb{Z}/p^{k+1-u})^* \) represents \( a \in L^{+,u}_{k+1,p} \), then with \( B_u = \{ m \ | \ 0 \leq m \leq p^{\max(k+1-2u,0)}-1 \} \):

\[
\begin{align*}
(2.11a) \ & \ c_{O^{p^{k+1-2u,a}}} = \{ [0, c' + mp^u, c' + (m-1)p^u, -p^u] \}_{m \in B_u}, \text{ and} \\
(2.11b) \ & \ (c_{O^{p^{k+1-2u,a}}} \mathbf{sh})_{\equiv} \{ [0, p^u, c' + (m+1)p^u, c' + mp^u] \}_{m \in B_u}.
\end{align*}
\]

We now count intersections of cusp orbits and their shifts; Table 1 has the matrix.

**Proposition 2.4.** Use the notation above. The intersection pairing gives these entries:

\[
\begin{equation}
(2.12) \quad (c_{O^{p^{k+1}}} c_{O^{p^{k+1-2u,a}}} c_{O^{p^{k+1-2u',a'}}}) = 0 \text{ if both } u, u' \neq 0; \text{ and}
\end{equation}
\]

\[
\begin{align*}
(2.13) \ & \ (c_{O^{p^{k+1}}} c_{O^{p^{k+1-2u,a}}} c_{O^{p^{k+1-2u',a'}}}) = p^{\max(k+1-2u',0)} \text{ if } u' \neq 0; \text{ and}
\end{align*}
\]

\[
\begin{equation}
(2.14) \quad (c_{O^{p^{k+1}}} c_{O^{p^{k+1-2u,a}}} c_{O^{p^{k+1-2u',a'}}}) = p^{k+1-p_k}.
\end{equation}
\]

Proof. We have normalized so that entry 1 in each Nielsen class rep. in a cusp is 0. Equality of two such absolute Nielsen classes requires that multiplication by some \( \alpha \in (\mathbb{Z}/p^{k+1})^* \) gives equality on the 2nd through 4th entries. For an \( \alpha \) to exist requires the two second entries be exactly divisible by the same power of \( p \).

As the \( \mathbf{sh} \)-incidence matrix is symmetric, we need only intersect \( c_{O^{p^{k+1-2u,a}}} \) and \( (c_{O^{p^{k+1-2u',a'}}} \mathbf{sh}), k+1 \geq u' \geq u \geq 0 \). The \( \mathbf{sh} \)-incidence entry is 0 unless \( u = 0 \):

\[
(2.15) \quad p^u \text{ divides the shifted, but not the unshifted, cusp second entry.}
\]

Now consider \( u = 0 \). Intersect \( c_{O^{p^{k+1}}} \) (its elements indexed by \( m \)), with \( (c_{O^{p^{k+1-2u',a'}}} \mathbf{sh}), u' \neq 0 \), its index parameters \( (c', m') \), with \( c' \) fixed in (2.11b).

Then, for each allowable \( m' \), there is an allowable \( t \) with \( m = tp^{u'}, (t, p) = 1 \) satisfying the equation \( (c' + m'p^{u'}) = -1 \mod p^{k+1-u'} \). So, the two cusps intersect \( p^{\max(k+1-2u',0)} \) times.

The case \( u' = 0 \) is similar, except you must stipulate that allowable \( m' \)'s are prime to \( p \). There are \( p^{k+1-p_k} \) of these, completing the count of intersections. \( \square \)

**Remark 2.5** (\( p \) cusp versus \( p \) dividing cusp widths). Prop. 2.4 is an absolute, not inner, Nielsen class case. So, we aren’t surprised the conclusion of Prop. 2.3 is wrong for the width 1 \( p \)-cusps corresponding to \( u \leq \frac{k+1}{2} \): on these \( j \)-line covers \( p \) does not divide their ramification index. Yet, by going up, say, \( k' \) levels, when \( u + k' \) exceeds, \( \frac{k+k'+1}{2} \), the cusps above them all have ramification divisible by \( p \).
To adjust for inner cusps, don’t apply Normalizations 2 and 3 (except allowing contributions to the cusp width).

For each particular, if $\nu(v) \leftrightarrow g$, multiply abs. contributions by $p^{k+1}$ and divide by $p^k$. The result, $p^{k+1} + p^k$ agrees, as it should, with $|\mathcal{N}|^{abs}$ and the cusp width sum in Table 1.

### Table 1. $\mathsf{sh}$-incidence for $\mathcal{N}_i(D_p^{k+1},24)_{abs,rd}$ listings for $cO_p^{k+1-2u}$

<table>
<thead>
<tr>
<th>Cusp orbit</th>
<th>$u = 0$</th>
<th>$a \in L_k^{k+1,p^2}$</th>
<th>$a \in L_k^{k+1,p^u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 0$</td>
<td>$p^{k+1} - p^0$</td>
<td>$p^{k+1-2u}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$a \in L_k^{k+1,p^u}$</td>
<td>$p^{k+1-2u}$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$a \in L_k^{k+1,p^u}$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$1 \leq u \leq k+1$ &amp; $</td>
<td>u</td>
<td>&lt; k+1$ &amp; $</td>
<td>u</td>
</tr>
</tbody>
</table>

### Remark 2.6 (Degree of $X_0(p^{k+1})/\mathbb{P}_1$). We traditionally count this degree as the number of copies of $\mathbb{Z}/p^{k+1}$ in $\mathbb{Z}/p^{k+1}$ modulo $\mathbb{Z}/p^{k+1}$. That is count elements of $V$ not in $pV - p^2(k+1) - p^2 -$ and divide by $p^{k+1} - p^k$. The result, $p^{k+1} + p^k$ agrees, as it should, with $|\mathcal{N}|^{abs}$ and the cusp width sum in Table 1.

2.4.2. Inner case. Suppose conjugacy classes $C$ in a group $H$ lift naturally to classes in a covering group $\psi : G \to H$, inducing $\psi^\ast : \mathcal{N}(G,C)^{\alpha} \to \mathcal{N}(H,C)^{\alpha}$, where $\alpha$ and $\beta$ need not be the same equivalence. Then, cusps orbits in $\mathcal{N}(G,C)^{\alpha}$ map naturally to cusp orbits in $\mathcal{N}(H,C)^{\beta}$; from which the following lemma follows.

**Lemma 2.7.** If $cO$ and $cO'$ are cusp orbits in $\mathcal{N}(G,C)^{\alpha}$, then any $\alpha$ class intersection in $cO \cap (cO')_{\mathsf{sh}}$ gives a $\alpha$ class intersection $\psi^\ast(cO) \cap (\psi^\ast(cO'))_{\mathsf{sh}}$. In particular, if $|\psi^\ast(cO) \cap (\psi^\ast(cO'))_{\mathsf{sh}}| = 0$, then so is $|cO \cap (cO')_{\mathsf{sh}}|$.  

As in §2.2.2, apply (2.6) to write a collection of inner reps. for $cO_{[0,0,c,c]}$ as $\{(0,t,t+c,c) \mid t \in \mathbb{Z}/p^{k+1}\}$. So $cO_{[0,0,c,c]}$ contains $[0,1,1+c,c]$ and $(cO_{[0,0,c,c]})_{\mathsf{sh}}$ contains $[1,1+c,c,0] = g''$. Apply (2.6) again: $cO_{g} = cO_{g_{H-M}} = cO_{g'}$.

(2.11a) has a normalized rep., $[0,c',c'-p^a,-p^a]$, of an absolute cusp orbit. To adjust for inner cusps, don’t apply Normalizations 2 and 3 (except allowing $\alpha = -1$) of §2.3. Then, the cusps attached to the parameter $u$ corresponding to (2.7) have parameters $(c',\alpha) \in (\mathbb{Z}/p^{k+1-1} \times (\mathbb{Z}/p^{k+1-1})^\ast$. Inner reps. of the cusp $cO_{p^{k+1-1},c',\alpha}$ attached to $(c',\alpha)$ then have the form

\[
\{(0,c + m\alpha p^a, c + (m-1)\alpha p^a, -\alpha p^a)\} \in \mathcal{B}_u.
\]

The cusp width associated to $u$ is now $p^{k+1-1}$; there is no $H_u$ to shorten it. For each $u$, multiply abs. contributions by $p^a\varphi(p^{k+1-1})/2 = \varphi(p^{k+1})/2$ to get inner contributions to the cusp width.

Lem. 2.7 simplifies the $\mathsf{sh}$-incidence for $\mathcal{N}(G_k,24)^{in,rd}$ from the computation for $\mathcal{N}(G_k,24)^{abs,rd}$ as we can restrict to where one is the cusp orbit of an H-M rep.

\[
W_{H-M} = \{[0,0,c,c] \mid c \in (\mathbb{Z}/p^{k+1})^\ast/\langle \pm 1 \rangle\}.
\]

As in §2.2.1 denote the cusp orbit of $g$ by $cO_{g}$. Notice that $\mathsf{sh}^2$ applied to $[1,1,0,0] = g'$ is $g_{H-M}$. (2.8a).

**Lemma 2.8.** The element $[1,1+c,c,0] = g''$ represents the unique inner (reduced) class in $cO_{g_{H-M}} \cap (cO_{[0,0,c,c]})_{\mathsf{sh}}$. So, all entries in the $(p^{k+1} + p^k) \times (p^{k+1} - p^k)$ matrix with entries for $cO_{g} \cap cO_{g'}$ are 1.

Also, if $c$ is the parameter for an element in $W_{H-M}$, and $(c',\alpha)$ are the parameters for a cusp corresponding to the parameter $u$, then the $\mathsf{sh}$-incidence entry corresponding to $(c,(c',\alpha))$ is 1 (resp. 0) if and only if $c \equiv c' \mod p^{k+1-1}$. 

DIHEDRAL GROUP COVERS

2.9. Theorem

Proof. Instead of (2.11b), count the intersection of \( cO_{[0, 0, c, c]} \) with reps. of inner classes of \( \mathfrak{sh} \) applied to (2.12):

\[
\{(0, \alpha p^u, \epsilon' + (m' + 1)\alpha p^u, \epsilon' + m' \alpha p^u)\}_{0 \leq m' \leq p^{k+1} - u - 1}.
\]

The resulting equations are \( mc = \alpha p^u \) and \(-c = \epsilon' + (m') \alpha p^u\), which has a solution in \( m' \) if and only if \( \epsilon' = -c \mod p^{k+1} - u \) modulo multiplication by \( \pm 1 \).

\[ \square \]

2.5. MT view of Modular curve monodromy. We now add to the opening discussion of [Fr95, §I.A] titled, “Focus on modular curves” how the action of \( H_3 \) on reduced Nielsen classes \( \text{Ni}(D_{p^{k+1}}, \mathbb{C}_{2s})^{\text{alt}, rd} \), identifies that action as \( \text{SL}_2(\mathbb{Z}_p)/\{\pm 1\} \) acting as the geometric monodromy group of the projective limit of the modular curve covers \( X_1(p^{k+1}) \) (or of \( X_1(p^{k+1}) \)). Again, \( p \neq 2 \) is a prime.

As in §1.1, choose a set of classical generators of \( \pi_1(\mathbb{P}_1 \setminus \{z_0\}, z_0) \). Denote the isotopy class of these by \( \bar{g}_1, \ldots, \bar{g}_4 \). To product-one (1.1c) add these relations:

\[
\begin{align*}
g_i^2 &= 1, \quad i = 1, \ldots, 4. 
\end{align*}
\]

Denote the resulting group as \( \text{Dih}(\mathbb{Z}^2) \): for \( \text{Dihedral action on } \mathbb{Z}^2 \). Denote the images of the \( \bar{g}_i \) s in \( \text{Dih}(\mathbb{Z}^2) \) by \( g_i^* \) s. Mapping all \( g_i^* \) s to -1 produces

\[ \text{pr}_4 : \text{Dih}(\mathbb{Z}^2) \rightarrow \mathbb{Z}/2 = \{\pm 1\}. \]

Denote the (multiplicative) group generated by \( v_1 = g_1^* g_2^* = g_3^* g_4^* \) and \( g_1^* g_3^* \) by \( V \).

[Fr95, p. 114] calculates precisely for what is \( \text{Dih}(\mathbb{Z}^2) \). We summarize.

2.14a) \( V \) is isomorphic to \( \mathbb{Z}^2 \), constituting the kernel of \( \text{pr}_4 \).

2.14b) Each \( g_i^* \) conjugates any element of \( V \) to its (multiplicative) inverse, and \( \text{Dih}(\mathbb{Z}^2) \) is isomorphic to \( \mathbb{Z}^2 \times \mathbb{Z}/2. \)

The dihedral group \( D_{p^{k+1}} \) is generated by two order 2 elements , with product \( p^{k+1} \). Indeed, that is its characterization. Each \( \mathfrak{g} \in \text{Ni}(D_{p^{k+1}}, \mathbb{C}_{2s})^{\text{alt}, rd} \) induces a cover \( \psi_\mathfrak{g} : \text{Dih}(\mathbb{Z}^2) \rightarrow D_{p^{k+1}}. \) Now we see how \( H_4 \) acts through \( \text{SL}_2(\mathbb{Z}/p^{k+1})/\{\pm 1\} \).

Let \( j_0 \in U_j \in \mathbb{P}_1^1 \setminus \{\infty\} \) be the \( j \)-invariant corresponding to \( z_0 \). Using the classical generators, above, any point \( p_1 \in X_1\left(p^{k+1}\right) \) over \( j_0 \) corresponds to an element \( g_1 \) in the Nielsen class. The monodromy action of a closed path \( P \) in \( \pi(U_j, j_0) \) takes \( p_1 \) to another \( p_2 \) over \( j_0 \). The action of such a \( P \) is the same as given by the image of a \( q \in H_4 \) acting on \( \text{Ni}(D_{p^{k+1}}, \mathbb{C}_{2s})^{\text{alt}, rd} \), and so that action corresponds to taking \( g_1 \) to another Nielsen class rep. \( g_2 \). We refer to the action of \( H_4 \) in Thm. 2.9 as the action on cohomology through branch cycles. The module \( V \) is the abelianized fundamental group, 1st homology, of \( E_{j_0} \).

Theorem 2.9. Any \( q \in H_4 \) has an action as an element of \( \text{SL}_2(\mathbb{Z}/p^{k+1})/\{\pm 1\} \) acting on \( H_1(E_{j_0}, \mathbb{Z}) = V \) extending its action Nielsen classes. Write \( q \) as a word in \( \mathfrak{sh} \) and \( v_2 \) to compute that explicitly.

Proof. By expressing \( V \) in generators and relations in the \( g_i^* \) s, we have on \( V \) a natural action of \( H_4 \), that factors through \( M_4 \). Finding the action of \( M_4 \) requires only finding how \( q_2 \) and \( \mathfrak{sh} \) act on \( v_1 = g_1^* g_2^*, v_2 = g_3^* g_4^* \). We write that action here.

\[
\begin{align*}
\mathfrak{sh} : \quad v_1 &\mapsto g_2^* g_3^* = (g_1^* g_2^*)^{-1} (g_1^* g_2^*) g_3^* = (g_1^* g_2^*)^{-1} (g_1^* g_2^*) = v_1^{-1} v_2 \\
\mathfrak{sh} : \quad v_2 &\mapsto g_1^* g_2^* = g_2^* g_3^* (g_1^* g_2^*) = v_1^{-2} v_2 \\
q_2 : \quad v_1 &\mapsto g_1^* g_2^* (g_1^* g_2^*)^{-1} = v_1 (v_2)^{-1} \\
q_2 : \quad v_2 &\mapsto g_1^* g_2^* = v_1
\end{align*}
\]

\[ (2.15) \]
Write \( V \) additively to see \( sh \mapsto \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \) and \( q_2 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

\[ \square \]

3. The Nielsen classes \( \text{Ni}(\text{Ker}_{p^{k+1}} \rtimes \mathbb{Z}/3, C_{2r'}) \)

Use the semidirect product notation from §1.3.2. In this section \( H = \mathbb{Z}/3 = A_3 \), the alternating group of group of degree 3. Then, \( G = G_{p^{k+1}} = \text{Ker} \rtimes \mathbb{Z}/3 \) in (1.9) with \( \text{Ker} = \text{Ker}_{p^{k+1}} = (\mathbb{Z}/p^{k+1})^2 \). Here, \( p \neq 3 \) is prime, and \( H = \mathbb{Z}/3 \) acts in its 2-dimensional rational representation on \( \mathbb{Z}^2 \), reduced modulo \( p^{k+1} \).

We follow closely the notation of §2.5, so as to identify the action of \( H_2 \) on reduced Nielsen classes and the monodromy action on the first homology of the Hurwitz space fiber. To simplify that, we will prepare a special fiber of the Hurwitz space using a classical set of generators and their isotopy classes.

3.1. Introducing \( \text{Ni}(G_{p^{k+1}}, C_{2r'}) \). The group \( \text{Ker}_{p^{k+1}} \rtimes GL_2(\mathbb{Z}/p^{k+1}) \) acts as permutations on \( \text{Ker}_{p^{k+1}} \); as in §1.3.2, as matrices acting on vectors. The representation has degree \((p^{k+1})^2\). In \( A_3 = \mathbb{Z}/3 \) there are two conjugacy classes of 3-cycles. These extend uniquely to classes of order 3 elements in \( G_{p^{k+1}} \).

We will take \( C \) to have \( r' \) elements in one conjugacy class, and \( r' \) in the other. When \( \mathbb{Z}/3 \) acts irreducibly, then

\[ \text{Ker}_{p^{k+1}} \rtimes \langle (\mathbb{Z}/p^{k+1})^*, \mathbb{Z}/3 \rangle \]

is the normalizer of \( G_{p^{k+1}} \) in \( \text{Ker}_{p^{k+1}} \rtimes GL_2(\mathbb{Z}/p^{k+1}) \). Here \((\mathbb{Z}/p^{k+1})^* \) acts as diagonal matrices. That is the case when \( p = 2 \) and it is where we start.

3.1.1. Notation for \( \text{Ni}(G_{2^{k+1}}, C_{2r'}) \). When \( p = 2 \) and \( k = 1 \), then \( G_2 = A_4 \). This special case, and its corresponding Nielsen class \( \text{Ni}(A_4, C_{23r'}) \), is serious for us. The notation indicates there are \( r' \) 3-cycles that map to \((123) \in A_3 \), and \( r' \) 3-cycles that map to \((132) \in A_3 \).

Recall from §1.2.1 an H-M rep. of form \((g_1, g_1^{-1}, \ldots, g_r, g_r^{-1})\). The key fact for our Nielsen classes is that there are exactly two braid orbits \( O_{\pm r'}^* \) on \( \text{Ni}(A_4, C_{23r'})^* \), with * either abs or in. This is a special case of a general result with \( A_4 \) replacing \( A_4 \); [Fr10a, Thm. 1.3]. The more important orbit to us is that labeled \( O_{+ r'}^* \).

§1.3.3 introduced the central 2-Frattini cover \( \text{SL}_2(\mathbb{Z}/5) \rightarrow A_5 \), which is the universal central extension of \( A_5 \). Denote the pullback of \( A_4 \leq A_5 \) in \( \text{SL}_2(\mathbb{Z}/5) \) by \( \tilde{A}_4 \). In the next statement let \( G^* \rightarrow G \) be a central \( p \)-Frattini extension.

DEF 3.1. Assume each entry, \( g_i \) of \( g \in \text{Ni}(G, C) \) is \( p' \). Then, it has a unique same order lift to \( \tilde{g}_i \in G^* \) and a lift invariant:

\[ s_{G^* \rightarrow G}(g) \overset{\text{def}}{=} \tilde{g}_1 \cdots \tilde{g}_4 \in \ker(G^* \rightarrow G). \]

The lift invariant is a general idea: More general than given here; but significantly applies only to central Frattini extensions \( G^* \rightarrow G \). [Fr10a, Proof of Thm. 1.3] includes an algorithm to compute it for \( g \in \text{Ni}(G, C) \) when \( G = A_n \), and \( G^* = \tilde{A}_n \), the Spin cover of \( A_n \), and related situations. Lem. 3.2 gives an example of what we mean by a related situation. The proof of Prop. 3.4 explains how to compute it when the covers in the Nielsen class have genus 0.

The first paragraph of Lem. 3.2 is a special case of a general result, and a small extension of a previous result. So, we have put the technical group point in App. D.
explaining that the Schur multiplier in \(G_{2k+1}\) is antecedent to that of \(A_4\). Ex. D.2 should clarify our application is not to a trivial situation.

**Lemma 3.2.** The universal central extension of \(\tilde{G}_{2k+1} \to G_{2k+1}\) has kernel \(\mathbb{Z}/2\), \(k \geq 0\). For all other primes \(p\), \(G_{pq+1}\) is its own universal central extension.

There are explicit elements \(*g', g'' \in Ni(G_{2k+1}, C_{132})\), on which the lift invariant \(s_{G_{2k+1}}/G_{2k+1}\) has respective values \(\pm 1\). So, these elements lie in distinct braid orbits on this Nielsen class.

**Proof.** We have only to show the second paragraph. Braids leave the lift invariant fixed [Fr95, Lem. 3.12]. So, elements of \(Ni(G_{2k+1}, C_{132})\) with resp. lift invariants \(\pm 1\), are in different braid orbits. In \(Ni(A_4, C_{132})\) consider:

\[
g' \overset{\text{def}}{=} ((123), (321), (134), (431)) = (g_1, g_1^{-1}, g_2, g_2^{-1})
g'' \overset{\text{def}}{=} ((123), (134), (124), (124)) = (g_1, g_2, g_3, g_3).
\]

Lift each of \(g_1, g_2, g_3\) to order 3 elements \(\hat{g}_i \in \hat{A}_4\) (resp. \(g_i \in G_{2k+1}\); defined up to conjugation by \(\text{Ker}(G_{2k+1} \to A_4)\)), \(i = 1, 2, 3\). We now choose the \(g_i\) so that

\[
g' = (g_1, g_1^{-1}, g_2, g_2^{-1}) \quad \text{and} \quad g'' = (g_1, g_2, g_3, g_3^{-1}),
\]

are in the Nielsen class \(Ni(G_{2k+1}, C_{132})\). Since \(G_{2k+1}\) is a Frattini cover of \(A_4\), (1.1a) (generation) is automatic for the \(g'\) and \(g''\) entries. The only condition left is (1.1c) (product-one). Entries of \(g'\) have product-one.

Induct on \(k\), using that \(\text{Ker}(G_{2k+1} \to G_2)\) is an irreducible \(A_4\) module (not \(1\)). So, the hypothesis of Prop. D.1 holds, and \(s_{2k+1}\) on them has the values \(\pm 1\). The elementary [FrK97, Obst, Lem. 3.2] suffices as \(1\) doesn’t appear in this module’s Loewy display (Ex. D.2).

There are exactly two Hurwitz space components, \(H(A_4, C_{2r^2})^*/\), in the Nielsen class defined by any of our natural equivalences [Fr10a, Thm. 1.3]. Both have definition field \(Q\). §3.1.2 outlines the appearance of these components by displaying the \(sh\)-incidence matrix. We enhance [Fr10a, §3.3.2] using the \(sh\)-incidence matrix to start our comparison with modular curves (Thm. 2.4).

3.1.2. \(sh\)-incidence for \(A_4\) and four 3-cycles. As in §1.3.2, permutations will act on the right of integers. The outer automorphism of \(A_4\), represented by conjugation by some 2-cycle, switches the conjugacy classes \(+C_3\) and \(-C_3\). Cor. 2.2 shows the action of \(Q''\) on the dihedral Nielsen classes is trivial. It acts differently here.

Label the six elements in \(Ni(A_3, C_{132})^{\text{in}}\), by pairs of \(+s\) and \(-s\), arranged in a row. By switching \(+s\) and \(-s\), each has a complement. The outer automorphism send an element with its complement: \(Ni(A_3, C_{132})^{\text{in}} \to Ni(A_3, C_{132})^{\text{abs}}\) is two-to-one. The next lemma uses the notation \(O_g^\ast\), with \(* = \text{abs or in}\) for the braid orbit of \(g\) in the Nielsen class with equivalence \(*\).

**Lemma 3.3.** The orbit length of \(Q''\) on \(O_g^{\text{in}}\) (resp. \(O_g^{\text{abs}}\)) is 2 (resp. 1). The numbers for for \(O_g^{\ast}\) (are \(O_g^{\text{abs}}\)) 2 (resp. 2).

In particular, \(O_g^{\text{in, rd}} = O_g^{\text{abs, rd}}, \) while the natural map \(O_g^{\ast,\text{rd}} \to O_g^{\text{abs,\ t}}\) is 2-to-1.

**Proof.** The resp. effect of \(sh^2, q_1q_3^{-1}\), and \(sh^2q_1q_3^{-1}\) takes \(g'\) to

\[
((134), (143), (123), (321)), ((321), (123), (143), (134)), \quad \text{and} \quad ((143), (134), (321), (123)).
\]
We want an element in $S_4$, if any, that conjugates these to $g'$: $(1\,3)(2\,4)$ (resp. $(1\,3)$ and $(2\,4)$) conjugates the 1st (resp. 2nd and 3rd) of them to $g'$.

The case of $g''$ is even easier, for $sh^2$ and $sh^2q_1q_3^{-1}$ both move the repetition of the 3rd and 4th positions, to a repetition in the 1st and 2nd positions. Then, $q_1q_3^{-1}$ maps $g''$ to a conjugate of it by a power of $(1\,2\,4)$.

By inspection modding out by $Q''$ also has the effect of equivalencing an element with its complement. So, there are three elements in

$$\text{Ni}(A_3, C_{\pm 3\varpi})^{\text{rd, rd}} = \text{Ni}(A_3, C_{\pm 3\varpi})^{\text{abs, rd}}.$$ Label these $[1]$, $[2]$ and $[3]$, represented resp. by $+$ and its complement. So, there are three elements in

$$\text{Ni}(A_3, C_{\pm 3\varpi})^{\text{rd, rd}} = \text{Ni}(A_3, C_{\pm 3\varpi})^{\text{abs, rd}}.$$ Label these $[1]$, $[2]$ and $[3]$, represented resp. by $+$ and its complement.

Now consider $\text{Ni}(A_4, C_{\pm 3\varpi})^{\text{rd, rd}}$. In any reduced class we may assume the first entry of a rep. is $(1\,2\,3)$ (allowing further inner conjugation by a power of $(1\,2\,3)$).

Denote $((1\,2\,3), (1\,3\,2), (1\,4\,3), (1\,3\,4))$, which lies over $[3]$, by $g_{3,1}$. There are five total classes over $[3]$ which we can label $g_{3,j}$, according to their 2nd and 3rd positions with $(1, 2, 3)$ in the first as follows:

$$g_{3,2} \mapsto ((1\,2\,4), (1\,3\,2)); g_{3,3} \mapsto ((1\,2\,4), (2\,3\,4)); g_{3,4} \mapsto ((1\,2\,4), (1\,2\,4))$$

**Proposition 3.4.** The $\gamma_\infty$ inner orbit of $g_{3,1}$ consists of $g_{3,1}$, $j = 1, 2, 3$, all having lift invariant $s_{A_3/A_4}$ value $+1$. The other two elements over $[3]$ are in the same absolute class and have length $1$ $\gamma_\infty$ inner orbits, each with lift invariant $-1$.

So, there are two braid orbits on $\text{Ni}(A_3, C_{\pm 3\varpi})^{\text{rd, rd}}$ of resp. lengths 9 ($+1$ lift invariant) and 6 ($-1$ lift invariant). Using Lem. 3.3, $O_{g''}^{\text{rd}} = O_{g'}^{\text{abs, rd}}$ identifies with the former, while $O_{g''}^{\text{rd}}$ (resp. $O_{g'}^{\text{abs, rd}}$) identifies with the latter (resp. $\text{Ni}(A_3, C_{\pm 3\varpi})$).

**Proof.** Since $\gamma_\infty$ maps the orbit over $[3]$ into itself, its orbit on $g_{3,1}$ consists of conjugating the middle two terms by powers of the product of the 2nd and 3rd entries, giving $g_{3,j}$, $j = 1, 2, 3$. The middle two terms of $g_{3,4}$ are the same, so $\gamma_\infty$ fixes them. It therefore must fix the last remaining class, represented by $g_{3,5}$, too.

We use a case of the lift invariant that applies to all Nielsen classes where $G = A_n$, the genus is 0 and $C$ consists of $2'$ (odd order) elements. For a $2'$ element $g \in A_n$, let $w(g)$ by the sum of $(\ell - 1)$ mod 2 over the length $\ell$ of disjoint cycles in $g$. Then, extend additively to define $w(g)$ for $g$ in a Nielsen class. Then, the value of the lift invariant is $(-1)^{w(g)}$. And $C$ consists of $r$ conjugacy classes of 3-cycles. $\text{Se90a}$ got this case from a preliminary version of $\text{Fr10a}$, Inv. Cor. 2.3 which reduces the general case to $r$ 3-cycles where the invariant is $(-1)^r$. To compute $s_{A_4/A_3}(g_{3,4})$, we show its value is that for a Nielsen class with genus 0, and $r = 3$.

For $g$ to have lift invariant $+1$ is exactly the same as its being in the image of $\text{Ni}(A_3, C_{\pm 3\varpi})$. For an H-M rep. the argument that $g_{3,1}$ lists to $g_{3,1} \in \text{Ni}(A_4, C_{\pm 3\varpi})$ is already in the proof of Lem. 3.2. For $g_{3,4}$, consider the 3-tuple $(4g_3, 4g_2, 4g_4)$ by replacing the middle two terms by their product. The result is still transitive on $A_4$. From R-H (App. B) the element belongs in a genus 0 Nielsen class. By the result above, its lift invariant is $-1$.

So, the unique order 3 lifts of the entries of $(4g_3^2, 4g_1^2, 4g_4)$ have entries with product $-1$. Clearly, then so do the entries of $(4g_1, 4g_2, 4g_3^2, 4g_4)$. Once we see $g_{3,5}$.
is absolute reduced equivalent to \( g_{3,4} \), then it too has lift invariant -1. To see this, conjugate \( g_{3,4} \) by (2 3), apply \( q_2q_1^{-1} \) and conjugate by \((2 3 4)^{-4} \) to get \( g_{3,5} \). Now use the identifiers in Lem. 3.3 to conclude the proposition.

Using the action by \( sh \) and \( q_2 \) in succession on the fiber over [3] gives a natural listing of elements \( g_{p,1}^j \), \( i = 1, \ldots, 5 \) and \( j = 1, 2, 3 \), over \([j] \). There are six total \( q_2 \) reduced orbits, labeled \( O_{1,1}, O_{1,3}, O_{3,1}, O_{3,4}, O_{3,4} \) and \( O_{3,5} \). Subscripts indicate elements seeding the cusp orbit; superscripts are for cusp widths. Unlike modular curves, there are two blocks \( N_i^+ \) and \( N_i^- \). The block for \( N_i^+ \) (absolute or inner reduced equivalence) corresponds to the braid orbit with all H-M reps., lift invariant +1. Then, \( N_0^- \) – the inner reduced case – is the orbit with lift invariant -1.

<table>
<thead>
<tr>
<th>( N_i^+ ) Orbit</th>
<th>( O_{1,1}^j )</th>
<th>( O_{1,3}^j )</th>
<th>( O_{3,1}^j )</th>
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<tr>
<td>( O_{1,1}^j )</td>
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<td>1</td>
<td>2</td>
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<tr>
<td>( O_{1,3}^j )</td>
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<td>1</td>
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<tr>
<td>( O_{3,1}^j )</td>
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**Remark 3.5.** [Fr06, §6.4.4] uses [Woh64] to show that neither corresponding Hurwitz space cover of \( P_1 \) is a modular curve. Unlike the modular curve case, the two cusp orbits of the H-M reps. \( g_{1,3} \) and \( g_{3,1} \) have different widths: resp. 4 and 3.

Read off the genuses of the reduced Hurwitz spaces from \( N_i^\pm \) using \((\gamma_0, \gamma_1, \gamma_\infty)\) from §2.1.2: Both are 0. [Fr10b] computes genuses of infinitely many \( A_n \) examples.

**3.2. Using an H-M rep. as a base point.** §3.1.2 considered a single Nielsen class \( Ni(A_4, C_{\pm 3^j}) \). It found there were two braid orbits. Even for \( p = 2 \), there are two directions to go: Increasing the conjugacy class multiplicity to consider \( Ni(A_4, C_{\pm 3^j}) \), \( r' \geq 2 \), and changing \( A_4 \) to \( G_{2k+1} \). In this section we go after \( Ni(A_4, C_{\pm 3^j}) \). One subtlety for \( r' = 2 \) is gone for \( r' \geq 3 \); the group \( Q'' \) is trivial.

3.2.1. **Explicitly picking a component.** Another case of [Fr10a, Thm. 1.3] gives the analog of Prop. 3.4: For each \( r' \geq 2 \), there are two braid orbits, giving reduced Hurwitz space components mapping to \( J_{2, r'} \) (as in (2.3)), separated exactly by their lift invariant under \( s_{A_4/A_4} \). Prop. 3.6 summarizes one reason to look at these spaces: to find canonical "automorphic" functions on these spaces.

**Proposition 3.6.** Each reduced Hurwitz spaces component \( H' \) corresponding to a braid orbit on \( Ni(G_{2k+1}, C_{\pm 3^j}) \), \( k \geq 0 \), supports a canonically defined \( \theta \)-null. That function is even (so potentially non-zero) only if the lift invariant is even.

**Proof.** [Fr10a, Prop. 6.12] produces a canonical \( \theta \) function (up to multiplication by a factor of automorphy involving a character, on any reduced Hurwitz space component of a Nielsen class \( Ni(G, C) \) where all conjugacy classes in \( C \) have odd order elements (are 2'). For any cover \( \varphi_p : X_p \to \mathbb{P}^1 \) representing a point \( p \) on such a component, the divisor of \( d\varphi_p \) has the form \( 2D_p \). The \( \theta \) will be even (versus odd) if and only if the linear system of \( D_p \) has even dimension. Serre showed this depends only on the Nielsen class and the lift invariant – our language, not his.

In [Se90b, Thm. 2] take the special case \( X = \mathbb{P}^1 \), so [Se90b, exp. (17)] applies. Serre notes this reverts to [Se90a, exp. (10)] referred to in the proof of Prop. 3.4, which also has the definition of \( w(g) \), for \( O_g \) a braid orbit of a Nielsen class. Then, we get even for the half-canonical class exactly when the product of the
lift invariant (3.1) and \((-1)^w(g)\) is 1. Our case has \(2r'\) equal values of \(w(g_i)\), so the last contribution is always 1, and the result depends only on the lift invariant. □

Use the notation for \(D_p\) and \(\mathcal{H}'\) in the proof of Prop. 3.6.

**Problem 3.7.** The precise condition in [Fr10a, Prop. 6.12] for the \(\theta\)-null attached to \(\mathcal{H}'\) to be nonzero is that some \(p \in \mathcal{H}'\) has the linear system \(D_p\) without a positive divisor. For which \(\mathcal{H}'\)'s with even lift invariant does this hold?

\[\text{3.2.2. Explicitly picking covers.} \]
It behooves us to consider what precisely are the components in Prop. 3.6 that have lift inv. +1, and from this point we concentrate only on tower levels above the orbit \(N_0^+\) in §3.1.2. §3.3 considers a technique for finding that for all levels \(k\), not just level 0.

To simplify, choose \(g' \in Ni(A_4, C_{\pm 3r'})\) to be an H-M rep. where the odd indexed entries map to \((123) \in A_3\), and the even entries map to \((132) \in A_3\). Again, it’s braid \(O_{g'}\); consists of the elements of \(Ni(A_4, C_{\pm 3r'})\) with lift invariant +1. This is compatible with the same notation in (3.2), and it makes sense for any \(p\), not just \(p = 2\). Denote the element

\[
(123), (132), \ldots, (123), (132) \in Ni(A_4, C_{\pm 3r'}) \text{ by } g'.
\]

Let \(\varphi_{g'} : X_{g'} \hookrightarrow \mathbb{P}^1\) be an H-M \(A_4\) cover with branch points \(z_0\). That is, from \(g_1^t, \ldots, g_{2r'}^t\), formed from classical generators at the top of this section, apply Riemann’s Existence to \((g_1^t, \ldots, g_{2r'}^t) \rightarrow g'\). The map \(O_{g'} \rightarrow Ni(A_4, C_{\pm 3r'})\) is then compatible with the degree 3 cover \(\varphi_{g'} : X_{g'} \hookrightarrow \mathbb{P}^1\) through which \(\varphi_{g'}\) factors, from \((g_1^t, \ldots, g_{2r'}^t) \rightarrow g'\). Equations for an affine description of this cover come from setting \(z\) to \(z_0\) here:

\[
y^3 = \prod_{i=1}^{r'} (x - z_i)(x - z_{r'+i})^2.
\]

Suppose \(g \in O_{g'}^{\text{abs}}\); entries of \(g\) are 3-tuples of elements of \(A_4\) in its standard, degree 4, representation \(T_{n4}\). Then, mapping \((P_1, \ldots, P_{2r'})\) (in order) to the entries of \(g \overset{\text{def}}{=} T_{n4}(g)\) gives a degree 4 cover \(\psi_{n4}(g) : X_{n4}(g) \rightarrow \mathbb{P}^1\). This gives a point of the absolute (reduced if we want it) Hurwitz space.

Let \(T_{\text{reg}}\) by the regular representation of \(A_4\). What corresponds to the Galois closure of that cover is sending \((P_1, \ldots, P_{2r'})\) to the entries of \(T_{\text{reg}}(g)\). Each entry of \(T_{\text{reg}}(g)\) is an element acting on the integers from 1 to 12 in the regular representation of \(A_4\) (a group of order 12). Then \((P_1, \ldots, P_{2r'}) \rightarrow T_{\text{reg}}(g)\) corresponds to a Galois, degree 12, cover \(X_{\text{reg}}(g) \rightarrow \mathbb{P}^1\).

There can be other important permutation representations. For \(Ni(A_4, C_{\pm 3r'})\), let \(T_1\) be the permutation representation of \(A_4\) on the cosets of the group generated by an order 2 \(h_1 \in A_4\) (say \(h_1 = (12)(34)\)). Then, \((P_1, \ldots, P_{2r'}) \rightarrow T_1(g)\) corresponds to a degree 6 cover \(X_{T_1}(g) \rightarrow \mathbb{P}^1\).

We can identify this as \(X_{\text{reg}}(g)/\langle h_1 \rangle \rightarrow \mathbb{P}^1\) using the identification of the covering group with \(A_4\). All three order 2 elements of \(A_4\) are conjugate in \(A_4\). So, up to absolute equivalence this cover is independent of the choice of \(h_1\).

**3.3. Coordinates for 1st homology.** As in §1.3.2, \(A_4\) is an extension of \(\mathbb{Z}/3\) by \(\text{Ker}_2\), the Klein 4-group. Consider \(X_{\text{reg}}(g)/\langle h_1 \rangle \rightarrow X_{\text{reg}}(Z)/\text{Ker}_2 = Y\). There are three different inequivalent covers of \(Y\) depending on the choice of \(h_1\).
Using \( z_0 \) and a particular H-M rep. \( g' \in \text{Ni}(\text{Ker}_2 \times \mathbb{Z}/3, C_{\pm 3'}) \), we now give explicit coordinates for \( H_1(X_{\text{reg}}(g'))/\text{Ker}_2, \mathbb{Z}) \) in terms of \( g' \). Then, we compute the braid group action on this presentation.

Since \( X_{\text{reg}}(g')/\text{Ker}_2 \rightarrow \mathbb{P}^1_z \) is the cover associated with \((P_1, \ldots, P_r) \rightarrow \mathbb{g}'\), we denote \( X_{\text{reg}}'(g')/\text{Ker}_2 \) by \( X_{\text{reg}}(g') \). From R-H (App. B) this is a Riemann surface of genus, \( g_{g', r} \), equal to \( 2r' - 2; 2(3 + g_{g', r} - 1) = 4r'\).

There is a natural map from \( H_r \rightarrow S_r \) corresponding to permuting the branch points. Denote by \( S_{\text{odd, even}} \equiv S_r \times S_r \) the subgroup of \( S_r \) that acts as permutations of the sets \( \{1, 3, \ldots, r-1\} \) and \( \{2, 4, \ldots, 2r'\} \). Then, \( H, g' \) identifies with the preimage in \( H_r \) of \( S_{\text{odd, even}} \).

We also need the preimage of \( \text{Ni}(G_{2k+1}, C_{\pm 3'})^* \) over \( \mathbb{g}' \). Denote this by \( \text{Ni}(G_{2k+1}, C_{\pm 3'})^*_{g'} \). §3.4 answers Prob. (3.5) for \( p = 2 \).

Problem 3.8. For all \( g \) in the braid orbit of \( H, g' \) do the following.

(3.5a) Decode triplets of homology vectors corresponding to degree 2 covers through which \( X_{\text{reg}}(g') \rightarrow X_{\text{reg}}(g')/\text{Ker}_2 \) factors.

(3.5b) Compute the image, \( M_{g', r} \), of the action of \( H, g' \) on \( H^1(X, g', Z) \).

(3.5c) Compute the image, \( R^*_{g', r} \) of \( M_{g', r} \) on \( \text{Ni}(\text{Ker}_{2k+1} \times \mathbb{Z}/3, C_{\pm 3'})^*_{g'} \), the reduced Nielsen classes corresponding to \( *= \) in or abs.

More generally, do this for or any prime \( p \neq 3 \), not just \( p = 2 \).

Goal (3.5b) is the analog for \( \mathbb{P}^1 / 3 \) of seeing the \( \text{PSL}_2(\mathbb{Z}) \) action on elliptic curve homology in the case of modular curves. This also shows the explicit action on the \( \theta \)-nulls in [Fr89, §6] on these spaces.

Goal (3.5c) is the analog of seeing the monodromy groups for the modular curves \( X_1(p^{k+1}) \) and \( X_0(p^{k+1}) \) as \( \mathbb{P}^1 \) covers — using the braid action. The difference is that the \( j \)-line monodromy was tagged by Galois before he died in 1841, while no one has written out this analog before.

### 3.4. Answering the \( \mathbb{Z}/3 \) Monodromy Problem

Consider a cover \( \varphi : X \rightarrow \mathbb{P}^1_z \) branched at \( z \) defined by a permutation representation \( T : G \rightarrow S_n \), and as previously \( G(1) \) is the stabilizer of 1. We will write the fundamental group of \( X \) using classical generators of \( \pi_1(\mathbb{P}^1_z, z_0) \). App. refers to these as \( P_1, \ldots, P_r \). It is isotopy classes of these we need, which we denote \( \tilde{g}_1, \ldots, \tilde{g}_r \), or for short, \( \mathbb{g}' \). Let \( F_r \) be the free group on elements \( \tilde{g}_1, \ldots, \tilde{g}_r \) equipped with the natural homomorphism \( F_r \rightarrow \mathbb{P}^1_z \) by \( \tilde{g}_i \rightarrow g^*, i = 1, \ldots, r \). Denote by \( F_r(1) \) the pullback of \( G(1) \).

3.4.1. Generators for \( H_1(X, g', Z) \). With branch cycles \( g \) for \( \varphi \) based on \( g' \), follow [Fr89, §3.1] to apply Schreier’s Thm. (Lem. 3.10). Let \( N \) be the smallest normal subgroup of \( F_r \) containing \( \tilde{g}_1 \cdots \tilde{g}_r \) and \( \tilde{g}_i^{\text{ord}(g)} \), \( i = 1, \ldots, r \). For \( \varphi \) a Galois cover, the fundamental group of \( X \) is isomorphic to the image of \( F_r(1) \) in \( F_r/N \).

Now take \( \varphi \) to be \( \varphi \cdot g' \). The \( g'_i \) are subject to these relations:

\[
(3.6) \quad (g'_i)^3 = 1, \quad i = 1, \ldots, r, \text{ and (product-one)} \quad g'_1 \cdots g'_r = 1.
\]

Since we care really about \( H_1(X, g', Z) \) and certain of its quotients, add this:

\[
(3.7) \quad \text{In addition to (3.6), the kernel of the homomorphism } \psi'' \text{ from } g' \rightarrow g' \text{ is abelian (its elements commute).}
\]

The superscripts \( l(\text{ef} \cdot t) \) and \( r(\text{gh} \cdot t) \) in these elements in the kernel of \( \psi'' \) refer to the position of \( g_1 \):

\[
(3.8) \quad g^*_1 g^*_u = w^l_u, \quad g^*_v g^*_1 = w^l_v, \quad g^*_1 g^*_u = w^l_u, \quad g^*_v g^*_1 = w^l_v, \quad u = 1, \ldots, r', \text{ and } \quad u = 1, \ldots, r'^{-1}.
\]
To simplify notation, denote \( H_1(X, g', Z) \) by \( \ker_0 \). Denote the group the \( g_i^* \)s, satisfying (3.6) and (3.7), generate by \( U_{g'} \). A surjective homomorphism, 
\[ \varphi^* : H_1(X, g', Z) \rightarrow A, \] so, \( \varphi^* \in H^1(X, g', A) \),
gives a Galois cover \( \varphi : Z \rightarrow X_{g'} \) with group \( A \) and \( H_1(Z, Z) = \ker(\varphi^*) \).

**Proposition 3.9.** The \( v^h_u \) s, \( v^h_u \) s, \( w^h_u \) s and \( w^h_u \) s generate \( \ker_0 \). Further, the relations among them are generated by

\[ (3.9) \quad v^1_h(w^1_1)^{-1}v^1_h(w^1_2)^{-1} \cdots v^1_h = 1 = v^1_h(w^1_1)^{-1}v^1_h(w^1_2)^{-1} \cdots v^1_h. \]

Then, \( U_{g'} \) is isomorphic to \( \ker_0 \times (g^1_*)^a \). The action of \( q \in H_r \) on \( k \in \ker_0 \) (or in \( \ker_0/2 \)) expressed in terms of the \( q \)'s and \( w \)'s is computable.

Suppose \( k_0 \in H^1(X_{g'}, Z/2) \) corresponds to a degree 2 cover \( Y \rightarrow X_{g'} \) through which \( X_{\text{reg}(g')} \) factors. Then, for \( q \in H_r \), we can explicitly calculate \( (k_0)q \). Then, the triples of degree 2 covers through which \( X_{\text{reg}(g')}q \rightarrow X_{g'}q \) factors correspond to \( (k_0)q \), and the images of this under \( g^*_{1} \) and \( (g^1_1)^2 \).

**Proof.** From the conditions (3.6), the \( v_u \) s and the \( w_u \) s are in \( \ker_0 \). Further, \( \alpha_1 = 1, \alpha_2 = \bar{g}_1 \) and \( \alpha_3 = \bar{g}_1^{-1} \) give coset reps. in \( F_r \) of \( F_r(1) \). Directly apply (3.12) to these coset reps. and the generators \( \bar{g}_1, \ldots, \bar{g}_r \) for \( F_r \). In that notation, the generators include \( \alpha_1 \bar{g}_1 \rho(\alpha_1 \bar{g}_1)^{-1} = \bar{g}_j \bar{g}_1^{-1} \) (resp. \( \bar{g}_j \bar{g}_1 \)) if \( j \neq 1 \) is odd (resp. even). For \( \alpha_3 \) replacing \( \alpha_1 \), the result is \( \bar{g}_1 \bar{g}_1 \bar{g}_1 \) (resp. \( \bar{g}_1 \bar{g}_1 \)) for \( j \neq 1 \) odd (resp. even).

So, replacing \( \bar{g}_j \) s with \( g^j_1 \) s in these words gives generators for \( H_1(X, g', Z) \). Further, the elements \( g^1_* g^j_1 g^1_* \) are already in the subgroup spanned by the other generators. Example:

\[ (3.10) \quad g^1_* (g^2_1) (g^2_1)^{-1} (g^1_1)^{-1} (g^1_1) = g^1_* g^j_1 g^1_1, j \text{ odd.} \]

It is similar for \( j \) even.

The commuting relation (3.7) gives

\[ (3.11) \quad g^1_* g^j_1 g^1_* = g^1_* (g^2_1)^{-1} (g^2_1)^{-1} (g^1_* (g^2_1)^{-1} (g^1_1)^{-1}) = (g^2_1)^{-1} (g^2_1)^{-1} (g^1_1)^{-1}. \]

Expand \( v^h_u(w^h_1)^{-1}v^h_u(w^h_2)^{-1} \cdots v^h_u \) as

\[ g^1_* g_2 (g^2_1)^{-1} g^1_* (g^1_1)^{-1} \cdots g^1_* g_2 \]

to see that it is 1. This shows the elements with \( h \) superscripts generate a subspace of rank \( r' - 2 \). Similarly for the elements with \( i \) superscripts. Since the genus of \( X_{g'} \) is \( r' - 2 \), among these generators we have the exact correct rank, and so the vectors listed in the statement of the Prop. generate the homology. \( \square \)

We now compute coset representatives for \( S_{\text{Odd,Even}} \) in \( S_{2r'} \). Their number is \( \frac{2r'!}{(r'!)} \). Any group will move, say, \( k \) evens (resp. \( k \) odds) to a specific subset of odds (resp. evens). For each \( k, 0 \leq k \leq r' \), you count the number of \( k \) subsets of target odd (resp. even) numbers from \( r' \) of them as \( \binom{r'}{k} \). Multiply those two numbers to get the number of cosets indexed by \( k \). The result is that

\[ \frac{2r'!}{(r'!)} = \sum_{k=0}^{r'} \binom{r'}{k}^2. \]

So, we get corresponding coset reps as follows. Take one of the subsets of even (resp. odd) integers \( \text{Even}' \in \text{Even}_k \) (resp. \( \text{Odd}' \in \text{Odd}_k \)), and form the permutation \( \sigma_{\text{Even}'} \) consisting of the product of transpositions \( \sigma_{2u-1, u'_{\text{Even}}} = (2u-1) u'_{\text{Even}} \)
where $1 \leq u \leq k$ and $u'_{\text{Even}}$ is the $u$th element in Even', etc. The transposition has a corresponding braid group element $q_{2u-1,u'_{\text{Even}}}$ that maps to it. Note: The generators $(1 \cdots 2r')^{-1}$ and $(1 2)$ of $S_{2r'}$ have natural lifts $s h$ and $q_1$ for $H_{2r'}$.

3.4.2. Generators for $H_1g'$. A famous Schreier formula gives a similar result for $S_{\text{Odd,Even}}$ and for $H_1g'$, based on the following lemma, and the idea that we have a natural way – given above – to lift elements from the former group to the latter.

**Lemma 3.10.** With $H \leq G$ and $G^* \to G$ a cover, let $H^*$ be the pullback of $H$ to $G^*$. Then, any lift of coset reps. of $H$ in $G$ to $G^*$ gives cosets reps. of $H^*$ in $G^*$.

Denote the coset reps. of $H$ in $G$ by $\{\alpha_i\}_{i \in I}$ and by $\rho : G \to G$ the map that takes $\alpha \in G$ to the element $\alpha_i$ representing its coset. Use $\{\alpha_i^\rho\}_{i \in I}$ and $\rho^\alpha$ for the corresponding elements for $(H^*, G^*)$. Suppose $g_1, \ldots, g_r$ are generators of $G$ that lift to generators $g_1', \ldots, g_r'$ of $G^*$. Then, the elements in $\{\alpha_i g_j \rho(\alpha_i g_j)^{-1}\}_{i,j,} \cap H$ (resp. the same elements with $\alpha$ superscripts), generate $H$ (resp. $H^*$).

**Proof.** Since $(G^* : H^*) = (G : H)$, no matter the lifts of the coset reps. of $H$ in $G$, you get the same number of distinct coset reps. for $H^*$ in $G^*$.

If $G^*$ is free on the $g_j$, then the formula of Schreier, as recounted in [FrJ86, Lem. 15.23], would give free generators for $G^*$. With no loss we can replace $G^*$ by a free group on those generators, and then the images of the resulting generators in our actual group $G^*$ or $G$ would be generators, our goal.

To get free generators of $H^*$ we need a function $\rho : G^* \to G^*$ representing right cosets of $H^*$, with the following properties: $\rho(1) = 1 : \rho(\alpha_i) \in H^* \alpha_i$; and $\rho(h^* \alpha) = \rho(\alpha)$ for each $h^* \in H^*$ and $\alpha^* \in G^*$. Furthermore $\rho$ may be selected to have the following property: $\text{length}_{g^*}(\rho(\alpha^*)) = \text{min}_{i,j \in H^*} \text{length}_{g^*}(h^* \alpha^*)$ for $\alpha^* \in G^*$ where length$_{g^*}$ denotes the length of a word in the $g^*$s (allowing $\pm 1$ exponents). With these conditions, the collection

$$(3.12) \quad H^* = \{\alpha_i^* g_j^* \rho(\alpha_i^* g_j^*)^{-1} | i, j \text{ and } \alpha_i^* g_j^* \notin \rho(G^*)\}$$

generates $H^*$ freely.

3.5. The spaces $\mathcal{H}(G_p, C_{\pm 3}^*)^{\text{rd}}, p \neq 3$ (or $3$). Use the action of $\mathbb{Z}/3$ on $\mathbb{Z}^3/\mathbb{Z} \equiv \mathbb{Z}^2 \overset{\text{def}}{=} V$ by abelianizing Ex. 1.7. So, the generator $\alpha$ of $\mathbb{Z}/3$ acts as the shift on a representative $(b_1, b_2, b_3)$ with $b_1 + b_2 + b_3 = 0$. Denote $V/pk+1V$ by $V_p$ for $k=1$. We'll start with $k=0$.

We use that the Jacobi symbol $\left(\frac{-3}{p}\right)$ is

$$(3.12) \quad (\frac{-3}{p}) = -1^{p-1} \left(\frac{p}{3}\right) = \left(\frac{p}{3}\right).$$

There are two cases:

(3.13a) When $-3$ is not a square mod $p$, the $\mathbb{Z}/[\alpha]$ module $V_p$ is irreducible.

From quadratic reciprocity, this corresponds to $p \equiv -1 \mod 3$.

(3.13b) Otherwise, $\mathbb{Z}/[\alpha]$ module $V_p$ is a sum of two 1-dimensional modules.

**Proposition 3.11.** When (3.13a) consider the elements of the Nielsen class $\text{Ni}(G_p, C_{\pm 3}^*)^{\text{in}}$ of the form

**Proof.** As in §3.4 consider Nielsen class reps. lying over $g' \in \text{Ni}(\mathbb{Z}/3, C_{\pm 3}^*)$. By writing out the product of the matrices

$$(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 \\ 2b & 1 \end{pmatrix} \cdots \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 \\ 2b & 1 \end{pmatrix} $$
conclude that product-one is equivalent to
\[ j_1b + \alpha j_2b + \cdots + 2r - 1j_2r - 1b + 2r = 0. \]

Denote \( ( \begin{smallmatrix} \alpha & 0 \\ 2 & 1 \end{smallmatrix} ) \) (resp. \( ( \begin{smallmatrix} \alpha & 0 \\ 2+i & 1 \end{smallmatrix} ) \)) by \( g_{2i} \) (resp. \( g_{2i+1} \)).

Generation: The elements \( j_0b - j_2b \) and \( j_0b + j_1b \) are in the group generated by the \( g_i \)s. \( \square \)

4. Frattini properties of \( j \)-line covers

Given a Nielsen class \( Ni(G_0, C) \) and a prime \( p \) dividing \(|G|\), but not \( N_C \), we can form a sequence of Nielsen classes \( Ni(G_k, ab, C) \) by replacing \( G_0 \) by \( G_k, ab \), and \( C \) by the unique conjugacy classes in \( G_k, ab \) that lift those from \( G_0 \). Consider the \( p \)-representation cover \( R_p \to G_0 \). The following is a corollary of \([Fr06, Cor. 4.19]\) by applying it to abelianized \( MTs \).

4.1. Nonempty MTs.

**Proposition 4.1.** There is an if and only if criterion that there is something in \( Ni(G_k, ab, C) \), for each \( k \geq 1 \), lying above a braid orbit \( O_0 \) of \( Ni(G_0, C) \):

(4.1) given \( g \in O_0, s_{R_p/G_0}(g) = 1 \).

When \( \ker(R_p \to G_0) \) has exponent \( p^u \), then above any braid orbit \( O_u \) of \( Ni(G_u, ab, C) \) there is a nonempty braid orbit of \( Ni(G_k, ab, C) \) for each \( k \geq u \).

4.2. \( \mathbb{Z}^2 \times \mathbb{Z} / 3 \) monodromy. The following problem extends Prob. 3.5.

**Problem 4.2.** For \( p \neq 3 \) a prime, look at the collection of groups \( R_p = \{ R_{p^{k+1}} \}_{k=0}^{\infty} \).

(4.2a) Show that for all but finitely many \( p \), the sequence of covers \( R_{p^{k+1}} \to R_p \) is a \( p \)-Frattini cover for all values of \( k \).

(4.2b) Show for all primes \( p \neq 3 \) that there is a \( k_0 \) (larger than 0 for only a finite explicit set of \( p \)) so that for \( k \geq k_0 \), \( R_{p^{k+1}} \to R_{p^{k_0+1}} \) is a \( p \)-Frattini cover.

The properties (4.2) is the exact analog for \( \mathbb{Z} / 3 \) of the beginning part of Serre’s *Open Image Theorem*, which in our continuing analog.

The sh-incidence matrix remains the same if we replace \( \gamma_1 = sh \) by \( \gamma_0 \) [Fr07b, Lem. 4.8]. So, fixed points of either are represented on the diagonal.

Problem: Use the \( MT \) method to compute which elements in H-M cusp are fixed points of \( \gamma_i, i = 0, 1 \).

4.3. Nielsen classes and the OIT.

4.4. Dependence of the cusp pairing on the Nielsen class. For any Nielsen class \( Ni(G, C)^* \) with \( * \) an equivalence relation like inner or absolute, we have the corresponding reduced space \( \mathcal{H}(G, C)^{*,rd} \). It will form a natural cover of the \( j \)-line \( \mathcal{H}(G, C)^{*,rd} \to \mathbb{P}^1 \), and on its cusps we have the sh-incidence pairing.

Suppose, however, we have two Nielsen class descriptions, from \( Ni(iG, iC) \) of the exact same space. How can we compare them?
Figure 1. Example Classical Generators
Appendix A. Classical Generators

Figure 1 gives the ingredients for classical generators of the fundamental group, \( \pi_1(U_z, z_0) \), of the \( r \)-punctured sphere, with the punctures given by \( z = \{ z_1, \ldots, z_r \} \). These are ordered closed paths \( \delta_i \sigma \delta_i^{-1} = \sigma_i, i = 1, \ldots, r \).

Here are their properties. There are discs, \( i = 1, \ldots, r \): \( D_i \) with center \( z_i \); all disjoint, each excludes \( z_0 \); \( b_i \) be on the boundary of \( D_i \). Their clockwise orientation refers to the boundary of \( D_i \). It is a path \( \sigma_i \) with initial and end point \( b_i \); \( \delta_i \) is a simple simplicial path with initial point \( z_0 \) and end point \( b_i \). We also assume \( \delta_i \) meets none of \( \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_r \), and it meets \( \sigma_i \) only at its endpoint.

There is a crucial condition on meeting the boundary of \( D_0 \). First: \( D_0 \), with center \( z_0 \), is disjoint from each \( D_1, \ldots, D_r \). Consider \( a_i \), the first intersection of \( \delta_i \) and boundary \( \sigma_0 \) of \( D_0 \). Then, \( \delta_1, \ldots, \delta_r \) satisfy these conditions:

- (1.1a) they are pairwise nonintersecting, except at \( z_0 \); and
- (1.1b) \( a_1, \ldots, a_r \) are in order clockwise around \( \sigma_0 \).

Since the paths are simplicial, the last condition is independent of \( D_0 \), for \( D_0 \) sufficiently small. For any ordering of the collection of points \( z \), there are many sets of classical generators whose ordering corresponds to the order of \( z \). Usually that means, given branch cycles of a cover there will be several branch cycles descriptions – up to, say, absolute equivalence – corresponding to a given cover of \( \mathbb{P}^1_z \) branched over \( z \).

Appendix B. R(ie mann)-H(urwitz)

R-H: Computes the genus \( g_X \) of a degree \( n \) cover \( \varphi : X \to \mathbb{P}^1_z \) as follows.

- \( z_\varphi \) are the branch points, and \((\bar{\sigma}_1, \ldots, \bar{\sigma}_r)\) are classical generators (App.A1) of \( \pi_1(U_{z_\varphi}) \).
- \( X^0 = \varphi^{-1}(U_{z_\varphi}) \). So, \( \varphi : X^0 \to U_z \) is unramified, giving \( \varphi_* : \pi_1(U_{z_\varphi}) \to S_n \).
- \( \varphi_* (\bar{\sigma}_1, \ldots, \bar{\sigma}_r) = (g_1, \ldots, g_r) \).

Branch cycles and the genus

With \( \text{ind}(g_i) = n - |g_i \text{ orbits}| \),

\[
2(n + g_X - 1) = \sum_{i=1}^{r} \text{ind}(g_i).
\]

Then, \((g_1, \ldots, g_r)\) are branch cycles of \( \varphi \).

Exercise: Compute genus of a cover with branch cycles \( g \in \text{Ni}(D_{p^{k+1}}, C_{2^4})^{\text{abs}} \) in §I.B (p. 7). Same for \( g \in \text{Ni}(D_{p^{k+1}}, C_{2^4})^{\text{in}} \).

Appendix C. Apply R-H to MT components \((r = 4)\)

\( O' \) is a \( \tilde{M}_4 \) orbit on a reduced Nielsen class \( \text{Ni}(G, C)^{\text{abs}}/Q'' \) (or \( \text{Ni}(G, C)^{\text{in}}/Q'' \)). Denote action of \( (\gamma_0 = q_1 q_2, \gamma_1 = \text{sh}, \gamma_\infty = (\gamma_0 \gamma_1)^{-1}) \) on \( O' \) by \( (\gamma_0', \gamma_1', \gamma_\infty') \). Branch cycles for a cover \( \mathcal{H}' \to \mathbb{P}^1_j \),

R-H gives genus, \( g_{\mathcal{H}'} \):

\[
2(\text{deg}(\mathcal{H}'/\mathbb{P}^1_j) + g' - 1) = \text{ind}(\gamma_0') + \text{ind}(\gamma_1') + \text{ind}(\gamma_\infty').
\]
Appendix D. Antecedent Schur multipliers

Let $H$ be a finite group of order divisible by $p$. §1.3.3 refers to its universal $p$-Frattini cover $\hat{\psi}: \hat{\psi}H \to H$. Let $R_p \to H$ be a maximal central extension of $H$ that is a quotient of $\hat{\psi}$. Then, $\ker(R_p \to H) \overset{\text{def}}{=} \mathcal{S}C_p(H)$ is the $p$-part of the Schur multiplier of $H$.

Consider an extension $\varphi: H_k \to H_0$, with kernel $(\mathbb{Z}/k)^n$, $k \geq 2$ of one of the following two types:

- (4.1a) $H$ is $p'$;
- (4.1b) $H_k$ is the $k$-th characteristic abelianized $p$-Frattini extension of $H_0$.

The symbol 1 stands for the trivial module for the action of $H$. We also consider $p'$ conjugacy classes $C$ in $H_0$ along with their natural extension to $H_k$.

**Proposition D.1.** All characteristic $p$-Frattini modules $G_1(H_t)$, $1 \leq t \leq k$, are isomorphic as $H$ modules, and all the groups $\mathcal{S}C_p(H_t)$, $1 \leq t \leq k$, are isomorphic.

Further, assume that 1 does not appear as a quotient of $M \ker(H_1 \to H_0)$ (as an $H_0$ module). Then, any element $g \in \Ni(H_t, C)$ extends to an element of $g \in \Ni(H_k, C)$ for $t = 1$ (resp. $t = 0$) in case (D.1a) (resp. (D.1b)) with the same value of the lifting invariant as given in (3.1).

**Proof.** [FrK97, Obst. Lem. 3.2] gives an elementary argument for the second paragraph, under the stronger hypothesis that 1 is not a subquotient of $M$. Otherwise the result is [Fr06, Cor. 4.19].

**Example D.2.**

**References**


On my home page http://math.uci.edu/~mfried, section: Ib. Definitions and discussions from Major Article Themes:

- * Book: Intro. and Chp.s on Riemann’s Existence Thm


What Gauss told Riemann about Abel’s Theorem


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