THE SHIFT INCIDENCE MATRIX 
AND ORBITS OF HURWITZ SPACES

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Abstract. Hurwitz spaces are parameter spaces for families of (non-
singular Riemann surface) covers of the Riemann sphere \( \mathbb{P}_1 \) uniformized
by a variable \( z \). Two types – differing by covering equivalences – play a
role in most investigations: absolute and inner. A equivalence – reduced
– figures strongly in applications connecting to classical problems. Cov-
ners have a monodromy group, \( G_\varphi \), and that is constant on a connected
component of a Hurwitz space. Another constant on such a connected
component is the set of conjugacy classes \( C \) – cardinality \( r_C \) – that ap-
pear when one identifies an element of a Nielsen class (defined by \( (G, C) \))
with those covers having a fixed (unordered) set of branch points.

Most applications eventually concentrate on a particular collection of
Nielsen classes that hone in on the conjugacy classes. When the Hurwitz
space defined by a Nielsen class has more than one component – several
braid orbits on Nielsen classes, \( r \geq 4 \) – that is significant. For one, for
finding the moduli definition field of those components. Two tools have
been successful at finding distinguishing component geometry.

1. A lift invariant when \( G \) has a nontrivial Schur multiplier.
2. The \( sh \)-incidence matrix of this paper’s title.

#2 is our topic (with examples connecting to #1): an algorithm for
computing components based on cusp orbits. This is especially revealing
when \( r = 4 \) for its comparison of reduced Hurwitz spaces with modular
curves, which are particular examples.

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1. THE BRAID ACTION, AND OUR USE OF EXAMPLES

This section reminds of details on Nielsen classes, and braid group ac-
tion on them. Between [FrV91] and [FrV92] and [BFr02], there was a great
expansion in the applications benefitting from analysis of Hurwitz spaces
and the moduli definition fields of their components. “Great!” Nevertheless,
we have seen a need for a tool to identify components, and telling accounts
for their differences, of a given Hurwitz space.

1.1. Introduction. The \(-sh\)-incidence (shift incidence) matrix is named
for its use of the shift, coming from the braid group, acting on elements of
a Nielsen class defined by a finite group, \( G \), and some generating conjugacy
classes, \( C \), of \( G \). The most elementary case, when the number of classes,
\( r \overset{\text{def}}{=} r_C \) is 3, does not seriously involve the braid group. We assume, except sometimes for induction purposes that \( r \geq 4 \). This tool automatically reveals properties of the space through an identification of cusps with cusp group orbits in the Nielsen class. We recall the basic definitions, especially how braids act on the Nielsen classes (§1.2), and the combinatorial cusps (§1.3). Then, (§2 produce the algorithm at the heart of the paper.

§3 illustrates the algorithm with examples. Several, which we only review, have already appeared in print. They therefore show precisely the value of the \( \text{sh} \)-incidence matrix. Other examples have not yet appeared in print, though they are related to results that have. These illustrate our three main themes.

(1.1a) How in practice to catch the appearance of the lift invariant (#1 in the abstract).
(1.1b) Why cusp orbit geometry alone often separates braid orbits.
(1.1c) While the lift invariant often distinguishes all braid orbits in a Nielsen class, in telling cases it cannot.

At this stage, almost any example must develop a theory to handle understanding those components. Especially for the arithmetic geometer first author who foremost is considering extending the main definition field tool, the Branch Cycle Lemma (BCL), to figure the module definition field when there is more than one component. We briefly remind of the old examples that guided our investigations.

At least when fine moduli holds, the BCL gives precisely the condition that a point on a Hurwitz space corresponding to \((G, C)\) with coordinates in a number field \( K \) corresponds to a cover defined over \( K \). The case of inner Hurwitz spaces give that if \( K \) contains a precisely defined field, \( \mathbb{Q}^{\text{in}}_{G,C} \), the result is a regular realization of \( G \) with branch cycles in \( C \). Our examples uniformly take the case \( \mathbb{Q}^{\text{in}}_{G,C} = \mathbb{Q} \). In practice the case where the equivalence of covers is called absolute is of equal significance. For that there is a corresponding field \( \mathbb{Q}^{\text{abs}}_{G,C} \) contained in \( \mathbb{Q}^{\text{in}}_{G,C} \).

It would be super if, for any Hurwitz space component, we could decorate the braid orbit corresponding to that component with what we glean from (1.1a) and (1.1b), and apply a variant on the BCL to get an analog of \( \mathbb{Q}^{\text{in}}_{G,C} \) for the inner and absolute cases. We conclude the paper by discussing what precisely that would entail, for which we regard using the
**sh-INCIDENCE**

sh-incidence matrix as a source of empirical data. The key practical example — still unresolved — is when a Hurwitz space has several components we call Harbater-Mumford.

For our new examples, from preprints, we mostly explain the sh-incidence matrices – illustrating they are graphic devices – pointing for details to the papers to be published, and the book [Fr18] the first author is completing. It gives a new look and generalization to Serre’s Open Image Theorem (as begun in [Se68]). There are new considerations even in Serre’s case.

These are designed to show how the generalization replacing the dihedral group $D_\ell$ by essentially any finite group $G$, and any prime $\ell$ dividing $|G|$, produces analogs — for the case $r_C = 4$, when reduced Hurwitz spaces have dimension 1, and are natural covers of the j-line minus the point at $\infty$ – of modular curves. We touch bases with that, referencing [Fr17] and [Fr18].

**Remark 1.1.** A complete elementary treatment of Riemann’s Existence Theorem, starting from [Ahl79], is at [Fr90]. A practical exposition, using the solution to Davenport’s Problem takes up the first sections of [Fr12]. [Vo96] is a treatise primarily aimed at group theory for those into the regular Inverse Galois Problem; in ways a complement to [Se92] ([Fr94] ties them together) and – though less technical– [MM99].

1.2. **Braid action on Nielsen Classes.** To simplify notation, unless otherwise said, always make these two assumptions.

(1.2a) Conjugacy classes, $C = \{C_1, \ldots, C_r\}$, in $G$ are generating.

(1.2b) $G$ is given as a transitive subgroup of $S_n$.

Meanings: (1.2a) $\implies$ the full collection of elements in $C$ generates $G$; and (1.2b) $\implies$ the cover generated by $g$ is connected. Even with (1.2a), it may be nontrivial to decide if there is $g \in G^r \cap C$ that generates.

Thm. 1.3 uses (1.2a) gives the combinatorial description of the equivalence class of $\mathbb{P}^1$ Riemann surface covers with branch points at a specific location $z \in \mathbb{P}^r \setminus D_r \overset{\text{def}}{=} U_r$, project $r$-space minus its discriminant locus. This represents unordered collections of $r$ distinct points on $\mathbb{P}^1 \overset{\text{def}}{=} \mathbb{C} \cup \{\infty\}$.

**Explaining Nielsen classes, $\text{Ni}(G, C)^1$:** This is based on choosing a collection of classical generators, $\mathcal{P} = \{P_1, \ldots, P_r\}$, for the fundamental group of the $r$-punctured sphere $U_z = \mathbb{P}^1 \setminus \{z\}$ as documented in Rem. 1.1.

The cover comes from the mapping $P_i \mapsto g_i$, $i = 1, \ldots, r$, producing a permutation representation $\pi(U_z, z_0) \to G \leq S_n$. As explained in [Fr17, §1.2.2], from the theory of the fundamental group, this gives a degree $n$ cover $f^0 : W^0 \to U_z$. Completing the converse to Thm. 1.3 is not immediate. You
fill in the holes in \( f_0 \) to get the desired \( f : W \to \mathbb{P}^1_z \). Starting from [Ahl79], [Fr80, Chap. 4] documents a full proof.

**Definition 1.2.** Two covers \( i f : i W \to \mathbb{P}^1_z \), \( i = 1, 2 \) are absolutely equivalent if there exists a continuous \( \varphi : i W \to 2W \) such that \( 2f \circ \varphi = 1f \).

Denote the subgroup of the normalizer, \( N_{S_n}(G) \), of \( G \) in \( S_n \) that permutes a given collection, \( C \), of conjugacy classes, by \( N_{S_n}(G, C) \).

**Theorem 1.3.** Assume \( z \overset{\text{def}}{=} z_1, \ldots, z_r \in \mathbb{P}^1_z \) distinct. Then, some degree \( n \) cover \( f : W \to \mathbb{P}^1_z \) with branch points \( z \), and \( G = G_f \leq S_n \) produces classes \( C \) in \( G \), if and only if there is \( g \in G^r \cap C \) with these properties:

\begin{align*}
(1.3a) (g) &= G \ (\text{generation}); \text{ and} \\
(1.3b) \prod_{i=1}^r g_i &= 1 \ (\text{product-one}).
\end{align*}

Two such \( g, 2g \) represent absolutely equivalent covers if, for \( h \in N_{S_n}(G, C) \), \( h_2gh^{-1} = 1g \). Indeed, \( r \)-tuples satisfying (1.3) give all possible Riemann surface covers – up to absolute equivalence – with these properties.

The index of an element \( g \in S_n \) is \( n \) minus the \# of disjoint cycles in \( g \).

Refer to one of those covers attached to \( g \) as \( f_g : W_g \to \mathbb{P}^1_z \).

(1.4) The genus \( g_g \) of \( W_g \) appears in \( 2(\text{deg}(f) + g_g - 1) = \sum_{i=1}^r \text{ind}(g_i) \).

**Definition 1.4.** Given \((G, C)\), the \( g \) satisfying (1.3) is the Nielsen class \( \text{Ni}(G, C) \), with \( \text{Ni}(G, C)^\dagger, \dagger = \text{abs referencing absolute equivalence from} \) the representation \( T : G \to S_n \).

Similarly for inner Nielsen classes, whose elements correspond to Galois closures, \( \hat{f} : \hat{W} \to \mathbb{P}^1_z \), of the covers labeled \( f_g \) above, with an explicit isomorphism \( \psi : \text{Aut}(\hat{f}/\mathbb{P}^1_z) \to G \). Suppose given two such \((\hat{f}_1, \psi_1), (\hat{f}_2, \psi_2)\) and a continuous \( \hat{\varphi} : \hat{W}_1 \to \hat{W}_2 \) for which \( \hat{f}_2 \circ \hat{\varphi} = \hat{f}_1 \). This induces

\[ \varphi^* : \text{Aut}(\hat{W}_2/\mathbb{P}^1_z) \to \text{Aut}(\hat{W}_1/\mathbb{P}^1_z) \text{ by} \]

\[ a \in \text{Aut}(\hat{W}_2) \mapsto (\hat{\varphi})^{-1} \circ a \circ \hat{\varphi} \in \text{Aut}(\hat{W}_1). \]

Then, \( \varphi \) is an inner equivalence if

(1.5) \[ \psi_2 \circ \varphi^* \circ \psi_1^{-1} \text{ is an inner automorphism of } G. \]

Equivalence classes of such pairs \((\hat{f}, \psi)\) correspond to elements of \( \text{Ni}(G, C) \), with \( \dagger = \text{in} \). Or, denote them \( \text{Ni}(G, C)/G \) with the \( g \in G \) action mapping \( g \in \text{Ni}(G, C) \) to conjugation by \( g \) distributed on all entries of \( g \).

A cover doesn’t include an ordering its branch points. Adding such an ordering would destroy most applications number theorists care about. This makes sense of saying a cover is in the Nielsen class \( \text{Ni}(G, C) \).
Explaining braid action on $\text{Ni}(G, C)^{\dagger}$: The rubric of [Fr17, §1.3.1] of dragging a cover by its branch points gives a sense of how the braid group enters. Here, though, we just give the basic facts and the actions of two generators of the Hurwitz monodromy group, $H_r$, in its action on Nielsen classes.

(1.6a) $H_r$ is the fundamental group $\pi_1(U_r, z^0)$ with $z^0 \in U_r$, a basepoint.

(1.6b) $H_r$ is generated by two elements:

$$q_i : g \overset{\text{def}}{=} (g_1, \ldots, g_r) \mapsto (g_1, \ldots, g_{i-1}, g_{i}g_{i+1}g_{i}^{-1}, g_{i}, g_{i+2}, \ldots, g_r);$$

$$\text{sh} : g \mapsto (g_2, g_3, \ldots, g_r, g_1)$$

and $H_r \overset{\text{def}}{=} \langle q_2, \text{sh} \rangle$ with

$$\text{sh} q_i \text{sh}^{-1} = q_{i+1}, \ i = 1, \ldots, r-1.$$

(1.6c) Braids, $B_r$, on $r$ strings give $H_r = B_r/\langle q_1 \cdots q_r - 1 q_r - 1 \cdots q_1 \rangle$.

The case $r = 4$ in (1.6b) is so important in examples, that in reduced Nielsen classes, we conveniently refer to $q_2$ as the middle twist. As usual, in notation for free groups modulo relations, (1.6c) means to mod out by the normal subgroup generated by the relation $q_1 \cdots q_r - 1 q_r - 1 \cdots q_1 = R_H$.

**Principle 1.5.** From (1.6), we get a permutation representation of $H_r$ on $\text{Ni}(G, C)^{\dagger}$. Given $\dagger$, that gives a cover $\Phi \overset{\text{def}}{=} \Phi^{\dagger} : \mathcal{H}(G, C)^{\dagger} \to U_r$: The Hurwitz space of $\dagger$-equivalences of covers.

The elements in $\langle q_1 \cdots q_r - 1 q_r - 1 \cdots q_1 \rangle$ have the affect

$$g \in \text{Ni}(G, C)^{\dagger} \mapsto ggg^{-1} \text{ for some } g \in G.$$ Indeed, for

$$g \in \text{Ni}(G, C), \{(g)q^{-1}R_nq \mid q \in B_r\} = \{g^{-1}gg \mid g \in G\}.$$

Circumstances dictate when to identify covers $i : \mathcal{H} \to \mathbb{P}^1_z$, $i = 1, 2$, branched at $\infty z$, obtained from any one cover using the dragging-branchpoints principle. One might regard Inner (resp. Absolute) equivalence as minimal (resp. maximal). Act by $H_r$ on either equivalence $\text{Ni}(G, C)^{\dagger}$.

### 1.3. Reduced Nielsen Classes and cusp orbits.

The Möbius transformations $\text{PSL}_2(\mathbb{C})$ act on $\mathbb{P}^1_\mathbb{C}$.

**Definition 1.6** (Reduced action). A cover $f : W \to \mathbb{P}^1_\mathbb{C}$ is reduced equivalent to $\alpha \circ f : W \to \mathbb{P}^1_\mathbb{C}$ for $\alpha \in \text{PSL}_2(\mathbb{C})$.

Also, $\alpha$ acts on $z \in U_r$ by acting on each entry. That extends to an action on any cover $\Phi^{\dagger} : \mathcal{H}(G, C)^{\dagger} \to U_r$, giving a reduced Hurwitz space cover:

$$\mathcal{H}(G, C)^{\dagger, \text{rd}} \to \mathbb{P}^1_{\mathbb{C}}/\text{PSL}_2(\mathbb{C}) \overset{\text{def}}{=} J_r.$$

#### 1.3.1. Genus formula for $r = 4$.

When $r = 4$, $U_r/\text{PSL}_2(\mathbb{C})$ identifies with $\mathbb{P}^1_\mathbb{C} \setminus \{\infty\}$. A reduced Hurwitz space of 4 branch point covers is a natural $j$-line cover. That completes to $\overline{\mathcal{H}}(G, C)^{\dagger, \text{rd}} \to \mathbb{P}^1_\mathbb{C}$ ramified over $0, 1, \infty$. Denote the group $\langle q_1 q_3^{-1}, \text{sh}^2 \rangle$ by $Q''$.  


Theorem 1.7. Suppose a component, $\overline{H}$, of $\overline{\mathcal{H}}(G, C)^{\dagger, \text{rd}}$ is given by a braid orbit, $O$, on the corresponding Nielsen classes $\text{Ni}(G, C)^{\dagger, \text{rd}}$. Then, the ramification, respectively over $0, 1, \infty$, of $\overline{H} \to \mathbb{P}^1$ is given by the disjoint cycles of $\gamma_0 = q_1q_2$, $\gamma_1 = q_1q_2q_1$, $\gamma_\infty = q_2$ acting on $O$.

The genus, $g_{\overline{H}}$, of $\overline{H}$, a la Riemann-Hurwitz, appears from

$$2(|O| + g_{\overline{H}} - 1) = \text{ind}(\gamma_0) + \text{ind}(\gamma_1) + \text{ind}(\gamma_\infty).$$

2. sh-incidence Algorithm

The sh-incidence matrix entwines the interaction of the braids and the group $G$, showing up in invariants of components of Hurwitz spaces through a labeling on cusps.

2.1. Twist orbits. The sh-incidence matrix is given in Def. 2.4. It is based on cusp orbits, which are listed in Lem. 2.2. Def. 2.1 gives the most easily identified Nielsen class elements defining a very common type of cusp orbit. It also turns out to be problematic, though rare, when more than two braid orbits contain such cusps. The name comes from combining the phenomena of [Ha84] and [Mu72] that appears on the boundary of a Hurwitz space component containing an HM rep. The significance was that under mild explicit conditions this appearance allowed showing the moduli definition field of such a component would be $\mathbb{Q}$, even if there was more than one component, as in [Fr95, Thm. 3.21].

Definition 2.1. An element $g \in \text{Ni}(G, C)$ is a Harbater-Mumford (HM) representative if it has the form $(g_1, g_1^{-1}, \ldots, g_s, g_s^{-1})$ (so $2s = r$). A braid orbit $O$ is said to be HM, if the orbit contains an HM rep.

The genus formula is key to the proven cases of the Main Conjecture on $\text{M}(\text{odular}) \text{T(ower)s}$. That high levels have general type: the canonical class (of holomorphic differentials) is ample. The proven case is when $r_C \leq 4$. This implies in particular that high tower levels have no rational point.

As a special case, high levels modular curve towers have no rational points, but as said previously the modular curve case is to MTs as the dihedral group $D_\ell$ and the prime $\ell$ are to all finite groups and primes dividing their orders. To initially address the Main Conjecture, [BFr02] inspected the properties of the MT for $\text{Ni}(A_5, C_{3^4}), \ell = 2$, with $C$ consisting of four repetitions of the 3-cycle class.
There is a superficial similarity between that Nielsen class, with one braid orbit, and \( \text{Ni}(A_4, C_{34}) \) (also with \( \ell = 2 \)) with two braid orbits. Both have braid orbits with \( \text{HM} \) reps. Yet, both are one case of a natural series of Nielsen classes, very different in their behavior. We will use them as an example for how to engage the \( \text{sh} \)-incidence algorithm.

Lem. 2.2, for all \( r \), lists \( q_i \) orbits. Note though, one index \( i \) suffices for the \( \text{sh} \)-incidence matrix. For historical reasons we choose \( i = 2 \).

**Lemma 2.2.** Let \( g \in \text{Ni}(G, C) \) be a Nielsen class representative. With \( \mu = g_i g_{i+1} \), the orbit of \( Q_i \) on \( g \) is the collection

\[
(g)q_i^k = \begin{cases} (g_1, \ldots, g_{i-1}, \mu^l g_i \mu^{-l}, \mu^l g_{i+1} \mu^{-l}, g_{i+2}, \ldots) & \text{for } k = 2l \\ (g_1, \ldots, g_{i-1}, \mu^l g_i g_{i+1} \mu^{-l}, \mu^l g_{i+2} \mu^{-l}, g_{i+2}, \ldots) & \text{for } k = 1 + 2l. \end{cases}
\]

If \( r = 4 \), and \( g \in \text{Ni}(G, C) \) is an \( \text{HM} \) rep. \( (g_1, g_1^{-1}, g_2, g_2^{-1}) \), then so are all elements in the \( Q'' \) orbit of \( g \). Then, the \( Q'' \) orbit length on \( Q_g \) is

\[ 4/(m+1) \]

with \( m \) the count of conjugacies, in \( \dagger \) equivalence, given by

\[ h g_1 h^{-1} = g_2 \text{ or } g_1 = h g_2^{-1} h^{-1} \text{ with } h^2 = 1; \text{ or } h(g_1, g_2)h^{-1} = (g_1^{-1}, g_2^{-1}). \]

If \( G \) (resp. \( N_{S_n}(G, C) \)) has no center (resp. element that centralizes \( G \)) for \( \dagger = \text{in} \) (resp. \( \dagger = \text{abs} \)) then \( h \) is determined by these conditions.

**Proof.** The formula of the 1st paragraph comes from the definition of \( q_i \). Similarly, if you apply \( q_1 q_3^{-1} \) or \( \text{sh}^2 \) to an \( \text{HM} \) rep, then it is immediate the result is another \( \text{HM} \) rep.

Now consider the \( Q'' \) orbit length, which is 4 divided by the number of elements \( q \in Q'' \) for which \( (g)q = hgh^{-1} \) for some \( h \in G \) (resp. \( h \in N_{S_n}(G, C) \) if \( \dagger = \text{in} \) (resp. \( \dagger = \text{abs} \)). The cases are similar, so we will just do the 1st.

If \( h^2 = 1 \) for which \( g_1 = h g_2 h^{-1} \), then \( g_1^{-1} = h g_2^{-1} h^{-1} \) and

\[
h((g)\text{sh}^2)h^{-1} = h(g_2, g_2^{-1}, g_1, g_1^{-1})h^{-1} = g.
\]

If \( h_1, h_2 \) both satisfied these conditions then \( h_1 h_2^{-1} \) would commute with \( g \), contrary to the centralizing assumption. Either one or all three of those conditions are satisfied. Add that number to 1 (for the trivial element in \( Q'' \)) to get \( m \).

**Remark 2.3.** General expectation from Lem. 2.2 is that \( Q_2 \) orbits would have length \( 2 \cdot \text{ord}(g_i g_{i+1}) \equiv 2 \). There is an important exception –Prop. 2.10 – for which the orbit length (even without concerns about centralizers) is half that expectation. The first condition is that \( o \) is odd. It applies to the cusp labeled \( \circ O_{1,3}^3 \) in the \( \text{Ni}_0^+ \) block of the case \( \text{Ni}(A_4, C_{34}) \) §3.1.
2.2. **Listing cusps for the sh-incidence matrix.** Now we give the algorithm using Def. 2.4, the sh-incidence matrix, for computing braid orbits on specific reduced Nielsen classes.

For, $S$, a set of representatives in $\text{Ni}(G, C)$ and any equivalence relation $\bullet$ on the Nielsen class, denote by $S^{q_2, \bullet}$ (resp. $S^{\text{sh}, \bullet}$) the collection of $\bullet$ equivalence classes of $q_2$ (resp. sh) orbits.

We have been using $O$ for braid orbits on a Nielsen class. For $q_2$ (cusp) orbits the notation will be $cO$, with the understanding that $\bullet$-equivalence has been specified.

If $r = 4$ and $\bullet$ is one of the reduced equivalences, then sh has order 2, thereby producing a symmetric matrix.

**Definition 2.4.** List $\bullet$-equivalence classes, $cO_1, \ldots, cO_u$, of $q_2$ (cusp) orbits. The sh-incidence matrix $A(G, C)$ has $(i, j)$ term $|(O_i)\text{sh} \cap O_j|$.

Denote the transpose of an $n \times n$ matrix by $^\text{tr}T$. Equivalence $n \times n$ matrices $A$ and $TA^\text{tr}T$ running over permutation matrices $T$ associated to elements of $S_n$. Refer to a matrix $A$ as in **block form** if there are matrices $B_1, \ldots, B_u$, for which $A$ has the form of a $u \times u$ diagonal matrix with diagonal entries $B_1, \ldots, B_u$. The proof of Lem. 2.5 is the algorithm. Rem. 2.7 and Rem. 2.9 have extra comments.

**Lemma 2.5.** For some $T$, $A(G, C)$ is in block form with the block rows (and columns) labeled by cusp orbits whose union of elements consists of a single braid orbit on $\text{Ni}(G, C)$ under $\bullet$-equivalence.

**Proof.** Start with any $q_2$ orbit and label it $cO_{1,1}$. Then form the sequence

\[ cO_{1,1} \rightarrow (cO_{1,1}^{\text{sh}, \bullet})^{q_2, \bullet} \rightarrow ((cO_{1,1}^{\text{sh}, \bullet})^{q_2, \bullet})^{q_2, \bullet} \ldots \]

until the sequence stops. The result will be a union of distinct $q_2$ orbits under $\bullet$-equivalence.

Denote this collection by $c\mathcal{O}_1 = \{cO_{1,1}, \ldots, cO_{1,b_1}\}$. Since $H_r = \langle q_2, \text{sh} \rangle$, together $c\mathcal{O}_1$ contains all elements – modulo $\bullet$-equivalence – in the Nielsen class that are in the $H_r$ orbit of any element of $cO_{1,1}$.

Label the rows and columns of the first block of your matrix, $B_1$, by the elements of $c\mathcal{O}_1$. In step 1 of (2.1) you iterate applications of sh on $cO_{1,1}$ and check for all new $q_2$ orbits. The $(i, j)$-entry is $|(cO_{1,i})\text{sh} \cap cO_{1,j}|$. Call $cO_i$ and $cO_j$ **neighbors** if $|(cO_{1,i})\text{sh}^t \cap cO_{1,j}|$ is nonzero for some $t$. If the process stops after one step, then all $q_2$ orbits are neighbors of $cO_{1,1}$. In step 2 you do the same thing to any of the new $q_2$ orbits, etc., until you stop getting new $q_2$ orbits. The resulting sh-incidence matrix is obtained from a maximal sequence of neighbors.
Therefore, this gives exactly one block, $B_1$ as described in the opening paragraph of the lemma. If further $q_2$ orbits in the Nielsen class haven’t been used, then start again until you have used them all. Eventually you get blocks $B_1, \ldots, B_u$ corresponding to unions of $q_2$ orbits $cO_1, \ldots, cO_u$. □

2.3. $r = 4$ and finishing the computation.

**Lemma 2.6.** Now assume $r = 4$, and $\bullet$-equivalence is one of the reduced equivalences, so $q_2$ orbits are $\gamma_\infty$ orbits. Then the $\text{sh}$-incidence matrix is symmetric.

Replacing $\text{sh}$ by either $\gamma_0$ or $\gamma_1$ acting on $\gamma_\infty$ orbits gives the same blocks in the resulting matrix. Then, fixed points of $\gamma_j$, $j = 0$ or $1$, on any $\tilde{M}_4$ orbit give nonzero entries along the diagonal of the corresponding block. Then,

$$|(cO_{1,t})\text{sh} \cap cO_{1,t'}| = |(cO_{1,t})\gamma_0^2 \cap (cO_{1,t'})\text{sh}| = |cO_{1,t} \cap (cO_{1,t'})\text{sh}|.$$ 

That shows the final matrix in block form, has each block symmetric.

**Proof.** Take $r = 4$ and for $\bullet$-equivalence one of the reduced equivalences, where $\text{sh}^2$ is the identity on braid orbits.

Use that on reduced classes $q_1 = q_3$, with relation (??), $q_1q_2q_1 = q_2q_1q_2$. Now consider what happens if we replace $\text{sh}$ by $\gamma_1$ represented by $q_1q_2$. Since we start with a $q_2$ orbit, say $cO$, the collection with $\text{sh}$ applied is given by

$$(cO)\text{sh} = (cO)q_1q_2q_1 = (cO)q_2q_1q_2 = (cO)q_1q_2.$$ 

That shows the matrix is the same with $\gamma_0$ replacing $\text{sh}$. Of course, $\gamma_1$ is $\text{sh}$. That finishes the proof of the lemma. □

**Remark 2.7 (The algorithm–Part 1).** Regard the sequence of expression (2.1) as iterated steps – it shows two steps – in computing one braid orbit on the $\bullet$-equivalence classes on $\text{Ni}(G, C)$. Our examples often have one step.

Notice in our $\text{Ni}(A_4, C_{4,32-32})$ example §3.1, the seed for each braid orbit– respectively an $\text{HM}$ rep. Def. 2.1, and a $\text{D(ouble)}\text{I(dentity)}$ rep. – end up giving their Hurwitz space components corresponding monikers.

It makes sense to list $q_2$ orbits, referring to the blocks, as

$$cO_1^{w_{1,1}}, \ldots, cO_1^{w_{1,b_1}}, cO_2^{w_{2,1}}, \ldots, cO_2^{w_{2,b_2}}, \ldots, cO_u^{w_{u,1}}, \ldots, cO_u^{w_{u,b_u}},$$

with the superscript $w_{i,j}$ the cusp width (cusp orbit length). Still, should there be more than one step, labeling within any one block probably should correspond to the step in which it appears. Especially if the seed orbit has been chosen well.
Lemma 2.8. For \( r = 4 \), in the \( i \)th block of the \( \text{sh} \)-incidence matrix, we can read off the degree of the component \( \mathcal{H}_i \) over \( \mathbb{P}^1 \) as \( \sum_{j=1}^{b_i} w_{i,j} \). Further,

\[
   w_{i,j} = \sum_{k=1}^{b_i} |(cO_{i,j}) \cap O_{i,k}|, \quad j = 1, \ldots, b_i.
\]

Proof. This follows by recognizing the cusps as the \( q_2 \) orbits on the \( i \)th braid orbit of the Nielsen classes under reduced equivalence. This is a piece of the proof of Thm. ???. The rest is from the combinatorics of the \( \text{sh} \)-incidence matrix.

Remark 2.9 (The algorithm-Part 2). Fixed points of \( \gamma_0 \) or \( \gamma_1 \) on a reduced orbit imply that the reduced Hurwitz space component does not have fine moduli. We can almost read that data off directly from the \( \text{sh} \)-incidence matrix. If there are no nonzero diagonal elements corresponding to that block in Lem. 2.5, then for certain \( \gamma_0 \) or \( \gamma_1 \) have no fixed point, and then the only test necessary for fine moduli is that \( \mathcal{Q}'' \) acts on that orbit as a Klein 4-group.

Yet, if there diagonal elements aren’t all 0, there may, or not, be \( \gamma_0 \) or \( \gamma_1 \) fixed points. Both cases occur in the \( \text{Ni}(A_4, C_{4,2}) \) §3.1.

For \( g_1, g_2 \) in a group, denote the centralizer of \( \langle g_1, g_2 \rangle \) by \( Z(g_1, g_2) \).

Let \( g_1g_2 = g_3 \), and \( g_2g_1 = g'_3 \). Let \( o(g_1, g_2) = o \) (resp. \( o'(g_1, g_2) = o' \)) be the length of the orbit of \( \gamma^2 \) (resp. \( \gamma \)) on \( (g_1, g_2) \). If \( g_1 = g_2 \), then \( o = o' = 1 \).

Proposition 2.10. Assume \( g_1 \neq g_2 \). The orbit of \( \gamma^2 \) containing \( (g_1, g_2) \) is

\[
   (g_3^{-1}g_1g_3^{-1}, g_3^{-1}g_2g_3^{-1}), \quad j = 0, \ldots, \text{ord}(g_3) - 1. \quad \text{So,}
   o = \text{ord}(g_3) / |\langle g_3 \rangle \cap Z(g_1, g_2)| \overset{\text{def}}{=} o(g_1, g_2).
\]

Then, \( o' = 2 \cdot o \), unless \( o \) is odd, and with \( x = (g_3)^{(o-1)/2} \) and \( y = (g_3)^{(o-1)/2} \)

\[
   (2.2) \quad (\text{so } g_1y = xg_1 \text{ and } yg_2 = g_2x), \ yg_2 \text{ has order } 2 \text{ and } o' = o.
\]

Proof. For \( t \) an integer,

\[
   (g_1, g_2)^{2t} = (g_3^{-1}g_1g_3^{-1}, g_3^{-1}g_2g_3^{-1}) \text{ and } (g_1, g_2)^{2t+1} = (g_3g_1g_2g_3^{-1}g_3^{-1}, g_3g_1g_1g_3^{-1}g_3^{-1}).
\]

The minimal \( t \) with \( (g_1, g_2)^{2t} = (g_1, g_2) \) is \( o(g_1, g_2) \). Further, the minimal \( j \) with \( (g_1, g_2)^j = (g_1, g_2) \) divides any other integer with this property. So \( j \text{ divides } 2o(g_1, g_2) \) and if \( j \) is odd, \( j \text{ divides } o(g_1, g_2) \).

From the above, if the orbit of \( \gamma \) does not have length \( 2o(g_1, g_2) \), it has length \( o(g_1, g_2) \). Use the notation around (2.2). The expressions \( g_1y = xg_1 \) and \( yg_2 = g_2x \) are tautologies. If \( o \) is odd, then \( (g_1, g_2)g_2^o = x(g_1, g_2)g_2x^{-1} \).

Assume this equals \( (g_1, g_2) \), which is true if and only if \( xg_1 = g_2x = yg_2 \). The
expression \((g_1g_2)^o = 1\) and \(xg_1yg_2 = 1\) are equivalent. Conclude \((yg_2)^2 = 1\).

So long as the order of \(yg_2\) is not 1, this shows (2.2) holds. If, however, \(yg_2 = xg_1 = g_2x = g_1y = 1\), then \(g_1 = g_2\), contrary to hypothesis.

This reversible argument shows the converse: \((g_1, g_2)q_2 = (g_1, g_2)\) follows from (2.2). This concludes the proof. □

3. Examples illustrating concepts of \(\S 2\)

With each example we comment on the nature of the cusps that occur in each orbit.

3.1. Braid orbits on \(\text{Ni}(A_4, C_{3^4})^*\).

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