WEIL'S DECOMPOSITION THEOREM, SIEGEL'S THEOREM AND NÉRON DIVISORS

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ABSTRACT. Let $\mathcal{C}$ be a nonsingular projective curve over a number field K (with ring of integers O_K) having genus $g = g(\mathcal{C})$. Let $O_{K,a}$ be the localization of O_K at $a \in K^*$. Suppose $z : \mathcal{C} \to \mathbb{P}^1_z$ is a nonconstant function. We outline a version of Siegel's Theorem on the finiteness of rational points on $\mathcal{C}$ at which $z(\boldsymbol{p}) \in O_{K,a}$ for infinitely many $\boldsymbol{p} \in \mathcal{C}(K)$. The conclusion is that $g(\mathcal{C}) = 0$ and there are at most two places over $\infty \in \mathbb{P}^1_z$ on $\mathcal{C}$.

This proof shows the role of Weil's Decomposition Theorem. Its first use is to allow base change of coordinates while preserving basic diophantine properties. Siegel used it to transform properties for the pair $(\mathcal{C}, z)$ to properties of new pairs $(\mathcal{C}_n, z_n)$, where n is a positive integer and $\psi_n : \mathcal{C}_n \to \mathcal{C}$ is a high degree cover constructed from the Jacobian of $\mathcal{C}$. The final contradiction giving Siegel's Theorem depends on the Thue-Siegel-Roth Theorem. The subtlety appears from an expression of the following form: $\mu n^{2g(\mathcal{C})}/n^{2g(\mathcal{C})-2} > 2$ for n sufficiently large. The expression μ is explicit. The quantity $n^{2g(\mathcal{C})-2}$ is the (sufficiently small) degree of a function Φ_1 on $\mathcal{C}_n$. The construction of Φ_1 is a nontrivial addition to the function theory of the Jacobian.

My notes come from lectures I gave at Bowdoin College, Summer of 1967, when I finished my degree. They benefited from a translation of Siegel's paper by William Leveque, while I was a graduate student at University of Michigan. I know of no appearance in English of this proof. Néron's version was what Lang used in his first edition of *Diophantine Geometry*. A sense of the Decomposition Theorem disappeared from this. Here it motivates Néron's improvement of Weil's distributions and his introduction of canonical heights.

CONTENTS

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1. Introduction to Siegel's Theorem

Let $\mathcal{C}$ be a nonsingular projective curve with infinitely many K rational places. If $g(\mathcal{C}) > 1$, this is impossible by Faltings' Theorem. We are not, however, assuming Faltings' Theorem. Rather, we follow the chain of ideas back to the 1920's. Call this collection of K rational points $\mathcal{N}$, though at any time we may replace it by a (still infinite) subset to normalize the situation. Write the function field $K(\mathcal{C})$ as $K(x, y)$, the quotient field of $K[X, Y]/(f)$ with f the equation of a plane curve birational to $\mathcal{C}$. Call $z \in K(\mathcal{C}) \setminus K$ *quasi-integral* with respect to $\mathcal{N}$ if for some $a \in K^*$, $z(\boldsymbol{p}) \in O_{K,a}$ for $\boldsymbol{p} \in \mathcal{N}$. Call z quasi-integral if there is an infinite set $\mathcal{N}$ for which z is quasi-integral with respect to $\mathcal{N}$. Tacitly, when referring to a quasi-integral function we assume $\mathcal{N}$ chosen so this property holds.

Theorem 1.1. *Suppose $z \in K(\mathcal{C})$ and z is quasi-integral with respect to an infinite set of K rational places $\mathcal{N}$. Then $g(\mathcal{C}) = 0$. Further, there at most two points on $\mathcal{C}$ above infinity for the map $z : \mathcal{C} \to \mathbb{P}^1_z$.*

1.1. Normalization conditions and basic notation.

Recall, there are r_1 real and $2r_2$ complex embeddings $\sigma : K \to \mathbb{C}$ where $r_1 + 2r_2 = [K : \mathbb{Q}] \stackrel{\text{def}}{=} l$. For any such embedding, apply σ to the places of $\mathcal{N}$ to get $\mathcal{N}^\sigma$. Without loss assume each $\mathcal{N}^\sigma$ has but one limit point $\bar{\boldsymbol{p}}_\sigma$. Further, take both x and y to be bounded (in absolute value) on each $\mathcal{N}^\sigma$. We say they are *totally bounded*. So, neither x nor y has $\bar{p}_\sigma$ as a pole. For simplicity assume $\mathcal{N}^\sigma$ contains no poles of either x or y.

Since $\mathcal{O}_K$ is a Dedekind domain (Noetherian, dimension 1 and integrally closed), each ideal decomposes into a unique product (up to order) of prime ideals: If $\mathfrak{A}$ is an ideal of $\mathcal{O}_K$, then $\mathfrak{A} = \prod \mathfrak{A}_i$. A *fractional ideal* is an element in the free abelian group generated by primes ideals. We write this multiplicatively: $\mathfrak{A} = \prod \mathfrak{A}_i / \prod \mathfrak{B}_j$.

The norm of a fractional ideal $\mathfrak{A}$ is

$$\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{A}) := \frac{\prod \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{A}_i)}{\prod \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{B}_j)}.$$

1.2. **Setup for the Decomposition Theorem.** The divisor of x is $(x) = N_x - D_x$: N_x (numerator of x) is the sum of zeros of x on $\mathcal{C}$ and D_x (denominator of x) is the sum of poles. This is not usual notation, which is $D_0 - D_\infty$. Rather, it is to indicate we use N_x as the numerator of an expression for the Weil distribution attached to the zeros of x. Let $\mathcal{D} = \mathcal{D}_{\mathcal{C},K}$ be the collection of functions from K points on $\mathcal{C}$ to integral ideals of O_K. Let $\mathcal{F}$ be the functions in $\mathcal{D}$ that are *finite* in the sense they take on only finitely many values: the *quasi-finite* functions.

There is a natural addition of functions in $\mathcal{D}$: $D_1, D_2 \in \mathcal{D}$ add to $D_1 + D_2$ with value $D_1(\boldsymbol{p})D_2(\boldsymbol{p})$ at $\boldsymbol{p}$. Extend this to differences: $D_1 - D_2$ has value $D_1(\boldsymbol{p})/D_2(p)$ at $\boldsymbol{p}$. We work within the quotient $\mathcal{D}/\mathcal{F} = \dot{\mathcal{D}}$. The relation $f \doteq g$ means $f, g \in \mathcal{D}$ have difference in $\mathcal{F}$. To each K point $\boldsymbol{p} \in \mathcal{C}(K)$, Weil attached a function we denote also by $\boldsymbol{p} \in \dot{\mathcal{D}}$ (see §4). It interpreted the values of functions arithmetically in a style similar to how we express Riemann's generalization of Abel's Theorem.

Theorem 1.2 (Weil's Decomposition Theorem)**.** *If $z \in K(\mathcal{C})$, then*

$$(z(p)) \doteq \frac{\prod_{i=1}^{\deg(z)} p_i(p)}{\prod_{i=1}^{\deg(z)} q_i(p)}$$

where p_i and q_i are the zeros and poles of z, respectively.

Combining the production of distributions with the Riemann-Roch Theorem shows the growth of the distribution of divisors on the set $\mathcal{C}(K)$ depends up to an error estimate only on the degree of the divisor. It is the roughness of this error estimate that Néron improved.

Recall $[K(\mathcal{C}) : K(x)] = \deg(x)$. The notation $A(x) \ll B(x)$ means there is a constant M so that $A(x) < MB(x)$ for all x.

Theorem 1.3. *For any $\delta > 0$, there exists an infinite subsequence $\mathcal{N}(\delta) \subset \mathcal{N}$ with*

$$\mathrm{N}_{K/\mathbb{Q}}(D_x(p))^{\frac{\deg(y)}{\deg(x)}-\delta} \ll \mathrm{N}_{K/\mathbb{Q}}(D_y(p)).$$

2. Proof of Theorem 1.3

Let $n > 0$. We produce a function ϕ_n, with divisor

$$(\phi_n) = nD_x - t_n D_y + \mathfrak{A}_n$$

where $\mathfrak{A} \geq 0$. Determine a minimal value t_n so $(\phi_n) - nD_x + t_n D_y \geq 0$. By Riemann-Roch:

$$\dim(-nD_x + t_n D_y) \geq -n\deg(x) + t_n \deg(y) + 1 - g. \tag{2.1}$$

For (2.1) to be greater than zero, choose t_n so $(n\deg(x) + 1 - g)/\deg(y) \leq t_n$. A suitable choice is the greatest integer in $M \stackrel{\text{def}}{=} \frac{n\deg(x)+g+\deg(y)}{\deg(y)}$. Then,

$$(\phi_n(\boldsymbol{p})) \doteq \frac{D_x(\boldsymbol{p})^n \mathfrak{A}_n(\boldsymbol{p})}{D_y(\boldsymbol{p})^M}.$$

Notice this says ϕ_n is integrally dependent on y. Since y is totally bounded, so is ϕ_n. So, $\mathrm{N}_{K/\mathbb{Q}}(\phi_n(\boldsymbol{p})) \ll 1$ on a subsequence of $\mathcal{N}$. Since $1 \ll \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{A}_n(\boldsymbol{p}))$, $\mathrm{N}_{K/\mathbb{Q}}(D_x(p))^n \ll \mathrm{N}_{K/\mathbb{Q}}(D_y(p))^M$. Now take nth roots. For any $\delta > 0$, if n is sufficiently large, $\mathrm{N}_{K/\mathbb{Q}}(D_x(p))^{\deg(y)/\deg(x)-\delta} \ll \mathrm{N}_{K/\mathbb{Q}}(D_y(p))$, the statement of the theorem.

2.1. **Quasi-integral z produces σ with $\bar{p}_\sigma$ algebraic.** As above, $\bar{p}_\sigma$ is a limit point of $\mathcal{N}^\sigma$. Let $z \in K(\mathcal{C})$ be quasi-integral on $\mathcal{N}$.

Theorem 2.1. *For some $\sigma : K \to \mathbb{C}$, $\bar{p}_\sigma$ is a pole of z^σ. So, $\bar{p}_\sigma$ is algebraic. Extend K to L to include coordinates for $\bar{p}_\sigma$. Assume j is the order of the pole of z^σ at $\bar{p}_\sigma$. Find $\phi \in L(\mathcal{C})$ that is 0 at $\bar{\boldsymbol{p}}_\sigma$. Then, there is an infinite subsequence $\mathcal{N}' \le \mathcal{N}$ so that for $\boldsymbol{p} \in \mathcal{N}$,*

$$|\phi(\boldsymbol{p}^\sigma)| \ll \frac{1}{\mathrm{N}_{K/\mathbb{Q}}(D_x(\boldsymbol{p}))^{\deg(z)/2l\deg(x)}}.$$

Proof. Replace z by a translate to assume $z^\sigma(\bar{\boldsymbol{p}}_\sigma) \ne 0$ for all σ. Set $y = 1/z$. Then, $y(\boldsymbol{p})$ is totally bounded.

Since z is quasi-integral, $D_z(\boldsymbol{p})$ is quasi-finite and $N_y(\boldsymbol{p})$ is quasi-finite. So,

$$\frac{1}{\mathrm{N}_{K/\mathbb{Q}}(D_y(\boldsymbol{p}))} = \frac{1}{\mathrm{N}_{K/\mathbb{Q}}((1/z(\boldsymbol{p})))} \ll \frac{1}{\mathrm{N}_{K/\mathbb{Q}}(D_x(\boldsymbol{p}))^{\frac{\deg(y)}{\deg(x)}-\delta}}.$$

Then, there exists $\sigma : K \to \mathbb{C}$ so that

$$\left|\frac{1}{z^\sigma(p^\sigma)}\right| \ll \frac{1}{\mathrm{N}_{K/\mathbb{Q}}(D_x(\boldsymbol{p}))^{\frac{\deg(y)}{l\deg(x)}-\delta}}. \tag{2.2}$$

By assumption, x is totally bounded. It can't be quasi-integral on $\mathcal{N}$ or it would take only finitely many values on $\mathcal{N}$; it would be a constant function. So, the righthand side of (2.2) goes to zero. This implies $z^\sigma(\bar{\boldsymbol{p}}_\sigma) = \infty$.

If $\phi \in L(\mathcal{C})$ vanishes at $\bar{\boldsymbol{p}}_\sigma$, then $\phi^j z^\sigma$ is bounded on $\mathcal{N}'^\sigma$. So,

$$|\phi(\boldsymbol{p}^\sigma)| \ll \frac{1}{\mathrm{N}_{K/\mathbb{Q}}(D_x(\boldsymbol{p}))^{\deg(z)/jl\deg(x)-\delta'}}$$

for $\boldsymbol{p} \in \mathcal{N}'$. Take $\delta' = \frac{\deg(y)}{2lj\deg(x)}$. □

3. Contradiction to the Thue-Siegal-Roth theorem

In Thm. 2.1 make the following replacements: K^σ by K, z by x and $\mathcal{C}$ by $\mathcal{C}_n$ with $K(\mathcal{C}_n) = K(\Phi_1, \Phi_2)$ with $[K(\mathcal{C}_n) : K(\Phi_i] \le \deg(x)g^3n^{2g-2}$ and Φ_i totally bounded.

3.1. **Properties of $\mathcal{C}_n$.** Siegel chose a maximal unramified geometrically abelian cover $\mathcal{C}_n \to \mathcal{C}$ with group of exponent n. So, on $\mathcal{C}_n$, x will have degree $\deg(x)n^{2g}$. Replace $\bar{\boldsymbol{p}}_\sigma$ by $\bar{\boldsymbol{p}}_n$. The pole $\bar{\boldsymbol{p}}_n = \bar{\boldsymbol{p}}$ of x on $\mathcal{C}_n$ has order at most $\deg(x)$ because $\mathcal{C}_n \to \mathcal{C}$ is unramified. Note: $\Phi_1(\bar{\boldsymbol{p}})$ is algebraic since $\bar{\boldsymbol{p}}$ is. For ϕ choose $\Phi_1 - \Phi_1(\bar{\boldsymbol{p}})$.

In §3.4 we find $[L : K] < \infty$ so there is an infinite system of L points $\mathcal{N}_n$ on $\mathcal{C}_n$ lifting the points $\mathcal{N}$. To get to the contradiction of TSR, we also want the values of distributions to be principal ideals. Assume we have found such an L.

If all ideals of K become principal in L we can write $\Phi_1(\boldsymbol{p})$, $\boldsymbol{p} \in \mathcal{N}_n$ as $\rho(\boldsymbol{p})/\eta(\boldsymbol{p})$ with $\rho(\boldsymbol{p}), \eta(\boldsymbol{p}) \in \mathcal{O}_L$ and $(\rho(\boldsymbol{p}), \eta(\boldsymbol{p})) = 1$. Apply Thm. 2.1:

$$|\Phi(\bar{p}_\sigma) - \rho(\boldsymbol{p})/\eta(\boldsymbol{p})| \ll \frac{1}{|\mathrm{N}_{L/\mathbb{Q}}(\eta(\boldsymbol{p}))|^{\deg(x)n^{2g}/l\deg(x)^2g^3n^{2g-2}}}.$$

Write the exponent on the right as μn^2 with μ a constant independent of n.

3.2. **Translating to heights of polynomials.** The height of a polynomial with integer coefficients is the maximum of the absolute value of its coefficients. The height of an algebraic number is the height of its minimal polynomial (with integer coefficients). Then, $\rho(\boldsymbol{p})/\eta(\boldsymbol{p}) = \rho/\eta$ satisfies an equation $\prod_\sigma(\eta^\sigma t - \rho^\sigma) = f(t) = 0$.

The coefficients of $f(t)$ are symmetric functions in $\{\rho^\sigma/\eta^\sigma\}_{\sigma:L\to\bar{\mathbb{Q}}}$ multiplied by $|\mathrm{N}_{L/\mathbb{Q}}(\eta)|$. The expression $\prod_\sigma(|\eta^\sigma| + |\rho^\sigma|)$ bounds the sum of all the coefficients. So, from Φ_1 being totally bounded,

$$H(\rho/\eta) \le \frac{\prod_\sigma(|\eta^\sigma| + |\rho^\sigma|)}{\mathrm{N}_{L/\mathbb{Q}}(\eta)}\mathrm{N}_{L/\mathbb{Q}}(\eta) \ll \mathrm{N}_{L/\mathbb{Q}}(\eta). \tag{3.1}$$

3.3. **Review of the contradiction that gives Siegel's Theorem.** Invert the relation above, and conclude

$$|\Phi(\bar{p}_\sigma) - n/\rho| \ll \frac{1}{H(\eta)^{\mu n^2}}.$$

For n large this contradicts TSR.

Let us go through that contradiction in more detail. Our assumption was that $\mathcal{C}$ has an infinite set of K rational points $\mathcal{N}$ on which x is quasi-integral. This led to the conclusion that $\mathcal{N}$ has an algebraic limit point. Then, assuming $\mathcal{C}$ has genus at least one, allows for each n constructing a finite extension L_n/K and a sequence of covers $\mathcal{C}_n$ of degree $n^{2g(\mathcal{C})}$ over $\mathcal{C}$, with a bunch of L_n rational points. Crucially, $\mathcal{C}_n$ has generators Φ_1, Φ_2 of relatively low degree an^{2g-2} to which we apply Thm. 2.1. That is, $\Phi_1(\boldsymbol{p}) - \Phi_1(\bar{\boldsymbol{p}})$ totally bounded for $\boldsymbol{p} \in \mathcal{N}_n$, and this gives $\Phi_1(\boldsymbol{p}) \in L$ as a good approximation to the algebraic number $\Phi_1(\bar{\boldsymbol{p}})$.

The contradiction is that the approximation is too good. It violates a generalization of Roth's Theorem which we now state formally here.

Theorem 3.1 (Lang-Leveque version of Roth's Theorem). *Suppose α is algebraic and not in the number field L. For any $\epsilon > 0$, there are only finitely many solutions of the equation $|\alpha - \beta| \ll \frac{1}{H(\beta)^{2+\epsilon}}$ as β runs over L.*

3.4. **Extending K to a suitable field L.** Suppose h is the order of the ideal class group of a number field K, $r_1 + r_2 - 1 = r$ is the rank of the group of units U_K of $\mathcal{O}_K$. Let $\mathcal{A}_1, \dots, \mathcal{A}_h$ be ideal representatives of the ideal class group. Then $\mathcal{A}_i^h = (\alpha_i)$. Let $u_1, \dots, u_r$ be representatives of U_K/U_K^h. Form the field

$$L_K = K((\alpha_i u_j)^{\frac{1}{h}}, 1 \le i \le h, 1 \le j \le r).$$

Lemma 3.2. *Each ideal of K becomes principal in L_K.*

Let $J_\mathcal{C}$ be the Jacobian variety of the curve $\mathcal{C}$. Recall it is birational to the projective g-dimensional variety $\mathcal{C}^g/S_g$.

Lemma 3.3. *Each point of $J_\mathcal{C}$ has a representation of the form $\sum_{i=1}^g \boldsymbol{p}_i$ of g points on $\mathcal{C}$. Some points have many representatives of this form, though generically the representation is unique.*

As usual L is a number field. The Mordell-Weil Theorem says the points of $J_\mathcal{C}(L)$ are finitely generated. Denote multiplication by n on $J_\mathcal{C}$ by $[n]$. Suppose $\mathcal{C}(L) \subset J_\mathcal{C}(L)$ has an infinite number of points $\mathcal{N}$. Then, for a given integer n, some coset of $[n]J_\mathcal{C}(L)$ contains an infinite number of points from $\mathcal{C}(L)$. Let $a + [n]J_\mathcal{C}(L)$ be such a coset.

Lemma 3.4. *Let $\mathcal{C}_n$ by the pullback by $[n]$ in $J_{\mathcal{C}}$ of the curve $\mathcal{C} - a$. Then, $\mathcal{C}_n(L)$ has an infinite set of points $\mathcal{N}_n$ lying above the points of $\mathcal{C}$.*

3.5. **Generators of the function field of $\mathcal{C}_n$.** Return to the functions x and y in $\mathcal{C}$ from our beginning discussion. Here is Siegel's way of finding functions on $\mathcal{C}_n$.

Use the notation above for representing points $\boldsymbol{q}_n \in \mathcal{C}_n$ as $\sum_{i=1}^{g} \boldsymbol{p}_i$ from points in $\mathcal{C}$. A generic point on $\mathcal{C}_n$ is suitably generic for this representation to be unique (yes, it requires proof, though it is easy). We use the functions

$$\Phi_{t,a,b,c}(\shortparallel_n) = \prod_{j=1}^{g} \Big(t + \frac{1}{ax(\boldsymbol{p}_j) + by(\boldsymbol{p}_j) + c}\Big).$$

Lemma 3.5. *For $t, a, b, c \in \mathbb{C}$ general, $[K(\mathcal{C}_n) : K(\Phi_{t,a,b,c})] = \deg(x) g^3 n^{2g-2}$.*

Let Φ_1 be one general choice, and then choose Φ_2 general enough to have distinct values on some fiber of $\Phi_1 : \mathcal{C}_n \to \mathbb{P}^1$. Then, $K(\Phi_1, \Phi_2) = K(\mathcal{C}_n)$.

Proof. The second statement is a general fact about finding functions to generate a function field. Given one, find another that has distinct values on the fiber map from the first.

For the first statement, count the number of poles of $\Phi_{t,a,b,c}$. These are from zeros of $ax(\boldsymbol{p}_j) + by(\boldsymbol{p}_j) + c$ on $\mathcal{C}$. [I haven't filled in the complete proof here.] □

4. Distributions and Weil functions

Here let $\mathcal{V}$ be a general nonsingular projective variety, though for most of what is done, it suffices for $\mathcal{V}$ to be normal. Again, K is a number field.

4.1. **Néron divisors on $\mathcal{V}(K)$.** Let M_K denote the valuations of K.

Example 4.1. Each prime p of $\mathbb{Q}$ defines a valuation by $|p^r m/n|_p = \frac{1}{p^r}$ for m, n prime to p. This is the p-adic absolute value.

For any prime $\mathfrak{p}$ of a number field K, form $\nu(\mathfrak{p}) = \nu$ as follows. For $\alpha \in \mathcal{O}_K$, $(\alpha) = \mathfrak{p}^r\mathfrak{A}$ with $\mathfrak{A}$ prime to $\mathfrak{p}$, then $|\alpha|_\nu = \frac{1}{p^{r[K_\mathfrak{p}:\mathbb{Q}_p]}}$. For archimedian places you take the absolute value if the embedding is in $\mathbb{R}$, and the square of the absolute value otherwise. For a finite extension F/K, $\alpha \in K$ and ν a valuation of K there are two product formulas:

$$\prod_{\nu'|\nu, \nu' \in M_F} |\alpha|_{\nu'} = |\alpha|_\nu^{[F:K]} \text{ and } \prod_{\nu \in M_K} |\alpha|_\nu = 1.$$

For $f \in K(\mathcal{V})$ and ν a valuation of K define a map

$$\boldsymbol{p} \in \mathcal{V}(K) \to \nu(f(\boldsymbol{p})) = -\log |f(\boldsymbol{p})|_\nu \in \mathbb{R}.$$

Suppose (U, f, α) on open $U \le V$, f defines a divisor on U, $\alpha : U \times M = M(\bar{K}) \to \mathbb{R}$ is locally bounded and continuous. Call $E \le U \times M$ affine M_K-bounded if there is an affine open $U \le V \le \mathbb{P}^n$ (coords $x_1, \dots, x_n$ for U) and M_K constant γ ($\gamma(\nu) = 0$ for a.a. ν) such that

$$\max_i |x_i|_\nu = |\boldsymbol{x}|_v \le e^{\gamma(\nu)}.$$

(Additively, [La83, p. 248], $\nu(\boldsymbol{x}) \ge -\gamma(\nu)$) for all $(\boldsymbol{x}, v) \in E$.

It is M_K-bounded if it is contained in a finite union of affine M_K-bounded subsets. If $\lambda(\boldsymbol{x}, \nu)$ is independent of $\boldsymbol{x}$ and is also an M_K-constant, call $\lambda(\boldsymbol{x}, \nu)$ a constant Néron function.

Consider (U, f, α) and (U_1, f_1, α_1) compatible [La83, p. 204] if (U, f) and (U_1, f_1) are compatible, and $\nu(f/f_1(\boldsymbol{p})) = \alpha_1(\boldsymbol{p}, \nu) - \alpha(\boldsymbol{p}, \nu)$ for each ν and $\boldsymbol{p} \in U \cap U_1$. If U_is cover V, such compatible triples represent a Néron divisor. (A maximal family is a called a Néron divisor.) Néron divisors form a group.

For D a Néron divisor, represented around P by (U, f, α) and $\boldsymbol{p}$ not in (D), define $\lambda_D(P, \nu) = \nu \circ f(\boldsymbol{p}) + \alpha(\boldsymbol{p}, \nu)$. So, $\lambda_D : (V \setminus |D|) \times M \to \mathbb{R}$ [La83, p. 254].

4.2. **Weil functions.** Lang does not use the word *distribution*. Weil was a good friend of Laurent Schwartz, and a recent obituary article on Schwartz pointed out that Weil inspired some aspect of Schwartz's distributions (several years after Weil's thesis where they appeared).

Definition 4.2. Let D be a Cartier divisor: A *Weil function* associated to D is $\lambda : (V \setminus |D|) \times M \to \mathbb{R}$ with this property. If (U, f) locally represents D, then $\alpha : U \times M \to \mathbb{R}$, locally bounded and continuous, satisfies $\lambda(\boldsymbol{p}, \nu) = \nu \circ f(\boldsymbol{p}) + \alpha(\boldsymbol{p}, \nu)$ gives (U, f, α).

Defining $\alpha(\boldsymbol{p}, \nu) = \alpha_i(\boldsymbol{p}, \nu) - \nu(f/f_i(\boldsymbol{p}))$ gives a bijection between Weil functions and Néron divisors [La83, p. 255].

Proposition 4.3. *The association $D \mapsto \lambda_D$ is a group isomorphism between Néron divisor and Weil functions. Also, if V is projective, λ a Weil function whose Cartier divisor is 0, then λ is a bounded continuous function.*

If V is nonsingular and there exists a proper Zariski closed $Z \leq V$ with λ bounded on $(V \setminus |Z|) \times M$, then (λ) (the associated Cartier divisor) is 0 and λ is bounded.

Proof. Suppose w lies in a single component of the support of $D = (\lambda) \neq (0)$. Let f be a function representing this divisor around w. Then, we can approximate w by $\boldsymbol{x}$, ν-adically close to w, but not on $|Z| \cup |D|$. So, $\nu \circ f(\boldsymbol{x})$ is large. Then $\lambda(\boldsymbol{x})$ large implies $(\lambda) = 0$. □

4.3. **Positive divisors.**

Definition 4.4. For each (U, f), f is a morphism on U. If the Cartier divisor D is positive, extend λ_D to all of V using $\lambda_D(\boldsymbol{x}) = \infty$ for $\boldsymbol{x} \in D$. Then say $\lambda_D(\boldsymbol{x}, \nu)$ for $\boldsymbol{x}$ close to D is large positive to regard it as continuous [La83, p. 258].

Proposition 4.5. *If V is projective (or λ on $V \times M$ is bounded), $(\lambda) \geq 0$, then there exists γ such that $\lambda(\boldsymbol{p}, \nu) \geq \gamma(\nu)$ for all $(\boldsymbol{p}, \nu) \in V \times M$.*

Proposition 4.6 ([La83, p. 259])**.** *Let λ_i be Weil functions with $(\lambda_i) = Y + X_i$, $X_i \geq 0$ for all i, and $X_1, \ldots, X_m$ have no points in common. Then $\lambda(\boldsymbol{p}, \nu) = \inf_i \lambda_i(\boldsymbol{p}, \nu)$ (for $\boldsymbol{p}$ outside Y) is a Weil function with Cartier divisor Y.*

Proof. For given $\boldsymbol{p} \in V$, $(U_i, f_i, \alpha_i) = \lambda_i$ on U_i around $\boldsymbol{p}$, $\boldsymbol{p} \notin |X_{i(\boldsymbol{p})}|$ and $U_{\boldsymbol{p}} = \cap_i U_i \cap (V \setminus X_{i(\boldsymbol{p})})$, let $f_{\boldsymbol{p}} = f_{i(\boldsymbol{p})}$ and $\alpha_{\boldsymbol{p}} : U_{\boldsymbol{p}} \times M \to \mathbb{R}$ by

$$\alpha_{\boldsymbol{p}}(\boldsymbol{x}, \nu) = \inf_i (v \circ (f_i/f_{i(\boldsymbol{p})})(\boldsymbol{x}) + \alpha_i(\boldsymbol{x}, \nu)).$$

□

Corollary 4.7. *Suppose V is projective and $X_1, \ldots, X_m$ are positive Cartier divisors with no intersection, $(\lambda_i) = X_i$, (λ_i Weil functions), $i = 1, \ldots, m$. Then there exists γ_1, γ_2 constants such that $\gamma_1(\nu) \leq \inf_i \lambda_i(\boldsymbol{p}, \nu) \leq \gamma_2(\nu)$ for all $\boldsymbol{p}, \nu$.*

Remark 4.8. Suppose V is projective and Y is a Cartier divisor. Then, there exists $X_i \geq 0$, $i = 1, \ldots, m$, and $Y_1, \ldots, Y_n$ such that $\cap_i X_i =$, $\cap_j Y_j =$ and $Y + X_i \sim Y_j$. This gives a Weil function associated to any Cartier divisor on a projective variety. Let $Y - Y_j + X_i = (f_{ij})$. From above, there exist Weil functions so $(\lambda_j) = Y - Y_j$ and $\lambda_j = \inf_i \lambda_{ij}$ with λ_{ij} the Weil function associated to f_{ij}.

4.4. **Weil's Decomposition Theorem and heights.** Suppose V is projective and nonsingular over K (just normal?), $\phi \in K(V)$ and $(\phi) = \sum m_W W$. Then, there exists γ_1, γ_2 constants with

$$\gamma_1(\nu) + \sum m_W \lambda_W(\boldsymbol{p}, \nu) \leq \nu \circ \phi(\boldsymbol{p}) \leq \gamma_2(\nu) + \sum m_W \lambda_W(\boldsymbol{p}, \nu).$$

Suppose λ is a Weil function and F is a field with product formula. Also, $[K : F] < \infty$ and V is a variety over K.

Definition 4.9. $h_\lambda(P) = \frac{1}{[K':F]} \sum_{\nu \in M(K')]} N_\nu \lambda(\boldsymbol{p}, \nu)$ with K'/K for which $\boldsymbol{p}$ is rational, and N_ν the residue extension degree.

Definition 4.10. Weil functions λ and λ' are linearly equivalent if there exists $f \in K(V)$ such that $\lambda = \lambda' + \lambda_f + \gamma$ with γ a constant Weil function [La83, p. 263].

If $\boldsymbol{p} \notin (\lambda) \sup(\lambda')$, this implies $h_\lambda(P) = h_{\lambda'}(P) + \gamma$ with γ a constant, by product formula. If $X = (\lambda)$, $\lambda' = \lambda - \lambda_f$. Given $X = (\lambda)$ and $\boldsymbol{p}$, there exists f with $\boldsymbol{p} \notin X \setminus (f)$. Define $h_\lambda(P) = h_{\lambda'}(P)$, so h_λ depends only on the linear equivalence class of λ.

For $\phi : V' \to V$, $h_{\lambda \circ \phi}(\boldsymbol{p}) = h_\lambda(\phi(\boldsymbol{p}))$ for $\boldsymbol{p} \in V'$. If λ, λ' are Weil functions with the same Cartier divisor, then $h_\lambda - h_{\lambda'}$ is bounded on $V(\bar{F})$ $(h_\lambda \sim h_{\lambda'})$: $h_{\lambda+\lambda'} = h_\lambda + h_{\lambda'}$.

Proposition 4.11. *Using the coordinate functions to define h on projective $V \leq \mathbb{P}^n$: X a hyperplane, $h_\lambda \sim h$.*

Proof. Let $X_0, \ldots, X_n$ be the hyperplane sections corresponding to the coordinate functions. Let $(f_i) = X_i - X$ for $\boldsymbol{p} \in X$,

$$h(\boldsymbol{p}) = \frac{1}{[K' : F]} \sum_{\nu \in M(K')]} N_\nu \sup_i \log |f_i(P)|_\nu.$$

Use that $\sup_i \log |(f_i(P)|_\nu = -\inf_i \nu \circ f_i(\boldsymbol{p})$. □

So for Cartier divisors c, c', associate equivalence classes h_c up to bounded functions on $V(\bar{F})$. So $h_{c+c'} = h_c + h_{c'} + O(1)$. If c is very ample and $\phi : V \to \mathbb{P}^n$, the projective embedding, then $h_c \sim h_\phi$.

5. Metrics on a line bundle

[Vo, p. 7]: Suppose $\{s_\alpha\} = D$ is a Cartier divisor on an algebraic variety V. Then a *metric* on $\mathcal{L}_D$ is a set of nowhere vanishing $\mathcal{C}^\infty$ functions $\rho_\alpha : U_\alpha \to \mathbb{R}$ with

$$\rho_\alpha / \rho_\beta = |s_\alpha / s_\beta|^2 = |\psi_{\beta,\alpha}|^2 \tag{5.1}$$

on the overlap $U_\alpha \cap U_\beta$.

5.1. **Translation of metrics into Weil functions.** On [Vo, p. 9], in the discussion of hermitian metrics, what is needed is a definition of ρ_α, composed from algebraic functions so that it makes sense to replace $||$ by $||_\nu$ when we want to refer to the ν-adic valuation. This would produce a function $\rho_{\alpha,\nu}$, even though this still produces mild singularities.

5.2. **Weil functions and metrics.** Suppose the functions $\{s_\alpha\}_{\alpha\in I}$ (typical notation) give a section s of the line bundle $\mathcal{L}_D$ (with divisor D). For $\boldsymbol{p} \in V$, define $|s(\boldsymbol{p})|_\rho^2$ to be $|s_\alpha(P)|^2/\rho_\alpha(P)$. Note: The transition functions are the inverse of the $\psi_{\beta,\alpha}$, and so the metric cancels the transition functions, to give a section in This gives a metric measuring the value of any pair of sections. (Vojta got this quite confused: He didn't have the square on the left, and the notation $||_\rho$ indicating it is a definition.)

The standard metric on $V = \mathbb{P}^n$ is $\rho_i(\boldsymbol{z}) = \sum_{j=0^n} |z_j/z_i|^2$ on the complement of the hyperplane $\{(\boldsymbol{z} \mid z_i = 0\}$. Let s be a section of $\mathcal{L}$ with divisor $(s) = D$. Then $\lambda_{s,\rho}(\boldsymbol{p}) = -\log|s(\boldsymbol{p})|_\rho$, $\boldsymbol{p} \notin D$. Let $\alpha_i(\boldsymbol{p}) = \log\rho_i(\boldsymbol{p})/2$ to get the corresponding Weil function from §4.2.

Definition 5.1. A canonical Néron family is an association (a homomorphism) $D \mapsto \lambda_D$ with the following properties.

(5.2a) If $D = (f)$, then $\lambda_D = v \circ f + A$ with A a constant.
(5.2b) If V has dimension 1, $\boldsymbol{p} \neq \boldsymbol{q}$, then $\lambda_{\boldsymbol{p}}(\boldsymbol{q}) = \lambda_{\boldsymbol{q}}(\boldsymbol{p})$.

We haven't yet shown how to get this canonical height, which comes from Néron's definition of a canonical height on an abelian variety. We do it for the archimedian place in §8.1. Néron defines $\lambda : \mathcal{C} \times \mathcal{C} \setminus \Delta$ and $\lambda_{\boldsymbol{p}}$ is pullback of $\boldsymbol{p} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

5.3. **Normalizing conditions for Néron functions.** [La88, p. 4]: If $D = (f)$, $\lambda_{(f),\nu} = \nu \circ f - \gamma_v(f)$. Suppose $K = \mathbb{C}$, and μ is a volume measure on $\mathcal{C}(K)$. Assume the following:

(5.3a) $\int_{\mathcal{C}(K)} d\mu = 1$; and
(5.3b) $\int_{\boldsymbol{q}\in\mathcal{C}(K)} \lambda(\boldsymbol{p},\boldsymbol{q})d\mu(\boldsymbol{q}) = 0$.

Then $\gamma_\nu(f) = \int_{\mathcal{C}(K)} -\log|f|_\nu d\mu$. [La88, p. 7] There is a bijection between Weil functions of divisor D and metrics on the bundle $\mathcal{O}_V(D)$.

6. Chern forms

Reminder from [Gu66, §7, p. 98-110]. Define the Chern class of a bundle using the short exact sequence

$$\mathbb{Z} \to \mathcal{O}_{\mathcal{C}} \xrightarrow{\exp^{2\pi i(\bullet)}} \mathcal{O}_{\mathcal{C}}^*. \tag{6.1}$$

Apply the long exact sequence of Čech cohomology, to send the transition functions of a line bundle, regarded as in $H^1(\mathcal{O}^*)$), to $H^2(\mathbb{Z})$. The image is the Chern class of the bundle.

6.1. **Differential form of Chern classes.** Extend (6.1) to $\mathcal{C}^\infty \to (\mathcal{C}^\infty)^*$ with the first a fine sheaf, so $H^2(\mathbb{Z})$ identifies with $H^1((\mathcal{C}^\infty)^*)$. Thus, the Chern class being 0 expresses topological triviality. The condition for topological (resp. analytic) triviality is that there exists a nowhere vanishing g_α with values in $\mathcal{C}^\infty$ (resp. $\mathcal{O}^*$) so the boundary gives the cocycle for the bundle.

6.2. **Laplacian and differential forms.** [La88, p. 10] Write $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}$ using $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$: $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ (apply them to z and $\bar z$). Then,

$$\partial f = \frac{\partial f}{\partial z}dz,\ \bar\partial f = \frac{\partial f}{\partial \bar z}d\bar z.$$

Then, $i\partial\bar{\partial}f = \frac{1}{2}\Delta f dx \wedge dy$, with Δf the *Laplacian*. This works to define the Laplacian on any complex manifold as a differential (1,1) form.

6.3. **A metric gives a 2-form** [Gu66, p. 101]**.** Suppose $\{\rho_\alpha\}_{\alpha\in I}$ is a metric. Form $\phi = \frac{1}{2\pi i}\partial\bar{\partial}\log\rho_\alpha$. We call this (1,1) form $c_1(\rho)$, the *Chern class form*. This is in contrast to the Chern class which is in $H^2(V,\mathbb{Z})$.

6.3.1. *A metric gives a 2-form.* Use the inverse of the transition functions $\psi_{\beta,\alpha}$ as in (5.1). Form a branch of log for $\frac{1}{2\pi i}\log\psi_{\beta,\alpha}$ for each, by assuming the cover is suitably fine. Since $\log\rho_\alpha$ is well-defined, $\log\rho_\alpha = \log\rho_\beta + \log\psi_{\beta,\alpha} + \log\bar{\psi}_{\beta,\alpha}$ (on $U_\alpha \cap U_\beta$). Note: ϕ is well-defined: $\partial\bar{\partial}\log|\phi_{\alpha\beta}|^2 = 0$.

6.3.2. *Describing the metrics.* Use any $\mathcal{C}^\infty$ differential forms τ_α where $\frac{1}{2\pi i}d\log\psi_{\beta,\alpha}$ is the boundary of τ_α. Then, $d\tau_\alpha = \phi$ is well-defined. Let $\mathcal{R}^*$ be the positive $\mathcal{C}^\infty$ functions. Then, $\log\mathcal{R}^* \to \mathcal{R}$ ($\mathcal{R}$ the sheaf of real valued functions) is an isomorphism. So, $H^1(\mathcal{R}^*) = 0$. This gives the existence of a metric by taking it as a 1-cocycle whose boundary is $|\psi_{\beta,\alpha}|^2$. Finally, Stokes Theorem [Gu66, p. 104] in the form of the residue theorem shows the degree of the bundle is what get from the Chern class of a bundle on a Riemann surface.

6.4. **A real operator and an alternative description of the Chern class.** Denote $i/4\pi(\bar{\partial}-\partial)$ by d^c. So $dd^c = i/4\pi\partial\bar{\partial}$. In polar coordinates:

$$d^c f = \frac{i}{4\pi}\frac{\partial f}{\partial r}rd\theta - \frac{1}{4\pi r}\frac{\partial f}{\partial\theta}dr.$$

[La88, p. 11] Define the Chern form of the metric ρ on the $\mathcal{C}^\infty$ bundle $\mathcal{L}$ representated by $(U_\alpha, \phi_{\alpha\beta}, \rho_\alpha)$ by $c_1(\rho) = dd^c\log(\rho_\alpha)$. Or $c_1(\rho) = -dd^c\log|s|^2_\rho$ if s is a holomorphic section.

7. Positive forms

Start with a general algebraic variety V. Call a rank k vector bundle $\mathcal{B}$ *holomorphic* if it has holomorphic transistion functions (in $\mathrm{GL}_k(\mathcal{O}_X)$).

7.1. **Hermitian inner products.** There is another bundle $\bar{\mathcal{B}}$ with transition functions that are complex conjugate to those of $\mathcal{B}$. Define a *Hermitian inner product* on $\mathcal{B}$ to be a bilinear map $\mathcal{B}\times\mathcal{B}\to\mathcal{C}^\infty_X$ by $<\boldsymbol{x},\boldsymbol{y}> = <\overline{\boldsymbol{y},\boldsymbol{x}}>$. Using the natural $^-$ map between $\mathcal{B}$ and $\bar{\mathcal{B}}$, represent $<\boldsymbol{x},\boldsymbol{y}>$ as $\boldsymbol{x}H^{\mathrm{tr}}\bar{\boldsymbol{y}}$. On basis elements locally spanning $\mathcal{B}$, restrict to a fiber $\mathcal{B}_v = \mathbb{C}$ for $v\in V$. Then, represent $<\boldsymbol{x},\boldsymbol{y}>_v$ as $\boldsymbol{x}{H_v}^{\mathrm{tr}}\bar{\boldsymbol{y}}$. Apply tr and $^-$ to this, to get $\boldsymbol{y}^{\mathrm{tr}}{\bar{H}_v}^{\mathrm{tr}}\bar{\boldsymbol{x}}$ Since the last is just $<\overline{\boldsymbol{x},\boldsymbol{y}}>_v = <\boldsymbol{y},\boldsymbol{x}>_v$, conclude $H_v = {}^{\mathrm{tr}}\bar{H}_v$: H_v is a *hermitian matrix.*

Example 7.1. Suppose $z_1,\dots,z_n$ are local complex coordinates on V. Then, the local sections of $\mathbb{T}_V = \mathcal{B}$ (the holomorphic tangent bundle) look like $\sum_{i=1}^n f_i(\boldsymbol{z})\frac{\partial}{\partial z_i}$ with the f_is $\mathcal{C}^\infty$ functions. So, $\bar{\mathbb{T}}_V$ has local sections $\sum_{i=1}^n f_i(\boldsymbol{z})\frac{\partial}{\partial\bar{z}_i}$.

From (1,1) forms in Ex. 7.1 we get a bilinear et pairing. Locally, H looks like

$$\omega = \frac{i}{2\pi}\sum h_{ij}dz_i\wedge d\bar{z}_j.$$

Call it positive if $(h_{ij}) = H$ is *Hermitian positive* (metric is positive if $c_1(\rho) > 0$): $\bar{\boldsymbol{x}}H^{\mathrm{tr}}\bar{\boldsymbol{x}}$ is positive for any complex vector $\boldsymbol{x}$. If V has dimension 1, ω is real if h is real. Then $\mathcal{L}_\rho \mapsto c_1(\rho)$ is a homomorphism from metrized line bundles to (1,1) forms. Given ρ, ρ' two metrics on $\mathcal{L}$, then $\rho' = \gamma\rho$ for some positive constant γ if they have the same Chern form.

7.2. **Properties of forms.** Let $f : V' \to V$ be a morphism. Then $f^*(\mathcal{L}_\rho)$ is semidefinite if ρ is a positive metric. If f is an embedding then $f^*(\mathcal{L}_\rho)$ is positive. If $\mathcal{L}^{\otimes m}$ has a positive metric, use $\rho^{1/m}$ as a metric on $\mathcal{L}$.

For projective space $\mathbb{P}^n$,

$$c_1(\rho) = \frac{i}{2\pi}\partial\bar\partial \log \rho(\boldsymbol{z}) = \frac{i}{2\pi\rho^2}\sum h_{ij} dz_i \wedge d\bar z_j$$

where $H = \rho(\boldsymbol{z}) - (\bar z_i z_j)$ is positive [La88, p. 14]. If $\mathcal{L}$ is ample, pullback $\mathcal{O}_\rho(1)$ to get a positive metric, and conversely: Book of Griffiths-Harris.

8. Green's Functions

Let D be a divisor on a curve $\mathcal{C}$. The θ function Lang choose here is that given by an even half-canonical divisor ([La83, p. 335]) in the usual diagram.

Definition 8.1. A Green's function G_D is a $\mathcal{C}^\infty$ except on the support of D. If e is a point on D uniformized by a variable z, the multiplicity appears in the behaviour of the Green's function, $G(D) \doteq -e\log|z|$, around the point. If $d\mu$ is the canonical volume form on $\mathcal{C}$, then

$$\frac{i}{\pi}\partial\bar\partial\, G = \deg(D)\, d\mu \text{ and } \int_{\mathcal{C}} G\, d\mu = 0.$$

The last condition eliminates the ambiguous constant.

Note: If $\deg(D) = 0$, then $\partial\bar\partial G$ is 0, and so G is harmonic except on the support of D. Two functions satisfying these conditions have difference that is harmonic everywhere, and so, using the last condition, they are equal.

More generally: If $\phi = f(z,\bar z) i dz \wedge d\bar z$, with ϕ positive, and $\int_{\mathcal{C}} \phi = 1$. A *Green's function* for D with respect to ϕ is a function $g_D : X \setminus D \to \mathbb{R}$ so

(8.1a) $g_D(\boldsymbol{p}) = -\log|f(\boldsymbol{p})|^2 + \alpha(\boldsymbol{p})$, with D is defined locally by f and α =continuous.

(8.1b) $dd^c g_D = \deg(D)\phi$ outside D.

(8.1c) $\int_X g_D\phi = 0$.

This gives $g_D = 2\lambda$ where λ is a Néron function, $g_{\boldsymbol{p}}$, $\boldsymbol{p} \in \mathcal{C}$ is the pullback from $g : \mathcal{C} \times \mathcal{C} \setminus \Delta$ with $g(\boldsymbol{p},\boldsymbol{q}) = g_{\boldsymbol{p}}(\boldsymbol{q})$.

8.1. **Néron Functions over the Complex Numbers.** [La83, p. 325] Let F_D be the normalized θ function with divisor D. Let $\mathcal{A}$ be an abelian variety. Define

$$\lambda_D(z) = -\log|F_D(\boldsymbol{z})| + \frac{\pi}{2}H(\boldsymbol{z},\boldsymbol{z}) \tag{8.2}$$

where $H(\boldsymbol{z},\boldsymbol{z}) = H_D$ is the Hermitian form attached to the canonical 2-form on $\mathcal{A}$. The same H comes with any divisor algebraically equivalent to D. (Read: Given by translation of D.)

The *normalized θ function* is meromorphic on $\mathbb{C}^n$ satisfying

$$F(\boldsymbol{z}+\boldsymbol{u}) = F(\boldsymbol{z})\exp[\pi H(\boldsymbol{z},\boldsymbol{u}) + \pi/2H(\boldsymbol{u},\boldsymbol{u}) + 2\pi\sqrt{-1}K(\boldsymbol{u})],$$

with $K(\boldsymbol{u})$ real valued [La83, p. 324]. To a divisor D on a complex torus with the Riemann period relations, there is a normalized θ called F_D. The axioms for $\lambda_D(\boldsymbol{z})$ on an abelian variety are these.

(8.3a) $D \mapsto \lambda_D$ is additive up to constants.
(8.3b) $\lambda_{D_a}(\boldsymbol{z}) = \lambda_D(\boldsymbol{z} - a) + c(a)$ for some constant $c(a)$ (invariant under translations).
(8.3c) $\lambda_D(\boldsymbol{z}) = -\log|f(\boldsymbol{z})| + \alpha(\boldsymbol{z})$ with $\alpha(\boldsymbol{z})$ continuous, locally $(f) = D$.

8.2. **Observations on Green's functions as Néron functions.** This translation invariance condition is what makes the Néron function canonical. Also, $\partial\bar{\partial}\ \lambda_D(\boldsymbol{z}) = \partial\bar{\partial}\ \frac{\pi}{2}H(\boldsymbol{z},\boldsymbol{z})$ [La83, p. 326]. The Green's function on $\mathcal{A}$ associated to D is the Néron function.

8.2.1. *Crucial points about λ_D on an abelian variety.* There are two pieces on the right side of (8.2). Despite appearances, neither is like the two pieces of a Néron function, $\lambda(\boldsymbol{p}) = \nu \circ f(\boldsymbol{p}) + \alpha(\boldsymbol{p})$. That is because each piece by itself is defined only on $\mathbb{C}^g$ and not on the complex torus analytically isomorphic to the Jacobian of $\mathcal{C}$.

The normalization (5.3b) works on an abelian variety. This is crucial to providing a family of Néron functions corresponding to a family of curves. The most direct applications we have in mind is to families of curves for which there is a canonical production of a θ divisor through half-canonical classes.

The pullback of $\psi : \mathcal{C} \to J(\mathcal{C})$ is what [La83, p. 334] uses to get the Green's function on $\mathcal{C}$. Suppose $\Phi : \mathcal{T} \to \mathcal{P}$ is a smooth family of curves. In particular, for each $x \in \mathcal{P}$, the fiber $\mathcal{T}_x$ is nonsingular curve of genus g. Applying the pullback directly even if the construction of the Green's function on the abelian varieties is canonical may fail. Reason: The failure of a section for Φ prevents applying the usual embedding of a Jacobian of a curve. The curve $\mathcal{C}$ naturally embeds in $\mathrm{Pic}^1(\mathcal{C})$, divisor classes of degree 1. At the minimum, embedding $\mathcal{T}_x$ in $\mathrm{Pic}(\mathcal{T}_x)^0$ requires coordinates in the field generated by x over the function field of $\mathcal{P}$. That means a divisor class of degree 1 defined over $K(\mathcal{P}, x)$. It is rare, and many applications don't have it. Still, the applications we have in mind allow using the embedding locally in x where it is near a *cusp* on the boundary of $\mathcal{P}$.

8.2.2. *Historical points from the '70s and '80s.* S. J. Arakelov, Families of algebraic curves with fixed degeneracies, Izv. Adak. Nauk. SSSR Ser. Mat. 35 No. 6 (1971). Intersection theory of divisors on an arithmetic surface, ... 38, No. 6 (1974), Theory of Intersections on the Arithmetic Surface, proc. Int. Congress Math. Vancouver (1974) 405–406.

Outlined how to write the Néron function over $\mathbb{C}$ in terms of θ functions, in terms of a Green's function on a curve, or in terms of differentials of 3rd kind. P. Hriljac, The Néron-Tate height and intersection theory on arithmetic surfaces, thesis, MIT 1982 showed how to get the Green's function as a pullback of the $\log(\theta)$.

[La83, p. 325] suggests using Morikawa's θ function when there is full multiplicative reduction. This would give an explicit form of λ as done by Tate for elliptic curves. The Hurwitz space component we deal with has points on its boundary that correspond to full multiplicative reduction. This is for the archimedean prime and (probably the prime 2).

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