Modular Towers: (Pro-)finite Groups and Cusp Geometry

We start with a prime p dividing the order of a finite group $G = G_0$. This produces a projective system of groups strikingly like the dihedral sequence $\{D_{p^{k+1}}\}_{k=0}^{\infty}$ (p. 7, 8). By recasting the regular Inverse Galois problem we produce Modular Towers (MTs). Its fundamental problems/results generalize those of modular curves (the case $G_0 = D_p$). MTs of *higher rank* (p. 7, 9), like modular curves, allow topics where p varies.

We use a rank two MT to show how two of Serre's largest projects combine: Spin covers of alternating groups; and his open image theorem for $G_{\mathbb{Q}}$ acting on projective systems of points on modular curves [Se90a], [Se68]. This works best applied to MTs for *p-perfect groups* in concert with *g-p' cusps* (p. 12), generalizing the previous notion of Harbater-Mumford cusps. Examples of reduced 4-branch point MTs (moduli dessins d'enfants) show how Schur multipliers applied to g-p' cusps control projective systems of components and cusp ramification from level to level. Only for dihedral-like groups do Schur multiplier effects disappear. We apply these to two well-known problems to strengthen the modular curve comparison (p. 24,25).

Main source of talk topics

[FV92] M. Fried and H. Völklein, *The embedding problem over an Hilbertian-PAC field*, Annals of Math **135** (1992), 469–481.

Our main example alludes to the technique relating inner and absolute Hurwitz spaces through outer automophisms of finite groups. This has as a corollary the only known presentation for $G_{\mathbb{Q}}$:

$$1 \to \tilde{F}_\omega \to G_{\mathbb{Q}} \to \prod_{n=2}^{\infty} S_n \to 1.$$

[BFr02] with P. Bailey, *Hurwitz monodromy, spin separation and higher levels of a Modular Tower*, in Proceedings of Symposia in Pure Mathematics **70** (2002) editors M. Fried and Y. Ihara, 1999 von Neumann Symposium on Arithmetic Fundamental Groups and Noncommutative Algebra, August 16-27, 1999 MSRI, 79–221.

[RIMS02] *Moduli of relatively nilpotent extensions*, Institute of Mathematical Science Analysis 1267, June 2002, Communications in Arithmetic Fundamental Groups, 70–94.

[FrS03] with D. Semmen, *Schur multiplier types and Shimura-like systems of varieties*, 20 pg. preprint, May 2003.

Key inspirational papers appearing in talk

[Se68] J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves*, 1st ed., McGill University Lecture Notes, Benjamin, New York • Amsterdam, 1968, in collaboration with Willem Kuyk and John Labute.

[Se90a] J.-P. Serre, *Relêvements dans $\tilde{A}_n$*, C. R. Acad. Sci. Paris **311** (1990), 477–482.

[Ser90b] J.-P. Serre, *Revêtements á ramification impaire et thêta-caractéristiques*, C. R. Acad. Sci. Paris **311** (1990), 547–552.

[IM95] Y. Ihara and M. Matsumoto, *On Galois actions on profinite completions of braid groups*, Proceedings AMS-NSF Summer Conference, vol. 186, 1995, Cont. Math series, Recent Developments in the Inverse Galois Problem, 173–200.

[Fr04] M. Fried, Profinite geometry: Higher rank Modular Towers (MTs): slides for a Luminy talk, March, 2004. Outlines a proof of the weak Main Conjecture, and conjectures how g-p' cusps describe limit projective sequences of MT components.

www.math.uci.edu/˜mfried/talkfiles/lum03-12-04.html

Others supporting sources

[An98] Y. André, *Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire*, Crelle's J. **505** (1998), 203–208.

[BTh03] P. Bailey, *Incremental ascent of a Modular Tower via branch cycle designs*, PhD Thesis, UCI Irvine 2003.

[GS78] R. Griess and P. Schmid, *The Frattini module*, Archiv. Math. **30** (1978), 256266.

[FrJ86] M. Fried and M. Jarden, *Field arithmetic*, Ergebnisse der Mathematik III, vol. 11, Springer Verlag, Heidelberg, 1986.

[DFr90] P. Dèbes and M. Fried, *Rigidity and real residue class fields*, Acta Arith. **56** (1990), 13–45.

[FrK97] M. Fried and Y. Kopeliovic, *Applying Modular Towers to the inverse Galois problem*, Geometric Galois Actions II Dessins d'Enfants, Mapping Class Groups . . . , vol. 243, Cambridge U. Press, 1997, London Math. Soc. Lecture Notes, pp. 172–197.

Modular Towers: Durham, Sept. 5 2003

Riemann sphere: $\mathbb{P}^1_z = \mathbb{C}_z \cup \{\infty\}$, for

$$\boldsymbol{z} = z_1, \dots, z_r \subset \mathbb{P}^1_z,\ \mathbb{P}^1_z \setminus \{\boldsymbol{z}\} = U_{\boldsymbol{z}}.$$

Modular Tower (MT): Sequence of moduli spaces; generalize sequences of modular curves. Levels are moduli spaces of covers from finite group G, prime p dividing $|G|$ and p' conjugacy classes **C** (in G, r of them; we take $r = 4$).

Construction depends on the universal p-Frattini cover of G (Pierre Debes' talk). This collects otherwise unknowable finite groups into a usable structure. Two tools allow comparison with general dessins d'enfant:

- The sh-incidence pairing on cusps;
- lifting invariants from Schur multipliers of the universal p-Frattini cover quotients.

Nielsen class combinatorics allow MT level computations using the geometry of their cusps.

Nielsen classes:

Group G with r conjugacy classes $\mathbf{C} = (\mathrm{C}_1, \dots, \mathrm{C}_r)$:

$$\mathrm{Ni}(G, \mathbf{C}) = \{\boldsymbol{g} \in \mathbf{C} \mid \langle \boldsymbol{g} \rangle = G, \prod_{i=1}^{r} g_i = 1\}$$

$\mapsto \mathrm{Ni}(G, \mathbf{C})/N_{S_n}(G, \mathbf{C})$ (absolute classes) or $\mapsto$ $\mathrm{Ni}(G, \mathbf{C})/G$ (inner classes).

Three example Nielsen classes:

- $\mathrm{Ni}(G_k(D_p), \mathbf{C}_{2^4})^{\mathrm{abs,rd}}$ (or $\mathrm{Ni}(G_k(D_p), \mathbf{C}_{2^4})^{\mathrm{in,rd}}$: 4 involutions in D_p (p odd). Case p of a rank 1 Modular Tower (modular curves).

- Simple group MT from $\mathrm{Ni}(G_k(A_5), \mathbf{C}_{3^4})$: Four 3-cycles in A_5. Rank 0 MT for $p = 2$.

- Tower akin to other two: $\mathrm{Ni}(G_k(A_4), \mathbf{C}_{\pm 3^2})$: Rational union of four 3-cycles in A_4. Rank 2 MT for $p = 2$.

Forming $G_k(G)$ for $p \,|\, |G|$

Finite group H acts on rank t lattice L or finitely generated free group F (L or F may be trivial): **C** generating conjugacy classes for H. Avoid p dividing order of elements in **C**. For serious results: Finite quotient groups are p-*perfect* (no $\mathbb{Z}/p$ quotient).

Pro-p group $\tilde{P}$ has a Frattini subgroup $\Phi(\tilde{P})$ generated by its pth powers and commutators. Consider the pro-p completion ${}_pF$ of F (or L).

Case 1: $p \not| \, |H|$, $\implies$ H action on $\tilde{P}/\Phi(\tilde{P})$ extends to $\tilde{P}$. ${}_pF \times^s H$ is Universal p-Frattini cover of ${}_pF/\Phi({}_pF) \times^s H = G = G_0$: p-slit case.

Case 2: For any finite group G and each prime p, $p \,|\, |G|$, there is a universal p-Frattini cover $\psi_p : {}_p\tilde{G} \to G$.

Example 1. Rank 1, D_∞: $\mathbb{Z} \times^s \{\pm 1\} = \mathbb{Z} \times^s H_2$.

Frattini Properties

1. Profreeness: $\ker(\psi_p) = \ker_0$ and a p-Sylow of ${}_p\tilde{G}$ are pro-free pro-p and ${}_p\tilde{G}$ is the minimal such profinite cover of G [FrJ; Chap. 21].

2. Characteristic sequences: $\{G_k\}_{k=0}^{\infty}$, $\{M_k\}_{k=0}^{\infty}$:

$$\ker_{k+1} = \Phi(\ker_k),\ G_k = {}_p\tilde{G}/\ker_k,$$

$$M_k = \ker(G_{k+1} \to G_k) \text{ a } G_k \text{ module.}$$

3. Subgroup properties:
   - p' classes of $G \mapsto p'$ classes of ${}_p\tilde{G}$.
   - Frattini: $G^* \le G_k, \psi_{k,0}(G^*) = G_0 \Longrightarrow G^* = G_k$.
   - Order p^u ($u \ge 1$) conj. classes of G_k lift to order p^{u+1} classes of G_{k+1}.

4. $M_0(G)$: p-Sylow P of G: Indecomposable summand of $\mathrm{Ind}_{N_G(P)}^{G}(M_0(N_G(P)))$ that maps to $M_0(N_G(P))$ [MT1-95, RIMS02]

5. Remaining Centerless: G_0 p-perect and centerless $\Longrightarrow$ so is G_k, $k \ge 0$ [FrK97]

Four 3-cycles

Example 2 (Rank two action). $H = H_3 = \mathbb{Z}/3$ acts on a free group F_2 with generators $\boldsymbol{v}_1, \boldsymbol{v}_2$: $\langle \mu \rangle \stackrel{\text{def}}{=} \mathbb{Z}/3$ by $\mu : (\boldsymbol{v}_1, \boldsymbol{v}_2) \mapsto (\boldsymbol{v}_2^{-1}, \boldsymbol{v}_1 \boldsymbol{v}_2^{-1})$.

Use the conjugacy classes $\mathbf{C}_{\pm 3^2}$: Four conjugacy classes of elements of order 3, two map to $\mu \in \mathbb{Z}/3$ and two map to $-\mu$. Avoid only $p = 3$: G_k above is $G_k((\mathbb{Z}/p)^2) \times^s H_3$. Use a copy of H_3 in $G_k((\mathbb{Z}/p)^2) \times^s H_3$ for each k ($p \neq 3$) to define absolute classes.

The collection of conjugacy classes in both examples is a *rational union*. All spaces formed from a Nielsen class $\mathrm{Ni}(G, \mathbf{C})$ where $\mathbf{C}$ is a rational union have equations over $\mathbb{Q}$ [FrV92].

Proposition 3. $\mathrm{Ni}(G_k((\mathbb{Z}/p)^2) \times^s H_3, \mathbf{C}_{\pm 3^2})$ *is nonempty. Covers in the inner classes form a space analogous to* $X_1(p^{k+1})$*; in absolute classes analogous to* $X_0(p^{k+1})$ *(cosets of* H_3*).*

Harbater-Mumford (H-M) reps. in Prop. 3

H-M reps. $k \geq 0$:

$$({}_kg_1, {}_kg_1^{-1}, {}_kg_2, {}_kg_2^{-1}) \in \text{Ni}(G_k, \mathbf{C}_{\pm 3^2}).$$

Since G_k is a Frattini cover of G_0, ${}_kg_1, {}_kg_2$ any order 3 lifts to G_k of generating order 3 $g_1, g_2 \in G_0$. No invariant subspaces for H on $(\mathbb{Z}/p)^2$. Take $g_1 = (0, \mu)$ and $g_2 = (v^\mu - v, \mu)$ for any v not commuting with μ.

Braid action on Nielsen classes

Combinatorics for $r = 4$ that allows computing properties of MT levels comes from the action of $\langle q_1, q_2, q_3 \rangle$ on $\text{Ni}(G, \mathbf{C})$ inducing an action on *reduced Nielsen classes*. Here is the action of

$$q_2 : (g_1, \ldots, g_4) \mapsto (g_1, g_2g_3g_2^{-1}, g_2, g_4).$$

Form: $\gamma_0 = q_1q_2$, $\gamma_1 = q_1q_2q_3 = \text{sh}$, $\gamma_\infty = q_2$.

Upper half plane quotient covers

Three important groups: $\mathcal{Q}'' = \langle \mathrm{sh}^2, q_1 q_3^{-1} \rangle$; the *cusp group* $\mathrm{Cu}_4 = \langle q_2, \mathcal{Q}'' \rangle$; and $\bar{M}_4 = \langle \gamma_0, \gamma_1 \rangle$ generated freely by elements of order 3 and 2.

$\gamma_0, \gamma_1, \gamma_\infty$ on $\mathrm{Ni}(G, \mathbf{C})^{\mathrm{rd}} = \mathrm{Ni}(G, \mathbf{C})/\mathcal{Q}''$ (reduced Niel. classes) gives $\mathcal{H}(G, \mathbf{C})^{\mathrm{rd}} \to \mathbb{P}^1_j$ branch cycles. γ s from Debes-Fried cuts [DFr90] [BFr02, §6] to match complex conjugation operator for two pairs of complex conjugate branch points.

$\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C})^{\mathrm{rd}}$ is an equivalence class of covers in $\mathrm{Ni}(G, \mathbf{C})$: ramification indices over 0 divide 3, over 1 divide 2. Orbits of γ_∞ correspond to cusps; orbits of $\bar{M}_4$ to components of $\mathcal{H}(G, \mathbf{C})^{\mathrm{rd}}$.

$G_0 = A_5$ and F trivial

Use $p = 2$ and $\mathbf{C} = \mathbf{C}_{3^4}$, four conjugacy classes of elements of order 3. Absolute equivalence: Cosets of A_4, genus 0 family with 4 ramified points $\boldsymbol{x}$ on each member.

$\boldsymbol{p} \in \mathcal{H}(G, \mathbf{C})^{\mathrm{abs,rd}} \mapsto (j(\boldsymbol{z_p}), j(\boldsymbol{x_p})) \in \mathbb{P}^1_j \times \mathbb{P}^1_j)$ embeds. Yet, not a modular curve (below).

The Main Conjectures

- Weak conjecture: If G is p-perfect, no rational points at high levels of $\text{Ni}(G, \mathbf{C})^{\text{in,rd}}$ Modular Tower $\Leftrightarrow$ No genus 0 or 1 components at high levels [BFr; Thm. 6.1].
- Strong conjecture: Running over p-perfect primes attached to a Modular Tower of arbitrary rank, only finitely many levels have genus 0 or 1 components.
- The structure of Frattini central extensions of finite group appears in properties of levels of MTs: Components and cusp growth.

Remainder of talk: How g-p'-cusps generalize H-M reps. to give structure for assigning cusps ($\langle\gamma_\infty\rangle$ orbits) to components ($\bar{M}_4$ orbits) in levels of the $(G_0, \mathbf{C})$ MT. Especially: For our main examples, how to account for all components at levels 0 and 1 from either Schur multipliers or the main technique behind [FrV92].

Types of cusps

A cusp corresponds to $(\boldsymbol{g})\mathrm{Cu}_4/\mathcal{Q}'' \subset \mathrm{Ni}(G,\mathbf{C})^{\mathrm{rd}}$: $\mathrm{wd}(\boldsymbol{g}) \stackrel{\text{def}}{=} |(\boldsymbol{g})\mathrm{Cu}_4/\mathcal{Q}''|$, *cusp width*. Cusp Types:

p' cusps: $p \not| (\boldsymbol{g})\mathrm{mp} \stackrel{\text{def}}{=} \mathrm{ord}(g_2g_3)$; and

g-p' cusps: $H_{2,3}(\boldsymbol{g}) = \langle g_2, g_3\rangle$ and $H_{1,4}(\boldsymbol{g}) = \langle g_1, g_4\rangle$ are p' groups. Usually: $H_{2,3}(\boldsymbol{g}) \cap H_{1,4}(\boldsymbol{g}) = \langle g_2g_3\rangle$.

- Both p' and g-p' cusps are Cu_4 invariants.
- In a MT, each g-p' cusp has a projective sequence of g-p' cusps over it (Schur-Zass.).

Example 4. For H-M rep. $\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$, $(\boldsymbol{g})\mathrm{sh}$ has width 1 or 2 ($p = 2$ and $k \geq 1$, $\mathrm{wd}((\boldsymbol{g})\mathrm{sh}) = 2$ for inner reduced classes) and $H_{2,3}(\boldsymbol{g}) \cap H_{1,4}(\boldsymbol{g}) = \langle 1\rangle$. Projective sequences of H-M reps. $\{{}_k\boldsymbol{g} \in \mathrm{Ni}(G_k,\mathbf{C})^{\mathrm{rd}}\}_{k=0}^{\infty}$ should have their width grow as cp^k (k large; while shifts $({}_k\boldsymbol{g})\mathrm{sh}$ have width ≤ 2). Checking width growth requires analysis with Schur multipliers.

Schur Multipliers

G_0 p-perfect $\implies$ ${}_p\tilde{G} \to G_0$ quotients have universal central p-extensions. Assume R^* in $G_{k+1} \to R^* \to G_k$ (head of $M_k = \ker(\psi_{k+1,k})$) and

$\langle h' \rangle = \ker(R^* \to G_k) = \mathbf{1}_{G_k} \leftarrow$ Schur quotient.

Generally, Schur quotients can occur anywhere between G_{k+1} and G_k as $R^* \to G^* \to G_k$, though those at the head play a special role [FrS03, §4]. Always: $\mathrm{Ni}(R^*, \mathbf{C}) \to \mathrm{Ni}(G^*, \mathbf{C})$ is injective.

Quotient with Antecedents: Lift h to $\hat{h}' \in {}_p\tilde{G}$. Gives list of Schur multipliers at higher levels:

$$\langle (\hat{h}')^{p^t} \mod \ker_{k+t} \rangle \stackrel{\text{def}}{=} R^*_{k+t} \to G_{k+t}.$$

Example 5. If $G_0 \le A_n$, $n \ge 4$, has nonsplit pullback to Spin_n, then G_k ($p = 2$; $k \ge 1$) has antecendent Schur multiplier from level $k = 0$.

Obstruction and cusp growth

Lemma 6. *No cusp in* $\mathrm{Ni}(G_{k+1}, \mathbf{C})^{\mathrm{rd}}$ *over* $\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{rd}} \implies$ *a Schur Multiplier between* G_k *and* G_{k+1} *and* $\boldsymbol{g}^* \in \mathrm{Ni}(G^*, \mathbf{C}) \mapsto \boldsymbol{g}$ *but*

[ ]* $\mathrm{Ni}(R^*, \mathbf{C}) \mapsto \mathrm{Ni}(G^*, \mathbf{C})$ *does not hit* $\boldsymbol{g}^*$ *(*$(\boldsymbol{g})\bar{M}_4$ *is obstructed).*

If $\boldsymbol{g} \in \mathrm{Ni}(G_k, \mathbf{C})^{\mathrm{rd}}$ *is* p'*, no* p' *cusp in* $\mathrm{Ni}(G_{k+1}, \mathbf{C})^{\mathrm{rd}}$ *over* $\boldsymbol{g} \implies$ *a Schur multiplier between* G_k *and* G_{k+1} *and either [*] holds, or over* $\boldsymbol{g}$

*[**]there is no* p' $\hat{\boldsymbol{g}}^* \in \mathrm{Ni}(R^*, \mathbf{C})$ *(wd(*$\boldsymbol{g}$*) grows).*

For k large, so long as M_0 has dimension at least 2 (basically from [GS78]), the multiplicity of Schur multipliers between G_k and G_{k+1} grows. Finding which affect obstruction and cusp width growth is necessary for a sequential genus computation.

Effective computations

1. Lifting Invariant: For C a p' conjugacy class: $\hat{g} \in \mathsf{C} \cap R^* \mapsto g \in \mathsf{C} \cap G_k$, $s_{R^*}(\boldsymbol{g}) = \hat{g}_1(\hat{g}_2\hat{g}_3)\hat{g}_4$ $(= \hat{g}_3\hat{g}_4(\widehat{g_2g_3})^{-1})(\widehat{g_4g_1})^{-1})\hat{g}_4\hat{g}_1$ if $\boldsymbol{g}$ is p').
2. sh-incidence symmetric matrix: List γ_∞ orbits $(\boldsymbol{g})\mathrm{Cu}_4/\mathcal{Q}'' = O = O_{\boldsymbol{g}}$:

$$\begin{pmatrix} \vdots & |O \cap (O)\gamma_1| \\ . & \cdots \end{pmatrix} = \begin{pmatrix} \vdots & |O \cap (O)\gamma_0| \\ . & \cdots \end{pmatrix}.$$

- Blocks give components of $\mathcal{H}(G, \mathbf{C})^{\mathrm{rd}}$.
- Fixed points of γ_0, γ_1 appear on diagonal.

3. Variants on formulas like those of [Se90a] to go from level k to k+1 when $p = 2$:

Example 7. $(\boldsymbol{g})\mathrm{Cu}_4$ is a p' cusp and orbits of $H_{2,3}(\boldsymbol{g})$ have genus 0 in $A_N \ge G_k$ with $R^* = G_k \times_{A_N} \mathrm{Spin}_N$. Subex: $g_{1,4} \in \mathrm{Ni}(A_4, \mathbf{C}_{\pm 3^2}) =$

$((1\,2\,3), (1\,3\,4), (1\,2\,4), (1\,2\,4))$:
$s_{R^*}(g_2, g_3, (g_2g_3)^{-1}) = +1$;
$s_{R^*}(g_4, g_1, (g_4g_1)^{-1}) = (-1)^{3\cdot(3^2-1)/8} = -1.$

$\bar{M}_4$ orbit of $g_{1,4}$ is obstructed.

$\text{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\text{abs,rd}} = \text{Ni}(A_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}, p = 2$

Nielsen class $\mapsto \text{Ni}(A_3, \mathbf{C}_{\pm 3^2})^{\text{rd}}$: Entries by sequences of conjugacy classes, $q_1 q_3^{-1}$ and sh switch these rows:

[1] + - + -   [2] + + - -   [3] + - - +
[4] - + - +   [5] - - + +   [6] - + + -

sh-incidence: In $O^k_{i,j}$, k is cusp width, i, j correspond to orbit reps. Diagonal entries $O^4_{1,1}$ (γ_1 fixes 1, γ_0 none) and $O^4_{1,4}$ aren't empty (γ_0 and γ_1 fix nothing).

H-M rep. $\mapsto \boldsymbol{g}_{1,1} = ((1\,2\,3), (1\,3\,2), (1\,3\,4), (1\,4\,3))$
$\boldsymbol{g}_{1,3} = ((1\,2\,3), (1\,2\,4), (1\,4\,2), (1\,3\,2))$
H-M rep. $\mapsto \boldsymbol{g}_{3,1} = ((1\,2\,3), (1\,3\,2), (1\,4\,3), (1\,3\,4))$

Ni_0^+ Orbit	$O^4_{1;1}$	$O^2_{1;3}$	$O^3_{3;1}$
$O^4_{1;1}$	1	1	2
$O^2_{1;3}$	1	0	1
$O^3_{3;1}$	2	1	0

Ni_0^- Orbit	$O^4_{1;4}$	$O^1_{3;4}$	$O^1_{3;5}$
$O^4_{1;4}$	2	1	1
$O^1_{3;4}$	1	0	0
$O^1_{3;5}$	1	0	0

Conclusions Ni$(A_4, \mathbf{C}_{\pm 3^2})^{\text{rd}}, p = 2$

1. Two components: $\mathcal{H}_0^+$ and the obstructed $\mathcal{H}_0^-$ (nothing above it at level 1), both genus 0 from $(\gamma_0, \gamma_1, \gamma_\infty)$ on reduced Nielsen classes.

2. Neither component is a Modular curve ([FrS03; Prop. 4.16] applying Wohlfahrt's Thm.).

3. Consider either as an absolute space: Moduli space of genus 1 curves. Birational embedding: $\boldsymbol{p} \in \mathcal{H}_0^{\text{abs},\pm,\text{rd}} \leftrightarrow \varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}_z^1$

$$\mapsto (j(\boldsymbol{p}), j(\text{Pic}(X_{\boldsymbol{p}})^{(0)})) \in \mathbb{P}_j^1 \times \mathbb{P}_j^1.$$

4. $\frac{1}{2}$-canonical classes: $(d\varphi_{\boldsymbol{p}}) = 2 \cdot D_{\boldsymbol{p}}$, with $D_{\boldsymbol{p}}$ $\frac{1}{2}$-canonical. Idea works for any odd ramification maps: $(d(\alpha \circ \varphi_{\boldsymbol{p}}))/2$, $\alpha \in \text{PGL}_2(\mathbb{C})$, lin. equiv. to $D_{\boldsymbol{p}}$.

Case of Serre formula [Ser90b]: For $\boldsymbol{p} \in \mathcal{H}_0^+$ (resp. $\mathcal{H}_0^-$), $D_{\boldsymbol{p}}$ is even (resp. odd).

5. $\mathcal{Q}''$ orbits on $\text{Ni}_0^{\pm}$ have length 2: $\mathcal{H}_0^{\pm,\text{rd}}$ not fine moduli, but higher levels of MT are.

$\text{Ni}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}} = \text{Ni}_0^{\text{in,rd}}$ sh-incidence matrix

We show there is one $\bar{M}_4$ orbit: $\mathcal{Q}''$ acts trivally.

Denote γ_∞ orbits of

$$\boldsymbol{g}_1 = ((1\,2\,3), (1\,3\,2), (1\,4\,5), (1\,5\,4)) \text{ and}$$

$$\boldsymbol{g}_2 = ((1\,2\,3), (1\,3\,2), (1\,5\,4), (1\,4\,5))$$

by $O_{5;1}$ and $O_{5;2}$; γ_∞ orbits of

$$((5\,1\,3), (2\,4\,5), (1\,5\,4), (1\,2\,3)) \text{ and}$$

$$((3\,2\,4), (5\,1\,3), (1\,5\,4), (1\,2\,3))$$

by $O_{3;1}$ and $O_{3;2}$; and of $(\boldsymbol{g}_1)$sh by $O_{1,2}$.

Orbit	$O5;1$	$O_{5;2}$	$O_{3;1}$	$O_{3;2}$	$O_{1,2}$
$O_{5;1}$	0	2	1	1	1
$O_{5;2}$	2	0	1	1	1
$O_{3;1}$	1	1	0	1	0
$O_{3;2}$	1	1	1	0	0
$O_{1,2}$	1	1	0	0	0

Conjugate by 2-cycle for $\text{Ni}_0^{\text{abs,rd}}$: 3 γ_∞ orbits (widths 5, 3, 1) $\Longrightarrow$ monodromy group is A_9.

Aim: Genus formulas from systems of g-p' cusps: Level 1 of $(A_5, \mathbf{C}_{3^4})[BFr; \S 9]$

$M_0(A_5) = \{(a_1, \dots, a_6) \in (\mathbb{Z}/2)^6\}/\langle(1,1,1,1,1,1)\rangle$:

Generated by six D_5 cosets, modulo sum of cosets; $M_0(A_5)_{|A_4} = M_0(A_4)$.

Three A_5 conj. classes on $M_0(A_5) \setminus \{\mathbf{0}\}$.

M_3' – centralizes some 3-cycle (3 $\neq 0 a_i$ s).

M_5' – centralizes some 5-cycle (1, or 5 $\neq 0 a_i$ s).

$V \setminus \{\mathbf{0}\} \to \ker(\mathrm{Spin}_5 \to A_5)$

Unique Schur mult. quotient $R_1^* \to G_1(A_5)$ is antecedent from $\mathrm{Spin}_5 = R_0^* \to A_5$

Fix $(g_1, g_2) \in (G_1(A_5))^2 \mapsto ((1\,2\,3), (1\,4\,5)) \in (A_5)^2$.

H-M Perturbations: $\{\boldsymbol{g}_{a,b} = (g_1, a g_1^{-1} a, b g_2 b, g_2^{-1})\}, a, b \in M_0 \setminus V$.

1. $\boldsymbol{g}_{a,b} \mapsto \hat{\boldsymbol{g}}_{a,b} \in \mathrm{Ni}(R_1^*, \mathbf{C}_{3^4})$,

   gives lifting invariant $s_{R_1^*}(\boldsymbol{g}_{a,b}) = \hat{a}^{g_1^{-1}} \hat{a} \hat{b} \hat{b}^{g_2^{-1}}$.

2. ${}_0\boldsymbol{g} = ((1\,2\,3), (1\,2\,3), (1\,4\,5), (1\,4\,5))$:

   Apply [Se90a]: $({}_0\boldsymbol{g})\mathrm{mp} = 10$ from
   $s_{R_0^*}((1\,2\,3)^{-1}, (1\,4\,5), (1\,3\,2\,4\,5)^{-1}) = (-1)^{2\cdot(3^2-1)/8 \cdot 5^2-1)/8} = -1$.

H-M reps. produce widest level 1 cusps (width 20).

Level 1 of $(A_5, \mathbf{C}_{34})$ continued [BFr; Prop. 9.8]

The assumptions of [Se90a] are no longer even close to valid. Yet we can compute:

1. $s_{R_1^*}(\boldsymbol{g}_{a,b}) = +1$ if $(a,b) \in M_3' \times M_3' \cup M_5' \times M_5'$; 8 γ_∞ orbits of width 20: 4 H-M reps., 4 near H-M reps. (2 each over ${}_0\boldsymbol{g}$). This accounts for $\mathcal{H}_1^+ \to \mathcal{H}_0$ of degree 16 and irreducible.

2. $s_{R_1^*}(\boldsymbol{g}_{a,b}) = -1$ if $(a,b) \in M_3' \times M_5' \cup M_5' \times M_3'$; 4 γ_∞ orbits of width 20 (complements of H-M reps.), 8 width 10 orbits (from $\mathcal{Q}''$ acts nontrivially on $(\boldsymbol{g}_{a,b}\langle q_2\rangle$, $\mathcal{Q}''$ orbit shortening).

Conclusions: Genus: $g(\bar{\mathcal{H}}_1^+) = 12$, $g(\bar{\mathcal{H}}_1^-) = 9$.

3. Real cusps are H-M or near H-M; one component of real points on $\bar{\mathcal{H}}_1^+$; and $\bar{\mathcal{H}}_1^-(\mathbb{R}) = \emptyset$.

4. For $N = 40$ (80, 120), $G_1(A_5) \le A_N$, $R_1^* = G_1(A_5) \times_{A_N} \text{Spin}_N$: Apply [Se90b]: $\frac{1}{2}$-canonical class for $\bar{\mathcal{H}}_1^+$ even, for $\bar{\mathcal{H}}_1^-$ odd.

Level 1 of $(A_4, \mathbf{C}_{\pm 3^2})$, $p = 2$

Using the Schur mult. of $G_1(A_4)$ is $(\mathbb{Z}/2)^2$: So, there are three Schur quotients: $R^* \to G_1(A_4)$. Again use H-M perturbations: $\ker(R^* \to G_1(A_4))$ is D_1, D_2, D_3.

Given a Schur quotient $D = \mathbb{Z}/p = \ker(R^*_D \to G_k)$ define $V_D \le M_k$: $v \in M_k \setminus \{0\}$ that lift to $\hat{v}$ in R^*_D of order p, $V_D = V_D^0 \cup \{0\}$, $\hat{M}_D = \ker(R^*_D \to G_k)$.

[RIMS02] Special case: Call D an *abelian* Schur quotient if $\hat{M}_D$ is abelian. For $k \ge 1$, D is abelian if and only if it is antecedent.

With $K_{4,H}$ a Klein 4-group ($H_3 = \mathbb{Z}/3$ acts), "Loewy displays" for this case [FSe02; §2.3]:

$\hat{M}_{D_1}$:$K_{4,H} \to K_{4,H} \oplus \mathbb{Z}/4$, $\hat{M}_{D_2}$:$K_{4,H} \to Q_8 \oplus \mathbb{Z}/2$,
$\hat{M}_{D_3}$: $K_{4,H} \to Q_8 \cdot \mathbb{Z}/4$.

Conclusions from level 1 of $(A_4, \mathbf{C}_{\pm 3^2})$

With $\hat{O}_4 = ((\mathbb{Z}/4)^2 \times^s H_3) \times_{A_4} \mathrm{Spin}_4$, $\mathcal{H}_O = \mathcal{H}(\hat{O}_4, \mathbf{C}_{\pm 3^2})^{\mathrm{rd}} = \mathcal{H}_O^+ \cup \mathcal{H}_O^-$: two components separated by lifting inv.: $\mathcal{H}_O^+$ is an H-M comp.

Three components in $\mathcal{H}(G_1(A_4), \mathbf{C}_{\pm 3^2})$ over $\mathcal{H}_O^+$ (resp. $\mathcal{H}_O^-$): $\mathcal{H}^{+,3}$, genus 3, $\mathcal{H}^{+,1}, \mathcal{H}^{+,2}$ both genus 1 (resp. $\mathcal{H}^{-,3}$ genus 3, $\mathcal{H}^{+,1}, \mathcal{H}^{+,2}$ both genus 0, but complex conjugate).

Facts about $\mathcal{H}^{+,1}, \mathcal{H}^{+,2}$: $\boldsymbol{p} = \mathcal{H}^{+,1}$ corresponds to $(\varphi_{\boldsymbol{p}} : X_{\boldsymbol{p}} \to \mathbb{P}^1_z, \alpha : G(X_{\boldsymbol{p}}/\mathbb{P}^1_z) \simeq G_1)$.

a. $\exists \beta$, outer automorphism of $G_1(A_4)$ with

$$\boldsymbol{p} = (\varphi_{\boldsymbol{p}}, \alpha) \in \mathcal{H}^{+,1} \mapsto \boldsymbol{p}' = (\varphi_{\boldsymbol{p}}, \beta \circ \alpha) \in \mathcal{H}^{+,2}.$$

a'. $\mathcal{H}^{+,1}$ and $\mathcal{H}^{+,2}$ are genus 1 degree 2 covers of genus 0 $\mathcal{H}_O^+$.

b. Only primes dividing discriminants of $\mathcal{H}^{+,1}$ and $\mathcal{H}^{+,2}$ are those dividing $|A_4|$ (they have fine moduli): $j(\bar{\mathcal{H}}^{+,1}) = j(\bar{\mathcal{H}}^{+,2}) = \frac{13^3}{3^4}$.

Summary: . . . it can't be an accident that from moduli space dessins d'enfant you get genus 0 and 1 spaces with applications

Schur multipliers separated all but one pair of components in these examples. A non-braided outer automorphism of $G_1(A_4)$ separated p. 22's unusual pair of H-M rep. components. [FrV02] realized group extensions by assuring braidings between outer autos. on Nielsen classes. We have solidly explained this talk's components.

Nonsolvable group application: There are similar spaces $\bar{\mathcal{H}}^{+,i}(A_5, \mathbf{C}_{\pm 5^2})$ and in a nonobvious way $\bar{\mathcal{H}}^{+,i}(A_5, \mathbf{C}_{\pm 5^2}) = \bar{\mathcal{H}}^{+,i}(A_4, \mathbf{C}_{\pm 3^2})$, $i = 1, 2$ [BTh03]. Rational points here give only possibility for ∞-ly many $(G_1(A_5), p = 2)$ regular $\mathbb{Q}$ realizations with $r = 4$ [BFr; §5].

moduli space dessins d'enfant with applications – Continued

Application of [An98]: From the level 0 embeddings in $\mathbb{P}^1_j \times \mathbb{P}^1_j$, only finitely many points of $\mathcal{H}(A_4, \mathbf{C}_{\pm 3^2})^{\text{in,rd}}$ and $\mathcal{H}(A_5, \mathbf{C}_{3^4})^{\text{in,rd}}$ $(p = 2)$ are special in the sense of Shimura. This hints that for these MTs, one can expect a Serre-type open image result on projective systems of points of the fiber, with only finitely many exceptions. [FrS03] explains the meaning.

Using Mazur's Theorem: $\bar{\mathcal{H}}(A_5, \mathbf{C}_{3^4})^{\text{abs,rd}} \times_{\mathbb{P}^1_j} \mathbb{P}^1_\lambda$ has genus 1 and exactly 12 $\mathbb{Q}$ points, generated by the cusps. So, there are precisely three $X \to \mathbb{P}^1_z$ degree 5 $(A_5, \mathbf{C}_{3^4})$ covers (up to $\text{PGL}_2(\mathbb{Q})$ action) with branch points in $\mathbb{Q}$.

Compare with Grothendieck-Teichmüller: Based at a g-p' (an H-M) cusp at level 0, consider all projective systems of g-p' components and the $G_{\mathbb{Q}}$ action (Ihara-Matusumoto-Wewers formulas) on projective systems of cusps. Example: Use to explain the p-adic nature of the near H-M reps. on the MTs presented in this talk [BFr; App. D.3].