Profinite geometry:
Higher rank Modular Towers (MTs)
Luminy, March 12, 2004

- Matthieu Romagny and Stefan Wewers introduced Nielsen classes and material on Hurwitz spaces.

- Helmut Voelklein and Kay Magaard introduced Hurwitz monodromy action, necessary for computations.

- Pierre Debes defined a (rank 0) Modular Tower (MT), comparing that with modular curves.

- The (weak; rank 0) Main Conjecture is that there are no rational points at suitably high levels. Pierre reduced this conjecture, for four branch point towers, to showing the genus rises with the levels.

- Darren Semmen presented the profinite Frattini category. This showed how Schur multipliers control properties of the Modular Tower levels.

Riemann sphere: $\mathbb{P}^1_z = \mathbb{C}_z \cup \{\infty\}$, for

$$\boldsymbol{z} = z_1, \dots, z_r \subset \mathbb{P}^1_z,\ \mathbb{P}^1_z \setminus \{\boldsymbol{z}\} = U_{\boldsymbol{z}}.$$

Introductory set up of Tasks

T_1. MTs is an answer to difficult questions:

T_1.a. Why is the Inverse Galois Problem so difficult?

T_1.b. How is the Inverse Galois Problem related to other problems people find deep or important?

T_2. There has been serious progress on MTs.

T_2.a. Structure shows cases of the Main Conjecture.

T_2.b. Specific MT levels give positive results.

Frattini extensions of a finite group G lie behind MTs. The key data, a *reduced* Nielsen class defined by a prime p dividing $|G|$ and r (p') conjugacy classes

$$\mathbf{C} = (\mathrm{C}_1, \dots, \mathrm{C}_r).$$

Characteristic sequence of p-Frattini covers of $G = G_0$:

$$\{G_k = {}_p\tilde{G}/\Phi^k(\tilde{P}_p)\}_{k=0}^{\infty} \text{ with } \tilde{P}_p = \ker({}_p\tilde{G} \to G).$$

[RIMS02; §2.2]: How to find rank of pro-p, pro-free $\tilde{P}_p$.

Pierre presented [FrK97; Thm. 4.4]:

Proposition 1. *Assume for some r_0 there are $\mathbb{Q}$ regular realizations of all of the $\{G_k\}_{k=0}^{\infty}$, with at most r_0 branch points. Then, there exists a MT with $r_{\mathbf{C}} \le r_0$ where every level has a $\mathbb{Q}$ point.*

Weak Main Conjecture: When G_0 is p-perfect, you can't have such $\mathbb{Q}$ points at every level.

Four Talk Parts

From here $r = 4$: MT levels are j-line covers.

I. Tools for computing cusp widths and elliptic ramification of the levels.

II. T_1, Outline $r = 4$: Main Conjecture holds.

III. Explain the Strong Main Conjecture.

IV. T_2: Produce MT components that apply to significant problems.

A rank 2 MT attached to $F_2 \times^s \mathbb{Z}/3$ shows structure on specific tower levels. [Dur03], [FrS03] go beyond examples.

App. A. has a MT attached to $F_2 \times^s \mathbb{Z}/2$: Interprets as Serre's *O(pen)I(mage)T(heorem)*.

MT levels are rarely modular curves: Not quotients of congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. Yet, modular curve thinking guides their use.

I. Knowing about a MT level

Pierre's talk, given $(G_0, \mathbf{C}, p)$ with p' classes $\mathbf{C}$:

- Gives projective sequence $\{\mathcal{H}_k\}_{k=0}^{\infty}$, of inner reduced Hurwitz spaces; each an upper half-plane quotient and $U_\infty = \mathbb{P}^1_j \setminus \infty$ cover.
- Weak conjecture reduction: For large k, all components of $\bar{\mathcal{H}}_k$ have genus exceeding 1.

What you need to compute the genuses!

- What are the $\mathcal{H}_k$ components.
- What are the cusp (points over ∞) widths in each component.
- What points ramify in each component over the elliptic points ($j = 0$ or 1).

Dictionary: *Reduced* Nielsen classes let us calculate components, cusp and elliptic ramification. We'll see how the Frattini property controls growth of cusp widths with k.

Nielsen classes

$$\begin{array}{rcl} \text{Ni}(G,\mathbf{C}) = & & \{\boldsymbol{g} \in \mathbf{C} \mid \langle \boldsymbol{g} \rangle = G, \prod_{i=1}^r g_i = 1\} \\ & \mapsto & \text{Ni}(G,\mathbf{C})/N_{S_n}(G,\mathbf{C}) \text{ (absolute)} \\ \text{or} & \mapsto & \text{Ni}(G,\mathbf{C})/G \text{ (inner) classes.} \end{array}$$

Elements q_i, $i = 1,2,3$. Each acts by a twisting on any 4-tuple in a Nielsen class. Example:

$$q_2 : \boldsymbol{g} \mapsto (\boldsymbol{g})q_2 = (g_1, g_2 g_3 g_2^{-1}, g_2, g_4).$$

Reduced equivalence: For $\alpha \in \text{PGL}_2(\mathbb{C})$, cover $\varphi : X \to \mathbb{P}^1_z \Rightarrow \alpha \circ \varphi : X \to \mathbb{P}^1_z$; preserves $j = j_{\boldsymbol{z}}$-invariant of branch point set $\boldsymbol{z} = \boldsymbol{z}_\varphi$.

Equivalence on Nielsen classes: Klein 4-group in $\text{PGL}_2(\mathbb{C})$ leaves the branch point set $\boldsymbol{z}$ fixed $\Leftrightarrow$ mod out by $\mathcal{Q}'' = \langle (q_1 q_2 q_3)^2, q_1 q_3^{-1} \rangle$.

Hurwitz monodromy: $H_4 = \langle q_1, q_2, q_3 \rangle$.

Mapping class: $H_4/\mathcal{Q}'' \equiv \text{PSL}_2(\mathbb{Z}) = \bar{M}_4 =$

$\langle \gamma_0, \gamma_1, \gamma_\infty \rangle, \gamma_0 = q_1 q_2, \gamma_1 = \text{sh} = q_1 q_2 q_3, \gamma_\infty = q_2,$
Satisfying product-one: $\gamma_0 \gamma_1 \gamma_\infty = 1$.

Reduced Nielsen classes and cusps

Absolute reduced (resp. inner reduced) reps.:

$$\begin{aligned}\mathrm{Ni}(G,\mathbf{C})/\langle N_{S_n}(G,\mathbf{C}),\mathcal{Q}''\rangle = &\ \mathrm{Ni}^{\mathrm{abs,rd}} \text{ and} \\ \mathrm{Ni}(G,\mathbf{C})/\langle G,\mathcal{Q}''\rangle = &\ \mathrm{Ni}^{\mathrm{in,rd}}.\end{aligned}$$

Cusp group: $\mathrm{Cu}_4 = \langle q_2, \mathcal{Q}''\rangle$.

Running over $\boldsymbol{g} \in \mathrm{Ni}(G_k,\mathbf{C})^{\mathrm{in,rd}}$:

- Cusps on $\bar{\mathcal{H}}_k \Leftrightarrow (\boldsymbol{g})\mathrm{Cu}_4$.
- Components on $\bar{\mathcal{H}}_k \Leftrightarrow (\boldsymbol{g})\bar{M}_4$.

$(\gamma_0,\gamma_1,\gamma_\infty)$ on a $\bar{M}_4$ orbit $\Leftrightarrow$ branch cycles for a component of $\bar{\mathcal{H}}(G,\mathbf{C})^{\mathrm{rd}} \to \mathbb{P}^1_j$.

- Ramified points over 0 $\Leftrightarrow$ orbits of γ_0.
- Ramified points over 1 $\Leftrightarrow$ orbits of γ_1.

II. Main Conjecture rubric on cusps

Assume projective system of components:

$$\{\bar{\mathcal{H}}'_k\}_{k=0}^{\infty} \Leftrightarrow \bar{M}_4 \text{ orbit } \mathrm{Ni}'_k.$$

Genus 0: Restricted γ s, $(\gamma'_{0,k}, \gamma'_{1,k}, \gamma'_{\infty,k})$, with

$$(*) \quad \begin{array}{l} 2(\deg(\bar{\mathcal{H}}'_k/\mathbb{P}^1_j) - 1) = \\ \mathrm{ind}(\gamma'_{0,k}) + \mathrm{ind}(\gamma'_{1,k}) + \mathrm{ind}(\gamma'_{\infty,k}). \end{array}$$

$\mathrm{ind}(\gamma'_{\infty,k})$ = sum over cusps $|(\boldsymbol{g})\mathrm{Cu}_4/\mathcal{Q}''| - 1$.

p' Cusp Types:g(roup)-p': $H_{2,3}(\boldsymbol{g}) = \langle g_2, g_3\rangle$ and $H_{1,4}(\boldsymbol{g}) = \langle g_1, g_4\rangle$ are p' groups.

o(nly)-p':$p \not| (\boldsymbol{g})\mathrm{mp} \stackrel{\mathrm{def}}{=} \mathrm{ord}(g_2 g_3)$, but not g-$p'$.

H(arbater)-M(umford) reps:$\boldsymbol{g} = (g_1, g_1^{-1}, g_2, g_2^{-1})$, $(\boldsymbol{g})$sh has width 1 or 2 ($p = 2$, $k \geq 1$: width 2 for inner reduced) and $H_{2,3}(\boldsymbol{g}) \cap H_{1,4}(\boldsymbol{g}) = \langle 1 \rangle$.

Frattini Cusp Principle 1 [BFr02; §8.1]:

$${}_k\boldsymbol{g} = ({}_kg_1, \ldots, {}_kg_4) \in \mathrm{Ni}'_k :$$

If $p^u | \mathrm{ord}({}_kg_2\,{}_kg_3) = ({}_k\boldsymbol{g})\mathbf{mp}$, $u \geq 1$, then $p^{u+1} | ({}_{k+1}\boldsymbol{g})\mathbf{mp}$ for any ${}_{k+1}\boldsymbol{g}$ over ${}_k\boldsymbol{g}$.

Cusp growth expections

Outline how cusp growth contradicts (*): Assume non-p' cusp $\boldsymbol{p}'_k \in \mathcal{H}'_k$, $\nu_k = [\mathcal{H}'_{k+1} : \mathcal{H}'_k]$.

Any of u_k primes $\boldsymbol{p}'_{k+1} \in \mathcal{H}'_{k+1}$ over $\boldsymbol{p}'_k$ has ramification order p ($\nu_k = p \cdot u_k$). Same for each $\boldsymbol{p}'_{k+2} \in \mathcal{H}'_{k+2}$ over a $\boldsymbol{p}'_{k+1}$.

- Index contribution of all $\boldsymbol{p}'_{k+2}$s to R-H from $\mathcal{H}'_{k+2}$ to $\mathcal{H}'_{k+1}$ is $u_k \cdot u_{k+1} \cdot (p-1)$.
- Exceeds $2(p \cdot u_{k+1} - 1)$ if u_k is "moderate."

Subtler points:

- What if $\mathcal{H}'_{k+1} \to \mathcal{H}'_k$ resembles maps from degree p polynomials $f_k \in \mathbb{Q}[x]$?
- What forces any non-p' cusps? How about contributions of γ'_0, γ'_1?
- What happens at low levels? Is it just like modular curves?
- What controls sequences $\{\bar{\mathcal{H}}'_k\}_{k=0}^{\infty}$; figuring p' cusps on them?

III. Strong Conjecture for $r = 4$

[FrS; §6.1] computes genus of $X_0(p^{k+1})$ and $X_1(p^{k+1})$ as MT levels. X_1 (inner) case:

1. One $\bar{M}_4$ orbit.
2. Role of H-M rep. (width p^{k+1}) and shift of H-M rep. cusps (g-p' cusps, width 1).
3. No fixed points to γ'_0 or γ'_1.
4. $\mathcal{Q}''$ acts trivially at all levels.

[FrS; Prop. 8.4] generalizes #3 and #4: MT version of Serre's abelian variety lemma: Only identity fixes many torsion points.

Group setup for Strong Main Conjecture:

F_u free of rank u: $F_u \times^s J$, J faithful action, $\mathbf{C} = (\mathrm{C}_1, \dots, \mathrm{C}_4)$ conjugacy classes in J.

Form $\tilde{F}_{u,p}$, pro-p, pro-free completion; $\Phi^t = \Phi^t_p$, tth Frattini quotient of $\tilde{F}_{u,p}$.

Nonempty Nielsen classes

$P_{\mathbf{C}}$ (resp. $P'_{\mathbf{C}}$) those primes p with

- $(p, |J|) \neq 1$ (resp. $p|\text{ord}(g)$) some $g \in \mathbf{C}$), and
- $\tilde{F}_{u,p}/\Phi^1 \times^s J$ is not p-perfect.

For $p \notin P_{\mathbf{C}}$, denote nontrivial (finite) J quotients of $\tilde{F}_{u,p}$ by $\mathcal{V}_p(J)$.

Problem 2. Running over $p \notin P_{\mathbf{C}}$, $V \in \mathcal{V}_p(J)$: Which $\text{Ni}(V \times^s J, \mathbf{C})^{\text{in}}$ are nonempty.

Form collection $\mathcal{G}_p$ of maximal projective (p-Nielsen) limits of groups over nonempty Nielsen classes. (There are $P'_{\mathbf{C}}$ versions of this.)

Problem 3. What are the $\tilde{G}^* \in \mathcal{G}_p$, $p \notin P_{\mathbf{C}}$? What are the $\bar{M}_4$ orbits on $\text{Ni}(\tilde{G}^*, \mathbf{C})^{\text{in,rd}}$?

Examples: $u = 2$, $|J|$ is a prime.

- $J_2 = \mathbb{Z}/2 = \{\pm 1\}$ generator maps $x_i \mapsto x_i^{-1}$, $i = 1, 2$; $\mathbf{C} = \mathbf{C}_{2^4}$, 4 reps. of -1. (App. A: nonempty $\Leftrightarrow V$ abelian; Serre's OIT.)
- $J_3 = \mathbb{Z}/3 = \langle \alpha \rangle$ generator maps $x_1 \mapsto x_2^{-1}$, $x_2 \mapsto x_1 x_2^{-1}$; $\mathbf{C} = \mathbf{C}_{\pm 3^2}$, two each of α, α^{-1}. (App. B: All Nielsen classes nonempty).

Strong Main Conjecture: $u \geq 0$, $r = 4$ MTs

$P_{\mathbf{C}}$ Conjecture:Over all $p \notin P_{\mathbf{C}}$, for only finitely many $V \in \mathcal{V}_p(J)$, does $\mathcal{H}(V \times^s J, \mathbf{C})^{\text{in,rd}}$ have genus 0 or 1 component.

Frattini Cusp Principle 2 [FrS03; Prop. 5.1]: p' and g-p' cusps don't depend on the choice of representative in $(\boldsymbol{g})\text{Cu}_4$. If $(\boldsymbol{g})\text{Cu}_4$ is g-p' in $\mathcal{H}(G, \mathbf{C})^{\text{rd}}$, level 0 of a MT, then:

- $\exists$ g-p' lifts ${}_k\boldsymbol{g} \in \text{Ni}(G_k, \mathbf{C})$ of $\boldsymbol{g}$;
- none of g-p' components is *obstructed*.

Modular curve comparison for Serre's OIT:

- $\{X_1(p^{k+1})\}_{k=0}^{\infty}$ proj. systems over $j' \in \mathbb{Q}_p$ near ∞: H-M reps.$\Longrightarrow$transvections in $G_{\mathbb{Q}_p}$.
- p-Frattini monodromy ($\text{PSL}_2(\mathbb{Z}/p^{k+1})$) for $X_1(p^{k+1}) \to \mathbb{P}^1_j$ crucial to OIT.

IV. Modular curve-like systems of components

- Proj. systems $\{\mathcal{H}(G_k, \mathbf{C})' \neq \emptyset\}_{k=0}^{\infty}$ of MT components should be g-p' components.
- For k large, outer autos. of G_k should conjugate all g-p' components on $\mathcal{H}(G_k, \mathbf{C})$.
- For k large, $\{\bar{\mathcal{H}}(G_k, \mathbf{C})' \to \mathbb{P}^1_j\}_{k=0}^{\infty}$ monodromy gps. should be a p-Frattini system.
- For k large, genuses of $\{\bar{\mathcal{H}}(G_k, \mathbf{C})'\}_{k=0}^{\infty}$ should have a modular curve-like formula, coming from clear understanding of g-p' cusps and disappearance of o-p' cusps.

We left out 3 biggest topics: [BFr02], [FrS03]

- Schur multipliers of G_ks produce many obstructed components, generalizing Serre's spin lifting project for alternating groups.
- Schur multipliers force sequences of p' (not g-p') cusps to become non-p'. Conjecture: This is a general phenomenon.
- Genus 0 and 1 components appearing at low levels have many applications (each of interest to their own specialists).

$F_2 \times^s \mathbb{Z}/3$, $p = 2$: Level 0, 1 components

Level 0: $\mathcal{H}(\tilde{F}_{2,2}/\Phi^1 \times^s J_3, \mathbf{C}_{\pm 3^2})^{\text{in,rd}} = \mathcal{H}_0^+ \cup \mathcal{H}_0^-$, both genus 0, one H-M, other not.

André's Thm. [An98],[Dur03, p. 15-18]: (*) Has $< \infty$ Shimura-special points (unlike J_2 case). Uses sh-incidence matrix *cusp pairing* on reduced spaces.

Conjecture: (*) is true for all other $p \neq 3$.

Level 1: $\mathcal{H}(\tilde{F}_{2,2}/\Phi^2 \times^s J_3, \mathbf{C}_{\pm 3^2})^{\text{in,rd}} \to \mathcal{H}_0^+$: 2 genus 0 $\mathcal{H}_1^{-,c}, \mathcal{H}_1^{-,c'}$, complex conjugate and *spin* obstructed; 2 genus 3, $\mathcal{H}_1^{+,3}, \mathcal{H}_1^{-,3}$, one spin obstructed, the other obstructed by another Schur multiplier; 2 genus 1, $\mathcal{H}_1^{+,\beta}$, $\mathcal{H}_1^{+,\beta^{-1}}$ both H-M comps ([BTh03], [FrS03]).

Significance of $\mathcal{H}_1^{+,\beta}$, $\mathcal{H}_1^{+,\beta^{-1}}$:

- $\text{Out}(\tilde{F}_{2,2}/\Phi^2 \times^s J_3)$ conjugates $\mathcal{H}_1^{+,\beta}$ to $\mathcal{H}_1^{+,\beta^{-1}}$.
- Only chance for ∞-ly many $r = 4$ regular realizations of maximal Frattini extension of A_5 by elementary 2-group.

App. A: Nielsen classes for $F_2 \times^s \mathbb{Z}/2$ (p. 10)

Complete $F_4 = \langle \boldsymbol{\sigma} = \{\sigma_1, \dots, \sigma_4\}\rangle$, modulo $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$ (product-one). Denote by $\hat{F}_{\boldsymbol{\sigma}}$.

Proposition 4. *Let $\hat{D}_{\boldsymbol{\sigma}}$ be quotient of $\hat{F}_{\boldsymbol{\sigma}}$ by*

$$\sigma_i^2 = 1,\ i = 1,2,3,4 \ (\textit{so } \sigma_1\sigma_2 = \sigma_4\sigma_3).$$

Then, $\prod_{p\neq 2} \hat{\mathbb{Z}}_p^2 \times^s J_2 \equiv \hat{D}_{\boldsymbol{\sigma}}$, $\hat{\mathbb{Z}}_p^2 \times^s J_2$ is the unique $\mathbf{C}_{2^4}$ p-Nielsen class limit.

Outline. We show $\hat{D}_{\boldsymbol{\sigma}}$ is $\hat{\mathbb{Z}}^2 \times^s J_2$ and $\sigma_1\sigma_2$ and $\sigma_1\sigma_3$ are independent generators of $\hat{\mathbb{Z}}^2$. Then, σ_1 acts on $\hat{\mathbb{Z}}^2$ by multiplication by -1. First: $\sigma_1(\sigma_1\sigma_2)\sigma_1 = \sigma_2\sigma_1$ shows σ_1 conjugates $\sigma_1\sigma_2$ to its inverse. Also,

$$(\sigma_1\sigma_2)(\sigma_1\sigma_3) = (\sigma_1\sigma_3)\sigma_3(\sigma_2\sigma_1)\sigma_3 = (\sigma_1\sigma_3)(\sigma_1\sigma_2)$$

shows the said generators commute. The maximal pro-p quotient is $\mathbb{Z}_p^2 \times^s \{\pm 1\}$.

Show $G = U \times^s J_2$, U a quotient of $\mathbb{Z}^2$, gives nonempty Nielsen classe: Use cofinal family of U s, $(\mathbb{Z}/p^{k+1})^2$, $p \neq 2$. Two proofs: Elliptic curves; pure Nielsen class [TVol04,§6.2.2]. □

App. B: Nielsen classes for $F_2 \times^s \mathbb{Z}/3$ (p. 11)

Use H-M reps. (example g-p' cusps) as example of when there are projective systems of nonempty Nielsen classes [TVol04, Prop. 6.5].

Proposition 5. *$\hat{F}_{2,p} \times^s J_3$, $p \neq 3$, is the unique $\mathbf{C}_{\pm 3^2}$ p-Nielsen class limit.*

Proof. Let $G = G_p = (\mathbb{Z}/p)^2 \times^s J_3$: $\langle\alpha\rangle = J_3$. Assume $g_1 = (\alpha, \boldsymbol{v}_1), g_2 = (\alpha, \boldsymbol{v}_2)$ generate G. The H-M rep. $(g_1, g_1^{-1}, g_2, g_2^{-1})$ is in $\mathrm{Ni}(G, \mathbf{C}_{\pm 3^2})$. Conjugate in G, so $\boldsymbol{v}_1 = \mathbf{0}$. To find such generators, consider $g_1 g_2^{-1} = (1, -\boldsymbol{v}_2)$ and $g_1^2 g_2 = (1, \alpha^{-1}(v_2))$. So, g_1, g_2 generate precisely when $\langle -\boldsymbol{v}_2, \alpha^{-1}(v_2)\rangle = (\mathbb{Z}/p)^2$. Such a $\boldsymbol{v}_2$ exists because the eigenvalues of α are distinct. So $(\mathbb{Z}/p)^2$ is a cyclic $\langle\alpha\rangle$ module.

Now consider Nielsen class with $G = U \times^s J_3$ and U having $(\mathbb{Z}/p)^2$ as a quotient. There is a surjective map $\psi : G \to (\mathbb{Z}/p)^2 \times^s J_3$: a Frattini cover. So, if g'_1, g'_2 generate $(\mathbb{Z}/p)^2 \times^s J_3$, then respective order 3 lifts of g'_1, g'_2 to $g_1, g_2 \in G$ automatically generate G. □

App. C: Additions to the Luminy talk

Disappearance of o-p' cusps (p. 12): At level 0 of the $(A_5, \mathbf{C}_{3^4})$ MT ($p = 2$), all the cusps are $2'$ cusps (widths are 2, 3,3, 5, 5). At level 1, all o-$2'$ cusps have disappeared, leaving only g-$2'$ cusps (H-M reps.) among the $2'$ cusps [BFr02; §9.1].

Serre's modular curve p-Frattini property (p. 11): $G_{\mathbb{Q}}$ acts on any projective sequence of points $\{\boldsymbol{p}_k \in X_0(p^{k+1})\}_{k=0}^{\infty}$, lying over $j' \in \mathbb{Q}$. This gives a map $G_{\mathbb{Q}} \to \lim_{\infty \leftarrow k} \mathrm{PSL}_2(\mathbb{Z}/p^{k+1}) = \mathrm{PSL}_2(\mathbb{Z}_p)$. If the induced map to $\mathrm{PSL}_2(\mathbb{Z}/p)$ is onto, the Frattini property says the image of $G_{\mathbb{Q}}$ is onto $\mathrm{PSL}_2(\mathbb{Z}_p)$.

Frattini limits $\mathbb{Z}/p^{k+1} \times^s \mathbb{Z}/2$ and $(\mathbb{Z}/p^{k+1})2 \times^s \mathbb{Z}/2$ in Serre's OIT (p. 9): We expect a simpler OIT theorem for the $\mathbb{Z}/3$ case for primes p where $\mathbb{Z}/3$ acts irreducibly because there is just one Frattini limit. André's Theorem is compatible with that when $p = 2$.

www.math.uci.edu/~mfried/talkfiles/lum03-12-04.html
links to Talk Source pdf files

[Dur03] durhamsh09-30-03.pdf: 25 slides from the Talk in Durham, Noncommutative aspects of Number Theory, Aug. 28–Sept. 5, 2003:

[BFr02] with P. Bailey, *Hurwitz monodromy, spin separation and higher levels of a Modular Tower*, in Proceed. of Symposia in Pure Math. **70** (2002) editors M. Fried and Y. Ihara, 1999 von Neumann Symposium, August 16-27, 1999 MSRI, 79–221.

[RIMS02] *Moduli of relatively nilpotent extensions*, Institute of Mathematical Science Analysis 1267, June 2002, Communications in Arithmetic Fundamental Groups, 70–94.

[FrS03] with D. Semmen, *Schur multiplier types and Shimura-like systems of varieties*, latest version January 2004.

[TVol04] *Two genus 0 problems of John Thompson*, To appear in the Cambridge University Press volume dedicated to John Thompson's 70th birthday, Conference November, 2002.

Key inspirational papers behind the talk

[Se68] J.-P. Serre, *Abelian ℓ-adic representations and elliptic curves*, 1st ed., McGill University Lecture Notes, Benjamin, New York • Amsterdam, 1968, in collaboration with Willem Kuyk and John Labute.

[Se90a] J.-P. Serre, *Relêvements dans $\tilde{A}_n$*, C. R. Acad. Sci. Paris **311** (1990), 477–482.

[Ser90b] J.-P. Serre, *Revêtements á ramification impaire et thêta-caractéristiques*, C. R. Acad. Sci. Paris **311** (1990), 547–552.

[FV92] with H. Völklein, *The embedding problem over an Hilbertian-PAC field*, Annals of Math **135** (1992), 469–481.

Our main example alludes to the technique relating inner and absolute Hurwitz spaces through outer automophisms of finite groups. This has as a corollary the only known presentation for $G_{\mathbb{Q}}$:

$$1 \to \tilde{F}_\omega \to G_{\mathbb{Q}} \to \prod_{n=2}^{\infty} S_n \to 1.$$

[IM95] Y. Ihara and M. Matsumoto, *On Galois actions on profinite completions of braid groups*, Proceedings AMS-NSF Summer Conference, vol. 186, 1995, Cont. Math series, Recent Developments in the Inverse Galois Problem, 173–200.

Others supporting sources

[An98] Y. André, *Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire*, Crelle's J. **505** (1998), 203–208.

[BTh03] P. Bailey, *Incremental ascent of a Modular Tower via branch cycle designs*, PhD Thesis, UCI Irvine 2003.

[GS78] R. Griess and P. Schmid, *The Frattini module*, Archiv. Math. **30** (1978), 256266.

[FrJ86] M. Fried and M. Jarden, *Field arithmetic*, Ergebnisse der Mathematik III, vol. 11, Springer Verlag, Heidelberg, 1986.

[DFr90] P. Dèbes and M. Fried, *Rigidity and real residue class fields*, Acta Arith. **56** (1990), 13–45.

[FrK97] M. Fried and Y. Kopeliovic, *Applying Modular Towers to the inverse Galois problem*, Geometric Galois Actions II Dessins d'Enfants, Mapping Class Groups . . . , vol. 243, Cambridge U. Press, 1997, London Math. Soc. Lecture Notes, pp. 172–197.