ENTIRE SOLUTIONS TO EQUATIONS OF MINIMAL SURFACE TYPE IN SIX DIMENSIONS

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Abstract. We construct nonlinear entire solutions in $\mathbb{R}^6$ to equations of minimal surface type that correspond to parametric elliptic functionals.

1. Introduction

A well-known theorem of Bernstein says that entire minimal graphs in $\mathbb{R}^3$ are planes. Building on work of Fleming [6], De Giorgi [5], and Almgren [1], Simons [12] extended this result to minimal graphs in $\mathbb{R}^{n+1}$ for $n \leq 7$. In contrast, there are nonlinear entire solutions to the minimal surface equation in dimension $n \geq 8$ due to Bombieri-De Giorgi-Giusti [3] and Simon [10].

In this paper we study the Bernstein problem for a more general class of parametric elliptic functionals. These assign to an oriented hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ the value

$$A_\Phi(\Sigma) := \int_\Sigma \Phi(\nu),$$

where $\nu$ is a choice of unit normal to $\Sigma$ and $\Phi \in C^{2,\alpha}(\mathbb{S}^n)$ is a positive even function. We say $\Phi$ is uniformly elliptic if its one-homogeneous extension to $\mathbb{R}^{n+1}$ has uniformly convex level sets. The case $\Phi = 1$ corresponds to the area functional. In the general case, the minimizers of $A_\Phi$ model crystal surfaces (see [9] and the references therein). Below we assume $\Phi$ is uniformly elliptic unless otherwise specified.

When a critical point of $A_\Phi$ can be written as the graph of a function $u$ on a domain $\Omega \subset \mathbb{R}^n$, we say that $u$ is $\Phi$-minimal. It solves an elliptic equation of minimal surface type (see Section 2). Jenkins [8] proved that global $\Phi$-minimal functions are linear in dimension $n = 2$. Simon [11] extended this result to dimension $n = 3$, using an important regularity theorem of Almgren-Schoen-Simon [2] for minimizers of the parametric problem. He also showed that the result holds up to dimension $n = 7$ when $\Phi$ is close in an appropriate sense to the area integrand.

The purpose of this paper is to construct a nonlinear entire $\Phi$-minimal function on $\mathbb{R}^6$, for an appropriate uniformly elliptic integrand (which is necessarily far from the area integrand). Our main theorem is:

**Theorem 1.1.** There exists a quadratic polynomial $u$ on $\mathbb{R}^6$ that is $\Phi$-minimal for a uniformly elliptic integrand $\Phi \in C^{2,1}(\mathbb{S}^6)$.

Theorem 1.1 settles the Bernstein problem for equations of minimal surface type in dimension $n \geq 6$, leaving open the cases $n = 4, 5$. It also answers the question whether or not there exists a nonlinear polynomial that solves such an equation. It remains an interesting open question whether or not there exists a nonlinear polynomial that solves the minimal surface equation.
Our approach to constructing entire solutions is different from the one taken by Bombieri-De Giorgi-Giusti, which is based on constructing super- and sub-solutions. We instead fix $u$, which reduces the problem to solving a linear hyperbolic equation for $\Phi$. It turns out that in $\mathbb{R}^6$, we can choose a quadratic polynomial $u$ such that the solutions to this hyperbolic equation are given by an explicit representation formula. By prescribing the Cauchy data carefully we obtain an integrand with the desired properties.

As a consequence of Theorem 1.1 we show that the cone over $S^2 \times S^2$ in $\mathbb{R}^6$ minimizes $A_{\Phi_0}$, where $\Phi_0$ is the restriction of the integrand $\Phi$ from Theorem 1.1 to $\mathbb{S}^6 \cap \{x_7 = 0\}$ (see Remark 3.3). In fact, each level set of the function $u$ from Theorem 1.1 minimizes $A_{\Phi_0}$. (This observation is what guided us to the example).

The analogue of the quadratic polynomial $u$ from Theorem 1.1 in dimension $n = 4$ is not $\Phi$-minimal for any uniformly elliptic integrand $\Phi$ that enjoys certain natural symmetries (see Remark 3.4). However, it is feasible that our approach could produce entire $\Phi$-minimal functions in the lowest possible dimension $n = 4$, that have sub-quadratic growth (see Remark 3.5).

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2. Preliminaries

2.1. Legendre Transform. Let $w$ be a smooth function on a domain $\Omega \subset \mathbb{R}^n$, and assume that $\nabla w$ is a diffeomorphism with inverse $X$. We define the Legendre transform $w^*$ on the image of $\nabla w$ by

$$w^*(p) := p \cdot X(p) - w(X(p)).$$

Differentiating two times we obtain

$$\nabla w^*(p) = X(p), \quad D^2w^*(p) = (D^2w)^{-1}(X(p)).$$

2.2. Euler-Lagrange Equation. Assume that $\Phi \in C^{2,\alpha}(\mathbb{S}^n)$ is a positive, uniformly elliptic integrand. Here and below we will identify $\Phi$ with its one-homogeneous extension to $\mathbb{R}^{n+1}$, and uniform ellipticity means that $\{\Phi < 1\}$ is uniformly convex.

If $\Sigma$ is the graph of a smooth function $u$ on a domain $\Omega \subset \mathbb{R}^n$ then we can rewrite the variational integral (1) as

$$A_{\Phi}(\Sigma) = \int_{\Omega} \varphi(\nabla u) \, dx,$$

where

$$\varphi(p) := \Phi(-p, 1).$$

Thus, if $\Sigma$ is a critical point of $A_{\Phi}$ then $u$ solves the Euler-Lagrange equation

$$\text{div}(\nabla \varphi(\nabla u)) = \varphi_{ij}(\nabla u)u_{ij} = 0$$

in $\Omega$. The function $\varphi$ is locally uniformly convex (by the uniform ellipticity of $\Phi$), but the ratio of the minimum to maximum eigenvalues of $D^2 \varphi$ degenerates at
infinity. Thus the equation (4) is a quasilinear degenerate elliptic PDE for $u$, known in the literature as a variational equation of minimal surface or mean curvature type (see e.g. Chapter 16 in [7] and the references therein).

Our approach is to rewrite (4) as a linear equation for $\varphi$. Assume that $\nabla u$ is a smooth diffeomorphism. Then using the relations in (2) we can rewrite the equation (4) as

$$(u^*)^{ij}(y)\varphi_{ij}(y) = 0$$

for $y$ in the image of $\nabla u$. Below we will fix $u^*$, and then solve the equation (5) for $\varphi$.

Remark 2.1. In parametric form, the Euler-Lagrange equation (4) for a critical point $\Sigma$ of $A_\Phi$ is

$$(6) \quad \text{tr}(D^2 \Phi(\nu^\Sigma(x)) \cdot II^\Sigma(x)) = \Phi_{ij}(\nu^\Sigma(x))II^\Sigma_{ij}(x) = 0,$$

where $\nu^\Sigma$ is the Gauss map of $\Sigma$ and $II^\Sigma$ is the second fundamental form of $\Sigma$. We note that (6) is invariant under dilations of $\Sigma$. Equation (4) can be viewed as the projection of the equation (6) onto a hyperplane.

Remark 2.2. The graph $\Sigma$ of an entire solution to (4) is not only a critical point, but a minimizer of $A_\Phi$. One way to see this is to observe that the translations of $\Sigma$ in the $x_{n+1}$ direction foliate either side of $\Sigma$. Another way is to extend the unit normal $\nu$ on $\Sigma$ to $\mathbb{R}^{n+1}$ by letting it be constant in the $x_{n+1}$ direction, and then show that $\nabla \Phi(\nu)$ is a calibration. Indeed, $\nabla \Phi(\nu)$ is divergence-free in $\mathbb{R}^{n+1}$ by the equation (6), and by viewing $\Phi$ as the support function of the uniformly convex hypersurface $K := \nabla \Phi(S^n)$ we see that

$$\nabla \Phi(\nu) \cdot \hat{\nu} \leq \Phi(\hat{\nu})$$

for any $\nu, \hat{\nu} \in S^n$, with equality if and only if $\nu = \hat{\nu}$.

3. Proof of Theorem 1.1

We denote points in $\mathbb{R}^6$ by $(p, q)$, with $p, q \in \mathbb{R}^3$. The polynomial $u$ from Theorem 1.1 is

$$(7) \quad u(p, q) := \frac{1}{2}(|p|^2 - |q|^2).$$

We note that $u = u^*$. Below we let $\Box$ denote the wave operator $\partial_x^2 - \partial_y^2$ on $\mathbb{R}^2$.

Lemma 3.1. To prove Theorem 1.1 it suffices to find an analytic function $\psi(x, y)$ on $\mathbb{R}^2$ that is even in $x$ and $y$, solves the PDE

$$(8) \quad \Box \psi + 2 \nabla \psi \cdot \left(\frac{1}{x}, -\frac{1}{y}\right) = 0$$

in the positive quadrant, and satisfies that the one-homogeneous function

$$\Psi(x, y, z) = |z| \psi\left(\frac{x}{z}, \frac{y}{z}\right)$$

on $\mathbb{R}^3 \setminus \{z = 0\}$ has a continuous extension to $\mathbb{R}^3$ that is positive and locally $C^{2,1}$ on $\mathbb{R}^3 \setminus \{0\}$, and has uniformly convex level sets.
Proof. Suppose we have found such a function $\psi$, and denote points in $\mathbb{R}^7$ by $(p, q, z)$ with $p, q \in \mathbb{R}^3$ and $z \in \mathbb{R}$. Then the function

$$\Phi(p, q, z) := \Psi(|p|, |q|, z)$$

satisfies the desired regularity and convexity conditions. Furthermore, if we define $\varphi$ by the relation (3), that is,

$$\varphi(p, q) := \Phi(-p, -q, 1) = \psi(|p|, |q|),$$

then by the definition (7) of $u$ and the equation (8) for $\psi$ we have

$$(u^*)_{ij} \varphi_{ij} = 0$$
on $\mathbb{R}^6$. Hence equation (5) holds and the function $u$ is $\Phi$-minimal. □

Proof of Theorem 1.1. We note that a function $\psi$ solves (8) in the positive quadrant if and only if

$$\Box(x y \psi) = 0.$$ 

The general solution to (8) is thus given by the formula

$$\psi(x, y) = \frac{f(x + y) + g(x - y)}{xy}.$$ 

We will show that the choice

$$f(s) = -g(s) = 2^{-\frac{3}{2}}(2 + s^2)^{3/2}$$
gives a function $\psi$ satisfying the remaining conditions of Lemma 3.1.

After rotating the plane by $\frac{\pi}{4}$ (and for ease of notation continuing to denote the coordinates by $x$ and $y$) we have for the above choices of $f$ and $g$ that

$$\psi(x, y) = \frac{(1 + x^2)^{3/2} - (1 + y^2)^{3/2}}{x^2 - y^2} = \frac{A^2 + AB + B^2}{A + B},$$

where

$$A := (1 + x^2)^{1/2}, \quad B := (1 + y^2)^{1/2}.$$ 

Hence $\psi$ is positive, analytic, and invariant under reflection over the axes and the diagonals. Furthermore, $\psi$ is locally uniformly convex. Indeed, after some calculation (which we omit) we arrive at

$$\det D^2 \psi = 3 (A + B)^{-4} \left( 2 + \frac{1}{AB} \right) > 0,$$

and since

$$D^2 \psi(0, 0) = \frac{3}{4} I$$

we conclude that $D^2 \psi$ is everywhere positive definite.

Now let

$$\Psi(x, y, z) := |z| \psi\left(\frac{x}{z}, \frac{y}{z}\right)$$

$$= \frac{(x^2 + z^2)^{3/2} - (y^2 + z^2)^{3/2}}{x^2 - y^2} = \frac{D^2 + DE + E^2}{D + E},$$
where
\[ D := (x^2 + z^2)^{1/2}, \quad E := (y^2 + z^2)^{1/2}. \]
By the local uniform convexity and analyticity of \( \psi \) and the one-homogeneity of \( \Psi \), we just need to check that \( \Psi \in C^{2,1} \) in a neighborhood of the circle \( S^2 \cap \{ z = 0 \} \), and that on this circle the Hessian of \( \Psi \) restricted to any plane tangent to \( S^2 \) is positive definite.

Restricting \( \Psi \) to the plane \( \{ x = 1 \} \) we get a function of \( y \) and \( z \) that is \( C^{2,1} \) in a neighborhood of the origin (and analytic away from the origin), and at \( (1, 0, 0) \) we have
\[ \Psi_{yy} = 2, \Psi_{yz} = 0, \Psi_{zz} = 3. \]
By the symmetries of \( \Psi \) this gives the result in a neighborhood of the points \( (\pm 1, 0, 0) \) and \( (0, \pm 1, 0) \). We may thus restrict our attention to the region
\[ \Omega_{\delta} := \{|x|, |y| \geq \delta\} \]
for \( \delta > 0 \) sufficiently small. In the region \( \Omega_{\delta} \cap \{|z| < \frac{\delta}{2}\} \) the function \( \Psi \) is analytic, and has the expansion
\[
\Psi(x, y, z) = \frac{x^2 + |xy| + y^2}{|x| + |y|} + \frac{3}{2} \frac{z^2}{|x| + |y|} - \frac{1}{|x| + |y|} \sum_{k \geq 2} a_k \left( \sum_{i=0}^{2k-4} \frac{1}{|x|^{i+1} |y|^{2k-3-i}} \right) z^{2k},
\]
where \( a_k \) are the coefficients in the Taylor series of \( (1 + s)^{3/2} \) around \( s = 0 \). Thus, for any unit vector \( e \in \{ z = 0 \} \) we have on \( \Omega_{\delta} \cap \{ z = 0 \} \) that
\[ \Psi_{ez} = 0, \Psi_{zz} = \frac{3}{|x| + |y|}, \]
It only remains to check that the first term in (9) is locally uniformly convex on lines in \( \{ z = 0 \} \) that don’t pass through the origin. By its one-homogeneity and symmetry in \( x \) and \( y \), it suffices to check this on the line \( \{ x = 1 \} \). Since
\[ \Psi(1, y, 0) = |y| + \frac{1}{1 + |y|} \]
is locally uniformly convex, we are done.

\[ \square \]

Remark 3.2. The integrand from Theorem 1.1 is given explicitly by the formula
\[
\Phi(p, q, z) = \frac{((|p| + |q|)^2 + 2z^2)^{3/2} - ((|p| - |q|)^2 + 2z^2)^{3/2}}{2^{5/2} |p||q|},
\]
where \( p, q \in \mathbb{R}^3 \) and \( z \in \mathbb{R} \).

Remark 3.3. Theorem 1.1 implies that the cone \( C \) over \( S^2 \times S^2 \) in \( \mathbb{R}^6 \) is a minimizer of \( A_{\Phi_0} \), where
\[ \Phi_0(p, q) = \frac{||p| + |q||^3 - ||p| - |q||^3}{2^{5/2} |p||q|} \]
is the restriction of \( \Phi \) (defined by (10)) to the hyperplane \( \{ z = 0 \} \). Indeed, the hypersurfaces \( \{ u = \pm 1 \} \) are critical points of \( A_{\Phi_0} \), and their dilations foliate either side of \( C \). To see e.g. that \( \{ u = 1 \} \) is a critical point of \( A_{\Phi_0} \), first note that \( Ru \) is \( \Phi \)-minimal for all \( R > 0 \) by the homogeneity of \( u \) and the invariance of the equation.
(4) under the rescalings \( u \to R^{-1}u(Rx) \). Then write the equation (6) for the graph of \( Ru \) over points in \( \{u = 1\} \), and pass to the limit as \( R \to \infty \).

Remark 3.4. The analogue of the quadratic polynomial (7) in \( \mathbb{R}^4 \), where \( p, q \in \mathbb{R}^2 \), is not \( \Phi \)-minimal for any uniformly elliptic integrand \( \Phi \) on \( S^4 \) that depends only on \( |p|, |q| \) and \( z \). To see this, we first observe that by the reasoning in Remark 3.3 it suffices to show that \( \{u = 1\} \) is not a critical point of \( A_{\Phi_0} \) for any uniformly elliptic integrand \( \Phi_0 \) on \( S^3 \) that depends only on \( |p| \) and \( |q| \). When we fix \( \Sigma := \{u = 1\} \) and impose that \( \Phi_0 \) depends only on \( |p| \) and \( |q| \), the equation (6) reduces to an ODE. By analyzing this ODE one can show that one eigenvalue of \( D^2\Phi_0 \) will tend to infinity on the Clifford torus \( S^1 \times S^1 \).

Remark 3.5. If we take \( u^*(p, q) = \frac{1}{m}(|p|^m - |q|^m) \) and \( \varphi(p, q) = \psi(|p|, |q|) \) with \( p, q \in \mathbb{R}^{k+1} \), then equation (5) is equivalent to the hyperbolic PDE

\[
\frac{1}{m-1} x^{2-m} \psi_{xx} + k x^{1-m} \psi_x = \frac{1}{m-1} y^{2-m} \psi_{yy} + k y^{1-m} \psi_y
\]

for \( \psi \) in the positive quadrant. The Cauchy problem for this equation can be solved in terms of certain hypergeometric functions (see [4] and the references therein). In special cases the representation formula is particularly simple, e.g. when \( k = m = 2 \) (treated above), or when \( k = 1 \) and \( m = 4 \), in which case the general solution is

\[
\psi(x, y) = \frac{f(x^2 + y^2)}{x^2 y^2} + g(x^2 - y^2).
\]

The corresponding integrand \( \Phi \) (constructed as in the proof of Lemma 3.1) is not uniformly elliptic for any choice of \( f \) and \( g \), because the maximum and minimum principal curvatures of the graph of

\[
u = \frac{3}{4} \left( \frac{1}{4} |p|^\frac{4}{3} - |q|^\frac{4}{3} \right)
\]

are not of comparable size near \( \{|p| = |q| = 0\} \). However, it is feasible that for a judicious choice of \( f \) and \( g \), one could make a small perturbation of the corresponding integrand and then use the method of super- and sub-solutions to construct an entire solution to a variational equation of minimal surface type in \( \mathbb{R}^4 \) that grows at the same rate as \( u \).

References


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