ENTIRE SOLUTIONS TO EQUATIONS OF MINIMAL SURFACE TYPE IN SIX DIMENSIONS

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Abstract. We construct nonlinear entire solutions in $\mathbb{R}^6$ to equations of minimal surface type that correspond to parametric elliptic functionals.

1. Introduction

A well-known theorem of Bernstein says that entire minimal graphs in $\mathbb{R}^3$ are planes. Building on work of Fleming [6], De Giorgi [5], and Almgren [1], Simons [13] extended this result to minimal graphs in $\mathbb{R}^{n+1}$ for $n \leq 7$. In contrast, there are nonlinear entire solutions to the minimal surface equation in dimension $n \geq 8$ due to Bombieri-De Giorgi-Giusti [3] and Simon [11].

In this paper we study the Bernstein problem for a more general class of parametric elliptic functionals. These assign to an oriented hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ the value

$$A_\Phi(\Sigma) := \int_\Sigma \Phi(\nu), \tag{1}$$

where ν is a choice of unit normal to Σ and $\Phi \in C^{2,\alpha}(\mathbb{S}^n)$ is a positive even function. We say Φ is uniformly elliptic if its one-homogeneous extension to $\mathbb{R}^{n+1}$ has uniformly convex level sets. The case $\Phi = 1$ corresponds to the area functional. In the general case, the minimizers of A_Φ model crystal surfaces (see [10] and the references therein). Below we assume Φ is uniformly elliptic unless otherwise specified.

When a critical point of A_Φ can be written as the graph of a function u on a domain $\Omega \subset \mathbb{R}^n$, we say that u is Φ-minimal. It solves an elliptic equation of minimal surface type (see Section 2). Jenkins [9] proved that global Φ-minimal functions are linear in dimension $n = 2$. Simon [12] extended this result to dimension $n = 3$, using an important regularity theorem of Almgren-Schoen-Simon [2] for minimizers of the parametric problem. He also showed that the result holds up to dimension $n = 7$ when Φ is close in an appropriate sense to the area integrand.

The purpose of this paper is to construct a nonlinear entire Φ-minimal function on $\mathbb{R}^6$, for an appropriate uniformly elliptic integrand (which is necessarily far from the area integrand). Our main theorem is:

Theorem 1.1. *There exists a quadratic polynomial u on $\mathbb{R}^6$ that is Φ-minimal for a uniformly elliptic integrand $\Phi \in C^{2,1}(\mathbb{S}^6)$.*

Theorem 1.1 settles the Bernstein problem for equations of minimal surface type in dimension $n \geq 6$, leaving open the cases $n = 4, 5$. It also answers the question whether or not there exists a nonlinear polynomial that solves such an equation.

It remains an interesting open question whether or not there exists a nonlinear polynomial that solves the minimal surface equation.

Our approach to constructing entire solutions is different from the one taken by Bombieri-De Giorgi-Giusti, which is based on constructing super- and sub- solutions. We instead fix u, which reduces the problem to solving a linear hyperbolic equation for Φ. It turns out that in $\mathbb{R}^6$, we can choose a quadratic polynomial u such that the solutions to this hyperbolic equation are given by an explicit representation formula. By prescribing the Cauchy data carefully we obtain an integrand with the desired properties.

As a consequence of Theorem 1.1 we show that the cone over $\mathbb{S}^2 \times \mathbb{S}^2$ in $\mathbb{R}^6$ minimizes A_{Φ_0}, where Φ_0 is the restriction of the integrand Φ from Theorem 1.1 to $\mathbb{S}^6 \cap \{x_7 = 0\}$ (see Remark 3.4). In fact, each level set of the function u from Theorem 1.1 minimizes A_{Φ_0}. (This observation is what guided us to the example). Morgan [10] previously showed that the cone over $\mathbb{S}^k \times \mathbb{S}^k$ in $\mathbb{R}^{2k+2}$ minimizes a parametric elliptic functional for each $k \geq 1$, using the method of calibrations.

Finally, we remark that the analogue of the quadratic polynomial u from Theorem 1.1 in dimension $n = 4$ is not Φ-minimal for any uniformly elliptic integrand Φ (see Remark 3.5). However, it is feasible that our approach could produce entire Φ-minimal functions in the lowest possible dimension $n = 4$, that have sub-quadratic growth (see Remark 3.6).

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2. Preliminaries

2.1. **Legendre Transform.** Let w be a smooth function on a domain $\Omega \subset \mathbb{R}^n$, and assume that ∇w is a diffeomorphism with inverse X. We define the Legendre transform w^* on the image of ∇w by

$$w^*(p) := p \cdot X(p) - w(X(p)).$$

Differentiating two times we obtain

$$\nabla w^*(p) = X(p), \quad D^2 w^*(p) = (D^2 w)^{-1}(X(p)). \tag{2}$$

2.2. **Euler-Lagrange Equation.** Assume that $\Phi \in C^{2,\alpha}(\mathbb{S}^n)$ is a positive, uniformly elliptic integrand. Here and below we will identify Φ with its one-homogeneous extension to $\mathbb{R}^{n+1}$, and uniform ellipticity means that $\{\Phi < 1\}$ is uniformly convex.

If Σ is the graph of a smooth function u on a domain $\Omega \subset \mathbb{R}^n$ then we can rewrite the variational integral (1) as

$$A_\Phi(\Sigma) = \int_\Omega \varphi(\nabla u)\, dx,$$

where

$$\varphi(p) := \Phi(-p, 1). \tag{3}$$

Thus, if Σ is a critical point of A_Φ then u solves the Euler-Lagrange equation

$$\mathrm{div}(\nabla \varphi(\nabla u)) = \varphi_{ij}(\nabla u) u_{ij} = 0 \tag{4}$$

in Ω. The function φ is locally uniformly convex (by the uniform ellipticity of Φ), but the ratio of the minimum to maximum eigenvalues of $D^2\varphi$ degenerates at infinity. Thus the equation (4) is a quasilinear degenerate elliptic PDE for u, known in the literature as a variational equation of minimal surface or mean curvature type (see e.g. Chapter 16 in [7] and the references therein).

Our approach is to rewrite (4) as a linear equation for φ. Assume that ∇u is a smooth diffeomorphism. Then using the relations in (2) we can rewrite the equation (4) as

$$(u^*)^{ij}(y)\varphi_{ij}(y) = 0 \tag{5}$$

for y in the image of ∇u. Below we will fix u^*, and then solve the equation (5) for φ.

Remark 2.1. In parametric form, the Euler-Lagrange equation (4) for a critical point Σ of A_Φ is

$$\mathrm{tr}(D^2\Phi(\nu^\Sigma(x)) \cdot II^\Sigma(x)) = \Phi_{ij}(\nu^\Sigma(x))II^\Sigma_{ij}(x) = 0, \tag{6}$$

where ν^Σ is the Gauss map of Σ and II^Σ is the second fundamental form of Σ. We note that (6) is invariant under dilations of Σ. Equation (4) can be viewed as the projection of the equation (6) onto a hyperplane.

Remark 2.2. The graph Σ of an entire solution to (4) is not only a critical point, but a minimizer of A_Φ. One way to see this is to observe that the translations of Σ in the x_{n+1} direction foliate either side of Σ. Another way is to extend the unit normal ν on Σ to $\mathbb{R}^{n+1}$ by letting it be constant in the x_{n+1} direction, and then show that $\nabla\Phi(\nu)$ is a calibration. Indeed, $\nabla\Phi(\nu)$ is divergence-free in $\mathbb{R}^{n+1}$ by the equation (6), and by viewing Φ as the support function of the uniformly convex hypersurface $K := \nabla\Phi(\mathbb{S}^n)$ we see that

$$\nabla\Phi(\nu) \cdot \tilde{\nu} \leq \Phi(\tilde{\nu})$$

for any ν, $\tilde{\nu} \in \mathbb{S}^n$, with equality if and only if $\nu = \tilde{\nu}$.

Remark 2.3. One can show in the same way as in Remark 2.2 that the graph Σ of a solution to (4) on a bounded domain $\Omega \subset \mathbb{R}^n$ minimizes A_Φ among hypersurfaces in $\mathbb{R}^{n+1}$ with boundary $\partial\Sigma$ that are contained in $\overline{\Omega} \times \mathbb{R}$. If in addition Ω is convex, or more generally, $\Omega = \Omega_1$ can be obtained by starting with a convex domain Ω_0 and continuously deforming through a path of disk-type domains Ω_t for $t \in [0, 1]$ whose boundaries have nonnegative mean curvature with respect to A_Φ (that is, $\Phi_{ij}(\nu^{\partial\Omega_t})II^{\partial\Omega_t}_{ij} \geq 0$ where $\nu^{\partial\Omega_t}$ is the inward unit normal) in a way that $\Omega \subset \Omega_t$ for all $t \in [0, 1]$, then Σ minimizes A_Φ among all hypersurfaces in $\mathbb{R}^{n+1}$ with the same boundary (see e.g. Theorem C in [8]).

3. Proof of Theorem 1.1

We denote points in $\mathbb{R}^6$ by (p, q), with $p, q \in \mathbb{R}^3$. The polynomial u from Theorem 1.1 is

$$u(p, q) := \frac{1}{2}(|p|^2 - |q|^2). \tag{7}$$

We note that $u = u^*$. Below we let $\Box$ denote the wave operator $\partial_x^2 - \partial_y^2$ on $\mathbb{R}^2$.

Lemma 3.1. *To prove Theorem 1.1 it suffices to find an analytic function* $\psi(x, y)$ *on* $\mathbb{R}^2$ *that is even in* x *and* y*, solves the PDE*

$$\Box\psi + 2\,\nabla\psi \cdot \left(\frac{1}{x}, -\frac{1}{y}\right) = 0 \tag{8}$$

in the positive quadrant, and satisfies that the one-homogeneous function

$$\Psi(x, y, z) = |z|\,\psi\left(\frac{x}{z}, \frac{y}{z}\right)$$

on $\mathbb{R}^3\backslash\{z = 0\}$ *has a continuous extension to* $\mathbb{R}^3$ *that is positive and locally* $C^{2,1}$ *on* $\mathbb{R}^3\backslash\{0\}$*, and has uniformly convex level sets.*

Proof. Suppose we have found such a function ψ, and denote points in $\mathbb{R}^7$ by (p, q, z) with $p, q \in \mathbb{R}^3$ and $z \in \mathbb{R}$. Then the function

$$\Phi(p, q, z) := \Psi(|p|, |q|, z)$$

satisfies the desired regularity and convexity conditions. Furthermore, if we define φ by the relation (3), that is,

$$\varphi(p, q) := \Phi(-p, -q, 1) = \psi(|p|, |q|),$$

then by the definition (7) of u and the equation (8) for ψ we have

$$(u^*)^{ij}\varphi_{ij} = 0$$

on $\mathbb{R}^6$. Hence equation (5) holds and the function u is Φ-minimal. $\Box$

Proof of Theorem 1.1. We note that a function ψ solves (8) in the positive quadrant if and only if

$$\Box(x\,y\,\psi) = 0.$$

The general solution to (8) is thus given by the formula

$$\psi(x, y) = \frac{f(x + y) + g(x - y)}{xy}.$$

We will show that the choice

$$f(s) = -g(s) = 2^{-\frac{5}{2}}(2 + s^2)^{3/2}$$

gives a function ψ satisfying the remaining conditions of Lemma 3.1.

After rotating the plane by $\frac{\pi}{4}$ (and for ease of notation continuing to denote the coordinates by x and y) we have for the above choices of f and g that

$$\begin{aligned}\psi(x, y) &= \frac{(1 + x^2)^{3/2} - (1 + y^2)^{3/2}}{x^2 - y^2} \\ &= \frac{A^2 + AB + B^2}{A + B},\end{aligned}$$

where

$$A := (1 + x^2)^{1/2}, \quad B := (1 + y^2)^{1/2}.$$

Hence ψ is positive, analytic, and invariant under reflection over the axes and the diagonals. Furthermore, ψ is locally uniformly convex. Indeed, after some calculation (which we omit) we arrive at

$$\det D^2\psi = 3\,(A + B)^{-4}\left(2 + \frac{1}{AB}\right) > 0,$$

and since

$$D^2\psi(0,\,0) = \frac{3}{4}I$$

we conclude that $D^2\psi$ is everywhere positive definite.

Now let

$$\begin{aligned}\Psi(x,\,y,\,z) &:= |z|\,\psi\left(\frac{x}{z},\,\frac{y}{z}\right)\\ &= \frac{(x^2+z^2)^{3/2}-(y^2+z^2)^{3/2}}{x^2-y^2}\\ &= \frac{D^2+DE+E^2}{D+E},\end{aligned}$$

where

$$D := (x^2+z^2)^{1/2}, \quad E := (y^2+z^2)^{1/2}.$$

By the local uniform convexity and analyticity of ψ and the one-homogeneity of Ψ, we just need to check that $\Psi \in C^{2,\,1}$ in a neighborhood of the circle $\mathbb{S}^2 \cap \{z = 0\}$, and that on this circle the Hessian of Ψ restricted to any plane tangent to $\mathbb{S}^2$ is positive definite.

Restricting Ψ to the plane $\{x = 1\}$ we get a function of y and z that is $C^{2,\,1}$ in a neighborhood of the origin (and analytic away from the origin), and at $(1,\,0,\,0)$ we have

$$\Psi_{yy} = 2,\ \Psi_{yz} = 0,\ \Psi_{zz} = 3.$$

By the symmetries of Ψ this gives the result in a neighborhood of the points $(\pm 1,\,0,\,0)$ and $(0,\,\pm 1,\,0)$. We may thus restrict our attention to the region

$$\Omega_\delta := \{|x|,\,|y| \geq \delta\}$$

for $\delta > 0$ sufficiently small. In the region $\Omega_\delta \cap \left\{|z| < \frac{\delta}{2}\right\}$ the function Ψ is analytic, and has the expansion

$$\begin{aligned}\Psi(x,\,y,\,z) = {} & \frac{x^2+|xy|+y^2}{|x|+|y|} + \frac{3}{2}\frac{z^2}{|x|+|y|}\\ & - \frac{1}{|x|+|y|}\sum_{k\geq 2} a_k\left(\sum_{i=0}^{2k-4}\frac{1}{|x|^{i+1}\,|y|^{2k-3-i}}\right)z^{2k},\end{aligned} \tag{9}$$

where a_k are the coefficients in the Taylor series of $(1+s)^{3/2}$ around $s = 0$. Thus, for any unit vector $e \in \{z = 0\}$ we have on $\Omega_\delta \cap \{z = 0\}$ that

$$\Psi_{ez} = 0,\ \Psi_{zz} = \frac{3}{|x|+|y|}.$$

It only remains to to check that the first term in (9) is locally uniformly convex on lines in $\{z = 0\}$ that don't pass through the origin. By its one-homogeneity and symmetry in x and y, it suffices to check this on the line $\{x = 1\}$. Since

$$\Psi(1,\,y,\,0) = |y| + \frac{1}{1+|y|}$$

is locally uniformly convex, we are done. $\square$

Remark 3.2. The integrand from Theorem 1.1 is given explicitly by the formula

$$\Phi(p,\, q,\, z) = \frac{\left((|p|+|q|)^2 + 2z^2\right)^{3/2} - \left((|p|-|q|)^2 + 2z^2\right)^{3/2}}{2^{5/2}|p||q|}, \tag{10}$$

where $p,\, q \in \mathbb{R}^3$ and $z \in \mathbb{R}$.

Remark 3.3. The convexity and regularity properties of Ψ can also be efficiently checked using the structure

$$\Psi(x,\, y,\, z) = G(\mathbf{w}(x,\, y,\, z)),$$

where

$$\mathbf{w}(x,\, y,\, z) = (|(x,\, z)|,\, |(y,\, z)|)$$

and

$$G(s,\, t) = s + t - \frac{st}{s+t}.$$

The key points are the convexity of the components of $\mathbf{w}$, and the convexity and monotonicity properties of G in the positive quadrant.

Remark 3.4. Theorem 1.1 implies that the cone C over $\mathbb{S}^2 \times \mathbb{S}^2$ in $\mathbb{R}^6$ is a minimizer of A_{Φ_0}, where

$$\Phi_0(p,\, q) = \frac{\big||p|+|q|\big|^3 - \big||p|-|q|\big|^3}{2^{5/2}|p||q|}$$

is the restriction of Φ (defined by (10)) to the hyperplane $\{z = 0\}$. In fact, every level set of u is a minimizer of A_{Φ_0}. To show this, it suffices to show that the hypersurfaces $\Sigma_{\pm} := \{u = \pm 1\}$ are critical points of A_{Φ_0}. Then all of the nonzero level sets of u, which are dilations of Σ_+ or Σ_-, would be critical points of A_{Φ_0} that foliate either side of $C = \{u = 0\}$.

To see e.g. that Σ_+ is a critical point of A_{Φ_0}, we first note that Ru is Φ-minimal for all $R > 0$ by the invariance of the equation (4) under the rescalings $u \to R^{-1}u(Rx)$ and the 2-homogeneity of u. Let II^R and ν^R denote the second fundamental form and upper unit normal to the graph of Ru in $\mathbb{R}^{n+1}$, and let II^{Σ_+} denote the second fundamental form and $\nu^{\Sigma_+} = -\nabla u/|\nabla u|$ the unit normal to Σ_+ in $\mathbb{R}^n$. At any point on Σ_+, we may choose a system of coordinates where $e_n = \nu^{\Sigma_+}$, so that an orthonormal basis of the tangent plane in $\mathbb{R}^{n+1}$ to the graph of Ru over this point contains the vectors e_1, ..., e_{n-1}. With respect to this basis we have

$$II^R_{ij} = O(R^{-1}) + \begin{cases} II^{\Sigma_+}_{ij}, & i,\, j < n \\ 0, & \text{otherwise.} \end{cases}$$

We also have that

$$\nu^R = (\nu^{\Sigma_+},\, 0) + O(R^{-1}).$$

On concludes by writing the equation (6) for the graph of Ru over points in Σ_+, and passing to the limit as $R \to \infty$, that Σ_+ is a critical point of A_{Φ_0}. This argument can in fact be used to show that the level sets of any homogeneous entire solution to (4) in $\mathbb{R}^n$ are critical points of A_{Φ_0}.

Remark 3.5. The analogue of the quadratic polynomial (7) in $\mathbb{R}^4$, where $p,\, q \in \mathbb{R}^2$, is not Φ-minimal for any uniformly elliptic integrand Φ on $\mathbb{S}^4$. To see this, we first note that by the invariance of u under rotations in p and in q, we may assume after averaging over these rotations that Φ depends only on $|p|$, $|q|$ and z. By the argument in Remark 3.4, it thus suffices to show that the level set $\{u = 1/2\}$ is not a critical point of A_{Φ_0} for any uniformly elliptic integrand Φ_0 on $\mathbb{S}^3$ that depends only on $|p|$ and $|q|$. When we fix $\Sigma = \{u = 1/2\}$, the equation (6) reduces to an ODE for Φ_0. By analyzing this ODE one can show that one eigenvalue of $D^2\Phi_0$ will tend to infinity on the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$.

More explicitly, we can write $\Sigma = \{|p| = \sigma(|q|)\}$, where

$$\sigma(t) := \sqrt{1+t^2},$$

and let $\varphi(s) := \Phi_0(1,\, s)$. When $p,\, q \in \mathbb{R}^{k+1}$, the equation (6) reduces to the ODE

$$\frac{\varphi''}{(\sigma^*)''} + k\frac{s\varphi' - \varphi}{s(\sigma^*)' - \sigma^*} + k\frac{\varphi'}{(\sigma^*)'} = 0 \tag{11}$$

on $(-1,\, 1)$, where

$$\sigma^*(s) = -\sqrt{1-s^2}$$

is the Legendre transform of σ. In the case $k = 2$ one checks directly that the even solutions to (11) are multiples of

$$\varphi(s) = 1 + \frac{1}{3}s^2.$$

This agrees with our construction from Theorem 1.1. However, in the case $k = 1$ one can see e.g. by expanding φ in a Taylor series that φ'' blows up near $s = \pm 1$. (In fact, for all even k, the even solutions to (11) are polynomials, and when $k \geq 2$ the second derivatives are bounded up to the endpoints.)

The cone over $\mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{R}^4$ is known to minimize a parametric elliptic functional (see [10]). It is natural to ask for a proof of this fact by foliation, and this remark shows that one cannot use level sets of the quadratic polynomial u, unlike in higher dimensions. However, it seems feasible that one can use surfaces that behave like level sets of functions that are homogeneous of degree smaller than two (see Remark 3.6). We intend to investigate this in future work.

Remark 3.6. If we take $u^*(p,\, q) = \frac{1}{m}(|p|^m - |q|^m)$ and $\varphi(p,\, q) = \psi(|p|,\, |q|)$ with $p,\, q \in \mathbb{R}^{k+1}$, then equation (5) is equivalent to the hyperbolic PDE

$$\frac{1}{m-1}x^{2-m}\psi_{xx} + k\,x^{1-m}\psi_x = \frac{1}{m-1}y^{2-m}\psi_{yy} + k\,y^{1-m}\psi_y$$

for ψ in the positive quadrant. The Cauchy problem for this equation can be solved in terms of certain hypergeometric functions (see [4] and the references therein). In special cases the representation formula is particularly simple, e.g. when $k = m = 2$ (treated above), or when $k = 1$ and $m = 4$, in which case the general solution is

$$\psi(x,\, y) = \frac{f(x^2+y^2) + g(x^2-y^2)}{x^2y^2}.$$

The corresponding integrand Φ (constructed as in the proof of Lemma 3.1) is not uniformly elliptic for any choice of f and g, because the maximum and minimum principal curvatures of the graph of

$$u = \frac{3}{4}\left(|p|^{\frac{4}{3}} - |q|^{\frac{4}{3}}\right)$$

are not of comparable size near $\{|p|\,|q| = 0\}$. However, it is feasible that for a judicious choice of f and g, one could make a small perturbation of the corresponding integrand and then use the method of super- and sub-solutions to construct an entire solution to a variational equation of minimal surface type in $\mathbb{R}^4$ that grows at the same rate as u.

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