A HALF-SPACE BERNSTEIN THEOREM FOR ANISOTROPIC MINIMAL GRAPHS

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ABSTRACT. We prove that an anisotropic minimal graph over a half-space with flat boundary must itself be flat. This generalizes a result of Edelen-Wang to the anisotropic case. The proof uses only the maximum principle and ideas from fully nonlinear PDE theory in lieu of a monotonicity formula.

1. INTRODUCTION

In this paper we prove that if Σ is an anisotropic minimal graph over a half-space and $\partial\Sigma$ is flat, then Σ is flat. More generally, we prove that if Σ is an anisotropic minimal graph over a convex domain that is not the whole space and Σ has linear boundary data, then Σ is flat.

We now state the result more precisely. We assume that $\Sigma \subset \mathbb{R}^{n+1}$ is the graph of a function $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is a convex domain that is not the whole space. We assume further that $u|_{\partial\Omega}$ agrees with a linear function L. Finally, we assume that Σ is a critical point of the functional

$$A_\Phi(\Sigma) := \int_\Sigma \Phi(\nu)\, d\mathcal{H}^n \qquad (1)$$

where $\mathcal{H}^n$ is n-dimensional Hausdorff measure, ν is the upper unit normal to Σ, and Φ is the support function of a smooth, bounded, uniformly convex set K (the Wulff shape) whose interior contains the origin. We prove:

Theorem 1.1. *Under the above conditions, u is linear.*

Our arguments in fact prove this result for a large variety of non-convex domains Ω contained in a half-space (see Remark 3.2). We state the result for convex domains for simplicity and to emphasize ideas. For bounded domains the result follows immediately from the maximum principle, so in what follows we always assume that Ω is unbounded.

We note that Theorem 1.1 holds in all dimensions, in contrast with Bernstein-type results for entire anisotropic minimal graphs (linearity is only guaranteed when $n \leq 3$ for general anisotropic functionals (see [14], [12], [9], [11]), and only when $n \leq 7$ in the case of the area functional $K = B_1$ (see [15], [3])). The linearity of the boundary data is thus quite powerful for rigidity.

Functionals of the form (1) are well-studied, both as natural generalizations of the area functional and as models e.g. of crystal formation ([2], [4], [5], [7], [8]). From a technical perspective, what distinguishes general anisotropic functionals from the area case is the absence of a monotonicity formula ([1]), so one cannot reduce regularity and Bernstein-type problems to the classification of cones. This requires the development of more general and sophisticated approaches. In the case of the area functional Theorem 1.1 was proven by Edelen-Wang in [6], and the monotonicity formula played an important role in the proof (particularly in the case that Ω is a half-space). In contrast, we use only the maximum principle, ideas from fully nonlinear PDE theory (namely, an ABP-type measure estimate (Lemma 2.1) and an argument reminiscent of the proof of Krylov's boundary Harnack inequality (Lemma 3.1), as exposed e.g. in Section 3 of [10]), and the minimizing properties of anisotropic minimal graphs.

The paper is organized as follows. In Section 2 we recall some useful facts about anisotropic functionals and about the minimizing properties of anisotropic minimal graphs, and we prove an ABP-type measure estimate. In Section 3 we prove Theorem 1.1.

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2. Preliminaries

2.1. **Anisotropic Minimal Surfaces.** First we recall a few useful identities related to the integrand Φ. First, we have

$$K = \nabla\Phi(\mathbb{S}^n), \quad \nu_K(\nabla\Phi(x)) = x \tag{2}$$

for $x \in \mathbb{S}^n$. Here ν_K is the outer unit normal to K. The second identity can be seen using the one-homogeneity of Φ, which implies that x is in the kernel of $D^2\Phi(x)$ for all $x \in \mathbb{R}^{n+1}\backslash\{0\}$. Differentiating the second identity we see that

$$II_K(\nabla\Phi(x)) = (D_T^2\Phi)^{-1}(x), \quad x \in \mathbb{S}^n. \tag{3}$$

Here $D_T^2\Phi(x)$ is the Hessian of Φ on the tangent plane to $\mathbb{S}^n$ at x, and here and below, II_S denotes the second fundamental form of a hypersurface S.

Next we recall that if S is a critical point of A_Φ with unit normal ν_S, then the Euler-Lagrange equation reads

$$\text{tr}(D_T^2\Phi(\nu_S(x))II_S(x)) = 0. \tag{4}$$

The property of being a critical point of A_Φ is dilation and translation invariant. Furthermore, isometries of S and ν_S by elements of $O(n+1)$ are critical points of anisotropic functionals obtained by performing the same isometries of K, and flipping the unit normal of S gives a critical point of the anisotropic functional obtained by replacing K with $-K$.

When S is the graph of a function w, and ν_S is the upper unit normal, the Euler-Lagrange equation (4) can be written

$$\text{tr}(D^2\phi(\nabla w)D^2 w) = 0, \tag{5}$$

where $\phi(z) = \Phi(-z, 1)$ for $z \in \mathbb{R}^n$. It follows from (3) and the uniform convexity of K that the equation (5) is uniformly elliptic when ∇w is bounded.

2.2. **Minimizing Properties of Graphs.** We will use the following minimizing property of anisotropic minimal graphs. Let $\Omega_S \subset \mathbb{R}^n$ be any domain and let S be a critical point of A_Φ given by the graph of a function $w \in C^\infty(\Omega_S) \cap C(\overline{\Omega_S})$, with upper unit normal ν_S. Let $E := \{x_{n+1} \leq w(x),\, x \in \overline{\Omega_S}\}$ be the subgraph of w. Finally, let $U \subset \mathbb{R}^{n+1}$ be any bounded open set that doesn't intersect the vertical sides $\{x_{n+1} \leq w(x),\, x \in \partial\Omega_S\}$. Then for any $U' \subset\subset U$, the anisotropic perimeter of $E \backslash U'$ (with respect to the outer unit normal) in U is at least the anisotropic area of S in U. This follows quickly from the observation that the vector field $\nabla\Phi(\nu_S)$ in the cylinder over Ω_S, extended to be constant in the x_{n+1} direction, is a calibration. Indeed, it is divergence-free (this follows from the Euler-Lagrange equation (4)), and satisfies $\nabla\Phi(\nu_S) \cdot a \leq \Phi(a)$ for all $a \in \mathbb{S}^n$, since $\Phi(a) = \max_{b \in \mathbb{S}^n} \nabla\Phi(b) \cdot a$.

2.3. **Measure Estimate.** Now we prove an ABP-type measure estimate reminiscent of the first step in the proof of the Krylov-Safonov Harnack inequality. The difference is that we do not deal with graphs. The following result is a generalization to the anisotropic case of a lemma proved for minimal surfaces in [13].

We first set some notation. We let $B_r(x)$ denote a ball in $\mathbb{R}^n$. We define $Q_{r,\,s,\,t}(x) \subset \mathbb{R}^{n+1}$ to be the cylinder $B_r(x) \times (s,\, t)$. For $\lambda \in (0,\, 1)$ we let the minimal Pucci operator $\mathcal{M}_\lambda^-$ on symmetric $n \times n$ matrices be defined by λ times the sum of positive eigenvalues plus λ^{-1} times the sum of negative eigenvalues.

The following lemma says that if an anisotropic minimal surface contained on one side of a hyperplane is very close at a point to the hyperplane, then it is very close at most points.

Lemma 2.1. *Assume that S is a smooth critical point of A_Φ in $Q_{1,\,-1,\,1}(0)$ given by the boundary of a set $E \subset Q_{1,0,1}(0)$. For all $\delta > 0$ small, there exists $\epsilon_0(\delta,\, n,\, K) > 0$ such that if $\epsilon e_{n+1} \in S$ and $\epsilon < \epsilon_0$, then S contains (and lies above) the graph of a function w on a set $G \subset B_{1/3}(0)$ such that*

$$|G| \geq |B_{1/3}| - C_1\delta^{1/2},\ 0 < w < C_1\delta^{3/2},\ \text{and } |\nabla w| < C_1\delta^{1/2}.$$

Here C_1 depends only on n, K.

Proof. We may assume that the unit normal to S is the inner unit normal to E, after possibly replacing K by $-K$. We claim that in each vertical cylinder over a ball of radius δ contained in $B_{1/3}(0)$ there is some point in S a distance at most $C_0(\delta,\, n,\, K)\epsilon$ from $\{x_{n+1} = 0\}$. Let $\lambda(K)$ be small enough that the eigenvalues of $D^2\phi$ are between λ and λ^{-1} in B_1, where $\phi(z) = \Phi(-z,\, 1)$. We can choose $M(n,\, K)$ large so that for

$$\varphi_0 := \min\{|x|^{-M},\, \delta^{-M}\} - (3/2)^M$$

we have $\mathcal{M}_\lambda^-(D^2\varphi_0) > 0$ outside of B_δ and $\varphi_0 > 1$ on $\partial B_{1/3}$. If $\epsilon_0(\delta,\, n,\, K)$ is small and $\epsilon < \epsilon_0$, then $\epsilon|\nabla\varphi_0| < 1$ outside B_δ, hence $\epsilon\varphi_0$ is a sub-solution to (5) outside B_δ. If the claim in the second sentence of the proof was false in the cylinder over some ball $B_\delta(x_0)$ for $C_0(\delta,\, n,\, K)$ sufficiently large, then we can slide the graph of $\epsilon\varphi_0(\cdot - x_0)$ from below until it touches S from one side outside of the cylinder over $B_\delta(x_0)$ (see Figure 1), and at the contact point we violate the equation (5).

Up to taking $\epsilon(\delta)$ smaller we may assume that $C_0\epsilon \leq \delta^{3/2}$. Below C_i, $i \geq 1$ will denote large constants depending on K. We let $r = C_2\delta^{1/2}$ and we slide copies of rK centered over points in $B_{1/3 - C_3\delta^{1/2}}$ from below until they touch S. By the first step, we can take C_2, C_3 such that the contact happens at points x that are

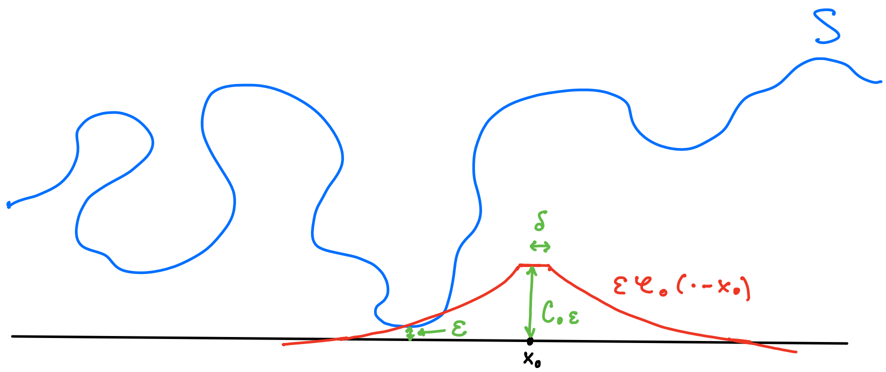

FIGURE 1. S gets $C_0\epsilon$ close to $\{x_{n+1} = 0\}$ at scale δ.

in the cylinder over $B_{1/3}$ and in $\{x_{n+1} < C_1\delta^{3/2}\}$, with upper unit normal $\nu(x)$ lying within $C_1\delta^{1/2}$ of e_{n+1} (see Figure 2). Here we are using that K is smooth and uniformly convex, hence has interior and exterior tangent spheres of universal radii (depending only on K) at all points on its boundary. The corresponding centers y can be found by the relation

$$y = x - r\nabla\Phi(\nu(x)).$$

Differentiating in x gives

$$D_x y = I + rD_T^2\Phi(\nu(x))II_S(x).$$

Since the second fundamental form of the rescaled Wulff shape at the contact point x is $r^{-1}(D_T^2\Phi)^{-1}(\nu(x))$ (see (3)), we have $II_S(x) \geq -r^{-1}(D_T^2\Phi)^{-1}(\nu(x))$, whence $D_x y \geq 0$. Since the second term is trace-free we have by the AGM inequality that

$$\det D_x y \leq 1.$$

Thus, the infinitesimal surface measure of centers y is smaller than that of contact points x. Since the tangent plane to the surface of contact points at x and the surface of centers at y is the same, the same inequality holds under projection in the x_{n+1} direction. Applying the area formula and recalling that the centers project in the x_{n+1} direction to $B_{1/3-C_3\delta^{1/2}}$ completes the proof.

□

After rescaling and covering arguments, Lemma 2.1 implies the following: For any $a,\, b > 0$, if a sequence of smooth critical points of A_Φ in $Q_{a,\,-b,\,b}(0)$ given by the boundaries of sets in $Q_{a,\,0,\,b}(0)$ with outer unit normal contain points that converge to a point in $B_a(0)\times\{0\}$, then they contain graphical portions over sets in B_a whose measure converges to $|B_a|$, such that the C^1 norms of these graphs tend to zero and the anisotropic area of these graphs approaches $|B_a|\Phi(-e_{n+1})$. We will use this fact below, appropriately translated and rotated.

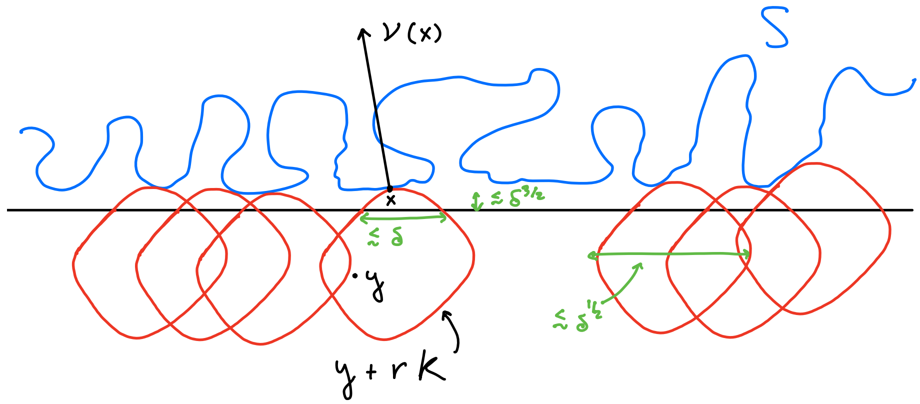

FIGURE 2. The contact points between copies of rK slid from below and S project to nearly the whole ball.

3. PROOF

Before proving Theorem 1.1 we establish some notation. After performing rigid motions, we may assume that $\Omega \subset \{x_1 > 0\}$, that $\{x_1 = 0\}$ is tangent to $\partial\Omega$ at the origin, and that $L(0) = 0$. We let

$$\Gamma = \text{graph}(L) \cap \{x_1 = 0\}.$$

There are three possibilities to consider:

(A) $\Omega = \{x_1 > 0\}$ (half-space case)
(B) $\Omega = \{0 < x_1 < c < \infty\}$ (slab case)
(C) $\overline{\Omega} \cap \{x_1 = 0\} \neq \{x_1 = 0\}$.

We define

$$A_+ := \inf\{A : u \leq L + Ax_1 \text{ in } \overline{\Omega}\}, \quad A_- := \sup\{A : u \geq L + Ax_1 \text{ in } \overline{\Omega}\} \tag{6}$$

where $A_+ \in \mathbb{R} \cup \{+\infty\}$ and $A_- \in \mathbb{R} \cup \{-\infty\}$. It is clear that $A_- \leq A_+$, and that $A_- \leq 0 \leq A_+$ in cases (B) and (C). To prove Theorem 1.1 it suffices to prove that $A_+ = A_-$.

We let $H_\pm$ denote the graphs of $L + A_\pm x_1$ in $\{x_1 \geq 0\}$. When $A_+ = \infty$ we interpret H_+ as the closed half-space in $\{x_1 = 0\}$ lying above Γ, and we understand H_- similarly when $A_- = -\infty$. Finally, we let

$$\Sigma_k := k^{-1}\Sigma \text{ and } u_k := k^{-1}u(k\cdot),$$

so that Σ_k are the graphs of u_k.

The following is a version of the Hopf lemma, and is reminiscent of a step in the proof of Krylov's boundary Harnack inequality.

Lemma 3.1. *Assume that A_- is anything in case (A) and nonzero in case (B) or (C). Then Σ_k contain points that converge as $k \to \infty$ to a point in $H_- \backslash \Gamma$. The same statement holds with "$-$" replaced by "$+$".*

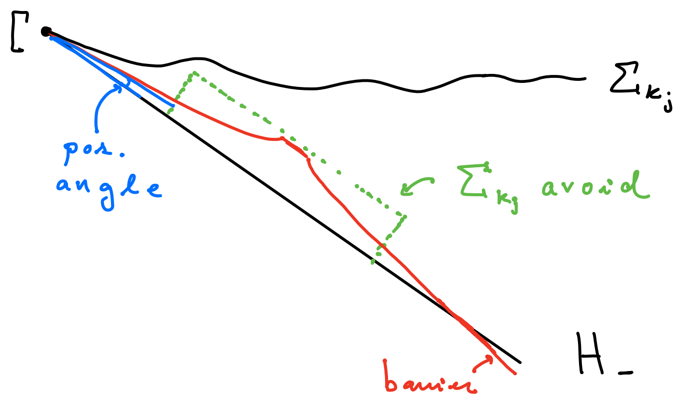

FIGURE 3. Hopf lemma type barriers.

Proof. Assume that Σ_k do not contain points that converge to something in $H_-\backslash\Gamma$. Then some subsequence $\{\Sigma_{k_j}\}$ avoids a neighborhood of the point q in $H_-\backslash\Gamma$ that is unit distance from the origin and orthogonal to Γ. We can find barriers similar to the one in the proof of Lemma 2.1 that are graphs over H_-, are centered near q, bound all Σ_{k_j} from below, and meet Γ at a positive angle (see Figure 3) to conclude that

$$u_{k_j} \geq L + \begin{cases} (A_- + \epsilon)x_1, & A_- \in \mathbb{R} \\ -\epsilon^{-1}x_1, & A_- = -\infty \end{cases}$$

in $B_\delta \cap k_j^{-1}\Omega$ for some $\epsilon,\, \delta > 0$. Here we used that $A_- < 0$ in cases (B) and (C) to guarantee that the barriers lie below Σ_{k_j} on the boundaries of Σ_{k_j}. From the definition of u_{k_j} and the invariance of the right hand side of the above inequality under Lipschitz rescalings, we see that the same inequality holds for u in $B_{k_j\delta} \cap \Omega$. After taking $j \to \infty$, we contradict the definition of A_-. After reflecting over $\{x_{n+1} = 0\}$, the same argument shows the result with "$-$" replaced by "$+$". □

Proof of Theorem 1.1. We first treat case (C). If $A_+ > 0$, then by Lemma 3.1 and Lemma 2.1 appropriately rescaled (in fact, just the proof of the first part using barriers) we get that Σ_k contains points close to H_+ that don't project in the x_{n+1} direction to $\overline{\Omega} \supset k^{-1}\overline{\Omega}$ for some k large, a contradiction of graphicality (see Figure 4). We conclude that $A_+ = 0$. The assertion that $A_- = 0$ follows from the same argument, after reflection over $\{x_{n+1} = 0\}$.

We now turn to case (B). If $0 < A_+ < \infty$, then by Lemma 3.1, Σ_k contain points converging to a point in $\{x_1 > 0\}$, thus we contradict the graphicality of Σ_k over $\{0 < x_1 < k^{-1}c\}$ for k large. Assume now that $A_+ = \infty$. Let B be a ball of radius one in $\{x_1 = 0\}$ that lies above Γ, and let $Q_h = \{-h < x_1 < h\} \times B$ for $h > 0$ small

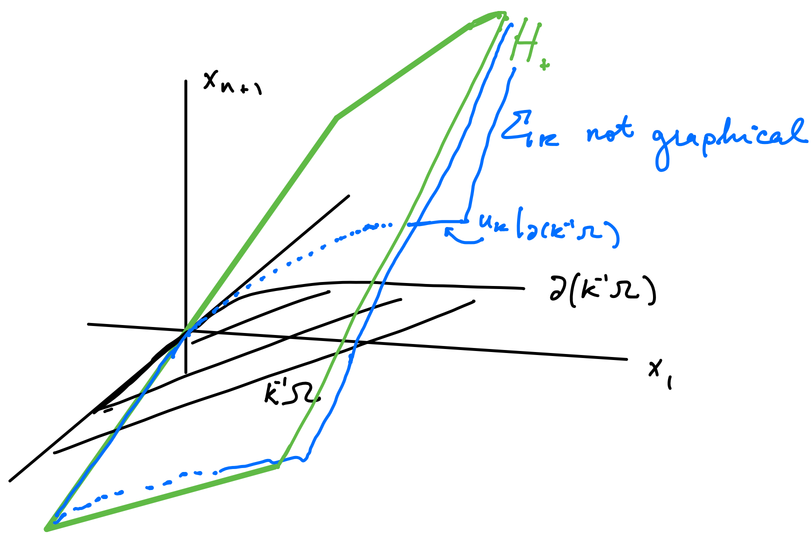

FIGURE 4. Σ_k is not graphical for k large.

to be determined. Lemmas 3.1 and 2.1 (appropriately rescaled) imply that, in Q_h, the hypersurfaces Σ_k contain a sheet of anisotropic area approaching $|B_1|\Phi(-e_1)$ as $k \to \infty$ (see Figure 5). Let $E_k = \{x_{n+1} < u_k(x),\, x \in k^{-1}\Omega\}$ and $F_k = E_k \backslash Q_h$. Then ∂F_k are competitors for ∂E_k in a neighborhood of Q_h which for k large have anisotropic area bounded above by that of ∂E_k minus $|B_1|\Phi(-e_1)/2$ plus $C(n,\, K)h$ (the last term coming from the thin sides of the cylinder Q_h). For $h(n,\, K)$ small we contradict the minimizing property of ∂E_k. We conclude that $A_+ = 0$. The claim that $A_- = 0$ follows in the same way, after reflecting over $\{x_{n+1} = 0\}$ (and changing the functional accordingly).

Finally we treat case (A). If A_+ and A_- are in $\mathbb{R}$ and $A_- < A_+$, then Lemma 3.1 and Lemma 2.1 imply that Σ_k are simultaneously close to $H_\pm$ in measure for k large, which contradicts the graphicality of Σ_k in the x_{n+1} direction. The problem is thus reduced (after possibly reflecting over $\{x_{n+1} = 0\}$) to ruling out the case that $A_+ = \infty$. We distinguish two sub-cases. The first is that $A_- \in \mathbb{R}$. Using Lemmas 3.1 and 2.1 near both H_+ and H_- we see that in Q_h, Σ_k contain a sheet of anisotropic area approaching $|B_1|\Phi(-e_1)$, and another portion that projects in the x_1 direction to nearly all of B. For this one uses that for k large, Σ_k are very close in measure to H_- on regions that get close to Γ (see Figure 6). Thus, the anisotropic area of Σ_k in Q_h is bounded from below by $|B_1|\Phi(-e_1) + c_0(n,\, K)$ as k gets large. Taking E_k and F_k as in case (B) we again contradict minimality for $h(n,\, K)$ small, since removing Q_h removes Σ_k in Q_h but adds at most the anisotropic area of the thin sides and one face of Q_h, which is $|B_1|\Phi(-e_1) + C(n,\, K)h$. The alternative is that $A_- = -\infty$. In this case Lemmas 3.1 and 2.1 imply that Σ_k have portions with anisotropic area nearly $|B_1|\Phi(-e_1)$ in Q_h and $|B_1|\Phi(e_1)$ in $-Q_h$ for k large. We may assume that we chose Q_h to be centered on the x_{n+1} axis. Using the graphicality of Σ_k in the x_{n+1} direction, we see by the pigeonhole principle that in at least one

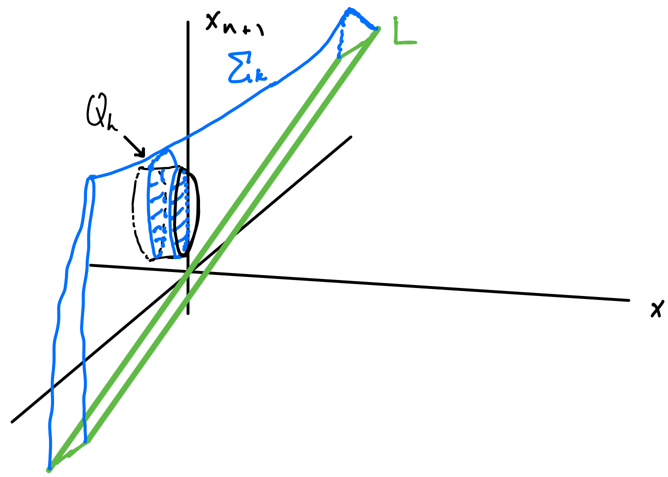

FIGURE 5. Σ_k has more anisotropic area in Q_h than the thin side of Q_h.

of Q_h, $-Q_h$, the hypersurface Σ_k contains another portion that projects in the x_1 direction to nearly half of B, $-B$. We may assume that this happens in Q_h, after possibly reflecting over $\{x_{n+1} = 0\}$. Then the anisotropic area of Σ_k in Q_h is again bounded from below by $|B_1|\Phi(-e_1) + c_0(n, K)$ for k large, and we contradict the minimizing property of Σ_k as in the previous sub-case to complete the proof.

□

Remark 3.2. Our arguments in fact show that Σ is flat for a large variety of non-convex domains Ω contained in a half-space. Examples include when Ω is contained in a convex cone that is not a half-space, or contained in a slab. Indeed, Lemma 3.1 holds for any $\Omega \subset \{x_1 > 0\}$ (with the assumption $A_\pm \neq 0$ in force when $\Omega \neq \{x_1 > 0\}$), and the arguments in the proof of Theorem 1.1 for cases (C) and (B) work in the same way for each of these cases, respectively.

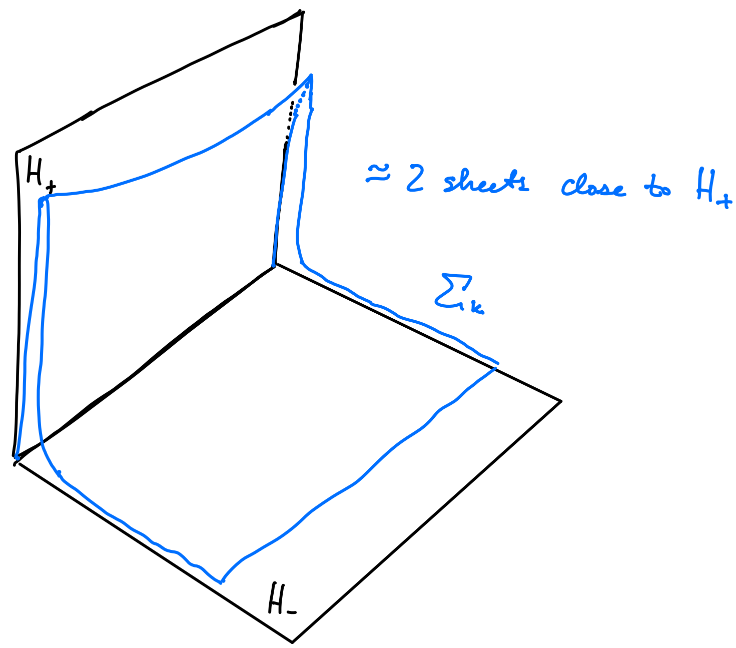

FIGURE 6. Σ_k contains nearly two vertical sheets for k large.

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