SINGULARITIES OF COMPLEX-VALUED SOLUTIONS TO LINEAR PARABOLIC EQUATIONS

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Abstract. We construct examples of blowup from smooth data for complex-valued solutions to linear uniformly parabolic equations in dimension $n \geq 2$, which are exactly as irregular as parabolic energy estimates allow.

1. Introduction

In this paper we consider linear uniformly parabolic equations of the form

$$u_t - \text{div}(A(x, t) \nabla u) = 0.$$  \hfill (1)

Here $u : \mathbb{R}^{n+1} \to \mathbb{C}$, and the coefficients are bounded measurable, complex-valued functions satisfying

$$\text{Re}(A_{kl}(x, t)p_k p_l) \geq \lambda |p|^2, \quad |A(x, t)(p)|^2 \leq \Lambda |p|^2$$  \hfill (2)

for some constants $\lambda, \Lambda > 0$, and for all $(x, t) \in \mathbb{R}^{n+1}$ and $p \in \mathbb{C}^n$. By a solution we mean that $u \in L^2_{\text{loc}, t}(H^1_{\text{loc}, x})$ solves (1) in the sense of distributions. We note that (1) can be viewed as a uniformly parabolic system of the form

$$\partial_t v^\alpha - \partial_k (B_{k\beta}(x, t)v^\beta_l) = 0, \quad 1 \leq k, l \leq n, \quad 1 \leq \alpha, \beta \leq 2.$$  \hfill (3)

Here $u = v^1 + iv^2$ and $B_{11} = B_{22} = \text{Re}(A), B_{12} = B_{21} = -\text{Im}(A)$.

We briefly discuss the elliptic case

$$\text{div}(A(x) \nabla u) = 0$$  \hfill (4)

in $\mathbb{R}^n$. Solutions to (4) are $C^\alpha$ when $n = 2$ by work of Morrey [Mo]. Real-valued solutions are $C^\alpha$ by fundamental work of De Giorgi [DG1] and Nash [Na]. There are classical counterexamples to continuity for solutions to elliptic systems when $n \geq 3$ (see [DG2], [GM], [Ma]). Discontinuous solutions to (4) were first constructed in dimension $n \geq 5$ [MNP], and later in dimension $n \geq 3$ [F]. In general, the best regularity we have for (4) is $u \in W^{1, 2+\delta}_{\text{loc}, t}$ for some $\delta(n, \lambda, \Lambda) > 0$ (see [Gi]), which is only slightly better than the energy class of the solutions. In fact, for each $\gamma > 2$ there are solutions to (4) that are not in $W^{1, \gamma}_{\text{loc}, t}$ (see [F]).

Interestingly, the parabolic problem (1) has resisted a similar understanding. Real-valued solutions are $C^\alpha$ [Na]. In general we have the higher-integrability results $\nabla u \in L^{2+\delta}_{\text{loc}, t}$ and $u \in L^{\infty}_{\text{loc}, t}(L^{2+\delta}_{\text{loc}, x})$ for some $\delta(n, \lambda, \Lambda) > 0$ (see [St], [NS]). There are also examples of discontinuity from smooth data when $n \geq 3$ ([FM], and [SJ1], [SJ2] for more general systems). However, the examples are in $L^\infty_{\text{loc}, t}(W^{1, 2+\delta}_{\text{loc}, x})$, and are thus significantly more regular than the higher-integrability results predict. When $n = 2$ the known results don’t imply continuity of solutions (unlike the elliptic case), which remained open for some time (see e.g. [SJ1], [SJ2]). We recently settled this problem with a counterexample [M1]. Still,
2 CONNOR MOONEY

the example in [M1] is barely irregular enough to develop a discontinuity (it is e.g. in $L^{\infty}_{loc}(L^p_{loc,x})$ for $p$ large), so the regularity gap between theory and examples remained large.

The purpose of this paper is to complete the picture for (1) by constructing solutions in dimension $n \geq 2$ that are exactly as irregular as the parabolic higher-integrability results allow. We also prove some Liouville theorems which explain why previous approaches only produced “elliptic” discontinuities. Our results connect the regularity problem for (1) in $\mathbb{R}^{n+1}$, in parabolic geometry, to that for the elliptic equation (4) in $\mathbb{R}^{n+2}$. We make this connection precise in the next section.

2. Results

In this section we state our results. We will deal with “spiraling” self-similar solutions to (1) of the form

$$(5) \quad u(x, t) = (-t)^{-\frac{\mu}{2}} e^{-\frac{t}{4} \log(-t)} w \left( \frac{x}{(-t)^{1/2}} \right).$$

These are invariant under $u \rightarrow \lambda \mu e^{i \log \lambda} u(\lambda x, \lambda^2 t)$. We obtain a solution to (1) on $\mathbb{R}^n \times (-\infty, 0)$ with coefficients $A(x/(-t)^{1/2})$ if $w$ solves the elliptic equation

$$(6) \quad \text{div}(A(x) \nabla w) = \frac{1}{2} (iw + \mu w + x \cdot \nabla w)$$

on $\mathbb{R}^n$, and $A$ satisfies (2) for some $\lambda, \Lambda > 0$. Furthermore, the solution defined by (5) is smooth up to $t = 0$ away from $x = 0$ and develops a “spiraling $-\mu$-homogeneous” discontinuity at $t = 0$ provided $\mu \geq 0$ and

$$(7) \quad w = |x|^{-\mu} g(x/|x|) e^{-i \log |x| (1 + \mathcal{E}(|x|^{-2}))} \text{ on } \mathbb{R}^n \setminus B_1.$$

Here $g \in C^\infty(S^{n-1})$ and $\mathcal{E}$ is a smooth function with $\mathcal{E}(0) = 0$. We can extend the solution to positive times e.g. by solving the heat equation with initial data $u(x, 0) := |x|^{-\mu} g(x/|x|) e^{-i \log |x|}$, provided $\mu < n$.

Our first result is:

**Theorem 2.1.** If $n \geq 2$ and $0 \leq 2\mu < n$, then there exists a nontrivial solution to (6) on $\mathbb{R}^n$ that satisfies (7).

By taking $\mu$ arbitrarily close to $\frac{n}{2}$ we obtain as a consequence:

**Corollary 2.2.** For all $n \geq 2$ and $\delta > 0$, there exists a solution to (1) on $\mathbb{R}^{n+1}$ such that

$$\lim_{t \rightarrow 0^-} ||u||_{L^{2+\delta}(B_1 \times \{-t\})} = \infty, \quad \lim_{t \rightarrow 0^-} ||\nabla u||_{L^{2+\delta}(B_1 \times (-1, -t))} = \infty.$$ 

(The ellipticity ratio $\lambda/\Lambda$ degenerates as $\delta \rightarrow 0$, in accordance with the higher-integrability results). We conclude, as in the elliptic case, that solutions to parabolic systems are only slightly better than their energy class.

Our remaining results are Liouville theorems for (6). It is natural to ask whether one can construct solutions that decay any faster than than we managed. Our first Liouville theorem shows this is not possible:

**Theorem 2.3.** Assume that $w \in H^1_{loc}(\mathbb{R}^n)$ solves (6), with $|w| = O(|x|^{-\mu})$ and $2\mu \geq n$. Then $w \equiv 0$. 
There are nontrivial $-\mu$-homogeneous solutions to elliptic systems of the form $\text{div}(A(x)\nabla u) = 0$ in $\mathbb{R}^n$ provided $2\mu < n - 2$, and there is a Liouville theorem for $-\mu$-homogeneous solutions on $\mathbb{R}^n \setminus \{0\}$ in the equality case (see [M2]). Thus, Theorems 2.1 and 2.3 mirror the elliptic results in dimension $n + 2$. This agrees with the observation that the parabolic energy $L^{\infty}_x(L^2_t) + L^2_t(H^1_x)$ in $\mathbb{R}^{n+1}$ and the elliptic energy $H^1$ in $\mathbb{R}^{n+2}$ are invariant under the matching rescalings $u \rightarrow \lambda^{n/2} u(\lambda x)$, resp. $u \rightarrow \lambda^{n/2} u(\lambda x)$.

Theorem 2.3 is a consequence of parabolic energy estimates. We can extend it to the “elliptic regime” $2\mu \geq n - 2$ when $w$ has the monotonicity property

$$
(2\mu + x \cdot \nabla)|w|^2 \geq 0:
$$

**Theorem 2.4.** Assume that $w \in H^1_{\text{loc}}(\mathbb{R}^n)$ solves (6), with $|w| = O(|x|^{-\mu})$ and $2\mu \geq n - 2$. If in addition $w$ satisfies (8), then $w \equiv 0$.

It is easy to check that previous examples ([FM], and [SJM], [SJ2] for more general systems) satisfy condition (8), which explains why they have “elliptic” discontinuities (that is, $n \geq 3$ and $2\mu < n - 2$).

The paper is organized as follows. In Section 3 we prove Theorem 2.1. In Section 4 we prove Theorems 2.3 and 2.4. Finally, in Section 5 we list some open questions.

### 3. Proof of Theorem 2.1

In this section we prove Theorem 2.1. We exploit the useful observation that if $\text{Im}(A)$ is symmetric, then the ellipticity condition (2) is satisfied provided $\text{Re}(A)$ is uniformly positive definite and $|A|$ is bounded (see [F]).

**Remark 3.1.** Heuristically, this structure allows strong coupling between components when we view (1) as the system (3). The example in [M1] has skew-symmetric imaginary coefficients, which corresponds to the symmetry $B_{\alpha\beta} = B_{\beta\alpha}$ of the system coefficients. In that case it is important to estimate the size of $\text{Im}(A)$ since it affects the ellipticity condition.

#### 3.1. Reduction to ODE System

We first reduce (6) to an ODE system. Let $r = |x|$ and let $\nu = r^{-1}x$ be the unit radial vector. We search for solutions of the form

$$
(9) \quad w = \varphi(r)g(\nu)e^{-i\log r}.
$$

Then

$$
(10) \quad \nabla w = ge^{-i\log r}(\varphi'(r) - i r^{-1}\varphi)e^{-i\log r} r^{-1} \nabla_{S^{n-1}} g.
$$

Here and below $\nabla_{S^{n-1}}$ and $\Delta_{S^{n-1}}$ denote the usual gradient and Laplace operators on the sphere. If

$$
B = f(r)\nu \otimes \nu + h(r)(I - \nu \otimes \nu)
$$

then we have

$$
B\nabla w = ge^{-i\log r} r^{-1}(f\varphi' - i r^{-1} f\varphi) \frac{\nu}{r^{n-1}} + h\varphi e^{-i\log r} r^{-1} \nabla_{S^{n-1}} g.
$$
We will choose \( \varphi \) such that \( \varphi' \) and \( r^{-1}\varphi \) are bounded. Using that \( \nu/r^{n-1} \) is divergence-free away from the origin we compute
\[
\text{div}(B\nabla w) = \left( \frac{r^{n-1}f \varphi'}{r^{n-1}} - \left( f - \frac{\Delta g}{g} \right) \frac{\varphi}{r^2} - i \left( \frac{r^{n-2}f \varphi'}{r^{n-1}} + \frac{f \varphi'}{r} \right) \right) .
\]

Let \( g \) be an eigenfunction of \( \Delta_{S^{n-1}} \) with eigenvalue \(-\lambda_g < 0\). Then the previous expression becomes
\[
\text{div}(B\nabla w) = ge^{-i\log r} \left[ \frac{r^{n-1}f \varphi'}{r^{n-1}} - (f + \lambda_g h) \frac{\varphi}{r^2} - i \left( \frac{r^{n-2}f \varphi^2}{r^{n-1}} \right) \right] .
\]

Thus, if we take coefficients
\[
A = \alpha I + i(\beta(r)\nu \otimes \nu + \gamma(r)(I - \nu \otimes \nu))
\]
with \( \alpha > 0 \) constant, and \( g \) is any linear function restricted to the sphere, we obtain
\[
\text{div}(A\nabla w) = ge^{-i\log r} \left[ \alpha \left( \frac{r^{n-1}\varphi'}{r^{n-1}} - n \frac{\varphi}{r^2} \right) + \frac{(r^{n-2}\beta\varphi^2)'}{r^{n-1}} \right.
\]
\[
+ i \left( \frac{(r^{n-1}\beta\varphi^2)'}{r^{n-1}} - (\beta + (n-1)\gamma) \frac{\varphi}{r^2} - \alpha \frac{(r^{n-2}\varphi^2)'}{r^{n-1}} \right) .
\]

Since
\[
iw + \mu w + x \cdot \nabla w = ge^{-i\log r} (\mu \varphi + r\varphi'),
\]
the equation (6) becomes the ODE system
\[
\begin{align}
\left( \frac{r^{n-2}\beta\varphi^2} {r^{n-1}\varphi} \right)' &= \frac{1}{2}(\mu \varphi + r\varphi') + n \alpha \frac{\varphi}{r^2} - \alpha \frac{(r^{n-1}\varphi^2)'}{r^{n-1}}, \\
(n-1)\gamma \frac{\varphi}{r^2} &= -\alpha \frac{(r^{n-2}\varphi^2)'}{r^{n-1}} + \frac{(r^{n-1}\beta\varphi^2)'}{r^{n-1}} - \beta \frac{\varphi}{r^2} .
\end{align}
\]

We will fix \( \varphi \sim r^{-\mu} \) and \( \alpha > 0 \) depending on \( \mu \). Then the first equation determines \( \beta \), and the second one \( \gamma \). By the remark at the beginning of the section, the point is to make choices such that \( \beta \) and \( \gamma \) are bounded.

### 3.2. Solving the ODE System

Integrating the first equation in (12) we obtain
\[
\beta = \frac{1}{4} \left( r^2 + 2\mu + n \right) \int_0^r \varphi^2(s)s^{n-1} ds
\]
\[
+ \frac{n \alpha}{r^{n-2}\varphi^2} \int_0^r \varphi^2(s)s^{n-3} ds
\]
\[
+ \frac{\alpha}{r^{n-2}\varphi^2} \int_0^r \varphi^2(s)s^{n-1} ds - \alpha \frac{r\varphi'}{\varphi} .
\]

\textbf{Remark 3.2.} It follows easily that if \( 2\mu \geq n \) and \( \varphi = O(r^{-\mu}) \), then \( \beta \) is unbounded (compare to Theorem 2.3).

We define
\[
\varphi(r) = \begin{cases} 
  r, & 0 \leq r < 3/4 \\
  r^{-\mu} + C_\mu r^{-\mu-2}, & r > 1
\end{cases}
\]
positive and smooth, \( 1/2 < r < 3/2 \)

where \( C_\mu \geq 0 \) will be chosen later.

\textbf{Remark 3.3.} By Theorem 2.4 it will be necessary to take \( C_\mu > 0 \) when \( 2\mu \geq n - 2 \) (and in particular, to generate discontinuities in the case \( n = 2 \)).
For \( r < 3/4 \) it is easy to check that \( \beta \) and \( \gamma \) are of the form \( c_1(n, \alpha) + c_2(n, \mu)r^2 \) (with \( c_i \) linear in \( \alpha \) and \( \mu \)) so we only need to analyze the solutions for \( r \) large. We divide into three cases.

**Case 1:** \( 2\mu < n - 2 \). We take \( C_\mu = 0 \) and \( \alpha = 1 \). It is easy to check that \( \beta \) and \( \gamma \) have the form \( c_1 + c_2 r^{2-n+2\mu} \) for \( r > 1 \), which is bounded.

**Case 2:** \( n - 2 < 2\mu < n \). Now the quantities

\[
D := \int_0^\infty (\varphi^2 - s^{-2\mu}) s^{n-1} ds, \quad E := \int_0^\infty \varphi^2 s^{n-3} ds, \quad F := \int_0^\infty \varphi'^2 s^{n-1} ds
\]

are bounded, for any fixed \( C_\mu \geq 0 \). The solution (13) becomes

\[
\beta = \left( -\frac{n-2\mu}{4} D + \alpha(nE + F) \right) r^{2\mu-n+2} + R(1).
\]

Here and below, \( R(1) \) denotes any smooth function on \( (1, \infty) \) whose \( j \)th derivative is \( O(r^{-j}) \) as \( r \to \infty \) for each \( j \geq 0 \). Using the definition of \( \varphi \) we estimate

\[
D \geq -\int_1^1 s^{n-1-2\mu} ds + 2C_\mu \int_1^\infty s^{n-3-2\mu} ds
= -\frac{1}{n-2\mu} + \frac{2C_\mu}{2\mu - n + 2}.
\]

We conclude that

\[
-\frac{n-2\mu}{4} D \leq \frac{1}{4} - \frac{n-2\mu}{2(2\mu - n + 2)} C_\mu < 0
\]

provided we choose \( C_\mu \) large. We may then choose \( \alpha > 0 \) small so that

\[
-\frac{n-2\mu}{4} D + \alpha(nE + F) = 0,
\]

hence

\[
\beta = R(1).
\]

Solving the second equation in (12) for \( \gamma \) gives

\[
\gamma = R(1),
\]

which completes this case.

**Case 3:** \( 2\mu = n - 2 \). This case is similar to the case \( 2\mu > n - 2 \), except to leading order \( \beta \) grows logarithmically. Computing (13) gives

\[
\beta = \left( -C_\mu + \alpha \left( n + \frac{1}{4}(n-2)^2 \right) \right) \log r + R(1).
\]

Choosing \( C_\mu \) and \( \alpha \) to satisfy the relation

\[
C_\mu = \left( n + \frac{1}{4}(n-2)^2 \right) \alpha
\]

we arrive at the same conclusion as in Case 2, completing the construction.
3.3. Proof of Theorem 2.1.

**Proof of Theorem 2.1:** For \(0 \leq 2\mu < n\), take \(\varphi, g, \alpha, \beta, \gamma\) as constructed above. Then the function

\[
 w = \varphi(r)g(\nu)e^{-i\log r}
\]

solves the equation (6) in \(\mathbb{R}^n\) with bounded coefficients

\[
 A = \alpha I + i(\beta(r)\nu \otimes \nu + \gamma(r)(I - \nu \otimes \nu))
\]

and has the asymptotics (7). Since \(\alpha > 0\) is constant and \(\text{Im}(A)\) is symmetric, the coefficients satisfy the ellipticity condition (2), completing the proof. \(\square\)

**Remark 3.4.** In our construction, \(w\) is Lipschitz but no better at 0, and smooth but not analytic away from 0. This is a consequence of choices we made for computational convenience. It is not hard to modify the construction so that \(w\) is analytic on \(\mathbb{R}^n\), e.g. by taking \(w = \varphi(r)g(\nu)e^{-\frac{1}{2}\log(1+r^2)}\) with \(g\) as above and

\[
 \varphi = r \left( (1 + r^2)^{-\frac{n+1}{2}} + C_\mu (1 + r^2)^{-\frac{n+3}{2}} \right).
\]

The coefficients \(A(x)\) also become analytic with these modifications.

4. Liouville Theorems

In this section we prove the Liouville theorems Theorem 2.3 and Theorem 2.4.

4.1. Proof of Theorem 2.3.

**Proof of Theorem 2.3.** Let \(\psi \in C_0^\infty(\mathbb{R}^n)\) be real-valued. Multiplying (6) by \(\bar{w}\psi^2\)

we obtain

\[
 (15) \quad 2\text{Re} \left( \text{div}(A\nabla w)\bar{w}\psi^2 \right) = \frac{1}{2}(2\mu|w|^2 + x \cdot \nabla |w|^2)\psi^2.
\]

Integrating by parts and using the ellipticity condition (2) we get

\[
 (16) \quad \int_{\mathbb{R}^n} (-\lambda|\nabla w|^2\psi^2 + C(\lambda, \Lambda)|w|^2|\nabla \psi|^2) \, dx \geq \frac{2\mu - n}{2} \int_{\mathbb{R}^n} |w|^2\psi^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^n} |w|^2x \cdot \nabla (\psi^2) \, dx.
\]

Since \(2\mu \geq n\), the first term on the right side is non-negative.

We now fix our choice of \(\psi\). Let \(\psi_1\) be a smooth, radially decreasing function supported in \(B_2\) with \(\psi_1 \equiv 1\) in \(B_1\), and let \(\psi_R := \psi_1(R^{-1}x)\). Take \(\psi = \psi_R\). Then the second term on the right side of (16) is non-negative, so the right side is non-negative. Using that \(|w|^2|\nabla \psi|^2 = O(R^{-2\mu-2})\) in \(B_{2R} \setminus B_R\) we conclude that

\[
 \int_{B_R} |\nabla w|^2 \, dx = O(R^{n-2\mu-2}) = O(R^{-2}),
\]

completing the proof. \(\square\)
4.2. Proof of Theorem 2.4.

**Proof of Theorem 2.4.** We start again with the identity (15). By (8) the right side of (15) is non-negative. Integrating by parts gives the Caccioppoli inequality
\[ \int_{\mathbb{R}^n} |\nabla w|^2 \psi^2 \, dx \leq C(\lambda, \Lambda) \int_{\mathbb{R}^n} |w|^2 |\nabla \psi|^2 \, dx. \]
Choosing \( \psi \) as before, we recover the inequality
\[ \int_{B_R} |\nabla w|^2 \, dx = O(R^{n-2\mu - 2}), \]
which proves the theorem when \( 2\mu > n - 2 \). In the critical case \( 2\mu = n - 2 \), use instead
\[
\psi = \begin{cases} 
1 & \text{in } B_1, \\
1 - \frac{\log(r)}{\log(R)} & \text{in } B_R \setminus B_1, \\
0 & \text{in } \mathbb{R}^n \setminus B_R 
\end{cases}
\]
to obtain
\[ \int_{B_{R\sqrt{n}}} |\nabla w|^2 \, dx = O\left(\frac{1}{\log R}\right). \]
\( \square \)

5. Some Questions

To conclude we list some open questions.

(1) Our examples have coefficients with symmetric imaginary part. Similar constructions might be possible with skew-symmetric imaginary coefficients, using techniques from [M1]. In this setting the imaginary coefficients play a role in ellipticity.

(2) For elliptic systems there is a sharp condition on the spectrum of the coefficients that guarantees continuity of solutions [Ko]. Sufficient conditions are known in the parabolic case ([Ko], [Ka]). It would be interesting to investigate how closely our counterexamples match these conditions.

(3) Solutions to parabolic systems in dimension \( n \geq 3 \) can be discontinuous on very large sets [SJ1]. It is natural to ask how large the discontinuity set can be when \( n = 2 \). Known results imply spatial continuity at almost every time, which is false when \( n \geq 3 \) by elliptic examples.

(4) Parabolic systems with the quasilinear structure
\[ u_t - \text{div}(A(u) \nabla u) = 0 \] (17)
have a well-developed partial regularity theory and are important in applications [GS]. Here the coefficients depend smoothly on \( u \). Constructing solutions to (17) becomes easier when \( u \in \mathbb{R}^m \) for \( m \) large because there is more room to “disperse \( u \)” [M1] contains examples of discontinuity formation for (17) when \( n = 2, m = 4 \). One can improve to \( n = 2, m = 3 \) using similar techniques [M3]. Continuity for solutions to (17) in the case \( n = m = 2 \) (in particular, the \( \mathbb{C} \)-valued scalar case) remains open. It seems possible in view of Theorem 2.4 that the restrictive geometry of the target could play in favor of regularity (see the discussion in [M3]).
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