SOME SINGULAR MINIMIZERS IN LOW DIMENSIONS IN THE CALCULUS OF VARIATIONS

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Abstract. We construct a singular minimizing map $\mathbf{u}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$ of a smooth uniformly convex functional of the form $\int_{B_1} F(D\mathbf{u})\,dx$.

1. Introduction

In this paper we consider minimizers of functionals of the form

$$\int_{B_1} F(D\mathbf{u})\,dx \tag{1.1}$$

where $\mathbf{u} \in H^1(B_1)$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ and F is a smooth, uniformly convex function on $M^{m\times n}$ with bounded second derivatives. By a minimizer we understand a map $\mathbf{u}$ for which the integral above increases after we perform any smooth deformation of $\mathbf{u}$, with compact support in B_1. If F satisfies these conditions then minimizers are unique subject to their own boundary condition. Moreover $\mathbf{u}$ is a minimizer if and only if it solves the Euler-Lagrange system

$$\text{div}(\nabla F(D\mathbf{u})) = 0, \tag{1.2}$$

in the sense of distributions.

The regularity of minimizers of (1.1) is a well-studied problem. Morrey [Mo] showed that in dimension $n = 2$ all minimizers are smooth. This is also true in the scalar case $m = 1$ by the classical results of De Giorgi and Nash [DG1],[Na]. In the scalar case, the regularity is obtained by differentiating equation (1.2) and treating the problem as a linear equation with bounded measurable coefficients. An example of De Giorgi [DG2] shows that these techniques cannot be extended to the case $m \geq 2$. Another example due to Giusti and Miranda [GM2] shows that elliptic systems do not have regularity even when the coefficients depend only on $\mathbf{u}$. On the other hand it is known that minimizers of (1.1) are smooth away from a closed singular set of Hausdorff $n-p$ dimensional measure zero for some $p > 2$, see [GM1], [GG]. (In fact, if F is uniformly quasi-convex then minimizers are smooth away from a closed set of Lebesgue measure zero, see Evans [E2]). However, the singular set may be non-empty. We will discuss some interesting examples below.

The main result of this paper is a counterexample to the regularity of minimizers of (1.1) when $n = 3$ and $m = 2$, which are the optimal dimensions in light of the previous results. The existence of such minimizing maps from $\mathbb{R}^3$ to $\mathbb{R}^3$ or from $\mathbb{R}^3$ to $\mathbb{R}^2$ is stated as an open problem in the book of Giaquinta (see [Gi], p. 61).

The first example of a singular minimizer of (1.1) is due to Nečas [Ne]. He considered the homogeneous degree one map

$$\mathbf{u}(x) = \frac{x \otimes x}{|x|}$$

from $\mathbb{R}^n$ to $\mathbb{R}^{n^2}$ for n large, and constructed explicitly a smooth uniformly convex F on $M^{n^2 \times n}$ for which $\mathbf{u}$ minimizes (1.1). Later Hao, Leonardi and Nečas [HLN] improved the dimension to $n = 5$ using

$$\mathbf{u}(x) = \frac{x \otimes x}{|x|} - \frac{|x|}{n} I. \tag{1.3}$$

The values of (1.3) are symmetric and traceless, and thus lie in a $n(n+1)/2 - 1$ dimensional subspace of $M^{n \times n}$. Šverák and Yan [SY1] showed that the map (1.3) is a counterexample for $n = 3$, $m = 5$. Their approach is to construct a quadratic null Lagrangian L which respects the symmetries of $\mathbf{u}$, such that $\nabla L = \nabla F$ on $D\mathbf{u}(B_1)$ for some smooth, uniformly convex F on $M^{5 \times 3}$. The Euler-Lagrange system $\text{div}(\nabla F(D\mathbf{u})) = \text{div}(\nabla L(D\mathbf{u})) = 0$ then holds automatically. In [SY2] they use the same technique to construct a non-Lipschitz minimizer with $n = 4$, $m = 3$ coming from the Hopf fibration. To our knowledge, these are the lowest-dimensional examples to date.

Our strategy is different and it is based on constructing a homogenous of degree one minimizer in the scalar case for an integrand which is convex but has "flat pieces".

An interesting problem about the regularity of minimizers occurs in the scalar case when considering in (1.1) convex integrands $F : \mathbb{R}^n \to \mathbb{R}$ for which the uniform convexity of F fails on some compact set $\mathcal{S}$. Assume for simplicity that F is smooth outside the degeneracy set $\mathcal{S}$, and also that F satisfies the usual quadratic growth at infinity. One key question is whether or not the gradient ∇u localizes as we focus closer and closer to a point $x_0 \in B_1$. In [DS] it was proved that, in dimension $n = 2$, the sets $\nabla u(B_\varepsilon(x_0))$ decrease uniformly as $\varepsilon \to 0$ either to a point outside $\mathcal{S}$, or to a connected subset of $\mathcal{S}$. In Theorem 1.1 below we show that this "continuity property" of ∇u does not hold in dimension $n = 3$ when the set $\mathcal{S}$ is the union of two disconnected convex sets. We remark that, as in the p-Laplace equation, it is relatively standard (see [E1, CF]) to obtain the continuity of ∇u outside the convex hull $\mathcal{S}^c$ of $\mathcal{S}$.

Let w be the homogeneous degree one function

$$w(x_1, x_2) = \frac{x_2^2 - x_1^2}{\sqrt{2(x_1^2 + x_2^2)}} = \frac{-1}{\sqrt{2}}\ r\ \cos 2\theta,$$

and let u_0 be the function on $\mathbb{R}^3$ obtained by revolving w around the x_1 axis,

$$u_0(x_1, x_2, x_3) = w\left(x_1, \sqrt{x_2^2 + x_3^2}\right).$$

We show that u_0 solves a degenerate elliptic equation that is uniformly elliptic away from the cone

$$K_0 = \{x_1^2 > x_2^2 + x_3^2\}.$$

Theorem 1.1. *For any $\delta > 0$ there exists a convex function $G_0 \in C^{1,1-\delta}(\mathbb{R}^3)$ which is linear on two bounded convex sets containing $\nabla u_0(K_0)$, uniformly convex and smooth away from these two convex sets, such that u_0 is a minimizer of the functional*

$$\int_{B_1} G_0(\nabla u_0)\, dx.$$

We use u_0 and G_0 to construct a singular minimizing map from $\mathbb{R}^3$ to $\mathbb{R}^2$. Rescaling u_0 we obtain a function u^1 that solves an equation that is uniformly elliptic away from a thin cone around the x_1 axis, and switching the x_1 and x_3 axes we get an analogous function u^2. Then $\mathbf{u} = (u^1, u^2)$ is a minimizing map for

$$F_0(p^1, p^2) := G_1(p^1) + G_2(p^2),$$

which is a convex function defined on $\mathbb{R}^6 \cong M^{2\times 3}$. Notice that the Euler-Lagrange system $\operatorname{div}(\nabla F_0(D\mathbf{u})) = 0$ is de-coupled, and F_0 fails to be uniformly convex or smooth in certain regions. However, a key observation is that F_0 separates quadratically from its tangent planes when restricted to the image of $D\mathbf{u}$. We obtain our example by making a small perturbation of F_0.

More specifically, let

$$u^1(x_1, x_2, x_3) = u_0(x_1/2, x_2, x_3), \quad u^2(x_1, x_2, x_3) = u^1(x_3, x_2, x_1)$$

and let

$$\mathbf{u} = (u^1, u^2). \tag{1.4}$$

Our main theorem is:

Theorem 1.2. *The map* (1.4) *is a minimizer of*

$$\int_{B_1} F(D\mathbf{u})\, dx$$

for some smooth, uniformly convex $F : M^{2\times 3} \to \mathbb{R}$.

The paper is organized as follows. In Section 2 we state a convex extension lemma and the key proposition, which asserts the existence of a suitable smooth small perturbation of G_0. We then use them to prove Theorem 1.2. In Section 3 we prove the key proposition. This section contains most of the technical details. In Section 4 we prove the extension lemma and some technical inequalities needed for the key proposition. Finally, at the end of Section 4 we outline how to prove Theorem 1.1.

2. Key Proposition and Proof of Theorem 1.2

In this section we state the extension lemma and the key proposition. We then use them to prove Theorem 1.2.

The function F_0 defined in the Introduction is not uniformly convex in $M^{2\times 3}$, but it separates quadratically from its tangent planes on the image of $D\mathbf{u}$ which, by the one-homogeneity of $\mathbf{u}$, is the two dimensional surface $D\mathbf{u}(S^2)$. The quadratic separation holds on this surface since G_1 is uniformly convex in the region where G_2 is flat and vice versa. We would like to find a uniformly convex extension of F_0 with the same tangent planes on $D\mathbf{u}(\partial B_1)$.

2.1. **Extension Lemma.** The extension lemma gives a simple criterion for deciding when the tangent planes on a smooth surface can be extended to a global smooth, uniformly convex function. Let Σ be a smooth compact, embedded surface in $\mathbb{R}^n$ of any dimension.

Lemma 2.1. *Let G be a smooth function and $\mathbf{v}$ a smooth vector field on Σ such that*

$$G(y) - G(x) - \mathbf{v}(x)\cdot(y - x) \geq \gamma|y - x|^2, \tag{2.1}$$

for any x, $y \in \Sigma$ and some $\gamma > 0$. Then there exists a global smooth function F such that $F = G$ and $\nabla F = \mathbf{v}$ on Σ, and $D^2 F \geq \gamma I$.

The idea of the proof is to first make a local extension by adding a large multiple of the square of distance from Σ. We then make an extension to all of $\mathbb{R}^n$ by taking the supremum of tangent paraboloids to the local extension. Finally we mollify and glue the local and global extensions. We postpone the proof to the appendix, Section 4. We also record an obvious corollary.

Definition 2.2. Let G be a smooth function on an open subset O of $\mathbb{R}^n$. We define the separation function S_G on $O \times O$ by

$$S_G(x, y) = G(y) - G(x) - \nabla G(x) \cdot (y - x).$$

Corollary 2.3. *Assume that G is a smooth function in a neighborhood of Σ such that $S_G(x, y) \geq \gamma |y - x|^2$ for any x, $y \in \Sigma$ and some $\gamma > 0$. Then there exists a global smooth, uniformly convex function F such that $F = G$ and $\nabla F = \nabla G$ on Σ.*

2.2. **Key Proposition.** In this section we state the key proposition. We first give the setup for the statement. Recall that $w = (x_2^2 - x_1^2)/\sqrt{2(x_1^2 + x_2^2)}$. Let

$$\Gamma = \nabla w(B_1 - \{0\}) = \nabla w(S^1).$$

We describe Γ as a collection of four congruent curves. The part of Γ in the region $\{p_2 \geq |p_1|\}$ can be written as a graph

$$\Gamma_1 = \{(p_1, \varphi(p_1))\}$$

for $p_1 \in [-1, 1]$, where φ is even, uniformly convex, tangent to $p_1^2 = p_2^2$ at ± 1, and separates from these lines like $(\text{dist})^{3/2}$. We will give a more precise description of φ in Section 3.

The other pieces of Γ can be written

$$\Gamma_2 = \{-\varphi(p_2), p_2\}, \quad \Gamma_3 = \{p_1, -\varphi(p_1)\}, \quad \Gamma_4 = \{\varphi(p_2), p_2\}$$

for $p_i \in [-1, 1]$, representing the left, bottom and right pieces of Γ (see figure 1).

Recall that $u_0 = w\left(x_1, \sqrt{x_2^2 + x_3^2}\right)$. Then

$$\Omega = \nabla u_0(S^2)$$

is the surface obtained by revolving Γ around the p_1 axis. Let $\Omega_R \subset \Omega$ be the surface obtained by revolving Γ_1 around the p_1 axis.

In the statement below, δ and γ are small positive constants depending on φ.

Proposition 2.4. *For any $\epsilon > 0$ there exists a smooth function G defined in a neighborhood of Ω such that*

$$div(\nabla G(\nabla u_0)) = 0 \qquad in \quad B_1 \setminus \{0\},$$

and

(1) *If $p \in \Omega_R \cap \{-1 + \delta \leq p_1 \leq 1 - \delta\}$ then $S_G(p, q) \geq \gamma |p - q|^2$ for all $q \in \Omega$,*
(2) *$S_G(p, q) \geq -\epsilon |p - q|^2$ otherwise for p, $q \in \Omega$.*

We delay the proof of this proposition to Section 3, and use it now to prove Theorem 1.2.

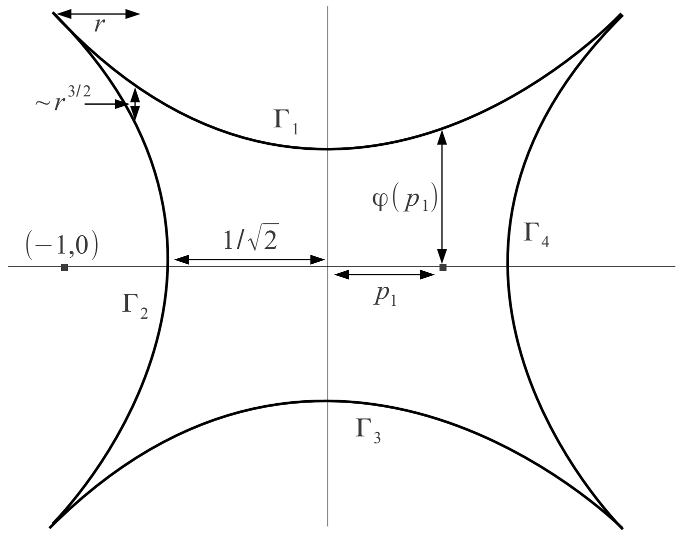

FIGURE 1. Γ consists of four identical curves separating from the lines $p_1^2 = p_2^2$ like $\text{dist}^{3/2}$.

2.3. **Proof of Theorem 1.2.** Recall that

$$u^1(x_1, x_2, x_3) = u_0(x_1/2, x_2, x_3), \quad u^2(x_1, x_2, x_3) = u^1(x_3, x_2, x_1),$$

and let

$$G_1(p_1, p_2, p_3) = G(2p_1, p_2, p_3), \quad G_2(p_1, p_2, p_3) = G_1(p_3, p_2, p_1).$$

Then by Proposition 2.4 we have $\text{div}(\nabla G_i(\nabla u^i)) = 0$. Let

$$\Sigma = D\mathbf{u}(B_1).$$

Since D^2u^1 has rank 2 away from the cone

$$K_1 = \left\{x_1^2 \geq 4(x_2^2 + x_3^2)\right\}$$

and similarly D^2u^2 has rank 2 away from

$$K_2 = \left\{x_3^2 \geq 4(x_1^2 + x_2^2)\right\},$$

it is easy to see that Σ is a smooth embedded surface in $\mathbb{R}^6$.

Let

$$\Omega_i = \nabla u^i(B_1 - K_i).$$

Note that Ω_1 is just Ω_R squeezed by a factor of $1/2$ in the p_1 direction. Let ν_i be the outer normals to Ω_i. Since u^i are homogeneous degree one we have $\nu_i(\nabla u^i(x)) = x$ on $(B_1 - K_i) \cap S^2$. Furthermore, the preimage $x \in S^2$ of any point in Σ satisfies either $|x_1| \leq |x_3|$ or vice versa. It follows from these observations that if $(p^1, p^2) \in \Sigma$ then either

$$p^1 \in \Omega_1 \cap \{-\beta/2 \leq p_1^1 \leq \beta/2\} \text{ or } p^2 \in \Omega_2 \cap \{-\beta/2 \leq p_3^2 \leq \beta/2\}$$

with β such that $\varphi'(\beta) = 1/2$, $\beta < 1 - \delta$ (see figure 2). Assume p^1 belongs to the set above.

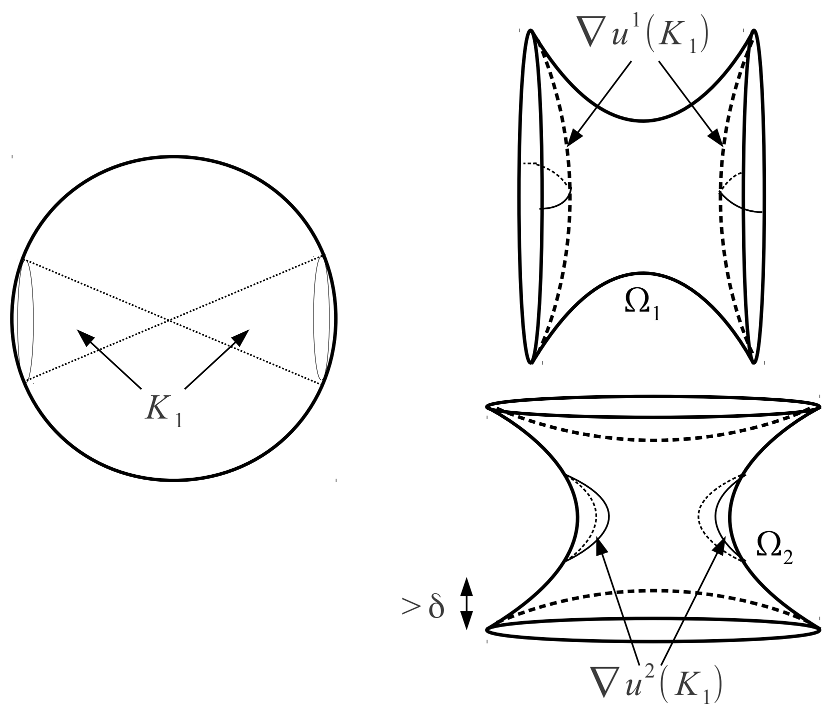

FIGURE 2. ∇u^1 maps the cone K_1 to a region where G_1 is slightly non-convex, but ∇u^2 maps it well inside Ω_2 where G_2 is uniformly convex.

Finally, let

$$F_0(p^1, p^2) = G_1(p^1) + G_2(p^2).$$

By rescaling Proposition 2.4 we have for (p^1, p^2), $(q^1, q^2) \in \Sigma$ that

$$\begin{aligned} S_{F_0}((p^1, p^2), (q^1, q^2)) &= S_{G_1}(p^1, q^1) + S_{G_2}(p^2, q^2) \\ &\geq \gamma |p^1 - q^1|^2 - \epsilon |p^2 - q^2|^2. \end{aligned}$$

Let $\omega_0 \in S^2$ be a preimage of p^1 under ∇u^1. Then $|\nabla u^1(\omega) - \nabla u^1(\omega_0)| > c|\omega - \omega_0|$ and $|\nabla u^i(\omega) - \nabla u^i(\omega_0)| < C|\omega - \omega_0|$ for any $\omega \in S^2$, so

$$|p^2 - q^2| \leq C|p^1 - q^1|,$$

giving quadratic separation. By Corollary 2.3 there is a smooth uniformly convex function F on $\mathbb{R}^6$ so that $F = F_0$ and $\nabla F = \nabla F_0$ on Σ, hence $\mathbf{u}$ satisfies the Euler-Lagrange system $\text{div}(\nabla F(D\mathbf{u})) = 0$ in $B_1 \setminus \{0\}$. Now it is straightforward to check that $\mathbf{u}$ is a weak solution of the system in the whole B_1. Indeed

$$\int_{B_1} \nabla F(D\mathbf{u}) \cdot D\psi = 0, \qquad \forall \psi \in C_0^\infty(B_1),$$

follows by integrating first by parts in $B_1 \setminus B_\epsilon$ and then letting $\epsilon \to 0$.

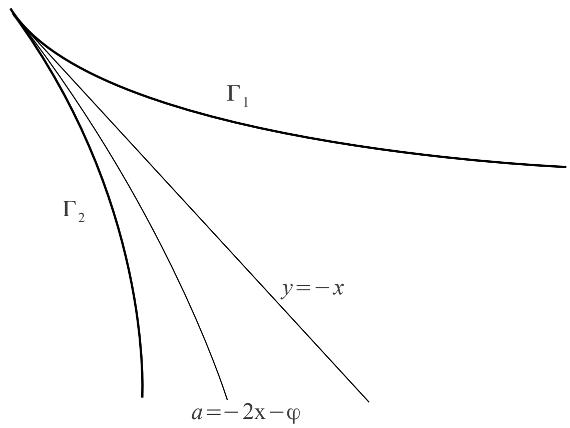

FIGURE 3. The graph $a = -2x - \varphi$ lies strictly above Γ_2.

3. Constructions

In this section we prove the key step, Proposition 2.4. Since $\Omega = \nabla u_0(B_1)$ is the surface obtained by revolving Γ around the p_1 axis, we can reduce to a one-dimensional problem on Γ and then revolve the resulting picture around the p_1 axis. Since all of our constructions will be on $\mathbb{R}^2$ in this section we use coordinates (x, y) rather than (p_1, p_2).

3.1. **Setup.** Define H to be an even function in x and y which has the form

$$H(x, y) = f(x) + h(x)(|y| - \varphi(x)), \tag{3.1}$$

and is defined in a neighborhood of every point on $\Gamma_1 \cup \Gamma_3$, for some smooth functions f and h on $[-1, 1]$. In our construction h will be identically zero and f linear near $x = \pm 1$, so H is linear in a neighborhood of the cusps of Γ. Notice that we can extend H to be a linear function (depending only on x) in a whole neighborhood of Γ_2 and similarly on Γ_4. Then H is defined and smooth in a neighborhood of Γ.

3.2. **Inequalities for φ.** We now record some useful properties of Γ. For proofs see Section 4. The first estimate gives an expansion for φ near $x = -1$.

Proposition 3.1. *The function φ is even, uniformly convex, and tangent to $y = |x|$ at $x = \pm 1$. Furthermore, φ'' is decreasing near $x = -1$ and we have the expansion*

$$\varphi''(-1 + \epsilon) = \sqrt{\frac{2}{3}}\epsilon^{-1/2} + O(1). \tag{3.2}$$

The second estimate says that the vertical reflection of φ over its tangent $y = -x$ lies above and separates from Γ_2 (see figure 3). It follows easily from the uniform convexity of φ.

Proposition 3.2. *The function $a(x) = -2x - \varphi$ is uniformly concave, tangent to Γ_2 at $x = -1$, and lies strictly above Γ_2 for $x > -1$.*

3.3. **Euler-Lagrange Equation.** Let

$$G(p_1, p_2, p_3) = H\left(p_1, \sqrt{p_2^2 + p_3^2}\right).$$

The condition that u_0 solves the Euler-Lagrange equation $\text{div}(\nabla G(\nabla u_0)) = 0$ is equivalent to

$$h(x) = \frac{f''(x)}{2\varphi''(x)}. \tag{3.3}$$

Indeed, since G is linear near the surfaces obtained by revolving Γ_2 and Γ_4, we just need to verify the Euler-Lagrange equation where ∇u_0 is on the surface Ω_R obtained by revolving Γ_1. By passing a derivative the Euler-Lagrange equation $\text{div}(\nabla G(\nabla u_0))$ is equivalent to

$$\text{tr}\left(D^2G(\nabla u_0) \cdot D^2 u_0\right) = 0.$$

Let Ω_R have outer normal ν and second fundamental form II. Since u_0 is homogeneous degree one we have $\nu(\nabla u_0(x)) = x$ on S^2. Let T be a frame tangent to S^2 at x, and differentiate to obtain $D^2_T u_0(x) = II^{-1}(\nabla u_0(x))$. In coordinates tangent to Ω_R at $p = (p_1, \varphi(p_1), 0)$ one computes

$$II = \frac{1}{\sqrt{1+\varphi'^2}}\begin{pmatrix} \frac{\varphi''}{1+\varphi'^2} & 0 \\ 0 & -\frac{1}{\varphi} \end{pmatrix}, \quad D^2G = \begin{pmatrix} \frac{f''-h\varphi''}{1+\varphi'^2} & 0 \\ 0 & \frac{h}{\varphi} \end{pmatrix}$$

and the Euler-Lagrange formula follows.

Remark 3.3. For a fast way to compute D^2G in tangential coordinates, differentiate the equation $G(p_1, \varphi(p_1), 0) = f(p_1)$:

$$\nabla G \cdot (1, \varphi') = f', \quad (1, \varphi')^T \cdot D^2G \cdot (1, \varphi') + h\varphi'' = f''.$$

The other eigenvalue comes from the rotational symmetry of G around the p_1 axis.

Remark 3.4. If we do the computation in $\mathbb{R}^n$ we have $n-1$ rotational principal curvatures and derivatives, giving the Euler-Lagrange equation $h = \frac{f''}{(n-1)\varphi''}$.

3.4. **Convexity Conditions.** Since most of our analysis is near a cusp, it is convenient to shift the picture by the vector $(1, -1)$ so that φ, f are defined on $[0, 2]$ and φ is tangent to $y = -x$ at zero. We assume this for the remainder of the section.

We examine convexity conditions between two points on Γ_1. Let $p = (x_0, \varphi(x_0))$ and $q = (x, \varphi(x))$. We first write the equation for the tangent plane L_p to H at $p = (x_0, \varphi(x_0))$:

$$L_p(x, y) = f(x_0) + f'(x_0)(x - x_0) + h(x_0)\left[y - (\varphi(x_0) + \varphi'(x_0)(x - x_0))\right].$$

Applying the Euler-Lagrange equation (3.3) we obtain

$$L_p = f(x_0) + f'(x_0)(x - x_0) - \frac{f''(x_0)}{2\varphi''(x_0)}\left[y - (\varphi(x_0) + \varphi'(x_0)(x - x_0))\right]. \tag{3.4}$$

By definition,

$$S_H(p, q) = f(x) - L_p(x, \varphi(x)).$$

Using equation (3.4) we obtain

$$S_H(p, q) = \int_{x_0}^{x} f''(t)(x - t)\, dt - \frac{f''(x_0)}{2\varphi''(x_0)}\int_{x_0}^{x} \varphi''(t)(x - t)\, dt. \tag{3.5}$$

Definition 3.5. For a nonnegative function $g:\mathbb{R}\to\mathbb{R}$ define the weighted average

$$s_g(x_0,x)=\frac{\int_{x_0}^{x} g(t)(x-t)\,dt}{g(x_0)(x-x_0)^2}.$$

With this definition we have

$$S_H(p,q)=f''(x_0)\left(s_{f''}(x_0,x)-\frac{1}{2}s_{\varphi''}(x_0,x)\right)(x-x_0)^2, \tag{3.6}$$

thus, the first qualitative convexity condition is

$$s_{f''}(x_0,x)\geq\frac{1}{2}s_{\varphi''}(x_0,x). \tag{3.7}$$

Remark 3.6. Notice that

$$\lim_{x\to x_0} s_g(x_0,x)=\frac{1}{2}.$$

It is easy to check that if g is increasing (decreasing) then $s_g(x_0,x)$ is increasing (decreasing) with x. With this observation one verifies that condition (3.7) holds for x_0, x near 0 if $f''(x)=Cx^{1-\alpha}$ for any $\alpha\in(0,1)$. Indeed, since f'' is increasing and φ'' is decreasing one only needs to check the condition at $x=0$, where one computes $s_{f''}(x_0,0)=\frac{1}{3-\alpha}$ and $\frac{1}{2}s_{\varphi''}(x_0,0)=\frac{1}{3}+O(\sqrt{x_0})$ which follows by Proposition 3.1.

We now examine convexity conditions between $p\in\Gamma_1$ and $q\in\Gamma_2$.

Let $p=(x_0,\varphi(x_0))$. In our construction we will have $h\geq 0$, and since H is linear near Γ_2, we see that $S_H(p,q)\geq 0$ if the intersection line of tangent planes to H at p and at 0 lies above the line $y=-x$ on $[0,2]$. Using equation (3.4) we compute the formula for the intersection line:

$$\begin{aligned} y&=\varphi(x_0)-\frac{2\varphi''(x_0)}{f''(x_0)}\int_0^{x_0} f''(t)(x_0-t)\,dt\\ &+\left(\varphi'(x_0)-\frac{2\varphi''(x_0)}{f''(x_0)}\int_0^{x_0} f''(t)\,dt\right)\cdot(x-x_0). \end{aligned} \tag{3.8}$$

If condition (3.7) holds at $x=0$, it means that the origin lies below the intersection line, thus $S_H(p,q)\geq 0$ for all $q\in\Gamma_2$ provided that the slope of the intersection line above is larger than -1:

$$\varphi'(x_0)-\frac{2\varphi''(x_0)}{f''(x_0)}\int_0^{x_0} f''(t)\,dt\geq -1=\varphi'(0).$$

Definition 3.7. For a nonnegative function $g:\mathbb{R}\to\mathbb{R}$ define

$$d_g(x)=\frac{\int_0^x g(t)\,dt}{xg(x)}.$$

With this definition the slope condition above can be written as

$$d_{f''}(x)\leq\frac{1}{2}d_{\varphi''}(x). \tag{3.9}$$

Remark 3.8. Near $x=0$ one computes $\frac{1}{2}d_{\varphi''}(x)=1+O(\sqrt{x})$. Thus, if $f''(x)=Cx^{1-\alpha}$ near $x=0$ then (3.9) holds. However, away from a small neighborhood of 0, condition (3.9) will not hold in our construction. We will use formula (3.4) more carefully, combined with Proposition 3.2, to deal with these cases.

Remark 3.9. Conditions (3.7) and (3.9) are independent of the linear part of f. Thus, when checking convexity conditions we only need to use the properties of f''.

3.5. **Preliminary Construction.** As a stepping stone to proving Proposition 2.4 we construct first a $C^{1,\alpha}$ function H_0 near Γ, that is globally convex. We will use this construction to prove Theorem 1.1 in Section 4. The function $H \in C^{\infty}$ is obtained by perturbing H_0. Below we define

$$G_0(p_1, p_2, p_3) = H_0\left(p_1, \sqrt{p_2^2 + p_3^3}\right).$$

Recall in the constructions below that we have shifted the picture by $(1, -1)$.

Proposition 3.10. *For any $\alpha \in (0,1)$ there exist a function H_0 near Γ such that*

1. *H_0 is a linear function depending only on x on Γ_2, and similarly on Γ_4.*
2. *H_0 is pointwise $C^{1,1-\alpha}$ on the cusps of Γ and smooth otherwise,*
3. *$div(\nabla G_0(\nabla u_0)) = 0$ away from the cone $\{x_1^2 = x_2^2 + x_3^2\}$,*
4. *$S_{H_0}(p,q) \geq 0$ for all p, $q \in \Gamma$,*
5. *If $p = (x, \varphi(x))$ then $S_{H_0}(p,q) \geq \eta(x)|p-q|^2$ for all $q \in \Gamma$, where η is some continuous function on $[0,2]$ with $\eta > 0$ on $(0,2)$ and $\eta(0) = \eta(2) = 0$.*

We will define f_0 by $f_0(0) = f_0'(0) = 0$ and prescribe f_0'', and then let H_0 be the function determined by f_0 through the Euler-Lagrange relation (3.3). It is easy to check that condition (3.7) holds if we take $f_0'' = \varphi''$. However, we want $h_0 = f_0''/(2\varphi'')$ to go to zero at the endpoints so that H_0 is linear on Γ_2 and Γ_4.

Motivated by the above and Remarks 3.6 and 3.8, define

$$f_0''(x) = \begin{cases} \delta^{\alpha-1}\varphi''(\delta)x^{1-\alpha}, & 0 \leq x \leq \delta \\ \varphi''(x), & \delta \leq x \leq 1 \\ f_0''(2-x), & 1 \leq x \leq 2 \end{cases}$$

(See figure 4). Assume δ is tiny so that φ'' is well approximated by its expansion (3.2). Let H_0 be the function as in (3.1) determined by f_0 through the Euler-Lagrange relation (3.3).

Proof of proposition 3.10. The first three items are clear by construction so we check the convexity conditions. By symmetry we only need to consider $p \in \Gamma_1 \cup \Gamma_2$.

If $p \in \Gamma_2$ the positive separation is a consequence of $H_0 \geq 0$. This follows from the definition of H_0 on $\Gamma_1 \cup \Gamma_3$. Also, by symmetry, the linear function on Γ_4 intersects the linear function on Γ_2 on the vertical line $\{x = 1\}$, and since $\Gamma_4 \subset \{x > 1\}$ we obtain $H_0 \geq 0$ on Γ_4 as well.

We now consider the situation when $p \in \Gamma_1$ and distinguish two cases depending whether $q \in \Gamma_1 \cup \Gamma_3$ or $q \in \Gamma_2 \cup \Gamma_4$.

Let $p = (x_0, \varphi(x_0))$.

First Case: Assume first that $q = (x, \varphi(x)) \in \Gamma_1$. By symmetry of f_0'' around $x = 1$ we may assume $x < x_0$.

If $x_0 \in [0, \delta]$ then by formula (3.6) and Remark 3.6 we have

$$S_{H_0}(p,q) \geq c(\alpha)f''(x_0)(x-x_0)^2.$$

If $x_0 \in [\delta, 2-\delta]$ we have $f_0'' = \varphi''$, so one computes

$$S_{H_0}(p,q) = \int_x^{x_0} (f''(t) - \frac{1}{2}\varphi''(t))(t-x)\,dt.$$

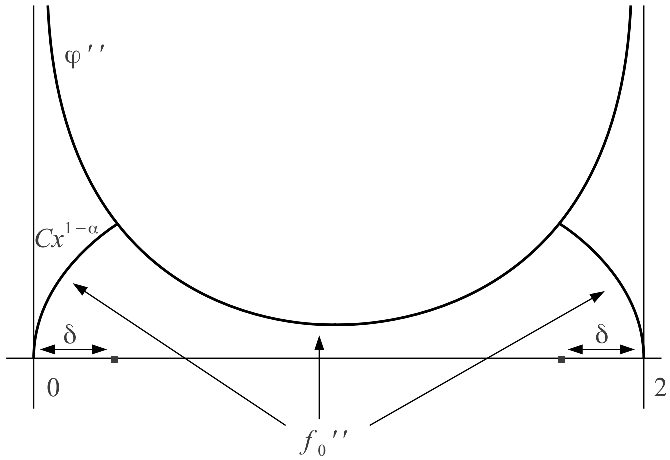

FIGURE 4. f_0'' agrees with φ'' on $[\delta, 2-\delta]$, behaves like $x^{1-\alpha}$ near zero, and is symmetric around $x = 1$.

If $x \geq \delta$ then this is clearly controlled below by $\frac{1}{4}\min(\varphi'')(x-x_0)^2$, and if $x < \delta$ then we have

$$S_{H_0}(p,q) = \int_x^\delta (f''(t) - \varphi''(t)/2)(t-x)\,dt + \frac{1}{2}\int_\delta^{x_0} \varphi''(t)(t-x)\,dt,$$

which is controlled below by

$$\varphi''(\delta)(s_{f''}(\delta,x) - s_{\varphi''}(\delta,x)/2)(\delta-x)^2 + \frac{1}{4}\min(\varphi'')(x_0-\delta)^2 \geq c(\alpha)(x-x_0)^2.$$

Finally, if $x_0 \geq 2-\delta$ then since f_0''/φ'' is decreasing on $[\delta, 2]$, we compute for $x \geq \delta$ that

$$S_{H_0}(p,q) \geq \frac{1}{2}\int_{x_0}^x f''(t)(x-t)\,dt \geq \frac{1}{4}\min\{f_0''(x_0), \min(\varphi'')\}(x-x_0)^2.$$

If $x < \delta$ then, since $f_0'' \leq \varphi''$ and they agree on $[\delta, 2-\delta]$, we have using expansion (3.2) that

$$S_{H_0}(p,q) \geq \frac{1}{2}\int_\delta^{2-\delta} \varphi''(t)(t-x)\,dt - C\sqrt{\delta} \geq c(x-x_0)^2.$$

If $q \in \Gamma_3$ then quadratic separation holds as well since

$$\partial_y H_0(x_0, \varphi(x_0)) = \frac{f_0''(x_0)}{2\varphi''(x_0)} > 0.$$

Second Case: By symmetry we may assume $q \in \Gamma_2$. If $x_0 \leq \delta$ we compute

$$d_{f_0''}(x_0) = \frac{1}{2-\alpha} < 1.$$

By Remark 3.8, inequality (3.9) holds strictly.

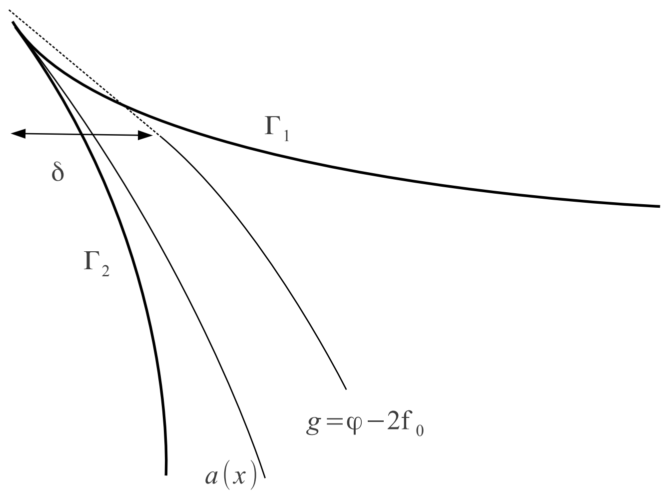

FIGURE 5. The tangent plane at $(x, \varphi(x))$ is negative on the curve $y = g(x)$, hence on Γ_2, for $x \in [\delta, 2-\delta]$.

Now assume $x_0 \in [\delta, 1]$. Define

$$g(x) = \varphi(x) - 2f_0(x).$$

Using the tangent plane formula (3.4) we compute

$$L_p(x, g(x)) = -\int_{x_0}^{x} f''(t)(x-t)\,dt + \frac{1}{2}\int_{x_0}^{x} \varphi''(t)(x-t)\,dt = -S_{H_0}(p, (x, \varphi(x))) \leq 0$$

by the computations in the first case. Furthermore, since $f_0'' \leq \varphi''$, the graph of g lies above the function

$$a(x) = -2x - \varphi(x)$$

defined in Proposition 3.2 (see figure 5). Since $a(x)$ lies strictly above Γ_2 for $x > 0$ and $\partial_y H_0(x_0, \varphi(x_0)) = 1/2$, we have strictly positive separation on Γ_2.

Finally, for $x_0 \in [1, 2]$, the intersection of the tangent planes at p and at $\tilde{p} = (2 - x_0, \varphi(2 - x_0))$ is the line $x = 1$ since f_0'' is symmetric around $x = 1$. By the previous computations, the tangent plane at $\tilde{p}$ is negative on Γ_2. Thus, the tangent plane at p is negative on Γ_2, completing the proof. □

3.6. **Proof of Key Proposition.** We can slightly modify the construction of H_0 from the previous section to make it smooth, at the expense of giving up a little convexity near the cusps of Γ. Below δ, $\gamma > 0$ are small constants depending only on φ. Let $G(p_1, p_2, p_3) = H\left(p_1, \sqrt{p_2^2 + p_3^2}\right)$.

Proposition 3.11. *For any $\epsilon > 0$ there exists a smooth function H defined on a neighborhood of Γ such that*

1. *H is linear (depending only on x) in a neighborhood of Γ_2, respectively Γ_4,*
2. *$div(\nabla G(\nabla u_0)) = 0$,*
3. *$H_y(x, \varphi(x)) \geq \frac{1}{2}$ for $x \in [\delta, 2-\delta]$, and $H_y \geq 0$ on Γ_1,*

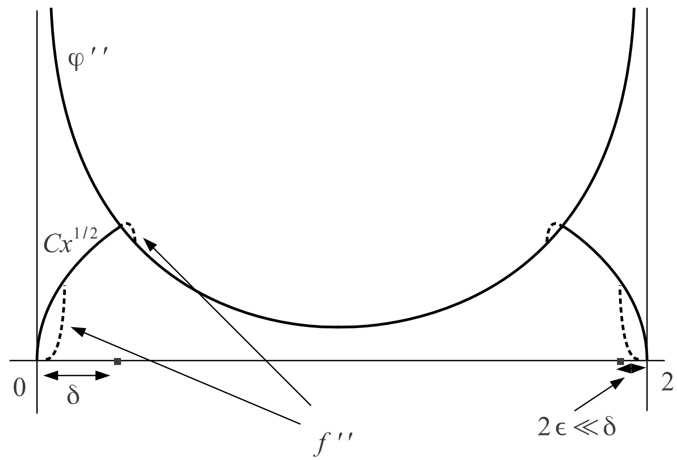

FIGURE 6. f'' is a small perturbation of f_0'' that connects smoothly to φ'' near $x = \delta$ and goes quickly to zero near $x = 0$.

(4) *If* $p = (x, \varphi(x))$ *with* $x \in [\delta, 2-\delta]$ *then* $S_H(p,q) \geq \gamma|p-q|^2$ *for all* $q \in \Gamma$,
(5) $S_H(p,q) \geq -\epsilon|p-q|^2$ *otherwise for* p, $q \in \Gamma$.

Note that the key Proposition 2.4 follows easily from Proposition 3.11 by defining G as above.

Let $\alpha = \frac{1}{2}$ in the construction of f_0'' from the previous section and let $\epsilon \ll \delta$. Let f'' be a smoothing of f_0'' defined by cutting it off smoothly to zero between ϵ and 2ϵ, gluing it smoothly to itself between δ and $\delta + \epsilon$, and making it symmetric over $x = 1$ (see figure 6). Let H be the function in (3.1) determined by f through the Euler-Lagrange relation (3.3).

Proof of proposition 3.11. The first three conclusions are clear by construction so we just need to check the convexity conclusions. Most of them will follow by continuity.

If $p \in \Gamma_2$ we have positive separation since $H \geq 0$, so assume $p = (x_0, \varphi(x_0))$.

If $x_0 \in [\delta, 2-\delta]$ then the conclusion holds by continuity from the arguments in the proof of Proposition 3.10 after taking ϵ small.

Next we may assume by symmetry that $x_0 \in [0, \delta]$.

Case 1: Assume that $x_0 \geq 10\epsilon$.

If $q = (x, \varphi(x))$ with $x > x_0$ then the positive separation follows again by continuity. If $x < x_0$ one computes

$$2s_{f''}(x_0, x) \geq 2s_{f''}(x_0, 0) \geq \frac{4}{5}(1 - (1/5)^{5/2}) > s_{\varphi''}(x_0, 0) \geq s_{\varphi''}(x_0, x)$$

so condition (3.7) holds and we have positive separation on Γ_1.

Since the cutoff is between ϵ and 2ϵ and f'' is increasing for $x < \delta$ we compute

$$d_{f''}(x_0) < \frac{2}{3}. \tag{3.10}$$

and by Remark 3.8 the condition (3.9) holds for $x_0 < \delta$. We thus have positive separation on Γ_2 and Γ_3.

Finally, for $q \in \Gamma_4$ positive separation follows again by continuity.

This establishes positive separation everywhere for $x_0 \in [10\epsilon, 2 - 10\epsilon]$.

Case 2: Assume $x_0 \leq 10\epsilon$.

The tangent plane at p is of order ϵ on Γ, so we have positive separation when $q \in \Gamma_4$.

Using that f'' is increasing and φ'' decreasing near 0, we obtain positive separation if $q = (x, \varphi(x))$ with $x \in [x_0, \delta]$. The same holds for $x > \delta$ by continuity.

If $q = (x, \varphi(x))$ for $x < x_0$ we compute

$$S_H(p, q) \geq -f''(x_0) s_{\varphi''}(x_0, x)(x - x_0)^2 \geq -C\sqrt{\epsilon}\ |p - q|^2,$$

since $s_{\varphi''}(x_0, x) \leq s_{\varphi''}(x_0, 0) \leq 1$. This gives the desired estimate on Γ_1.

Next we bound $S_H(p, q)$ with $q \in \Gamma_2$. For this we estimate the location of the intersection line l_p of the tangent plane at p with 0. By (3.8), l_p passes through

$$\left(x_0, \varphi(x_0) - \frac{2\varphi''(x_0)}{f''(x_0)} \int_0^{x_0} (x_0 - t) f''(t)\, dt\right).$$

We first claim that this point lies above the line $y = -x$. Indeed, since f'' is increasing in $[0, x_0]$, the second component is larger than $\varphi(x_0) - \varphi''(x_0)x_0^2$, and using the expansion (3.2) we see that

$$\varphi(x_0) + x_0 \geq \left(\frac{4}{3}\varphi''(x_0) + O(1)\right) x_0^2 > \varphi''(x_0)x_0^2.$$

By (3.10) the slope of l_p is between -1 and 0, so we have positive separation for $q \in \Gamma_3$ and $q \in \Gamma_2 \cap \{y < -x_0\}$.

Finally, from (3.8) we see that the slope of l_p is less than $\varphi'(x_0)$. Thus, for $x < x_0$, l_p lies above the line

$$y = l(x) = -x_0 + \varphi'(x_0)(x - x_0).$$

A short computation using the expansion (3.2) shows that $l(x)$ crosses $a(x)$, hence Γ_2, at some $x < \xi x_0$ where

$$\xi + \frac{2}{3}\xi^{3/2} = 1 + O(\sqrt{\epsilon}).$$

In particular, $\xi < 1 - c$. This gives that the separation is positive on $\Gamma_2 \cap \{x > \xi x_0\}$, and otherwise the separation is at worst $-C\sqrt{\epsilon}x_0^2 \geq -C\sqrt{\epsilon}|p - q|^2$ (see figure 7). □

Remark 3.12. The proof shows in fact that $S_H(p, q)$ is only negative for p, q very close to the same cusp.

4. Appendix

4.1. **Convex Extension Lemma.**

Proof of lemma 2.1. Let $\mathbf{v}_T$ be the tangential component and $\mathbf{v}_\perp$ be the normal component, and let $\nabla^\Sigma G$ be the gradient of G on Σ. Note that condition 2.1 implies $\mathbf{v}_T = \nabla^\Sigma G$. For $x \in \Sigma$ let T_x, N_x be the tangent and normal subspaces to Σ at x. Let $d_\Sigma(y)$ be the distance from y to Σ and let

$$\Sigma^r = \{y : d_\Sigma(y) < r\}.$$

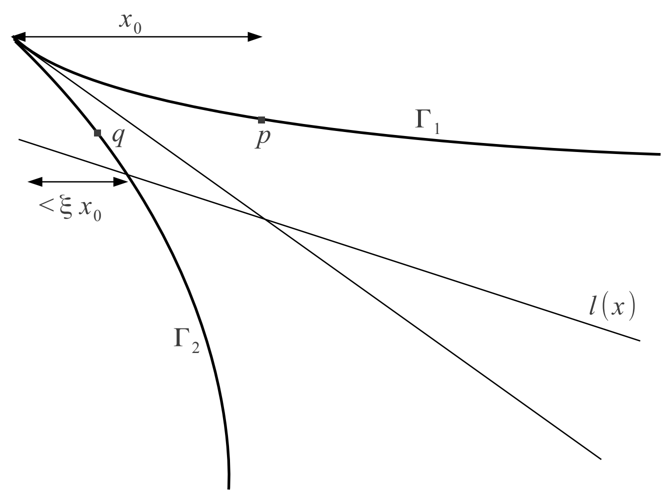

FIGURE 7. The separation is positive if q is below the line $l(x)$, and if the separation is negative then $|p-q|$ is of order x_0.

Finally, for $x \in \mathbb{R}^n$ let $y(x)$ be the closest point in Σ to x. It is well-known that y and d_Σ^2 are well-defined and smooth in a neighborhood of Σ, and for $x \in \Sigma$, $D_x y(x)$ is the projection to T_x and $D^2(d_\Sigma^2/2)(x)$ is the projection to N_x. (For proofs, see for example [AS]).

Step 1: We claim that the function

$$F(x) = G(y(x)) + \mathbf{v}(y(x)) \cdot (x - y(x)) + \frac{A}{2} d_\Sigma^2(x)$$

with A large lifts quadratically from its tangent planes in Σ^σ for σ sufficiently small. We first compute for $x \in \Sigma$ that

$$\begin{aligned} F(x + \epsilon z) &= G(x) + \epsilon \left(\nabla^\Sigma G(x) \cdot z_T + \mathbf{v}(x) \cdot z_\perp \right) + O(\epsilon^2) \\ &= G(x) + \epsilon \mathbf{v}(x) \cdot z + O(\epsilon^2) \end{aligned}$$

giving that $F = G$ on Σ and $\nabla F = \mathbf{v}$ on Σ.

Now, for ϵ small and $\nu \in N_x$ we have $y(x + \epsilon\nu) = x$ and $d_\Sigma(x + \epsilon\nu) = \epsilon$, so $F_{\nu\nu}(x) = A$. In addition, if $x \in \Sigma$ and $x + \epsilon z \in \Sigma$ for some unit vector z then by hypothesis we have

$$\begin{aligned} F(x + \epsilon z) &= F(x) + \epsilon \nabla F(x) \cdot z + \frac{\epsilon^2}{2} z^T \cdot D^2 F(x) \cdot z + O(\epsilon^3) \\ &\geq F(x) + \epsilon \nabla F(x) \cdot z + \gamma \epsilon^2. \end{aligned}$$

Taking ϵ to zero we see that $F_{\tau\tau}(x) > 2\gamma$ for any tangential unit vector τ.

Take any unit vector e and write $e = \alpha\tau + \sqrt{1-\alpha^2}\nu$ for some unit $\tau \in T_x\Sigma$ and $\nu \in N_x\Sigma$. Since $D^2(d_\Sigma^2/2)$ is the projection matrix onto N_x at $x \in \Sigma$, we have

$$\begin{aligned} F_{ee}(x) &= \alpha^2 F_{\tau\tau} + (1-\alpha^2)F_{\nu\nu} + 2\alpha\sqrt{1-\alpha^2}(F - Ad_\Sigma^2/2)_{\tau\nu} \\ &\geq 2\alpha^2\gamma + (1-\alpha^2)A - C\alpha\sqrt{1-\alpha^2} \end{aligned}$$

for some C independent of A. We conclude that $D^2F > \frac{3}{2}\gamma I$ on Σ for A sufficiently large, and in particular, $D^2F > \frac{3}{2}\gamma I$ on a neighborhood $\Sigma^{2\rho}$ of Σ.

Finally, we show that the tangent planes to F in Σ^σ separate quadratically for σ small. Let $x,\, z \in \Sigma^\sigma$. We divide into two cases.

If $|z-x| < \rho$ then x and z can be connected by a line segment contained in $\Sigma^{2\rho}$, so it is clear that

$$F(z) > F(x) + \nabla F(x)\cdot(z-x) + \frac{3}{4}\gamma|z-x|^2.$$

If on the other hand $|z-x| > \rho$, we use that

$$F(y(z)) > F(y(x)) + \nabla F(y(x))\cdot(y(z)-y(x)) + \gamma|y(z)-y(x)|^2.$$

Replacing $y(z)$ by z and $y(x)$ by x changes these quantities by at most $C\sigma$, and since and $|z-x| > \rho$ we have that

$$F(z) > F(x) + \nabla F(x)\cdot(z-x) + \frac{3}{4}\gamma|z-x|^2$$

for all $x,\, z \in \Sigma^\sigma$ for σ small.

Step 2: From now on denote the open set Σ^σ by N. Let N_ϵ denote $\{x \in N : B_\epsilon(x) \subset N\}$. Finally, let ρ_ϵ denote the standard mollifier $\epsilon^{-n}\rho(x/\epsilon)$ where ρ is supported in B_1, nonnegative, smooth and has unit mass.

We define a global uniformly convex function that agrees with F on N. Let

$$H_0(y) = \sup_{x\in N}\left\{F(x) + \nabla F(x)\cdot(y-x) + \frac{3}{4}\gamma|y-x|^2\right\}.$$

Then H_0 is a uniformly convex function on $\mathbb{R}^n$ with $D^2H_0 \geq \frac{3}{2}\gamma I$ and furthermore by construction we have that $H_0 = F$ on N.

To finish we glue H_0 to a mollification. Fix δ so that $\Sigma \subset N_{2\delta}$. Let

$$H_\epsilon = \rho_\epsilon * H_0$$

for some ϵ small. In N_δ we have

$$|H_\epsilon - H_0|,\, |\nabla H_\epsilon - \nabla H_0| < C\epsilon.$$

Finally, since $D^2H_0 \geq \frac{3}{2}\gamma I$ we have $D^2H_\epsilon > \frac{3}{2}\gamma I$.

Let η be a smooth cutoff function which is 1 on $N_{2\delta}$ and 0 outside of N_δ. Then let

$$H = \eta H_0 + (1-\eta)H_\epsilon.$$

We compute

$$D^2H = \eta D^2H_0 + (1-\eta)D^2H_\epsilon + 2\nabla\eta\otimes\nabla(H_0 - H_\epsilon) + D^2\eta(H_0 - H_\epsilon).$$

Then H is smooth, $H = F$ on $N_{2\delta}$ and taking ϵ small we have $D^2H > \gamma I$, completing the construction. □

4.2. **Expansion of φ.**

Proof of proposition 3.1. The symmetries of φ follow from the symmetries of w.

The curve Γ_1 is parametrized by $\nabla w(\theta)$ for $\theta \in [\pi/4, 3\pi/4]$. Let ν be the upward normal to Γ_1. Since w is homogeneous degree one we have $\nu(\nabla w(\theta)) = \theta$. Differentiating we get the the curvature $\kappa = \frac{1}{g''+g}$ where $g(\theta) = \frac{-1}{\sqrt{2}} \cos 2\theta$ are the values of w on S^1. Thus, φ is uniformly convex and its second derivatives blow up near $x = \pm 1$. To quantify this we compute

$$\begin{aligned}\nabla w(\theta) &= g(\theta)(\cos\theta, \sin\theta) + g'(\theta)(-\sin\theta, \cos\theta) \\ &= \frac{1}{\sqrt{2}}(-\cos\theta(1 + 2\sin^2\theta), \sin\theta(1 + 2\cos^2\theta)).\end{aligned}$$

Expanding around $\theta = \frac{\pi}{4}$ (which gets mapped to the left cusp on Γ_1) we get

$$\varphi\left(-1 + \frac{3}{2}\theta^2 + \theta^3 + O(\theta^4)\right) = 1 - \frac{3}{2}\theta^2 + \theta^3 + O(\theta^4). \tag{4.1}$$

Differentiating implicitly one computes

$$\varphi''(-1+\epsilon) = \sqrt{\frac{2}{3}}\epsilon^{-1/2} + O(1)$$

and that φ'' is decreasing near -1. □

4.3. **Theorem 1.1.** In [DS] the authors show that if u is a scalar minimizer to a convex functional $\int_{B_1} F(\nabla u)\, dx$ on $\mathbb{R}^2$ and F is uniformly convex in a neighborhood of $\nabla u(B_1) \cap \{|p_1| < 1\}$ then ∇u cannot jump arbitrarily fast across the strip. In particular, $\nabla u(B_\gamma)$ localizes to $\{p_1 < 1\}$ or $\{p_1 > -1\}$ for some γ small. In this final section we use the preliminary construction H_0 from section 3 to indicate why this result is not true in three or higher dimensions.

Make a global extension of H_0 by taking

$$\bar{H}_0(x) = \sup_{p \in \Gamma_1 \cup \Gamma_3} \{H_0(p) + \nabla H_0(p) \cdot (x - p) + \eta(p_1)|x - p|^2\}.$$

The resulting extension is smooth near any non-cusp point of Γ. It is uniformly convex near each point on $(\Gamma_1 \cup \Gamma_3) \cap \{|p_1| < 1\}$ with the modulus of convexity decaying towards the cusps. Furthermore, $\bar{H}_0$ is flat in a neighborhood of every point on $(\Gamma_2 \cup \Gamma_4) \cap \{|p_2| < 1\}$. Finally, if p is a cusp of Γ then it is straightforward to check that $\bar{H}_0$ is pointwise $C^{1,1-\alpha}$ at p, i.e. $S_{\bar{H}_0}(p, x) < C|x - p|^{2-\alpha}$ for all x near p. By iterating a mollification and gluing procedure similar to those used in the proof of lemma 2.1 near the cusps we can get a global convex extension $\bar{H}$ that is smooth away from the cusps, uniformly convex on $\Gamma_1 \cup \Gamma_3$ away from the cusps, flat on convex sets containing Γ_2 and Γ_4, and $C^{1,1-\alpha}$ at the cusps.

Remark 4.1. In dimension n the Euler-Lagrange equation allows us to take $f_0''(x) = x^{n-2-\alpha}$ near the cusp, which gives $\bar{H}$ an extra derivative for each dimension.

Let G_0 be the function on $\mathbb{R}^3$ obtained by revolving $\bar{H}$ around the p_1 axis (see figure 8). By construction u_0 solves the Euler-Lagrange equation $\text{div}(\nabla G_0(\nabla u_0)) = 0$ away from the cone $C_0 = \{|x_1| = r\}$ where $r = \sqrt{x_2^2 + x_3^3}$. Thus, it is not

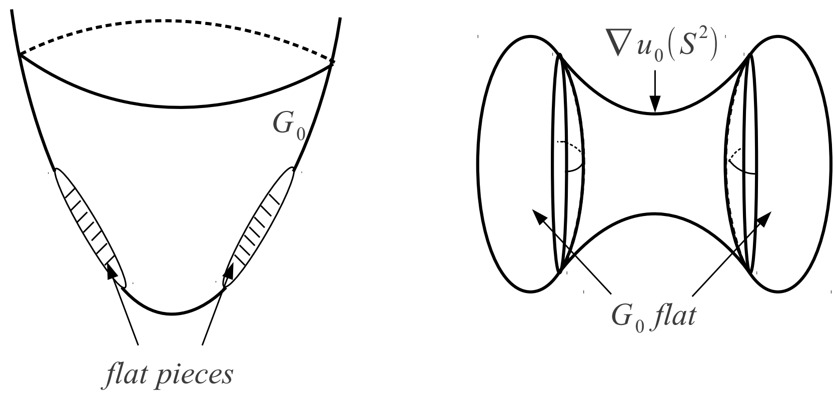

FIGURE 8. G_0 is linear on two bounded convex sets containing $\nabla u_0(\{|x_1| > r\})$.

immediate that u_0 minimizes $\int_{B_1} G_0(\nabla u_0)\, dx$. However, we claim u_0 is a minimizer. To show this we must establish

$$\int_{B_1} \nabla G_0(\nabla u_0) \cdot \nabla \psi \, dx = 0$$

for any $\psi \in C_0^\infty(B_1)$. The contribution from integrating in B_ϵ and a thin cone $\{(1-\epsilon)r < |x_1| < (1+\epsilon)r\}$ is small. Integrating by parts in the remaining region with boundary S, we get a boundary term of the form $\int_S \psi \nabla G_0(\nabla u_0) \cdot \nu \, ds$ where ν is the outer normal. The cones $\{|x_1| = (1 \pm \epsilon)r\}$ are ϵ close, and the outward normals on these cones are ϵ close to flipping direction, so by the continuity of ∇G_0 the contribution from this term is also small. Taking ϵ to zero we get the desired result.

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