GRADIENT ESTIMATES FOR THE LAGRANGIAN MEAN CURVATURE EQUATION WITH CRITICAL AND SUPERCRITICAL PHASE

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ABSTRACT. In this paper, we prove interior gradient estimates for the Lagrangian mean curvature equation, if the Lagrangian phase is critical and supercritical and C^2. Combined with the a priori interior Hessian estimates proved in [Bha21, Bha22], this solves the Dirichlet boundary value problem for the critical and supercritical Lagrangian mean curvature equation with C^0 boundary data. We also provide a uniform gradient estimate for lower regularity phases that satisfy certain additional hypotheses.

1. Introduction

In this paper, we study a priori interior gradient estimates in all dimensions for the Lagrangian mean curvature equation

$$F(D^2u) = \sum_{i=1}^{n} \arctan \lambda_i = \psi(x), \qquad x \in B_1(0) \subset \mathbb{R}^n, \tag{1.1}$$

under the assumption that $|\psi| \geq (n-2)\frac{\pi}{2}$. Here, $u : B_1 \to \mathbb{R}$ has gradient Du and Hessian matrix D^2u, with eigenvalues λ_i. We will denote $B_r = B_r(0)$ throughout.

When the phase ψ is constant, denoted by c, u solves the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = c, \tag{1.2}$$

or equivalently,

$$\cos c \sum_{1\leq 2k+1\leq n} (-1)^k \sigma_{2k+1} - \sin c \sum_{0\leq 2k\leq n} (-1)^k \sigma_{2k} = 0.$$

Equation (1.2) originates in the special Lagrangian geometry by Harvey-Lawson [HL82]. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the argument of the complex number $(1+i\lambda_1)...(1+i\lambda_n)$, or the phase ψ, is constant, and it is special if and only if $(x, Du(x))$ is a (volume minimizing) minimal surface in $(\mathbb{R}^n \times \mathbb{R}^n, dx^2+dy^2)$ [HL82].

More generally, for (1.1), it was shown in [HL82, (2.19)] that the mean curvature vector $\vec{H}$ of the Lagrangian graph $(x, Du(x))$ is $J\nabla_g\psi$, where ∇_g is the gradient, and J is the almost complex structure on $\mathbb{R}^n \times \mathbb{R}^n$. Note that $|\vec{H}|_g$ is bounded for $\psi \in C^1$. In the complex setting, a local version of the deformed Hermitian-Yang-Mills equation for a holomorphic line bundle over a compact Kähler manifold is represented by equation (1.1).

The notions of critical and supercritical phases were introduced by Yuan [Yua06]. The Lagrangian angle $\theta(\lambda) = \sum_i \arctan \lambda_i$ is critical if $|\theta| = (n-2)\pi/2$ and supercritical if $|\theta| > (n-2)\pi/2$. We recall that the variable phase $\psi(x)$ is called critical and supercritical if $|\psi(x)| \geq (n-2)\pi/2$, and supercritical if $|\psi(x)| \geq (n-2)\pi/2+\delta$ for some $\delta > 0$. It was shown in [Yua06, Lemma 2.1] that the level sets $\{\lambda : \theta = c\}$ are convex for critical and supercritical phases. In particular, there are Evans [Eva82]-Krylov [Kry83]-Safonov[Saf84, Saf89] $C^{2,\alpha}$ estimates if D^2u is bounded, and $\psi(x)$ is Hölder continuous.

In this paper, for C^2 critical and supercritical phases, we solve the Dirichlet problem for C^0 boundary data by establishing the missing interior gradient estimates. Interior Hessian estimates for supercritical $C^{1,1}$ phases were shown by Bhattacharya in [Bha21, Theorem 1.1]; interior Hessian estimates for critical and supercritical phases follow verbatim from the calculations done in [Bha21] (see [Bha22, Remark 2.1]); interior gradient estimates for supercritical C^1 phases were derived in [Bha21, Theorem 1.2].

Our main result is a gradient estimate for arbitrary C^2 critical and supercritical phases.

Theorem 1.1. *Let u be a $C^3(\overline{B_1})$ solution of (1.1) on $B_1(0) \subset \mathbb{R}^n$, where $\psi \in C^2(B_1)$ satisfies $\psi \geq (n-2)\frac{\pi}{2}$. Then*

$$|Du(0)| \leq C(n, ||D^2\psi||_{L^\infty(B_1)}) \left(1 + (osc_{B_1} u)^2\right). \tag{1.3}$$

We state the following Hessian estimate combining [Bha21, Theorem 1.1] and [Bha22, Remark 2.1].

Theorem 1.2. *Let u be a C^4 solution of (1.1) on $B_R(0) \subset \mathbb{R}^n$, where $\psi \in C^2(B_R)$, and $\psi \geq (n-2)\frac{\pi}{2}$. Then we have*

$$|D^2u(0)| \leq C \exp[C \max_{B_R(0)} |Du|^{2n-2}/R^{2n-2}] \tag{1.4}$$

where C is a positive constant depending on $||\psi||_{C^2(B_R)}$ and n.

As an application, we solve the following Dirichlet boundary value problem with C^0 boundary data.

Corollary 1.1. *Suppose that $\phi \in C^0(\partial\Omega)$ and $\psi : \overline{\Omega} \to [(n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ is in $C^2(\overline{\Omega})$, where Ω is a uniformly convex, bounded domain in $\mathbb{R}^n$. Then there exists a unique solution $u \in C^3(\Omega) \cap C^0(\overline{\Omega})$ to the Dirichlet problem*

$$\begin{cases} F(D^2u) = \sum_{i=1}^n \arctan \lambda_i = \psi(x) \text{ in } \Omega \\ u = \phi \text{ on } \partial\Omega \end{cases} \tag{1.5}$$

The solution u is, in fact, in $C^{3,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$, by classical uniformly elliptic theory.

The Dirichlet problem for a broad class of fully nonlinear, elliptic equations of the form $F(\lambda[D^2u]) = f(x)$ was first studied by Caffarelli-Nirenberg-Spruck in [CNS85], where they proved the existence of classical solutions under various hypotheses on the function F and the domain Ω. In [HL09], Harvey-Lawson studied the Dirichlet problem for fully nonlinear, degenerate elliptic equations of the form $F(D^2u) = 0$ on a smooth bounded domain in $\mathbb{R}^n$. The existence and uniqueness of continuous viscosity solutions to the Dirichlet problem for (1.2) with continuous boundary data was shown in [HL09, Yua08]; see also [Bha20]. In [BW10], Brendle-Warren studied a second boundary value problem for the special Lagrangian equation.

For subcritical phases $|c| < (n-2)\pi/2$, interior regularity is not understood. For critical $|c| = (n-2)\pi/2$ and supercritical $|c| > (n-2)\pi/2$ phases, interior gradient estimates were established by Warren-Yuan [WY09a, WY10], and also Yuan's unpublished notes from 2015. Interior Hessian estimates for dimension $n = 2$ were shown by Heinz, for $|c| = \pi/2$ in dimension $n = 3$ by Warren-Yuan [WY09b], and for general dimension $|c| \geq (n-2)\pi/2$ by Wang-Yuan [WY14]; see also Li [Li19] for a compactness approach and Zhou [Zho22] for estimates requiring Hessian constraints which generalize criticality.

Because the level set of the PDE is convex for critical and supercritical phases, the Evans-Krylov theory yields interior analyticity. The singular $C^{1,\alpha}$ subcritical phase solutions by Nadirashvili-Vlăduţ [NV10] and Wang-Yuan [WY13] show that interior regularity is not possible for subcritical phases, without an additional convexity condition, as in Bao-Chen [BC03], Chen-Warren-Yuan [CWY09], and Chen-Shankar-Yuan [CSY22], and that the Dirichlet problem is not classically solvable for arbitrary smooth boundary data. Interior gradient estimates for continuous boundary data are widely open. Global gradient estimates requiring Lipschitz boundary data were shown by [Lu22]. Homogeneous viscosity solutions of degree less than two were shown to not exist by Nadirashvili-Yuan [NY06]. The non-existence result of Mooney [Moo22] shows that counterexamples for interior C^1 regularity may be difficult to construct.

If the Lagrangian angle is not necessarily constant, then less is understood. In [HL19], Harvey-Lawson introduced a condition called "tameness" on the operator F, which is a little stronger than strict ellipticity and allows one to prove comparison. In [HL21], tamability was established for the supercritical Lagrangian mean curvature equation. In [CP21], Cirant-Payne established comparison principles for the Lagrangian mean curvature equation provided the Lagrangian phase is restricted to the intervals $((n-2k)\frac{\pi}{2},(n-2(k-1))\frac{\pi}{2})$ where $1\leq k\leq n$, which in turn solves the Dirichlet problem on these intervals as shown in [HL21, Theorem 6.2]. Hessian estimates for convex smooth solutions with $C^{1,1}$ phase $\psi=\psi(x)$ were obtained by Warren [War08, Theorem 8]. For convex viscosity solutions, interior regularity was established for C^2 phases; see Bhattacharya-Shankar [BS20b, BS20a]. For supercritical phases $|\psi(x)|\geq(n-2)\pi/2+\delta$, there is a comparison principle, and the Dirichlet problem was solved in Collins-Picard-Wu [CPW17], Dinew-Do-Tô [DDT18], Bhattacharya [Bha20], and interior gradient estimates were established in [Bha21]. Interior Hessian estimates for supercritical phases were established in [Bha21]. Interior Hessian estimates for critical and supercritical phases $|\psi(x)|\geq(n-2)\frac{\pi}{2}$, follow verbatim from the calculations done in [Bha21] (see [Bha22, Remark 2.1]): The proof of the Hessian estimate in [Bha21, Theorem 1.1] does not require a negative lower bound on the lowest eigenvalue. For supercritical phases in dimension $n=2$, a simplified proof [Bha22] was given for interior Hessian estimates using the super-isoperimetric inequality of Warren-Yuan [WY09a], avoiding the Michael-Simon mean value inequality [MS73]. In the case that ϕ is Lipschitz, Corollary 1.1 can be obtained by proving a global gradient estimate, as in [Lu22]. The existence of interior gradient estimates for the challenging borderline case of critical and supercritical phase has until now remained open. In this paper, we successfully solve this problem for C^2 phases.

Our approach to prove interior gradient estimate Theorem 1.1 accounts for the smallness of the gradient of the phase near its minimizing, critical values, using a pointwise interpolation inequality [NT70, Equation (3.11), pg. 19], see also [Hor83, Lemma 7.7.2], valid for C^2 phases. For constant phases, the gradient estimate is established using a maximum principle inequality. The variable phase contribution to the inequality is a "bad term" depending on the phase's gradient. Although the PDE's ellipticity degenerates at the critical phase, making the bad term large, the smallness of the gradient at such points provides a balance. Our proof, more generally, shows that an interior gradient estimate holds when ψ satisfies a certain first order differential inequality; see Remark 2.2. Such an inequality is valid when the phase is any of C^2, semi-concave, concave, or a supersolution of the infinity-Laplace equation; see Remark 2.3.

On the other hand, the gradient vanishes at slower rates for $C^{1,\alpha}$ phases, and does not appear to balance the degeneration of the ellipticity in our proof. But we note that certain Hölder continuous phases allow for gradient estimates; see Remark 2.4. In such cases, the phase separates from the critical value at such a large speed that the solution is nearly semi-convex, as in the supercritical case.

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2. Proof of the gradient estimate

We modify the pointwise proof of [WY09a] and [Bha21] to bridge the constant critical phase estimate of [WY09a] with the supercritical estimate of [Bha21]. The difference comes from how to treat the bad term involving $D\psi$.

Notation: we denote $a \sim b$ if $ca \le b \le Ca$, and $a \lesssim b$ if $a \le Cb$. Here, c and C are positive constants depending on n. We will denote $a \lesssim_\psi b$ if the constant also depends on $\|D^2\psi\|_{L^\infty(B_1)}$. We denote $a \ll_n 1$ if we are choosing a small fixed constant a depending on n. We assume summation under repeated indices unless otherwise indicated.

Let $M = \text{osc}_{B_1} u > 0$; replacing u with $u - \min_{B_1} u + M$, we assume that

$$M \le u \le 2M \qquad \text{in } B_1. \tag{2.1}$$

Let $w = \eta|Du| + Au^2/2$, where $\eta = 1 - |x|^2$, and $A = 3\sqrt{n}/M$. Let $x_0 \in B_1$ be where w is maximized. After a rotation, we assume that D^2u is diagonal, with $u_{ii} = \lambda_i$. Let us assume that $u_n \ge |Du|/\sqrt{n} > 0$. Then for each k, at the max point x_0,

$$0 = \partial_k w(x_0) = \eta \frac{u_k \lambda_k}{|Du|} + \eta_k |Du| + Auu_k. \tag{2.2}$$

Since $A = 3\sqrt{n}/M$ is sufficiently large, it follows that

$$\eta \lambda_n \frac{u_n}{|Du|} \in -(c(n), C(n))|Du|. \tag{2.3}$$

It follows that $\lambda_n < 0$ and

$$\eta \sim \frac{|Du|}{|\lambda_n|}. \tag{2.4}$$

Since $|Du| \lesssim \eta|\lambda_n|$, we may assume that $|\lambda_n| > 1$, since otherwise the estimate is done. Moreover, as shown in [WY14, Lemma 2.5], we know that $\lambda_k \ge |\lambda_n|$ for $k < n$ follows from $\psi \ge (n-2)\pi/2$.

We now proceed to the second derivatives of w. Let $g = I + (D^2u)^2$ be the induced metric $dx^2 + dy^2$ on $(x, Du(x))$, with $g^{-1} = (g^{ij})$ its inverse, and $g^{ij} = (1+\lambda_i^2)^{-1}\delta_{ij}$ at x_0. Then at x_0,

(2.5)
$$\begin{aligned} 0 &\geq g^{ij}\partial_{ij}w(x_0) \\ &= \underbrace{g^{ij}\eta_{ij}|Du|}_{(C1)} + \underbrace{2g^{ij}\eta_i|Du|_j}_{(C2)} + \underbrace{\eta g^{ij}|Du|_{ij}}_{(B)} + \underbrace{Aug^{ij}u_{ij}}_{(C3)} + \underbrace{Ag^{ij}u_iu_j}_{(G)}. \end{aligned}$$

The good term (G) absorbs the constant-phase terms (C1), (C2), and (C3) in the constant phase and supercritical cases, as in [WY09a] and [Bha21]. The bad term (B) contains variable phase contributions and will require closer examination.

The good term (G):

$$Ag^{ij}u_iu_j \geq A\frac{u_n^2}{1+\lambda_n^2} \sim A\frac{|Du|^2}{\lambda_n^2} \sim \frac{A\eta|Du|}{|\lambda_n|}. \tag{2.6}$$

The first constant phase term (C1):

$$g^{ij}\eta_{ij}|Du| = -2\sum_i \frac{1}{1+\lambda_i^2}|Du| \gtrsim -\frac{1}{\lambda_n^2}|Du| \sim -\frac{\eta}{|\lambda_n|}. \tag{2.7}$$

The second constant phase term (C2):

$$2g^{ij}\eta_i|Du|_j \overset{x_0}{\gtrsim} -\sum_i \frac{1}{1+\lambda_i^2}\frac{u_i\lambda_i}{|Du|} \gtrsim -\frac{1}{|\lambda_n|}. \tag{2.8}$$

The third constant phase term (C3):

$$Aug^{ij}u_{ij} = Au\sum_i \frac{\lambda_i}{1+\lambda_i^2} \gtrsim -\frac{1}{|\lambda_n|}. \tag{2.9}$$

The bad term (B), using third derivative calculation [Bha21, (2.4)]:

$$\begin{aligned} \eta g^{ij}|Du|_{ij} &= \eta g^{ij}\frac{u_{ijk}u_k}{|Du|} + \eta\sum_i g^{ii}\frac{(|Du|^2-u_i^2)\lambda_i^2}{|Du|^3} \\ &\geq \eta\psi_k\frac{u_k}{|Du|} \gtrsim -\eta|D\psi|. \end{aligned} \tag{2.10}$$

We thus need to bound this inequality at x_0:

$$\eta|Du| \leq C(n)M(1+\eta\,|D\psi|\,|\lambda_n|). \tag{2.11}$$

Letting $\phi = \psi - (n-2)\pi/2 \geq 0$, we apply the pointwise interpolation inequality for nonnegative C^2 functions in [Hor83, Lemma 7.7.2] on

$B_\delta(x_0)$, where $\delta = 1 - |x_0|$;

$$
\begin{aligned}
|D\phi(x_0)|^2 &\le \frac{\phi(x_0)^2}{(1-|x_0|)^2} + 2\|D^2\phi\|_{L^\infty(B_1)}\phi(x_0) \\
&\lesssim_\psi \frac{\phi(x_0)^2}{\eta^2} + \phi(x_0),
\end{aligned} \tag{2.12}
$$

where $\|D^2\phi\|_{L^\infty(B_1)}$ denotes the maximum of the absolute values of the eigenvalues of $D^2\phi$. Let us now recall the following algebraic inequality, valid for $\lambda_n < 0$ and $\lambda_k > 0$ for $k < n$:

$$
\begin{aligned}
\psi &= (n-1)\frac{\pi}{2} - \sum_{i<n} \arctan(\frac{1}{\lambda_i}) - \frac{\pi}{2} + \arctan(-\frac{1}{\lambda_n}) \\
&\le (n-2)\frac{\pi}{2} + \frac{1}{|\lambda_n|}.
\end{aligned} \tag{2.13}
$$

Substituting this information into (2.11), combined with (2.4), yields

$$
M^{-1}\eta|Du| \lesssim_\psi 1 + \eta|\lambda_n|^{1/2} \lesssim 1 + (\eta|Du|)^{1/2}. \tag{2.14}
$$

It follows that

$$
\eta|Du| \lesssim_\psi M + M^2 \lesssim C(n, \|\psi\|_{C^2(B_{1.5})})(1 + (\mathrm{osc}_{B_1} u)^2). \tag{2.15}
$$

The $Au^2/2$ term in $w(x_0)$ and the estimate of w on ∂B_1 are subordinate to this estimate, so we conclude the proof. $\square$

Remark 2.1. It is straightforward to refine the $|\lambda_n| < 1$ case and thereby improve the estimate (1.3) to the following

$$
|Du(0)| \le C(n, \|D^2\psi\|_{L^\infty(B_1)})(\mathrm{osc}_{B_1} u + (\mathrm{osc}_{B_1} u)^2). \tag{2.16}
$$

Remark 2.2. More generally, let $\psi \in C^1(B_1)$ be critical and supercritical, or $\psi - (n-2)\pi/2 =: \phi \ge 0$, and also satisfy the following first order differential inequality on B_1:

$$
|D\phi| \le \eta f(\eta^{-2}\phi), \tag{2.17}
$$

where $f(t) \searrow 0$ as $t \searrow 0$, and $\eta = 1 - |x|^2$. Then a C^1 estimate is valid for $C^3(\overline{B_1})$ solutions of (1.1):

$$
|Du(0)| \le C(n, f, \mathrm{osc}_{B_1} u). \tag{2.18}
$$

To prove the estimate, we insert (2.17) in the determining inequality (2.11) and use (2.13) and (2.4). We obtain at x_0, using that f is increasing,

$$\begin{aligned} M^{-1}\,\eta|Du| &\lesssim 1+\eta^2|\lambda_n| f\left(\frac{1}{\eta^2|\lambda_n|}\right) \\ &\lesssim 1+\eta|Du| f\left(\frac{C(n)}{\eta|Du|}\right). \end{aligned} \tag{2.19}$$

If $\eta|Du| \geq C(n,f,M) =: H$ for large enough H such that $f(C(n)/H) \ll_n M^{-1}$, then $M^{-1}\eta|Du| \lesssim 1$, and the estimate follows. In the alternative case that $\eta|Du| \leq C(n,f,M)$, the estimate (2.18) is already done.

Remark 2.3. Let us list some examples of phases which satisfy a first order inequality of the form (2.17).

1. For $f(t)^2 = t^2 + Ct$, we recover the C^2 interpolation inequality (2.12). Note that general $C^{1,\alpha}$ phases fail to satisfy the inequality (2.17).

2. Interpolation inequality (2.12) can be generalized to phases $\psi \in C^1(B_1)$ which are semi-concave, with $D^2\psi \leq KI$ for some $K > 0$. In this case, the dependence on $\|D^2\psi\|_{L^\infty(B_1)}$ is replaced with K. Indeed, by semi-concavity, there holds for $x_0 \in B_1(0)$ and $x \in B_\delta(x_0)$:

$$0 \leq \phi(x_0) + (x-x_0)\cdot D\phi(x_0) + K|x-x_0|^2/2. \tag{2.20}$$

The proof in [Hor83, Lemma 7.7.2] can then be repeated verbatim. This generalizes Theorem 1.1 to semi-concave phases.

3. The choice $f(t) = 2t$ corresponds to $\psi \in C^1(B_1)$ concave. Choosing $x - x_0 = -(1-|x_0|)D\phi(x_0)/|D\phi(x_0)|$ with $K = 0$ in (2.20) gives

$$|D\phi(x_0)| \leq \frac{\phi(x_0)}{1-|x_0|} \leq \frac{2\phi(x_0)}{\eta}. \tag{2.21}$$

This is the first term in (2.12), so as in (2.14), we obtain $\eta|Du| \leq C(n)M$. We thus obtain the linear estimate

$$|Du(0)| \leq C(n)(1+\mathrm{osc}_{B_1} u). \tag{2.22}$$

This can be improved to $|Du(0)| \leq C(n)\mathrm{osc}_{B_1} u$, as in Remark 2.1. One novelty here is the independence of ψ. For example, if

$$\psi(x) = (n-2)\frac{\pi}{2} + \epsilon(1-|x|^{1+\alpha}) \tag{2.23}$$

for some $\epsilon, \alpha \in (0,1)$, then (2.22) is independent of ϵ. The interior gradient estimate for C^1 supercritical phases in [Bha21] would degenerate as $\epsilon \to 0$.

4. Suppose that $\phi(x) \geq 0$ is a $C^1(B_1)$ viscosity supersolution of the infinity-Laplace equation/Aronsson's equation:

$$D^2\phi(D\phi, D\phi) = \phi_{ij}\phi_i\phi_j \leq 0. \tag{2.24}$$

Then using comparison with cones, there is a pointwise estimate [CEG01, Lemma 2.5] for the gradient:

$$|D\phi(x)| \leq \frac{\phi(x)}{1-|x|}. \tag{2.25}$$

In fact, this is concavity inequality (2.21), and this corresponds to $f(t) = 2t$ in (2.17). We conclude that a linear gradient estimate (2.22) is valid.

Remark 2.4. If u is a viscosity solution of (1.1) for Hölder phase

$$\psi = (n-2)\frac{\pi}{2} + |x|^\alpha, \tag{2.26}$$

where $0 < \alpha < 1$, the function

$$u(x) + C|x|^{2-\alpha} \tag{2.27}$$

is convex, if $C(\alpha)$ is large enough. This follows from the algebraic relation (2.13), which gives $|\lambda_{min}| < |x|^{-\alpha}$. It follows that $u(x)$ is locally Lipschitz continuous.

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