Notes on Elliptic Partial Differential Equations

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1 Introduction and Acknowledgements

I wrote these notes the summer after my first year at Columbia. They grew out of many discussions with Professor Ovidiu Savin, supplementing material from Gilbarg-Trudinger, Caffarelli-Cabré and Evans-Gariepy (see References). I am very grateful for Professor Savin's insights and patient guidance.

2 Harmonic Functions

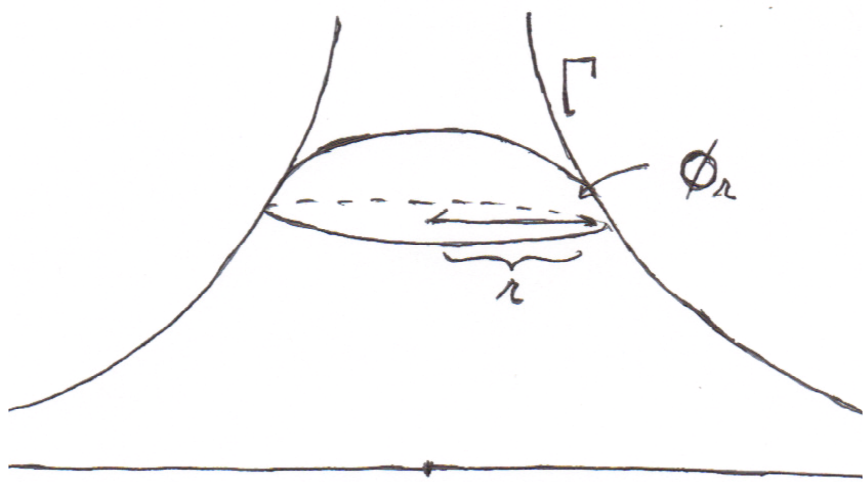
Reading: Chapter 2 of Gilbarg-Trudinger, familiarity with the mean value property, weak maximum principle, Harnack inequality, interior gradient estimates, weak solutions, and the Green's function and Poisson integral.

2.1 Mean Value Property for Weak Solutions

Here we give a proof of the mean value property for weak solutions and guide the reader to several proofs that weak solutions are classical. Let $u \in L^1_{loc}$ be a weak solution to $\Delta u = 0$, i.e.

$$\int_{B_1} u\Delta\eta = 0$$

for all $\eta \in C_0^\infty(B_1)$. (Actually, test functions in $W_0^{2,\infty}(B_1)$ are acceptable by approximation with smooth functions). The idea is to make a good choice of test function. Take the fundamental solution $\Gamma = |x|^{2-n}$ in $\mathbb{R}^n$, and cap it off with a paraboloid at radius r. The resulting function ϕ_r is $C^{1,1}$, hence $W^{2,\infty}$.



Furthermore, one easily computes $\Delta\phi_r$ in B_r using the divergence theorem:

$$\Delta\phi_r|B_r| = \int_{B_r} \Delta\phi_r = \int_{\partial B_r} D_\nu\Gamma = -C(n).$$

Since ϕ_r is harmonic outside B_r, this gives

$$\Delta\phi_r = -\frac{C(n)}{|B_r|}\chi_{B_r}.$$

We choose our test function to be

$$\eta = \phi_{r_1} - \phi_{r_2}$$

where $r_1 < r_2 < 1$. Using that u is weakly harmonic and the previous computation, we obtain

$$0 = \int_{B_1} u\Delta\eta = C(n)\left(\frac{1}{|B_{r_2}|}\int_{B_{r_2}} u - \frac{1}{|B_{r_1}|}\int_{B_{r_1}} u\right).$$

Finally, we can take $r_1 \to 0$ and apply the Lebesgue differentiation theorem to conclude that the mean value property is satisfied almost everywhere.

Remark: Since the average of u over balls of a fixed radius varies continuously, we see that u is continuous up to redefinition on a set of measure 0.

Exercise: Show that u is classically harmonic by comparing u on a small ball with the harmonic function agreeing with u on the boundary of this ball (from Gilbarg-Trudinger).

Exercise: Show that a mollification u_ϵ of u is classically harmonic. Using the continuity of u, show that u_ϵ converges uniformly to u on compact subsets. Conclude that u is classically harmonic using the previous exercise.

Exercise: Convolve u with a radially symmetric mollifier. Using the mean value property, show that this mollifier agrees with u. Conclude that u is smooth.

Exercise: Suppose that u and $D_\nu u$ vanish identically on some open subset of ∂B_1. Show that u vanishes identically in B_1 (assume analyticity).

Exercise: Draw a picture giving a heuristic proof that the mean value property gives an interior estimate for Du in terms of $\|u\|_{L^\infty}$.

Exercise: A geometric interpretation of Δ. Let $u \in C^2$; one way to understand Δu is the average growth of u from its linear part. Show that

$$\Delta u(x) = c \lim_{r\to 0} \frac{1}{r^2}\left(\frac{1}{|\partial B_r|_{H^{n-1}}}\int_{\partial B_r(x)} u - u(x)\right).$$

Why do we expect the $\frac{1}{r^2}$? This gives another proof that MVP implies harmonic.

2.2 Harnack Inequality

We guide the reader to several regularity results which rely only on the Harnack inequality for harmonic functions: Let u be a positive harmonic function in B_1. Then

$$u|_{B_{1/2}} \leq C(n)u(0).$$

Exercise: Prove the above Harnack inequality using the mean value property, and show that $C(n) \leq 2^n$.

Exercise: Using the Harnack inequality, show oscillation decay of harmonic functions:

$$osc_{B_{1/2}} u \leq \frac{C}{C+1} osc_{B_1} u$$

where u is not necessarily positive and C is the constant from the Harnack inequality.

Exercise: Using oscillation decay, show that

$$\|u\|_{C^\alpha}(B_{1/2}) \leq M\|u\|_{L^\infty(B_1)}$$

where

$$\left(\frac{1}{2}\right)^{\alpha} = \frac{C}{C+1}$$

and M depends only on n.

Exercise: Using the Harnack inequality, show that harmonic functions bounded below on $\mathbb{R}^n$ are bounded. Using oscillation decay, show that bounded harmonic functions on $\mathbb{R}^n$ are constant.

Exercise: Think about why the constant appearing in the Harnack inequality blows up as our subset fills B_1. Using the Poisson Kernel, show that the constants blow up no faster than $(1-r)^{1-n}$ for a Harnack inequality in B_r.

Exercise: Show that if we have a one-sided bound on a global harmonic function u, e.g. $u \geq f(|x|)$, then we have an upper bound $g(|x|)$ where g has the same growth as f at ∞. Show that if u is harmonic and we can touch it below by a paraboloid of opening M, then we can touch it by above with a paraboloid of opening CM.

Remark: The Harnack inequality can be interpreted in several ways. The one I like is that if we have two solutions, one on top of the other, that are close at a point, then they are close in a neighborhood. In particular, if we have translation-invariant equations we can compare a solution to a translation of itself to get $C^{1,\alpha}$ regularity.

2.3 Fun Exercises

Exercise: For $n = 2$ we often have stronger theorems than for $n \geq 3$. Here is an example: positive superharmonic functions in dimension 2 are constant, while there are nontrivial positive superharmonic functions in higher dimensions. Prove it.

Hint: The fundamental solution for $n = 2$ is is $-\log(|x|)$, which is unbounded in both directions. Compare your positive superharmonic function u to a family of harmonic functions obtained by flattening $-\log(|x|)$.

Exercise: One can show that harmonic functions which vanish on an open set vanish identically without knowing analyticity. Using the divergence theorem, show that if u is harmonic, then

$$\frac{d}{dr}\int_{|\omega|=1} u(r\omega)u(\omega/r)d\omega = 0.$$

Conclude by scaling that

$$\int_{|\omega|=1} u(a\omega)u(b\omega)d\omega = \int_{|\omega|=1} u^2(c\omega)d\omega$$

where $ab = c^2$. Using this result, one sees that if u vanishes in a neighborhood of the origin then it vanishes identically.

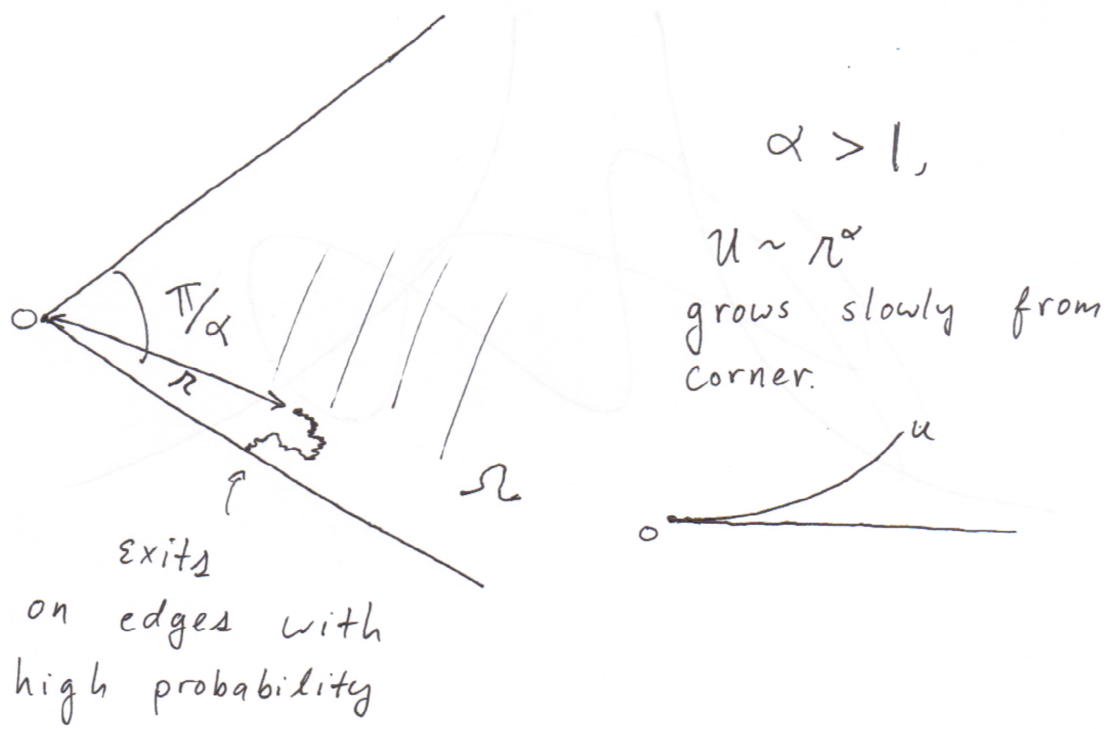
Exercise: Show that if u is harmonic on $B_1 - 0$ and $o(|x|^{2-n})$ near 0, then u is identically 0.

Growth of Harmonic Functions from Corners: We briefly discuss the growth of harmonic functions from corners in a domain. Consider the function

$$re(z^{\alpha}) = r^{\alpha}\cos(\alpha\theta)$$

on the plane. Note that this is harmonic on the domain $-\frac{\pi}{2\alpha} \leq \theta \leq \frac{\pi}{2\alpha}$, and grows like r^α from the corner. Hence, we can expect that harmonic functions:

- Grow quickly away from obtuse corners,
- Grow linearly away from flat edges,
- Grow slowly away from acute corners.



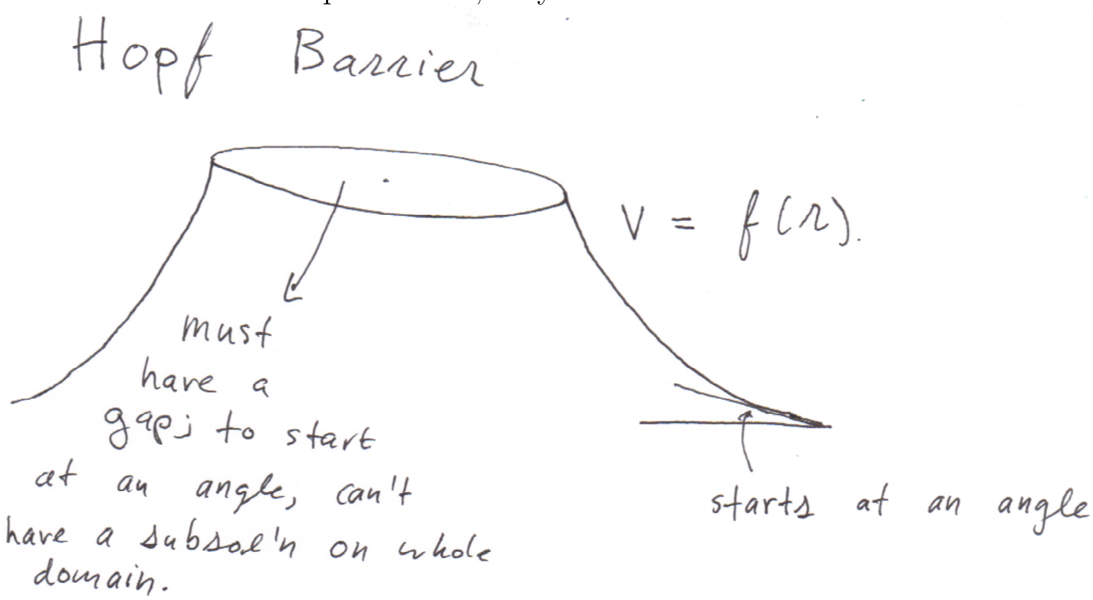
Some intuition for this behavior comes from the Brownian motion. Start a particle close to the domain and let it wander under the Brownian motion. Then the value of our solution is roughly the expectation that the particle will exit away from the edges of this corner. The intuition is that as the corner gets more acute this expectation gets smaller, which agrees with the previous computation.

3 Maximum Principle

Reading: Chapter 3, Sections 1 and 2 from Gilbarg-Trudinger. In this section I will try to illustrate how to think of the comparison principle as "making a movie of subsolutions," giving an easy way to visualize the Hopf lemma and more generally the comparison principle for fully nonlinear elliptic equations.

3.1 Hopf Lemma

The Hopf Lemma is, roughly, the construction of a radially symmetric subsolution barrier that always intersects horizontal planes at an angle. Ellipticity tells us that if we slide this function from beneath a supersolution, they cannot touch on the interior.



Exercise: Let $f(r)$ be a radially symmetric function. Show that at a point p in coordinates with e_1 in the radial direction, the Hessian matrix is:

$$\begin{pmatrix} f''(r) & 0 & ... & 0 \\ 0 & \frac{1}{r}f'(r) & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \frac{1}{r}f'(r) \end{pmatrix}$$

Conclude that for an elliptic operator $L = a^{ij}D_{ij}$ with ellipticity constants λ, Λ, and f a Hopf-Lemma type function with $f''(r) > 0$ and $f'(r) < 0$, we have

$$\lambda f''(r) + \Lambda\frac{n-1}{r}f'(r) \leq Lf \leq \Lambda f''(r) + \lambda\frac{n-1}{r}f'(r).$$

Exercise: Using the inequality from the previous exercise show that there is a universal α such that $r^{-\alpha}$ is a strict subsolution of $Lu = 0$. This is a typical Hopf Lemma barrier.

Exercise: A converse to the previous exercises: Suppose we are given an inequality "balancing" the eigenvalues $e_1, ..., e_n$ of the Hessian of u:

$$\lambda \sum_{e_k>0} e_k + \Lambda \sum_{e_j<0} e_j \leq 0 \leq \Lambda \sum_{e_k>0} e_k + \lambda \sum_{e_j<0} e_j.$$

Show that we can cook up some measurable $a^{ij}(x)$ so that

$$\lambda I \leq (a^{ij}) \leq \Lambda I$$

and

$$a^{ij}(x) u_{ij} = 0.$$

Thus, u solves some elliptic equation with no regularity assumption on the coefficients.

Now let $F(M)$ be a function on symmetric matrices satisfying the ellipticity condition:

$$F(M) > F(N)$$

if $M > N$. A way to think of the maximum principle is by "making a movie" of subsolutions which cannot touch a solution, for otherwise this would contradict ellipticity. More precisely:

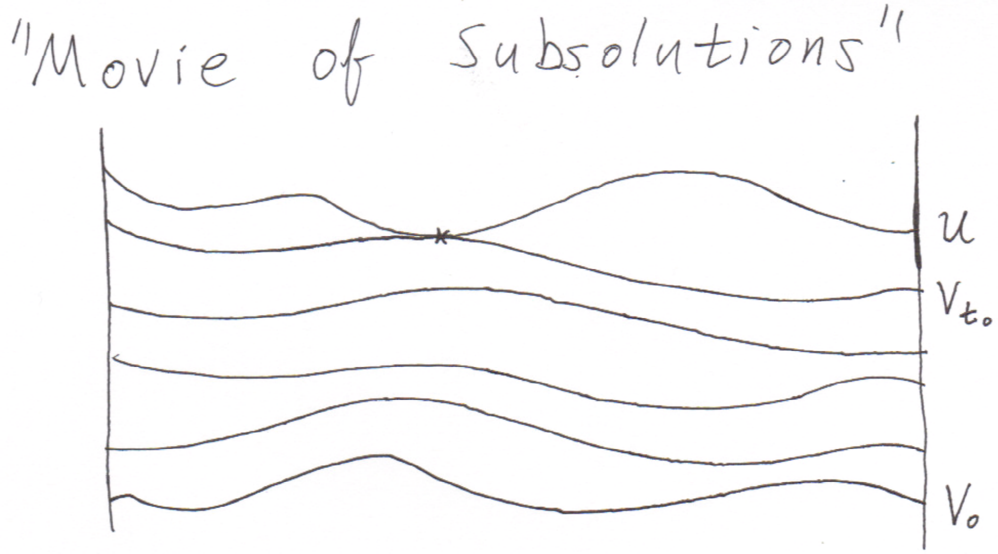
Exercise: Suppose $\{v_t\}$ is a family of strict subsolutions and u is a solution with

$$u \geq v_0$$

in B_1 and

$$u \geq v_t$$

on ∂B_1. Suppose that for some first time t_0, v_{t_0} touches u by below at some point in B_1. Why is this a contradiction? Conclude that $v_t < u$ for all t. Does anything change if we have $F(D^2u, Du, u, x) = 0$ and $F(D^2v_t, Dv_t, v_t, t) > 0$?



Exercise: Let $Lf = a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)$ and suppose that $Lu \leq 0$, with $u_{\partial B_1} \geq 0$. Observe that for $c \leq 0$, the subsolution property is preserved by sliding down. Suppose that $u(0) = 0$. Show using a Hopf barrier and making a movie that $u \equiv 0$. Think about why this is false for $c > 0$.

3.2 Comparison Principle for Fully Nonlinear Equations

The comparison principle, as discussed above, holds for solutions to very general elliptic equations. In particular, the difference between solutions to a fully nonlinear elliptic equation

$$F(D^2u, Du, u, x) = 0$$

(assume F is smooth) satisfies a linear elliptic equation:

Exercise: If u and v solve the fully nonlinear elliptic equation above, show that $w = u - v$ satisfies the linear equation

$$\left(\int_0^1 F_{ij}(D^2v + tD^2w, Dv + tDw, v + tw)dt\right) w_{ij} + ... = 0.$$

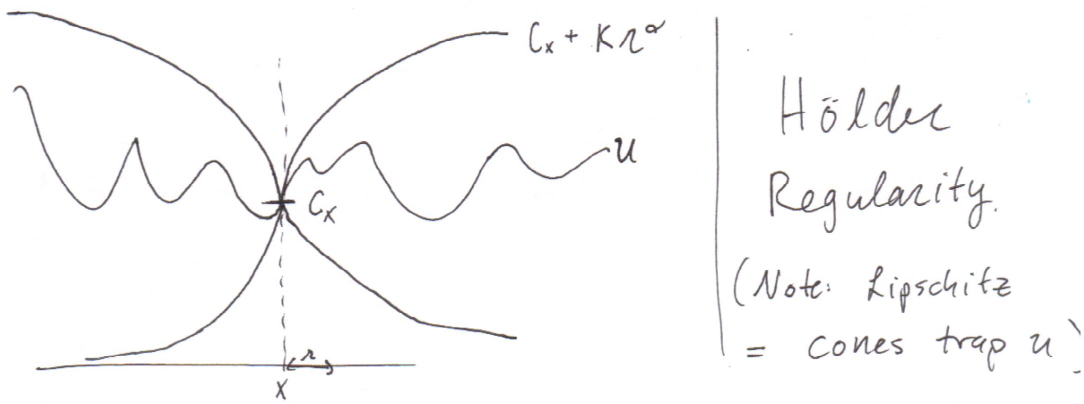
Conclude that a comparison principle holds provided F is decreasing in the third variable.

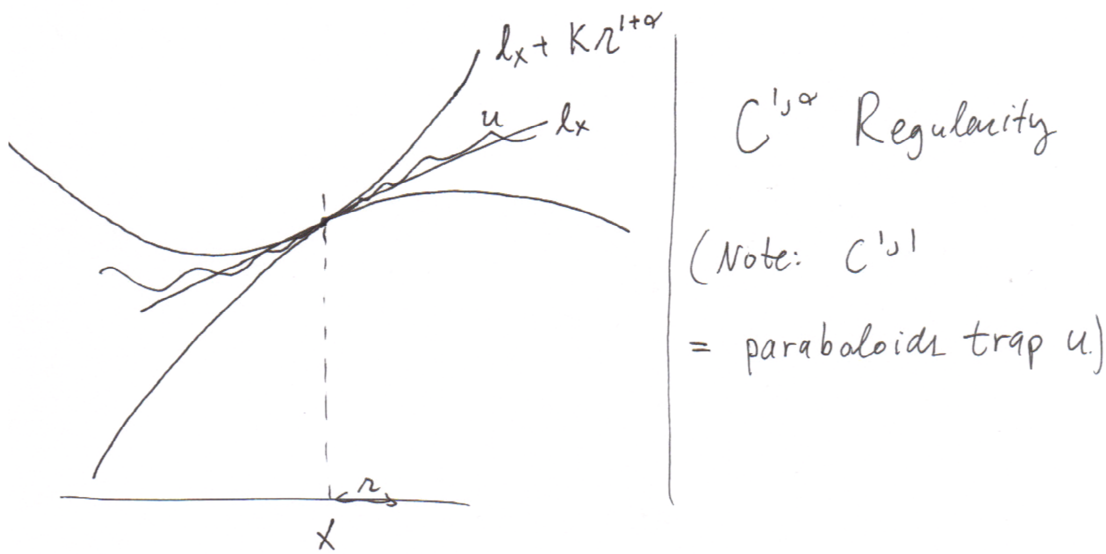
4 Schauder Theory

Reading: Chapters 4 and 6 of Gilbarg-Trudinger and later Chapter 8 of Caffarelli-Cabré.

4.1 Hölder Regularity

One way to approach $C^\alpha, C^{1,\alpha}$ and $C^{2,\alpha}$ estimates is via approximation by constants, planes and paraboloids respectively.





Example: A function u is in $C^\alpha(\Omega)$ if and only if for all $x \in \Omega$, there is some constant C_x and a uniform constant K such that

$$\|u - C_x\|_{L^\infty(B_r(x))} \leq K r^\alpha.$$

Furthermore, if $|C_x| + K \leq M$ for all $x \in \Omega$, then

$$\|u\|_{C^\alpha(\Omega)} \leq M.$$

This is an immediate consequence of the definition of Hölder regularity. The picture is that u is trapped between two α-polynomials of opening K at every point.

Example: A function u is in $C^{1,\alpha}(\Omega)$ if and only if for all $x \in \Omega$, there is some linear approximation $l_x(y) = a_x + \langle b_x, y - x\rangle$ and a uniform constant K such that

$$\|u - l_x\|_{L^\infty(B_r(x))} \leq Kr^{1+\alpha}.$$

Furthermore, if $|a_x| + |b_x| + K \leq M$ for all $x \in \Omega$, then

$$\|u\|_{C^{1,\alpha}(\Omega)} \leq M.$$

The picture is that u is trapped between two $\alpha+1$-polynomials of opening K at every point.

The forward direction is a consequence of Taylor's theorem. For the other direction, let $f_r(x) = \frac{1}{r}f(rx)$ be the linear rescaling of any function f. Let x and y be two points a distance r apart, and by translation assume that x and y are symmetric about 0. By hypothesis,

$$\begin{aligned}\|l_{x,r} - l_{y,r}\|_{L^\infty(B_1)} &\leq \|u_r - l_{x,r}\|_{L^\infty(B_1)} + \|u_r - l_{y,r}\|_{L^\infty(B_1)} \\ &\leq 2Kr^\alpha.\end{aligned}$$

Observing that the L^∞ norm of any linear function on B_1 controls its coefficients, we see that $|Dl_x - Dl_y| \leq Kr^\alpha$. The last statement follows because $\sup_\Omega |a_x| + \sup_\Omega |b_x| + K$ is the $C^{1,\alpha}$ norm of u.

A Picture for $\|u - l_x\|_{L^\infty(B_r(x))} \leq Kr^{1+\alpha} \Rightarrow u \in C^{1,\alpha}$:

If Du jumps too fast, contradiction of trapping.

Exercise: Formulate and prove a statement relating $C^{2,\alpha}$ regularity to approximation by paraboloids and draw a picture illustrating this claim.

Exercise: Think about why $\|u - l_x\|_{L\infty(B_r(x))} \le Kr$ implies $C^{0,1}$, not C^1. Similarly, why $\|u - P_x\|_{L^\infty(B_r(x))} \le Kr^2$ implies $C^{1,1}$ and not C^2. More generally, why we have to take $0 < \alpha < 1$.

It is often easier to show Hölder regularity by producing sequence of polynomials that approximate a function u around a point.

Proposition 1. *Suppose we can find a sequence of paraboloids $P_k(x) = a_k + \langle b_k, x\rangle + \frac{1}{2}\langle x, M_k x\rangle$ and an $r < 1$ such that*

$$\|u - P_k\|_{L^\infty(B_{r^k})} \le K r^{k(2+\alpha)}.$$

Then u is in $C^{2,\alpha}(0)$ with constant $C(r)K$. Here we require $0 < \alpha < 1$.

Proof. The idea is that this sequence of polynomials converges to a polynomial with the desired properties. Let $f_r(x) = \frac{1}{r^2} f(rx)$ be the quadratic rescaling of a function f. We have by hypothesis that

$$\begin{aligned}\|P_{k+1} - P_k\|_{L^\infty(B_{r^{k+1}})} &\le \|u - P_{k+1}\|_{L^\infty(B_{r^{k+1}})} + \|u - P_k\|_{L^\infty(B_{r^{k+1}})} \\ &\le 2Kr^{k(2+\alpha)}.\end{aligned}$$

Quadratic scaling gives

$$\|P_{k+1,r^k} - P_{k,r^k}\|_{L^\infty(B_1)} \le 2Kr^{k\alpha}.$$

Since the coefficients of polynomials on B_1 are controlled by the L^∞ norm, we have

$$|a_{k+1} - a_k| \le Kr^{k(2+\alpha)}, |b_{k+1} - b_k| \le Kr^{k(1+\alpha)}, |M_{k+1} - M_k| \le Kr^{k\alpha}.$$

It follows that P_k converge uniformly to a polynomial $P = a + \langle b, x\rangle + \frac{1}{2}\langle x, Mx\rangle$. Using these estimates, we obtain that P satisfies

$$\begin{aligned}&\|u - P\|_{L^\infty(B_{r^k})} \le \|u - P_k\|_{L^\infty(B_{r^k})} + |a_k - a| + r^k|b_k - b| + r^{2k}|M_k - M| \\ &\le CKr^{k(2+\alpha)}\end{aligned}$$

where C depends only on r. □

Exercise: Here we give an example of a function u approximated by paraboloids with $\alpha = 0$, but $u \notin C^2$. Let P be a quadratic polynomial and take η a cutoff function with $\eta|_{B_{1/2}} \equiv 1$. Consider the function

$$u(x) = \sum_k 2^{-2k}(\eta P)(2^k x).$$

Let P_l be the first l terms of the series. Show that

$$\|u - P_l\|_{L^\infty} \le C2^{-2l},$$

but u is not C^2. Where does the above proof go wrong?

4.2 Schauder Estimates for Laplacian

We prove the Schauder estimate for Laplacian using comparison with harmonic functions and approximation by paraboloids.

Theorem 1. *Suppose* $\Delta u = f$ *in* B_1 *where* $f \in C^\alpha(B_1)$*. Then* $u \in C^{2,\alpha}(B_{1/2})$ *and*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(n,\alpha)(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}).$$

Proof. **Simplifying Assumptions:** By subtracting $\frac{f(0)}{2n}|x|^2$ assume that $f(0) = 0$. Dividing by $\|u\|_{L^\infty(B_1)} + \frac{1}{\epsilon}\|f\|_{C^\alpha(B_1)}$, assume that $|u| \leq 1$ and $\|f\|_{C^\alpha(B_1)} \leq \epsilon$ to be chosen later depending only on n and α. The goal now is to prove that $u \in C^{2,\alpha}(0)$ with constant $C(n,\alpha)$. The idea is that for small data, u is very close to a harmonic function with the same boundary values. Hölder regularity and scaling allow us to use this idea at all scales.

Take the function w which agrees with u on ∂B_1 and is harmonic. Then $\Delta(u - w) = f$ and $|f| \leq \epsilon$. By the maximum principle,

$$|u - w| \leq C(n)\epsilon.$$

Furthermore, the (harmonic) quadratic approximation P_1 to w satisfies

$$\|w - P_1\|_{L^\infty(B_r)} \leq C(n)r^3$$

with coefficients boudned by $C(n)$ because $|w| \leq 1$. Putting these together, we obtain

$$\|u - P_1\|_{L^\infty(B_r)} \leq C(n)(r^3 + \epsilon).$$

Now, using that $0 < \alpha < 1$, choose r_0 small enough that $2Cr_0^3 \leq r_0^{2+\alpha}$ and then choose ϵ small enough that $2C\epsilon < r_0^{2+\alpha}$, giving that

$$\|u - P_1\|_{L^\infty(B_{r_0})} \leq r_0^{2+\alpha}.$$

Now take the $2+\alpha$ rescaling $v(x) = \frac{1}{r_0^{2+\alpha}}(u - P_1)(r_0 x)$, which satisfies $|v| \leq 1$ and $\Delta v(x) = \frac{1}{r_0^\alpha} f(r_0 x) = g(x)$. The right side again satisfies $|g| \leq \epsilon$, so we may repeat to find a polynomial P_2 with coefficients bounded by $C(n)$ so that

$$\|v - P_2\|_{L^\infty(B_{r_0})} \leq r_0^{2+\alpha}.$$

Scaling back, this becomes

$$\|u(x) - P_1(x) - r_0^{2+\alpha}P_2(x/r_0)\|_{L^\infty(B_{r_0^2})} \leq r_0^{2(2+\alpha)}.$$

Iterating, we find a sequence of polynomials approximating u with coefficients controlled by $C(n,\alpha)$, proving the claim. □

Exercise: Note that the proof works for $\alpha = 0$, but by the previous section approximation with $\alpha = 0$ doesn't imply C^2. Adjust the function at the end of the last section as follows: let $P = x_1 x_2$ so that P is harmonic, and take

$$u(x) = \sum_k \frac{2^{-2k}}{k}(\eta P)(2^k x).$$

Show that Δu is a well-defined continuous function that is not C^α at 0 for any α, giving a counterexample for $\alpha = 0$.

Exercise: Where does the proof break down for $\alpha = 1$?

4.3 Schauder Estimates for Hölder Coefficients

Exercise: Draw a picture that explains the interpolation inequality used by Gilbarg-Trudinger.

Exercise: (Difficult) Try to prove the Schauder estimates using the scale-invariant norm $\|\cdot\|^*_{C^{2,\alpha}}$ introduced in Gilbarg-Trudinger as follows:

Assume that $\|a^{ij}-\delta^{ij}\|_{C^\alpha(B_1)} \leq \delta$ to be chosen later depending only on universal constants, α and $\|a^{ij}\|_{C^\alpha}$. This is accomplished by stretching (0^{th} order scaling), i.e. $u_\rho(x) = u(\rho x)$, $a^{ij}_\rho(x) = a^{ij}(\rho x)$, $f_\rho(x) = \rho^2 f(\rho x)$. Freezing the coefficient at 0 we see that u satisfies the equation

$$a^{ij}(0)u_{ij} = f + (a^{ij}(0) - a^{ij})u_{ij}.$$

Let $a_k = \|u_{ij}\|^*_{C^\alpha(B_{1-2^{-k}})}$. Using the estimates for Laplacian, show that

$$a_k \leq C\|f\|_{C^\alpha(B_1)} + \epsilon_k a_{k+1} + a_{k-1},$$

Where $\epsilon_k < 2^{-k\alpha}$ for δ small. Given that a_k are bounded, show that a sequence satisfying this relation must satisfy $a_1 \leq C(n,\alpha)$.

Proof Using Approximation by Paraboloids

A lot of work in the following proof is hidden in the Krylov-Safonov Harnack inequality for viscosity solutions, which I treat in a different section.

Theorem 2. *Suppose $a^{ij}u_{ij} = f$ in B_1 in the viscosity sense, with $\lambda I \leq (a^{ij}) \leq \Lambda I$. Assume also that $a^{ij}, f \in C^\alpha(B_1)$ for $0 < \alpha < 1$. Then $u \in C^{2,\alpha}(B_1)$ with*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right),$$

with $C(n,\lambda,\Lambda,\alpha,\|a^{ij}\|_{C^\alpha})$.

Simplifying Assumptions:

1. $a^{ij}(0) = \delta^{ij}$ by an affine change of coordinates, universal.
2. $\|a^{ij}-\delta^{ij}\|_{C^\alpha(B_1)} \leq \epsilon_0$ to be chosen later depending only on universal constants, α and $\|a^{ij}\|_{C^\alpha}$. Again, this is accomplished by (0^{th} order scaling).
3. $f(0) = 0$ by letting $A = tr(a^{ij})$ universal and subtracting $\frac{f(0)}{2A}|x|^2$.
4. $\|u\|_{L^\infty(B_1)} \leq 1$ and $\|f\|_{C^\alpha(B_1)} \leq \epsilon_0$, dividing by $\|u\|_{L^\infty(B_1)} + \epsilon_0^{-1}\|f\|_{C^\alpha}$.

The goal now is to produce a quadratic polynomial Q such that $\|u - Q\|_{L^\infty(B_r)} \leq Cr^{2+\alpha}$ where C and the coefficients of Q are controlled by $n, \lambda, \Lambda, \alpha$.

Proof. First take w with $\Delta w = 0$ and $w = u$ on $\partial B_{3/4}$, and take the quadratic approximation Q_1 to w, which satisfies

$$\|Q_1 - w\|_{L^\infty(B_r)} \leq C(n)\|u\|_{L^\infty} r^3 \leq C(n)r^3$$

and coefficients bounded by $C(n)$. Observe that $a^{ij}(u-w)_{ij} = f - (a^{ij} - \delta^{ij})w_{ij} = g$. By interior derivative estimates we have

$$|g| \leq \epsilon_0(1 + \delta^{-2}),$$

so applying ABP (actually only requires Hölder growth of L^n norms) we have

$$\|u - w\|_{L^\infty(B_{3/4-\delta})} \leq \|u - w\|_{L^\infty(\partial B_{3/4-\delta})} + C\epsilon_0(1 + \delta^{-2}).$$

By Harnack we have $u \in C^{2\beta}$ so by barrier $w \in C^\beta(\bar{B}_{3/4})$ for some $0 < \beta < 1$. Putting it all together, we obtain

$$\|u - Q_1\|_{L^\infty(B_r)} \leq C(r^3 + \delta^\beta + \epsilon_0(1 + \delta^{-2})).$$

To summarize, the first term polynomial error from expansion of w, the second is growth of difference between u and w from boundary, and the last is from small data. Taking $3Cr_0^3 \leq r_0^{2+\alpha}$, $3C\delta^\beta \leq r_0^{2+\alpha}$ and $3C\epsilon_0(1+\delta^{-2}) \leq r_0^{2+\alpha}$, we have

$$\|u - Q_1\|_{L^\infty(B_{r_0})} \leq r_0^{2+\alpha}.$$

We used that $\alpha < 1$ in the previous step.

Now we show how to inductively produce the next polynomial. Let $v(x) = \frac{1}{r_0^{2+\alpha}}(u - Q_1)(r_0x)$, the $2+\alpha$ scaling. One computes

$$a^{ij}(r_0x)v_{ij} = \frac{1}{r_0^\alpha}(f(r_0x) + (a^{ij} - \delta^{ij})(r_0x)(Q_1)_{ij}).$$

We are in the same situation as before with the right side bounded by $\epsilon_0(1 + C(n)\sum_j r_0^{j\alpha})$. Repeat the above steps (with perhaps smaller ϵ_0) to find $Q_2 = Q_1 + r_0^{2+\alpha}P_1(\cdot/r_0)$ where P_1 has coefficients bounded by $C(n)$ and $\|u - Q_2\|_{L^\infty(B_{r_0^2})} \leq r_0^{2(2+\alpha)}$. Iterating, we find

$$Q_k = Q_1 + r_0^{2+\alpha}P_1(\cdot/r_0) + r_0^{2(2+\alpha)}P_2(\cdot/r_0^2) + ... + r_0^{k(2+\alpha)}P_k(\cdot/r_0^k)$$

where all the P_i have coefficients bounded by $C(n)$ and $\|u - Q_k\|_{L^\infty(B_{r_0^k})} \leq r_0^{k(2+\alpha)}$. Note that we can choose ϵ_0 to work for every stage because $(Q_k)_{ij} \leq C(n)\sum_k r_0^{k\alpha}$. □

Exercise: Show that we actually only need

$$\left(\frac{1}{|B_r|}\int_{B_r}|f|^n\right)^{\frac{1}{n}} \leq \epsilon_0 r^\alpha$$

and

$$\left(\frac{1}{|B_r|}\int_{B_r}|a^{ij} - \delta^{ij}|^n\right)^{\frac{1}{n}} \leq \epsilon_0 r^\alpha$$

for the above proof to work.

Exercise: Notice that the only place where we use that a^{ij} are C^α is when they appear on the right side of the equation for the difference between u and its approximations. Approximating by polynomials changes the equation slightly, and this requirement on a^{ij} is thus crucial to keeping a small right side when we iterate.

This observation also suggests that we have $C^{1,\alpha}$ estimates (approximation by linear functions) if $\|a^{ij} - \delta^{ij}\|_{L^\infty(B_1)}$ is small, since changing by planes doesn't change the equation. That is the content of the Cordes-Nirenberg estimate:

Theorem 3. *Suppose* $a^{ij}(x)u_{ij} = 0$ *in* B_1 *and that* $\|a^{ij} - \delta^{ij}\|_{L^\infty(B_1)} \leq \epsilon$. *Then for* ϵ *sufficiently small depending only on* $n, \lambda, \Lambda, \alpha$ *we have*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

Prove this estimate by approximation with tangent planes. Show that it also for the nonhomogeneous equation if

$$\left(\frac{1}{|B_r|}\int_{B_r}|f|^n\right)^{\frac{1}{n}} \leq \epsilon r^{\alpha-1}.$$

Remark: The Harnack inequality is used in the above proof, which hides a lot of the work. We use it because we need a pointwise estimate on the modulus of continuity of u from the boundary of balls. For divergence equations, there are mostly integral estimates which makes the perturbation theory simpler. When familiar with Sobolev spaces and Morrey spaces (in the Function Spaces section) and weak solutions, do the following exercise.

Exercise: Suppose $u \in H^1(B_1)$ and $\partial_i(a^{ij}u_j) = 0$. Solve $\Delta v = 0$ with $v|_{\partial B_1} = u$ and let $h = u - v \in H_0^1(B_1)$. Suppose

$$\|a^{ij} - \delta^{ij}\|_{C^\alpha(B_1)} \leq \epsilon.$$

Using the equation, show that

$$\int_{B_1}|Dh|^2 \leq C\epsilon\int_{B_1}|Du|^2.$$

The above estimate is the key point: for integral estimates of the error, we don't need a Harnack inequality (a pointwise estimate on the rate of growth of u from the boundary.) Using the interior derivative estimates for harmonic functions show that

$$\int_{B_\delta}|Dv|^2 \leq C\delta^n\int_{B_1}|Dv|^2.$$

Conclude that, by choosing δ and then ϵ small enough,

$$\int_{B_\delta}|Du|^2 \leq C\delta^{n-2+2\alpha}\int_{B_1}|Du|^2.$$

Conclude a C^α interior estimate for divergence-form equations using a result from Morrey spaces.

4.4 Boundary Estimates and the Neumann Condition

Boundary estimates can often be reduced to interior estimates by first reducing to the case of 0 boundary data and reflecting.

Exercise: Suppose that $\Delta u = f$, $f \in C^\alpha(B_1^+)$ and we impose the Neumann condition $D_\nu u = 0$ on $\{x_n = 0\}$. Prove uniqueness for the mixed Dirichlet/Neumann problem using a Hopf lemma type argument. Prove the boundary Schauder estimate by taking even reflections and applying uniqueness.

Estimates up to the boundary are crucial for solving the Dirichlet problem via the method of continuity.

4.5 Method of Continuity and the Dirichlet Problem

Exercise: This exercise gives a "hands-on" approach to the Dirichlet problem, illustrating how the method of continuity works. We want to solve the equation

$$a^{ij}(x)u_{ij} = f(x)$$

$$u|_{\partial B_1} = 0$$

where a^{ij}, f are Hölder continuous. To do so, start by solving

$$\Delta u_0 = f$$

$$u_0|_{\partial B_1} = 0.$$

Produce u_k for $k > 1$ inductively by solving

$$\Delta u_k = (\delta^{ij} - a^{ij})(u_{k-1})_{ij},$$

$$u_k|_{\partial B_1} = 0.$$

Show that the function $v_k = \sum_{j=0}^{k} t^k u_k$ satisfies

$$(\Delta + t(a^{ij}(x)D_{ij} - \Delta))(v_k)_{ij} = f + t^{k+1}(a^{ij}(x)D_{ij} - \Delta)(u_k)_{ij}.$$

Using the Schauder estimates and the maximum principle, show that

$$\|u_k\|_{C^{2,\alpha}(B_1)} \leq C(n,\alpha)^k \|f\|_{C^\alpha(B_1)}.$$

Conclude that for t sufficiently small depending only on n, α, the series v_k converges to a solution of

$$(\Delta + t(a^{ij}(x)D_{ij} - \Delta))u_{ij} = f$$

$$u|_{\partial B_1} = 0.$$

The key point is that with Schauder estimates up to the boundary, we can iterate this procedure finitely many times (with the same t) to solve the Dirichlet problem.

4.6 The Newtonian Potential

In this section we wish to address the regularity by considering the newtonian potential

$$Nf(y) = \int \Gamma(x - y)f(y)dy.$$

Example: For $f \in L^p(B_1)$ for $p > n$, the Newtonian potential is $C^{1,\alpha}$ for $\alpha = 1 - \frac{n}{p}$. Assume that x and z are symmetric about 0 and $|x - z| = \delta$. Applying Hölder's inequality to $|Nf(x) - Nf(z)|$ in B_δ, we obtain

$$\begin{aligned}\int_{B_\delta} |(D\Gamma_z - D\Gamma_x)f| &\leq C(n)\|f\|_p \left(\int_{B_\delta} |x|^{(1-n)q}\right)^{1/q} \\ &\leq C(n,\alpha)\|f\|_p \left(\int_0^\delta r^{(1-n)q+n-1} dr\right)^{1/q} \\ &\leq C(n,\alpha)\|f\|_p \delta^{1-n+n/q} \\ &= C(n,\alpha)\|f\|_p \delta^\alpha.\end{aligned}$$

Outside B_δ, we take advantage of the smallness of δ by applying the mean value theorem:

$$\begin{aligned}\delta \int_{B-B_\delta} |D^2\Gamma_{\tilde{x}} f| &\leq C(n)\delta\|f\|_p \left(\int_{\mathbb{R}^n - B_\delta} |x|^{-nq}\right)^{1/q} \\ &\leq C(n,\alpha)\delta\|f\|_p \left(\int_\delta^\infty r^{-nq+n-1} dr\right)^{1/q} \\ &\leq C(n,\alpha)\|f\|_p \delta^\alpha.\end{aligned}$$

(See the $W^{2,p}$ estimates for a deeper understanding of this. For $p > n$, they are like $C^{1,\alpha}$ estimates.)

Exercise: Using a similar technique to the example, show that if $f \in L^\infty(B_1)$ then $Nf \in C^{1,\alpha}$ for any $\alpha < 1$.

Exercise: Show that for $f \in L^p$ with $p > \frac{n}{2}$ we have Nf bounded.

5 Function Spaces

Reading: Chapter 7 of Gilbarg-Trudinger, Chapters 4 and 6 of Evans-Gariepy, and Chapter 1 of Caffarelli-Cabré.

5.1 Sobolev Inequality

Theorem 4. *Let $u \in W_0^{1,p}(B_1)$ where $1 \leq p < n$. Then*

$$\|u\|_{L^{p^*}(B_1)} \leq C(n,p)\|\nabla u\|_{L^p(B_1)}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Exercise: Show that the above value of p^* is the only possible value by scaling. (Consider the "stretches" of u, $u_r(x) = u(rx)$.)

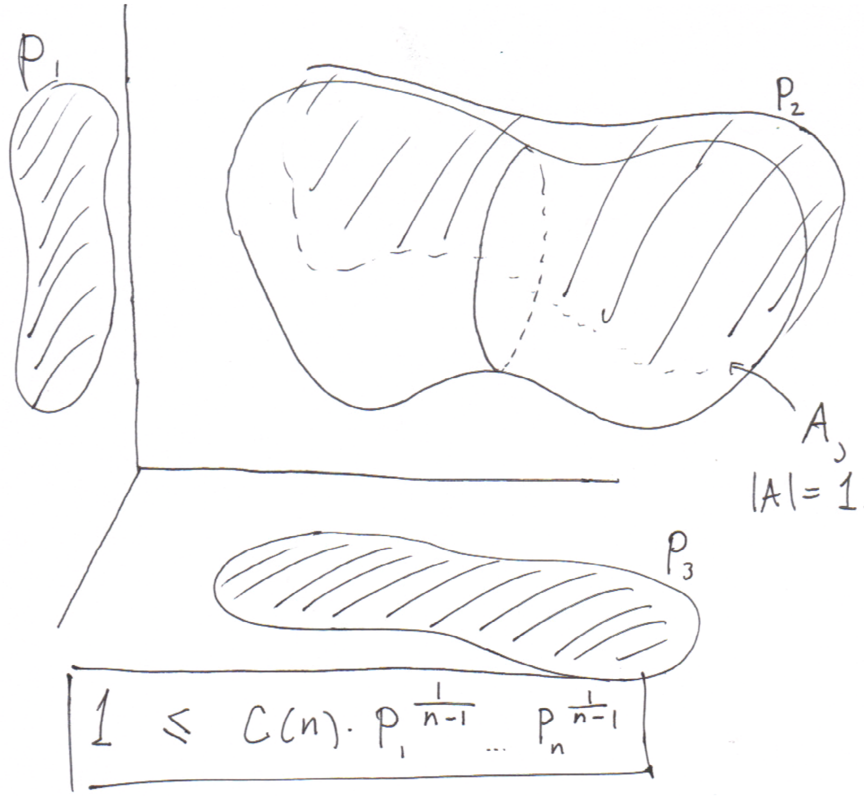
Proof. By approximation it suffices to consider $u \in C_0^\infty(\mathbb{R}^n)$. For simplicity we work with $p = 1$ and $n = 3$. Note that

$$u(x_1, x_2, x_3) \leq \int_{-\infty}^{\infty} |D_1 u(t_1, x_2, x_3)| dt_1.$$

We then compute, using Hölder,

$$\begin{aligned}\int_{-\infty}^{\infty} u(x)^{3/2} dx_1 \leq & \left(\int_{-\infty}^{\infty} |D_1 u(t_1, x_2, x_3)| dt_1\right)^{1/2} \\ & \cdot \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |D_2 u(x_1, t_2, x_3)| dt_2 dx_1\right)^{1/2} \\ & \cdot \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |D_3 u(x_1, x_2, t_3)| dt_3 dx_1\right)^{1/2}.\end{aligned}$$

Integrate over x_2 and x_3, applying the same inequality, to obtaining the desired result. Morally, for $u = \chi_A$, this tells us that the product of areas of projections to the $\frac{1}{n-1}$ controls the volume (see the picture). $\square$



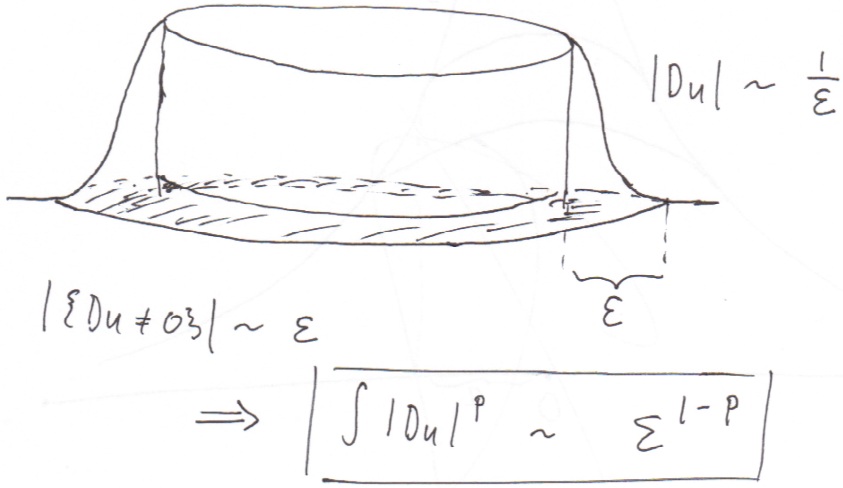
Exercise: Using the $p = 1$ case, prove the Sobolev inequality for $1 < p < n$.

An important feature of $W^{1,p}$ for $p > 1$ is that functions must “pay in measure” to jump from 0 to 1, while $W^{1,1}$ functions can have arbitrarily fast jumps.

Exercise: Consider the functions u_ϵ which are 1 on B_1 and 0 outside $B_{1+\epsilon}$, with $|Du_\epsilon| \leq \frac{1}{\epsilon}$. Show that

$$\int |Du_\epsilon|^p = C\epsilon^{1-p}.$$

Conclude that u_ϵ have bounded $W^{1,1}$ norm with arbitrarily quick jumps, but for $p > 1$ the $W^{1,p}$ norms blow up.



Exercise: Let $u \in W^{1,p}(B_1)$ with $0 \leq u \leq 1$. Let $A = \{u = 1\}$, $B = \{0 < u < 1\}$ and $D = \{u = 0\}$. Fix x in A and y in D, let $r = |x - y|$ and let ω be the unit vector in the $y - x$ direction. Then

$$1 = |u(x) - u(y)| \leq \int_0^r |Du(x + t\omega)| dt.$$

Integrate over $y \in D$ to obtain

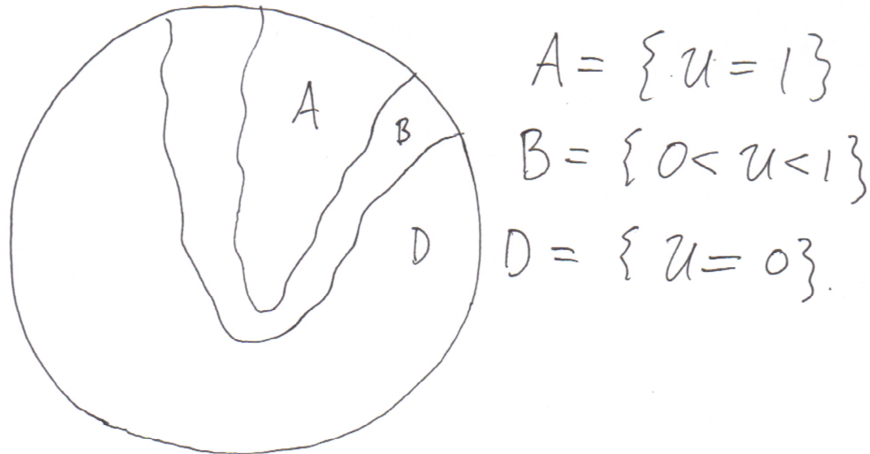
$$|D| \leq C(n) \int_B \frac{|Du(y)|}{|x - y|^{n-1}} dy.$$

Now integrate over $x \in A$ to obtain

$$|D||A|^{1-\frac{1}{n}} \leq C|B|^{1-\frac{1}{p}} \|Du\|_{L^p(B)}.$$

(Hint: The integral of $|x|^{1-n}$ over all regions with measure $|A|$ is maximized by integrating over the ball of radius $c|A|^{\frac{1}{n}}$.)

Conclude that all functions with $\|u\|_{W^{1,p}(B_1)} \leq 1$ must pay some measure to jump from 0 to 1, i.e. $|A|$ and $|D|$ cannot both be big.



$$|D|\,|A|^{1-1/n} \leq C(n)\,|B|^{1/2}\left(\int_B |Du|^2\right)^{1/2}$$

5.2 Morrey Inequality and Differentiability

Theorem 5. *Let $u \in W_0^{1,p}(B_1)$ where $p > n$. Then u is in fact Hölder, with $\alpha = 1 - \frac{n}{p}$, and*

$$\|u\|_{C^\alpha(B_1)} \leq C(n,p)(\|u\|_{L^p(B_1)} + \|\nabla u\|_{L^p(B_1)}).$$

Lemma 1. *Let $u \in C^1(B_1)$. Let u_{B_1} denote the average of u. Then*

$$|u(x) - u_{B_1}| \leq C(n)\int_{B_1} \frac{|\nabla u(y)|}{|x-y|^{n-1}}dy.$$

Proof. Let $x, y \in B_1$, $\omega = \frac{y-x}{|y-x|}$ and $r = |x-y|$. First, we have the inequality.

$$|u(x) - u(y)| \leq \int_0^r |\nabla u(x + t\omega)|.$$

Integrate in y using polar coordinates and change the order of integration:

$$\begin{aligned}
\int_{B_1} |u(x) - u(y)|dy &\leq \int_{|\omega|=1}\int_0^{R(\omega)} \left(\int_0^r |\nabla u(x_t\omega)|dt\right) r^{n-1}drd\omega \\
&= \int_{|\omega|=1}\int_0^{R(\omega)} \left(\int_t^{R(\omega)} |\nabla u(x+t\omega)|r^{n-1}dr\right) dtd\omega \\
&\leq C(n)\int_{|\omega|=1}\int_0^{R(\omega)} \frac{|\nabla u(x+t\omega)|}{t^{n-1}}t^{n-1}dtd\omega.
\end{aligned}$$

□

Exercise: Recall the proof that Nf is $C^{1,\alpha}$ for $\alpha = 1 - \frac{n}{p}$. Prove Morrey's inequality in a similar way by examining the above singular integral. (The key is that for $p > n$ we can apply Hölder's inequality).

Exercise: Morrey's inequality also implies that for $p > n$ the Sobolev functions $W^{1,p}$ are almost everywhere classically differentiable. Assume that 0 is a Lebesgue point for Du and let $h(x) = u(x) - u(0) - Du(0) \cdot x$. Let $h_r(x) = \frac{1}{r}h(rx)$ be the linear scaling of h. Show that

$$\int_{B_1} |Dh_r|^p \to 0.$$

Using the proof of Morrey's inequality show that

$$\|h_r\|_{L^\infty(B_1)} \leq C\|Dh_r\|_{L^p(B_1)},$$

which tells us that $h(x) = o(|x|)$.

Exercise: Use a similar technique to show that $W^{2,p}$ functions are classically twice differentiable for $p > \frac{n}{2}$. Show that $W^{k,p}$ functions are classically k times differentiable for $p > \frac{n}{k}$. (Hint: For such p, if we take the Sobolev conjugate $k-1$ times $p^{*\cdots*} > n$, so $u \in W^{1,q}$ for some $q > n$. Use Morrey once and take care of the rest using the Sobolev inequality).

5.3 The Case $p = n$, BMO, and John-Nirenberg Inequality

For $p = n$ we do not quite get an L^∞ bound for u. Such functions can have slow blowups:

Exercise: Show that in $\mathbb{R}^2$, $\log\log(1 + \frac{1}{|x|})$ is in $W^{1,2}(B_1)$.

The space $W^{1,n}$ is in fact continuously imbedded in the space of bounded mean oscillation (BMO) functions. By the basic lemma of the previous section,

$$\int_{B_r} |u(x) - u_{B_r}| \leq Cr \int_{B_r} |\nabla u| dx \leq cr^n \|\nabla u\|_n.$$

BMO functions can have logarithmic blowups:

Exercise: Show that $\log(|x|)$ is BMO.

The John-Nirenberg Lemma for BMO functions tells us that BMO functions can blow up at worst logarithmically:

Lemma 2. *If* $\frac{1}{|B_r|}\int_{B_r} |u - u_{B_r}| < 1$ *for all* B_r *then*

$$\frac{1}{|B_r|}\int_{B_r} e^{\delta(n)|u-u_{B_r}|} \leq C(n).$$

We guide the reader to a proof using the Calderón-Zygmund decomposition. For simplicity, we work with cubes Q instead of balls.

Exercise: (Decay of distribution function). Show that for a positive function u,

$$\int_{B_1} e^{\delta u(x)} dx = \delta \int_0^\infty e^{\delta t}|\{u > t\}| dt.$$

Using this result, show that the John-Nirenberg lemma will be proven if we show

$$|\{|u - u_Q| > kC(n)\} \cap Q| \leq 2^{-k}|Q|$$

for all k and cubes Q.

Fix Q. By hypothesis, we have

$$\frac{1}{|Q|}\int_Q |u-u_Q| < 1.$$

Dyadically decompose Q into cubes and keep a cube $\tilde{Q}$ if

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |u-u_Q| \geq 2,$$

but keep decomposing otherwise. By Lebesgue differentiation,

$$\{|u-u_Q| > 2 \cap Q\} \subset \cup\tilde{Q}.$$

Exercise: Using the above inequalities, show that

$$\sum |\tilde{Q}| \leq \frac{1}{2}|Q|.$$

Exercise: Using the fact that we stopped decomposing at $\tilde{Q}$ for the first time, show that

$$|u_Q - u_{\tilde{Q}}| \leq 2^{n+1}.$$

Exercise: Repeat the above steps for $|u-u_{\tilde{Q}}|$ in the $\tilde{Q}$ to conclude that

$$|\{|u-u_Q| > 2\cdot 2^{n+1}\} \cap Q| \leq 2^{-2}|Q|.$$

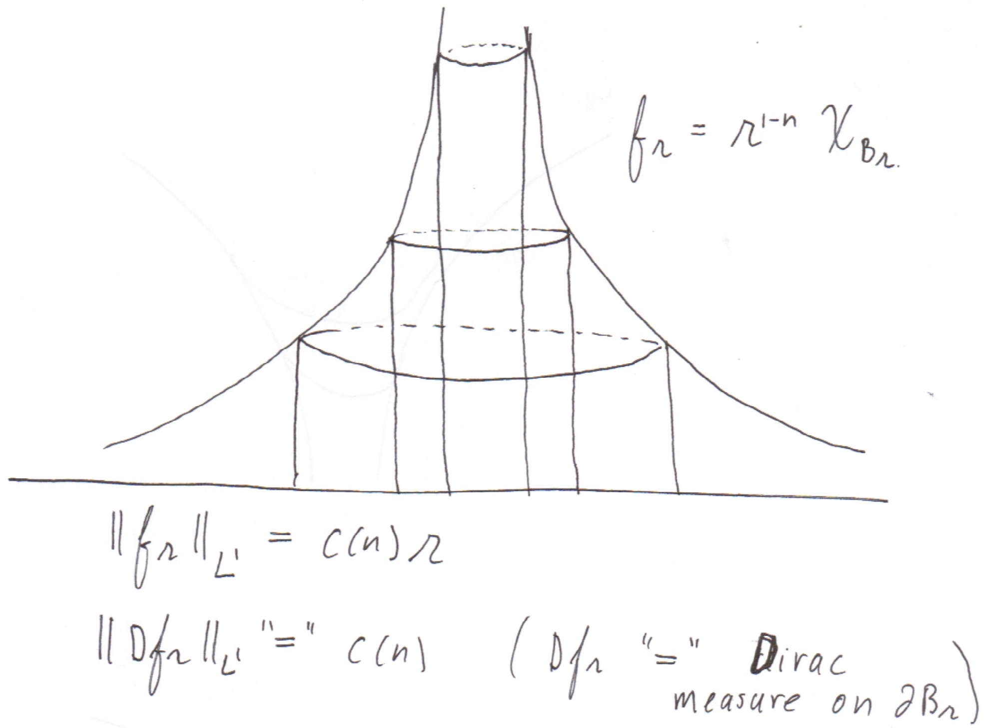
(Hint: $|u-u_Q| \leq |u-u_{\tilde{Q}}| + |u_Q - u_{\tilde{Q}}|$.) Repeat iteratively to prove the lemma.

5.4 Compactness and Poincaré Inequalities

Theorem 6. *$W_0^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for $1 \leq q < p^*$.*

Consider a bounded sequence $\{u_m\}$ in $W_0^{1,p}(\Omega)$. Then the mollifiers u_m^ϵ are compactly supported smooth functions. First, one checks easily that this is an equicontinuous family for fixed ϵ using boundedness in L^1. Second, check that $u_m^\epsilon \to u_m$ in L^q uniformly in m, doing first for L^1 and interpolating to use the Sobolev inequality. Conclude with diagonal argument.

Exercise: Take ρ_1 standard mollifier, and let $\eta_\epsilon(x) = \epsilon^{1-n}\rho(x/\epsilon)$. Show that this family is bounded in $W_0^{1,1}(B_1)$, but $\eta_\epsilon^{n/(n-1)}$ has no convergent subsequence. (Alternatively, consider the sequence $r^{1-n}\chi_{B_r}$ and interpret its derivative as a Dirac measure on ∂B_r.)



Exercise: (Poincaré Inequality). Prove the following inequality by compactness:

$$\|u - u_{B_1}\|_{L^p(B_1)} \leq C(n,p)\|Du\|_{L^p(B_1)}.$$

More precisely, suppose there is a sequence of zero mean functions u_m with $\|Du_m\|_{L^p(B_1)} \to 0$ but $\|u_m\|_{L^p(B_1)} = 1$.

Exercise: Show that the Poincaré Inequality scales as follows:

$$\|u - u_{B_r}\|_{L^p(B_r)} \leq C(n,p)r\|Du\|_{L^p(B_r)}.$$

Exercise: Use the Sobolev inequality and then the Poincaré inequality to show the following improvement:

$$\|u - u_{B_1}\|_{L^{p^*}(B_1)} \leq C\|Du\|_{L^p(B_1)}.$$

5.5 Morrey Spaces

Hölder regularity can also be characterized by growth of local integrals. The Morrey space consists of the functions in H^1 satisfying

$$\int_{B_r} |Du|^2 \leq Kr^{n-2+2\alpha}.$$

The goal is to show that Morrey functions are $C^\alpha(B_1)$ with

$$\|u\|_{C^\alpha(B_1)} \leq C(n,\alpha)K,$$

giving a useful criterion for proving Hölder estimates in PDE.

Exercise: Using the Poincaré inequality, show that functions u in the Morrey space satisfy

$$\frac{1}{|B_r|}\int_{B_r} |u - u_{B_r}|^2 \leq Cr^{2\alpha}.$$

Exercise: Show that

$$|u_{B_r} - u_{B_{r/2}}| \leq C(n)Kr^\alpha.$$

Conclude from the Lebesgue differentiation theorem that

$$|u(x) - u_{B_r(x)}| \leq C(n,\alpha)Kr^\alpha.$$

(In particular, such functions are the uniform limit of continuous functions on compact subsets, and are thus continuous).

Exercise: Show that for $|x-y| = r$,

$$|u_{B_r(x)} - u_{B_r(y)}| \leq C(n)r^\alpha.$$

Using this and the previous results, conclude that Morrey functions are C^α with the desired control on the C^α norm.

5.6 $W^{2,p}$ and Touching by Paraboloids

Read the proof of Aleksandrov's theorem in Evans-Gariepy:

Theorem 7. *Let $f : \mathbb{R}^n \to R$ be convex. Then f is second-order differentiable almost everywhere in the sense that there is a quadratic approximation P_x with*

$$\|f - P_x\|_{L^\infty(B_r(x))} = o(r^2)$$

for almost every x.

Read Chapter 1 of Caffarelli-Cabré. The function $\Theta(x)$ is the smallest opening of parabolas that can trap u at x. Know why $\Theta \in L^p$ implies $u \in W^{2,p}$ for $p > 1$ with

$$\|D^2u\|_{L^p} \leq C\|\Theta\|_{L^p}.$$

Thus, to obtain $W^{2,p}$ estimates, one studies the level sets of Θ.

Exercise: Geometrically, what is the distribution function $\mu(t)$ of Θ for some graph u? (Hint: slide paraboloids of opening t until they touch u from above and below.)

6 De Giorgi Harnack Inequality

Reading: Chapter 8 of Gilbarg-Trudinger.

6.1 Local Boundedness

Theorem 8. *Let $u \in H^1(B_1)$ satisfy*

$$D_i(a^{ij}u_j) \geq 0$$

weakly in B_1, where a^{ij} are bounded measureable and $0 < \lambda \leq (a^{ij}) \leq \Lambda$. Then

$$\|u^+\|_{L^\infty(B_{1/2})} \leq C\|u^+\|_{L^2(B_1)},$$

where $C(n, \lambda, \Lambda)$.

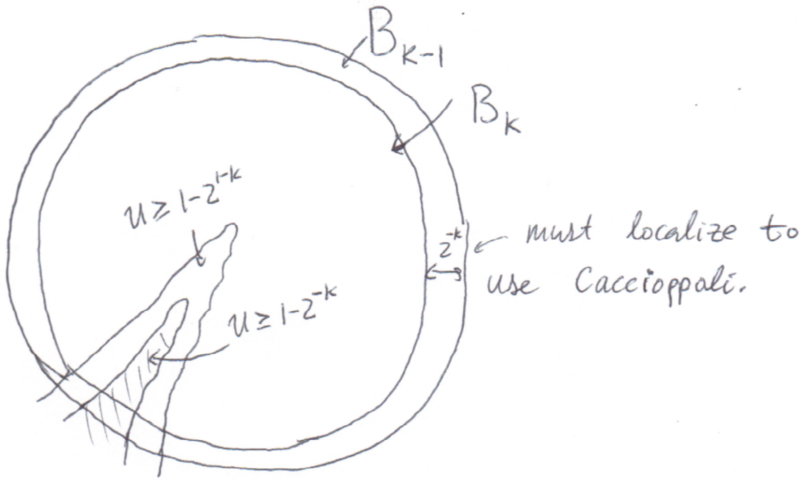
This says that there are "no spikes" on the interior for subsolutions of divergence form equations.

Proof. **Exercise: Caccioppoli Inequality**. Using the structure of the equation and ellipticity we obtain a reverse Poincaré inequality. Let $\eta \in C_0^\infty(B_1)$, use the test function $u^+\eta^2$ and apply ellipticity to obtain the Caccioppoli inequality:

$$\int_{B_1} |D(u\eta)|^2 \leq C(n, \lambda, \Lambda) \int_{B_1} u^2|D\eta|^2.$$

Roughly, this says that u cannot jump too quickly.

Step 2: Sobolev Inequality. Let $u_k = (u - (1 - 2^{-k}))^+$, $B_k = B_{1/2+2^{-k}}$ and let $a_k = \int_{B_k} u_k^2$. The claim is that $a_k \to 0$ if a_1 is sufficiently small. Multiplication by constants would give the desired result. The Sobolev inequality (morally isoperimetric inequality) along with the Cacciopoli inequality (structure of equation) will give us the iteration formula. Let η be a test function 1 on B_k and 0 outside of B_{k-1} with $|\nabla\eta| \leq C2^k$.



$\{u \geq 1-2^{-k}\} \cap B_{1/2}$ shrink to nothing; must shrink at every stage because of Caccioppoli.

Cacciopoli inequality gives $\int_{B_{k-1}} |\nabla(u_k\eta)|^2 \leq C2^{2k}a_{k-1}$.
Sobolev inequality gives $\left(\int_{B_k} u_k^{2^*}\right)^{2/2^*} \leq \int_{B_{k-1}} |\nabla(u_k\eta)|^2$.
Hölder inequality gives $a_k \leq \left(\int_{B_k} u_k^{2^*}\right)^{2/2^*} |\{u_k > 0\} \cap B_k|^{2/n}$.
Finally, by definition of a_k we have $|\{u_k > 0\} \cap B_k| \leq 2^{2k}a_{k-1}$.
Putting it all together, we get the iteration formula

$$a_k \leq C2^{(2+4/n)k}a_{k-1}^{1+2/n}.$$

□

Exercise: Show that a sequence satisfying $a_{k+1} \leq C^k a_k^{\gamma}$ for some $\gamma > 1$ will tend to 0 if a_1 is sufficiently small depending on C, γ.

Exercise (Moser Iteration): For simplicity assume that u is bounded and take the test function $\phi = \eta^2(u^+)^{\beta+1}$. As an exercise, show that

$$\|u\|_{L^{\beta\chi}(B_r)} \leq \left(\frac{C}{(R-r)^2}\right)^{\frac{1}{\beta}} \|u\|_{L^{\beta}(B_R)},$$

where $\chi = \frac{2^*}{2} > 1$. Iterate this inequality starting with $\beta = 2$ and using B_k as above to prove local boundedness.

6.2 Hölder Regularity

Theorem 9. ***Density Estimate.*** *Let u satisfy $D_i(a^{ij}u_j) \geq 0$ in B_2 with $0 \leq \lambda \leq (a^{ij}) \leq \Lambda$ and assume further that $|\{u \leq 0\} \cap B_1| \geq \delta |B_1|$. Then there exists a universal constant ϵ such that*

$$osc_{B_{1/2}} u^+ \leq (1-\epsilon) osc_{B_1} u^+.$$

Since u^+ is a subsolution, we drop the $+$ for simplicity. By local boundedness and multiplication by a constant assume $\sup_{B_1} u = 1$. Observe that by DeGiorgi for η sufficiently small and universal, if $|\{u = 0\} \cap B_1| \geq (1-\eta)|B_1|$ then we have the desired conclusion.

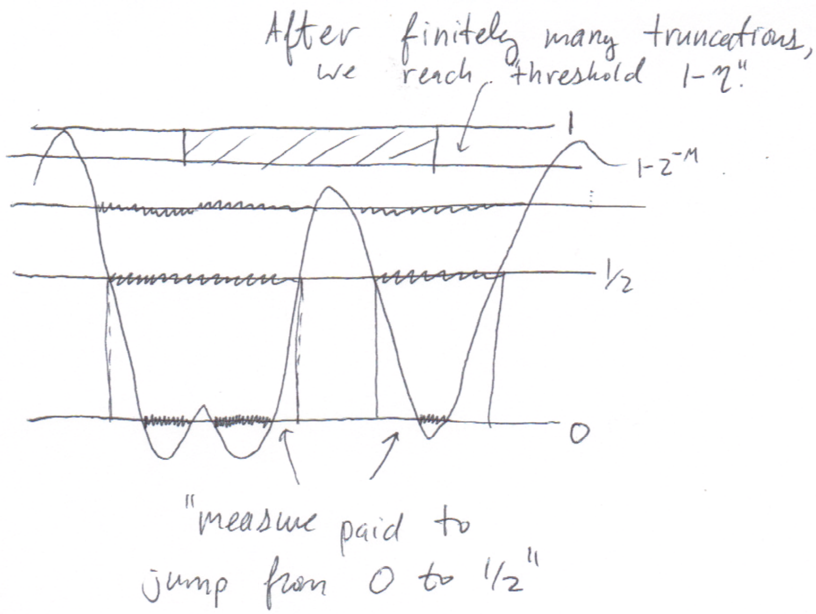
Claim: There is a constant $\delta_1(n, \lambda, \Lambda, \delta, \eta) > 0$ such that if

$$\delta |B_1| \leq |\{u = 0\} \cap B_1| \leq (1-\eta)|B_1|,$$

we have

$$|\{u \leq 1/2\} \cap B_1| \geq (\delta + \delta_1)|B_1|.$$

By iterating this result, after finitely many steps we are done.



Exercise: Truncate u above at $\frac{1}{2}$ and below at 0 to obtain $\tilde{u}$. Define $A = \{\tilde{u} = \frac{1}{2}\}$, $B = \{0 < u < \frac{1}{2}\}$ and $D = \{\tilde{u} = 0\}$. Recall the "pay in measure" exercise for $W^{1,p}$ from the function spaces section:

$$|D||A|^{1-\frac{1}{n}} \leq C(n)|B|^{\frac{1}{2}} \|D\tilde{u}\|_{L^2(B)}.$$

By localizing to the slightly smaller ball $B_{1-\mu}$ and applying the Cacciopoli inequality, show that

$$|D||A|^{1-\frac{1}{n}} \leq C|B|^{\frac{1}{2}}$$

for some $C(n, \lambda, \Lambda, \delta)$ up to replacing D, A, B by their intersections with $B_{1-\mu}$. Why may we assume that

$$|D| \geq \frac{\eta}{2}|B_1|?$$

Conclude the claim from this and the pay in measure inequality.

Exercise: Let u be a positive supersolution in B_2 with $|\{u \geq 1\} \in B_1| \geq \delta|B_1|$. Apply the previous result to $1-u$ to conclude that

$$u|_{B_{1/2}} \geq \epsilon.$$

Exercise: Let u be a solution to $D_i(a^{ij}u_j) = 0$ in B_2. Show that there is some $\gamma \in (0,1)$ with

$$osc_{B_{1/2}} u \leq \gamma osc_{B_1} u$$

by applying the previous results. (Hint: let $m = \inf_{B_1} u$ and $M = \sup_{B_1} u$. Then either $|\{2(u-m)/(M-m) > 1\} \cap B_1| \geq \frac{1}{2}|B_1|$ or its complement is. Applying the density estimates, conclude.)

Hölder continuity of solutions follows immediately:

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^2(B_1)},$$

where $\gamma = 2^{-\alpha}$.

6.3 Weak Harnack Inequality

Theorem 10. *Let $u \geq 0$ in B_1 satisfy*

$$D_i(a^{ij}u_j) \leq 0$$

where $\lambda I a^{ij} \Lambda I$. Then for some small p,

$$\|u\|_{L^p(B_{1/2})} \leq Cu(0).$$

Exercise: Assume that $u(0) = 1$ since this estimate is invariant under multiplication by constants. Show that it suffices to prove the decay of level sets

$$|\{u \geq M^k\} \cap B_{1/2}| \leq 2^{-k}|B_{1/2}|$$

for some universal M.

Exercise: We use the Calderón-Zygmund decomposition in a cube $Q = Q_1$ for simplicity. Using the density estimate with $\delta = \frac{1}{2(n+1)}$ show that for M large universal, prove that

$$|\{u \geq M\} \cap Q_{1/2}| < \frac{1}{2}|Q_{1/2}|,$$

given that $u(0) = 1$. Now decompose the $Q_{1/2}$ dyadically and keep Q_j if for some subcube $\tilde{Q}_j$,

$$|\{u \geq M^2\} \cap \tilde{Q}_j| \geq \frac{1}{2}|\tilde{Q}_j|.$$

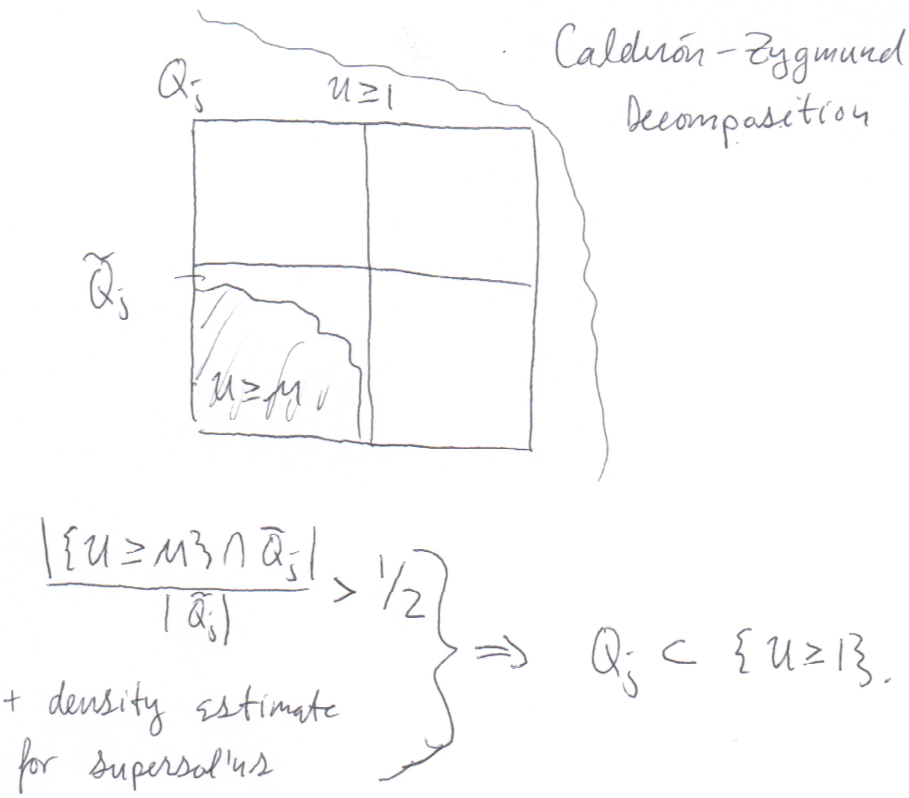
Using the density estimate for $\delta = 2^{-(2n+1)}$, show that

$$u|_{Q_j} \geq M.$$

Since $|\{u \geq M^2\} \cap Q_j| \leq \frac{1}{2}|Q_j|$, conclude from Lebesgue differentiation that

$$|\{u \geq M^2\} \cap Q_{1/2}| \leq \frac{1}{2}|\{u \geq M\} \cap Q_{1/2}|.$$

Iterate this procedure to prove the weak Harnack inequality.



Exercise: Prove the following more general form of the Calderón-Zygmund decomposition. Suppose that $A \subset B \subset Q$ satisfy: There is a $\delta < 1$ such that if

$$|A \cap Q| < \delta |Q|$$

and for some dyadic subcube $\tilde{Q}$

$$|A \cap \tilde{Q}| \geq \delta |\tilde{Q}|,$$

then the predecessor $\tilde{Q}_p$ of $\tilde{Q}$ satisfies

$$\tilde{Q}_p \subset B.$$

Show that

$$|A| \leq \delta |B|.$$

6.4 Harnack Inequality in 2 Dimensions

The Harnack inequality on $\mathbb{R}^2$ can be obtained in an elementary fashion. The key step is the following estimate:

Lemma 3. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be in $C^1(B_1)$ and let $\omega(r) = osc_{\partial B_r} u$. Assume also that ω is increasing. Then*

$$\omega(r)^2 \leq \frac{\pi}{\log(1/r)} \int_{B_1} |Du|^2.$$

Proof. For some $\theta_1(r)$ and $\theta_2(r)$ differing by an angle less than π, we have

$$\begin{aligned}
\omega(r) &= \int_{\theta_1}^{\theta_2} \frac{d}{d\theta} u(r\theta) d\theta \\
&\leq r \int_{\theta_1}^{\theta_2} |Du(r\theta)| d\theta \\
&\leq r\sqrt{\pi} \left(\int_0^{2\pi} |Du(r\theta)|^2 d\theta \right)^{\frac{1}{2}}.
\end{aligned}$$

Rewriting this inequality as

$$\frac{\omega(r)^2}{r} \leq \pi \int_0^{2\pi} |Du(r\theta)|^2 d\theta,$$

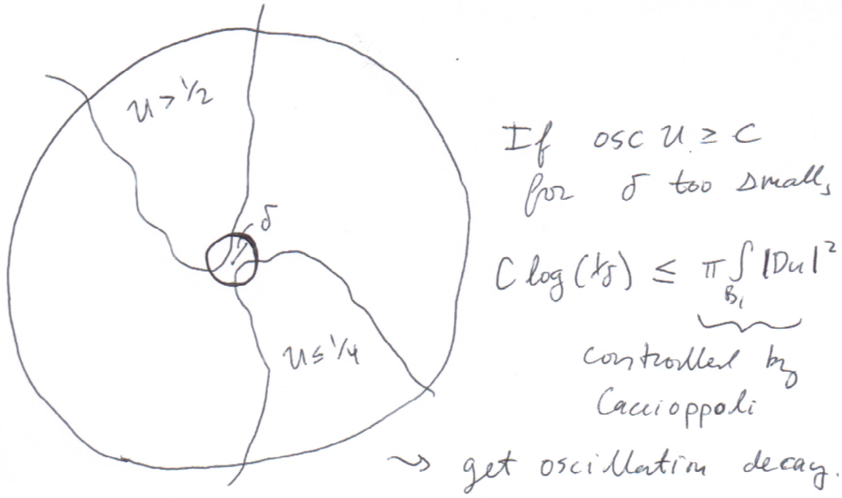
we may integrate from ρ to 1 and apply oscillation monotonicity to get the desired estimate. $\square$

Exercise: Why is this result only true in dimension 2? What goes wrong in higher dimensions?

The structure of the equation roughly gives us control of

$$\int_{B_1} |Du|^2$$

which suggests that if the oscillation got too large on a ball of small radius, we would have a contradiction.



Exercise: Let $u > 0$ in $B_1 \subset \mathbb{R}^2$ solve $D_i(a^{ij}u_j) = 0$. Show that $v = \log(u)$ satisfies

$$\int_{B_1} a^{ij} v_i \phi_j = \int_{B_1} a^{ij} v_i v_j \phi.$$

Choose a test function $\phi = \eta^2$ for $\eta \in C_0^\infty(B_1)$ with $\eta \equiv 1$ in $B_{3/4}$. Show that

$$\int_{B_{3/4}} |Dv|^2 \leq C$$

for some universal C.

Exercise: By the maximum principle, the oscillation of v above is increasing. Using the previous exercises, show the Harnack inequality in 2 dimensions.

7 $W^{2,p}$ Estimates

Reading: Chapter 9 of Gilbarg-Trudinger and later Chapter 7 of Caffarelli-Cabré.

7.1 Maximal Function and Marcinkiewicz Interpolation

The maximal function for a function $f : \mathbb{R}^n \to \mathbb{R}$ intuitively captures how much values of f around a point x influence $f(x)$. It is defined as

$$mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f|,$$

the largest average of f over balls around x.

Recall the Vitali covering lemma. Let F be a collection of closed balls of diameter $0 < d < 1$. Let F_j be those balls in F of diameter $2^{-j} \leq d < 2^{1-j}$. Take a maximal disjoint subcollection C_1 of F_1. (Uses axiom of choice). Then take a maximal disjoint subcollection C_2 of F_2 which is also disjoint from C_1. Proceed iteratively.

Exercise: Draw a picture illustrating this process.

Exercise: Let $\bar{B}$ be the 5-times dilation of a ball B. Show that for any $A \in F$, there is some $B \in \cup_i C_i$ with $B \cap A$ nonempty and

$$A \subset \bar{B}.$$

(Hint: $2^{-k} \leq diam(A) < 2^{1-k}$. If A was not chosen, then $A \cap_{i=1}^{k} C_i$ is nonempty by maximality).

Exercise: Let $f \in L^1(\mathbb{R}^n)$ and let $A_t = \{mf(x) > t\}$. Using a Vitali cover of A_t, show that

$$|A_t| \leq \frac{5^n}{t} \|f\|_{L^1(\mathbb{R}^n)}.$$

Thus, the maximal operator maps L^1 continuously to L^1_W.

Exercise: Consider $f = \chi_{[-1,1]}$. Show that mf is L^1_W but not L^1, and check the previous estimate.

Exercise: Show that if $f \in L^\infty$ then

$$\|mf\|_{L^\infty} \leq \|f\|_{L^\infty}.$$

Exercise (Interpolation): Let $f \in L^p(B_1)$ for $p > 1$ and let $g = f\chi_{|f|>\frac{t}{2}}$. Show that if $mf(x) > t$ then $mg(x) > \frac{t}{2}$. Conclude from the L^1 estimate that

$$|\{mf > t\}| \leq \frac{C(n)}{t} \int_{2|f|>t} |f|.$$

Noting by change of variable that $\int_{B_1} |mf|^p = p \int_0^\infty t^{p-1} |\{|mf| > t\}| dt$, show that

$$\|mf\|_{L^p(B_1)} \leq C(n,p) \|f\|_{L^p(B_1)}.$$

Thus, the maximal operator maps L^p to L^p continuously for $p > 1$.

7.2 Singular Integral Approach

Here we give some exercises to complement reading the Calderón-Zygmund estimate from Chapter 9 of Gilbarg-Trudinger.

Exercise: Suppose $u \in C_0^\infty(\mathbb{R}^n)$. Integrate by parts to show the formula

$$\int |D^2u|^2 = \int (\Delta u)^2$$

How fast must u, Du, D^2u decay for this formula to hold? This is the L^2 estimate for Laplace's equation.

Exercise: To get the L^1_W estimate we decompose f into a good part g comparable in size to t, and a bad part b which has average 0 on the Q_k, where

$$t \leq \frac{1}{|Q_k|} \int_{Q_k} |f| \leq 2^n t.$$

Then Tg is easy to control using the L^2 estimate. Why can we disregard a neighborhood of $\cup_k Q_k$ to get the L^1_W estimate? How is the key property that b has average 0 on Q_k used to estimate Tb?

7.3 Estimating the Distribution Function of Θ

The approach used to obtain L^p estimates ($p > n$ for viscosity solutions of fully nonlinear equations in Caffarelli-Cabré is to control the distribution function of Θ via ABP, giving an L^ϵ_W estimate. Assuming C^2 interior estimates for the constant coefficient equation, one can strengthen this result.

Exercise: Read Chapter 8 of Caffarelli-Cabré and do the $W^{2,\delta}$ estimate for linear equations. Why is the maximal function of f^n important? Where and how is the L^1_W estimate for the maximal function used?

Exercise: Using a perturbation procedure similar to that used for the Schauder estimates, show carefully how interior derivative estimates for harmonic functions accelerate the decay of $\mu_\Theta(t)$.

Exercise: Why is it important that $p > n$ and $f \in L^p(B_1)$? How is the $L^{\frac{p}{n}}$ estimate for the maximal function of f^n used in the covering argument?

8 Viscosity Solutions and ABP Estimate

Reading: Chapters 2 and 3 of Caffarelli-Cabré.

8.1 Visualizing Ellipticity

Let $F : Sym(n) \to \mathbb{R}$ be a function on symmetric matrices. The uniform ellipticity condition is that

$$\lambda\|N^+\| - \Lambda\|N^-\| \leq F(M+N) - F(M) \leq \Lambda\|N^+\| - \lambda\|N^-\|.$$

Identify $Sym(2)$ with $\mathbb{R}^3$ by making the following identifications of matrices to vectors:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

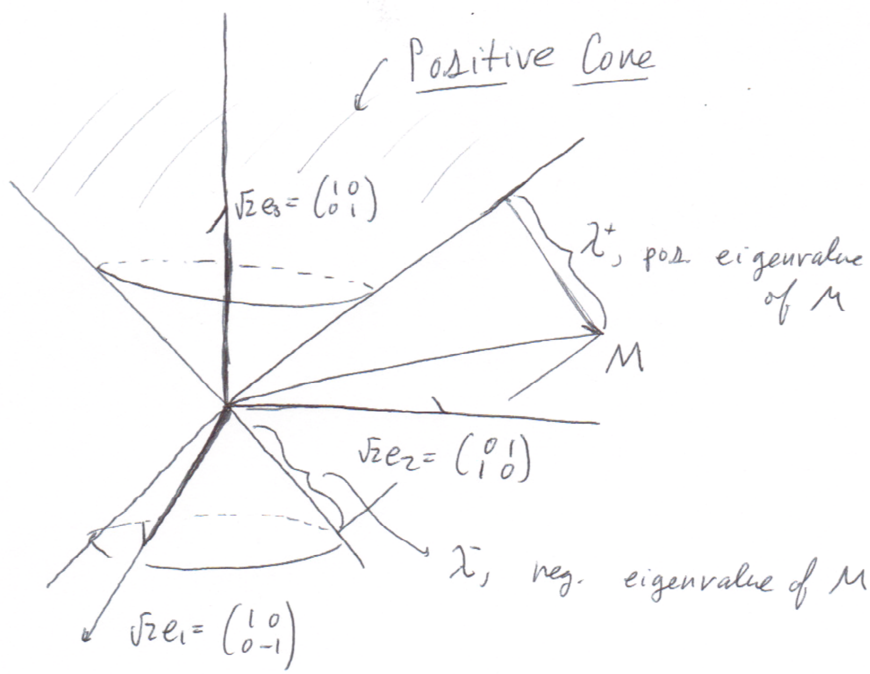
to $\sqrt{2}e_3$,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to $\sqrt{2}e_1$ and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

to $\sqrt{2}e_2$.



Exercise: Show that this identification is isometric where the inner product on matrices is $tr(AB)$.

Exercise: What is the process of diagonalization in this visualization?

Exercise: Find the level sets of trace and determinant and draw them.

Exercise: Geometrically, what are the eigenvalues of a matrix in this picture? (Hint: consider the "positive and negative cones" $z = \pm\sqrt{x^2 + y^2}$.

Geometrically, uniform ellipticity of F tells us that the level sets of F are trapped by cones of wider opening than the positive and negative cones at every point. Quantitatively, if

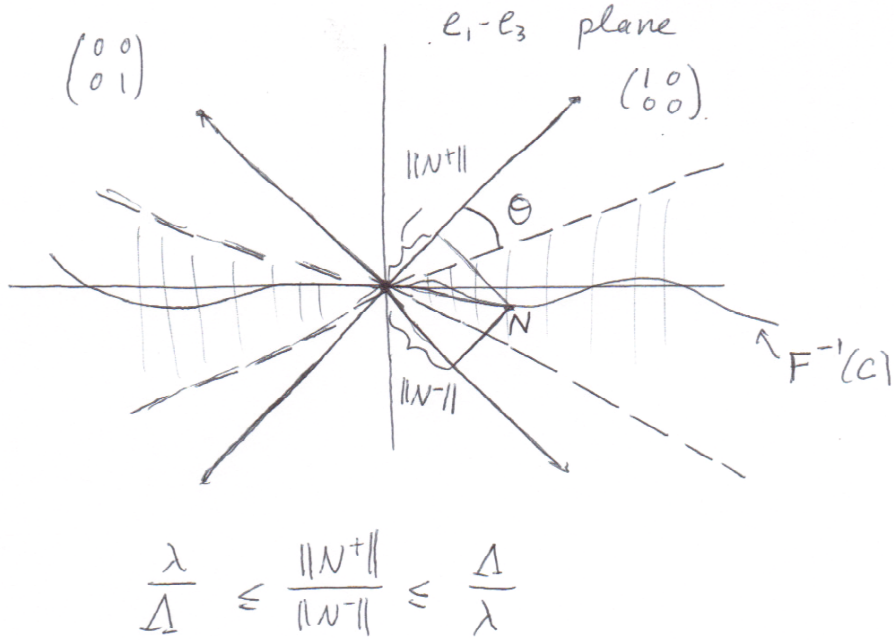
$$F(M + N) - F(M) = 0$$

then we can conclude that

$$\frac{\lambda}{\Lambda} \leq \frac{\|N^+\|}{\|N^-\|} \leq \frac{\Lambda}{\lambda},$$

so that the trapping cones contain lines at

$$\tan^{-1}\left(\frac{\lambda}{\Lambda}\right) \leq \theta \leq \tan^{-1}\left(\frac{\Lambda}{\lambda}\right).$$



Exercise: What are the level sets of the "Laplace operator" $\Delta(M) = tr(M)$? Of the "Monge-Ampére" operator $\det(M)$? What happens to ellipticity as one of the eigenvalues gets very large for a solution to the Monge-Ampére equation $\det(D^2u) = 1$? How is this reflected in the picture?

Intuitively, solutions to non-divergence elliptic equations have a "pointwise balancing of eigenvalues" of the Hessian; the equation controls the bending of solutions at a point. This is the idea behind the class S introduced in Caffarelli-Cabré.

Exercise: How is the balancing of eigenvalues reflected in the visualization of ellipticity above?

Exercise: Show that if $F(0, x) = 0$ for all $x \in B_1$ with F uniformly elliptic in the matrix variable (i.e. 0 is a solution) and u solves $F(D^2u(x), x) = 0$ classically then $u \in S$. Draw a picture using the identification above.

Exercise: Show that C^2 functions in S satisfy an equation of the form

$$a^{ij}(x)u_{ij} = 0$$

where $\lambda I \leq a^{ij} \leq \Lambda I$ and there is no regularity assumption on the coefficients.

8.2 Viscosity Solutions and Comparison Principle

A viscosity solution u to the equation

$$F(D^2u(x), x) = f(x)$$

has comparison to C^2 functions built into the definition. Namely, if a paraboloid P touches u by above at x_0 then

$$F(D^2P, x_0) \geq f(x_0),$$

and if it touches by below then we have the opposite inequality.

Exercise: Show that classical solutions to the elliptic equation $F(D^2u(x), x) = f(x)$ are viscosity solutions. Why is ellipticity important?

Exercise: Show that we may replace paraboloids by any C^2 function in the above definition.

Exercise: Show that C^2 viscosity solutions are classical solutions.

Exercise: Show the weak maximum principle for viscosity solutions of $\Delta u = 0$: if $u > 0$ on ∂B_1 the $u > 0$ in B_1.

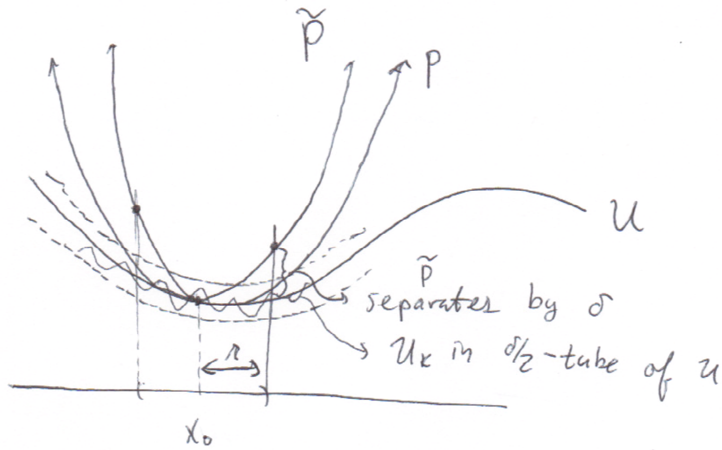
Exercise: Show the strong maximum principle for viscosity solutions of $\Delta u = 0$: if $u \geq 0$ on ∂B_1 and $u(x_0) = 0$ for some $x_0 \in B_1$ then $u \equiv 0$. (Hint: use a Hopf lemma type barrier).

Exercise: Show that a viscosity solution of $\Delta u = 0$ in B_1 is classical.

8.3 Closedness

Viscosity solutions are closed under uniform limits. More precisely, if u_k are viscosity subsolutions of $F(D^2v, x) = f(x)$ and $u_k \to u$ uniformly then u is also a viscosity subsolution. Here we must assume that F and f are continuous in x. (Ellipticity guarantees that F is Lipschitz in the matrix variable).

Exercise: Let P be a paraboloid touching u by above at x_0 and fix $r > 0$ small enough that $P \geq u$ in $B_r(x_0)$. Note that $\tilde{P} = P + \epsilon|x - x_0|^2$ separates from u on ∂B_r. Use this to show that for k sufficiently large, a shift of $\tilde{P}$ touches u_k by above at $x_1 \in B_r(x_0)$. Draw a picture illustrating this claim.



Exerise: Using the previous exercise, prove the closedness theorem.

8.4 ABP Estimate

Here we discuss the Alexandrov-Bakelman-Pucci estimate for viscosity supersolutions. The convex envelop of a graph u is denoted by Γ_u.

Theorem 11. *Let $u \in \bar{S}(f)$ (morally, $a^{ij}(x)u_{ij} \leq f$) and assume*

$$u|_{\partial B_1} \geq 0.$$

Then

$$|\inf_{B_1} u| \leq C \left(\int_{u=\Gamma_u} (f)^n \right)^{\frac{1}{n}}.$$

The ABP estimate roughly says that the measure of the contact set is bounded below by the minimum of a supersolution; there are no sharp corners from below.

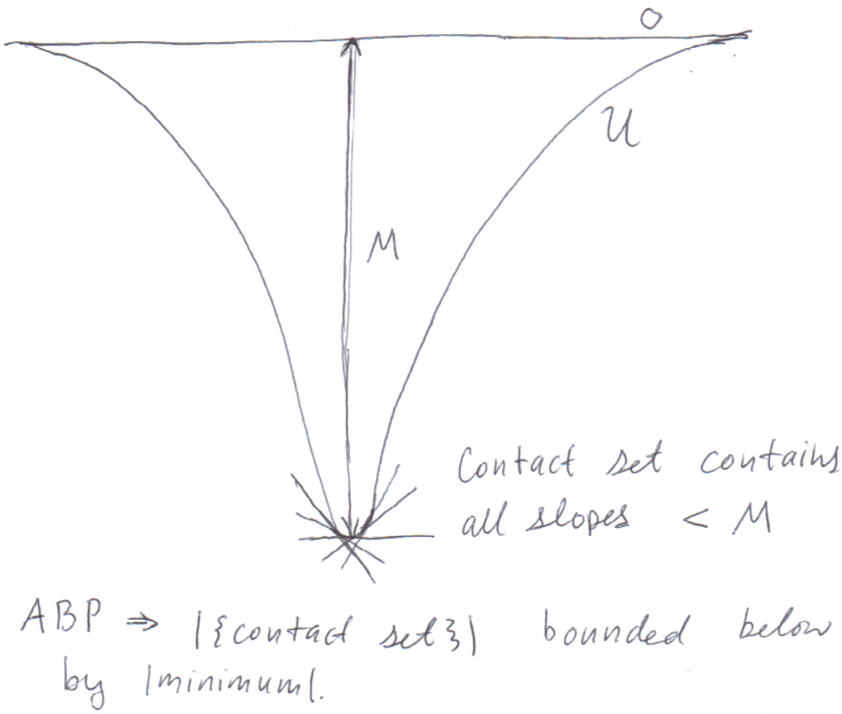
Exercise: Show that the ABP estimate is "equality" for the fundamental solution of Δ; Let $u = 1 - |x|^{2-n}$ in $B_1 - B_\rho$ and cap it off with a paraboloid from below so that u is $C^{1,1}$. Show that both sides of the inequality grow like ρ^{2-n}. (Hint: the contact set is not all of B_ρ.)

Exercise: Assume that Γ_u is $C^{1,1}$ on the contact set and let $M = |\inf_{B_1} u|$. Draw a picture showing that if we slide planes l of slope size $|Dl| \leq \frac{M}{2}$ from below, they touch on the contact set $\{u = \Gamma_u\}$. Hence,

$$B_{M/2}(0) \subset D\Gamma_u(\{u = \Gamma_u\}).$$

Using the area formula (change of variable), show that

$$M^n \leq C(n) \int_{u=\Gamma_u} \det(D^2\Gamma_u).$$



Exercise: If u is C^2 and $a^{ij}(x)u_{ij} \leq f(x)$ for $\lambda I \leq a^{ij} \leq \Lambda I$, show that

$$\det(D^2\Gamma_u) \leq Cf^n$$

on the contact set.

In the case of viscosity solutions, we must show that Γ_u is in fact $C^{1,1}$ and that

$$\det(D^2\Gamma_u) \leq Cf^n$$

on the contact set. Assume that $f^+ \leq 1$ in B_1, and that ϕ is a convex function satisfying

$$0 \leq \phi \leq u$$

and

$$\phi(0) = u(0) = 0.$$

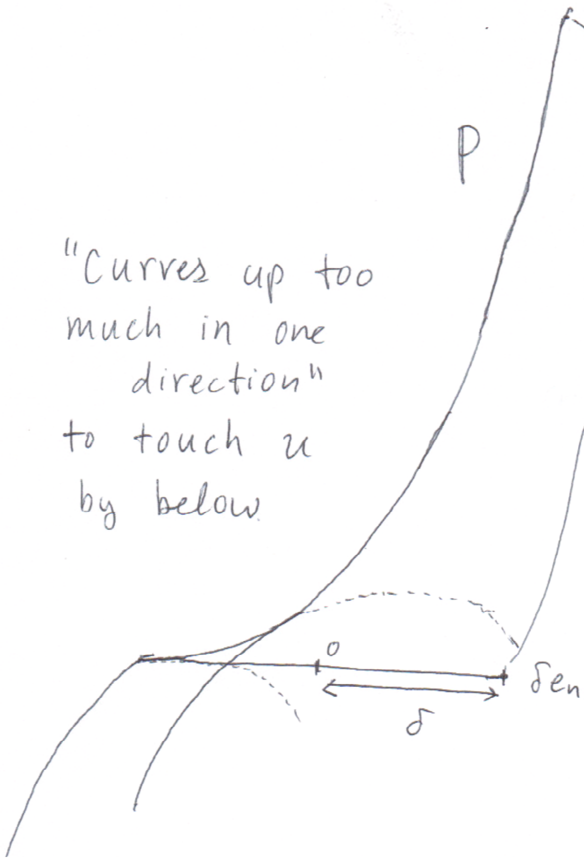
We claim that there exist C large and δ small universal such that $\phi \leq C$ in B_δ. If not, assume that the maximum of ϕ on ∂B_δ is at δe_n.

Exercise: Using the convexity of ϕ, show that $\phi \geq C$ at $(x', \delta e_n)$ in B_1.

Exercise: Consider the polynomial

$$P = \frac{C}{4\delta^2}(x_n + \delta)^2 - 2C|x'|^2.$$

Using the previous exercise, that $P(0) > 0$ if we slide P from below it touches u on the interior.



Exercise: Prove that

$$a^{ij}P_{ij} \geq C\left(\frac{\lambda}{2\delta^2} - 4\Lambda(n-1)\right).$$

Conclude that for δ small universal and C large universal, we have a contradiction of the previous exercise.

Exercise: Using quadratic scaling $u_r(x) = \frac{1}{r^2}u(rx)$ and $\phi(rx) = \frac{1}{r^2}\phi(rx)$ and dividing by $\sup_{B_r}(f^+)$,

$$\phi(x) \leq C(\sup_{B_r} f^+)r^2$$

in $B_{r\delta}$.

Exercise: Using the previous exercise, show that u is trapped between a plane and a paraboloid of opening $Cf(x)$ at contact points x. Conclude that Γ_u is $C^{1,1}$ on the contact sets, and that

$$\det(D^2\Gamma_u) \leq Cf^n$$

on the contact set. (Hint: $D^2\Gamma_u(x) \leq 2Cf(x)Id$ by letting $r \to 0$ above).

Exercise: Prove the ABP estimate using Jensen's regularization. The key point is to show that the contact sets "converge" in the appropriate sense.

9 Krylov-Safonov Harnack Inequality

Reading: Chapter 4 of Caffarelli-Cabré.

9.1 Basic Measure Estimate

The following estimate in measure, which holds "at all scales," is the heart of the Harnack inequality for nondivergence equations with bounded measurable coefficients.

Theorem 12. *Assume $u \in \bar{S}(0)$ (morally, $a^{ij}(x)u_{ij} \leq 0$ for some bounded measurable and uniformly elliptic a^{ij}) in B_2, that $u \geq 0$ and that $u(x) \leq 1$ for some $x \in \partial B_1$. Then*

$$\frac{|\{u \leq M\} \cap B_{1/2}|}{|B_{1/2}|} \geq \mu$$

for some large M and small μ universal.

The proof of this estimate uses the ABP inequality, which ties pointwise information to information in measure.

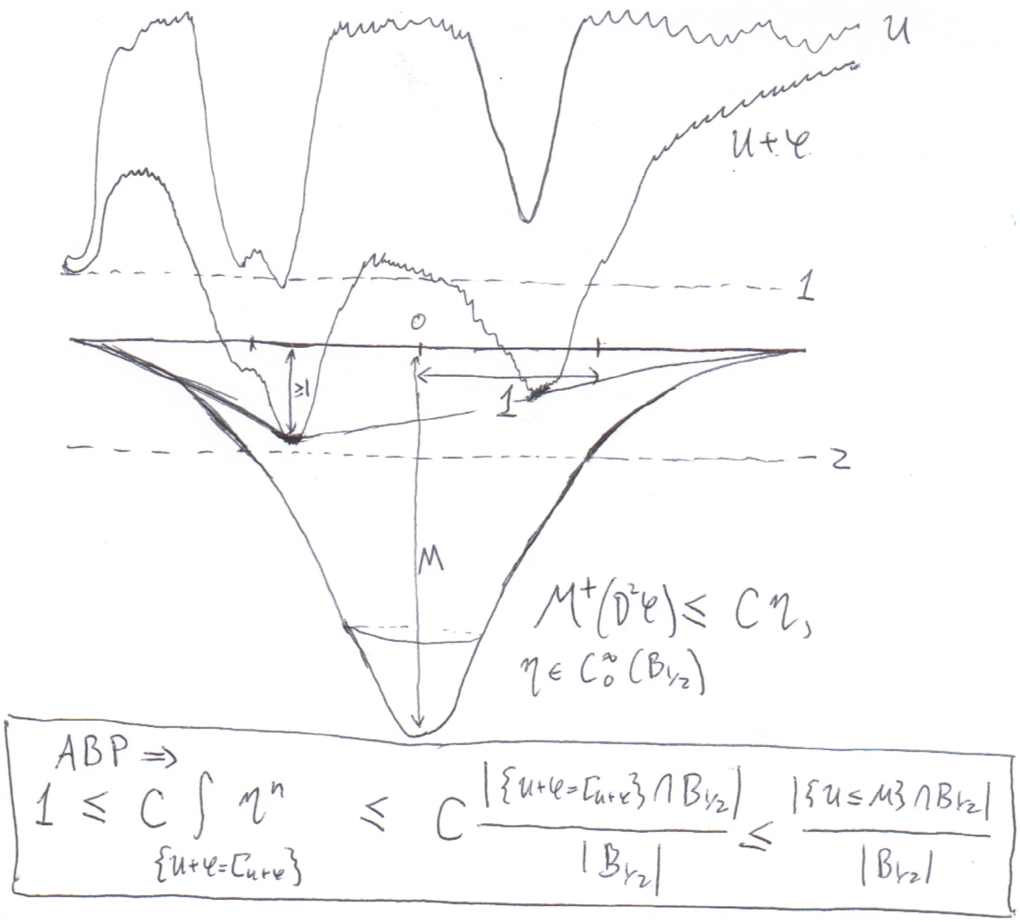
Proof. Take a "localizing" barrier ϕ of the form $A - Br^{-\alpha}$ in $B_2 - B_{1/2}$, with A and B chosen so that $\phi|_{\partial B_2} = 0$ and $\phi|_{\partial B_1} = -2$. Cap ϕ off in $B_{1/2}$ and smooth it out.

Exercise: Show that $M^+(D^2\phi) \leq C\eta$ for some universal C and $\eta \in C_0^\infty(B_{1/2})$ with $0 \leq \eta \leq 1$. Show also that $|\phi| \leq M$ universal. Conclude, since ϕ is smooth, that $u + \phi \in \bar{S}(C\eta)$. (Recall that comparing viscosity solutions to C^2 functions is easy from the definition).

Exercise: Note that $w = u + \phi \leq -1$ somewhere on ∂B_1. Conclude from the ABP estimate that

$$1 \leq C \int_{\{w=\Gamma_w\}\cap B_1} \eta^n \leq C|\{w = \Gamma_w\} \cap B_{1/2}|.$$

Exercise: Show that $\{w = \Gamma_w\} \subset \{u \leq M\}$. (Hint: the contact set is part of where $w \leq 0$.) Conclude the basic measure estimate.



□

Exercise: Show that this estimate is scaling and multiplication invariant: if u is defined on B_{2r} and $u \leq \alpha$ somewhere on ∂B_r, then

$$\frac{|\{u \leq M\alpha\} \cap B_{r/2}|}{|B_{r/2}|} \geq \mu.$$

Exercise: Show that for $u \in \bar{S}(f)$, the above estimate holds provided $\|f\|_{L^n(B_2)} \leq \epsilon_0$ for some universal ϵ_0.

9.2 Covering Argument

Let u satisfy the same conditions as above. The object of this section is to extend basic measure estimate (using it at "all scales") via a covering argument to prove that

$$|\{u > M^k\} \cap B_{1/2}| \leq (1-\mu)^k |B_{1/2}|.$$

From this we can conclude the weak Harnack inequality

$$\|u\|_{L^\epsilon(B_{1/2})} \leq C \inf_{B_1} u$$

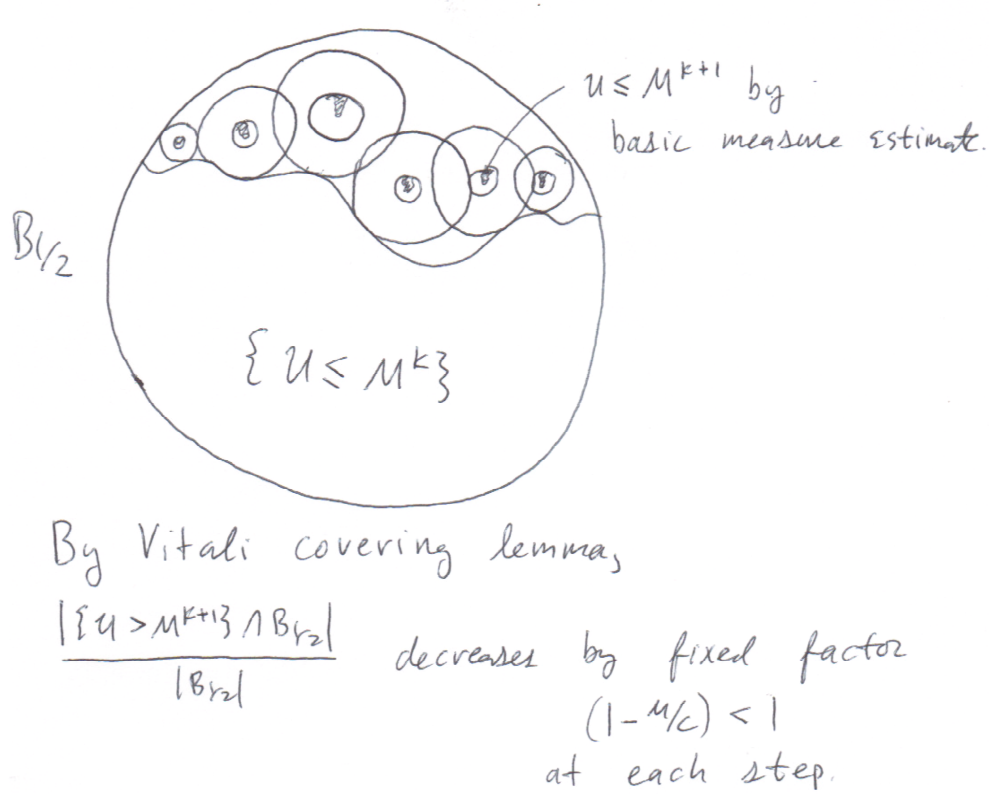
for some small ϵ and large C universal.

Exercise: The above estimate holds for $k = 1$ by the basic measure estimate. Proceed inductively using the Calderón-Zygmund decomposition, with cubes for convenience. Decompose $Q_{1/2}$ into dyadic cubes, keeping Q_j if $\{u > M^k\}$ has density exceeding $(1 - \mu)$. Using the basic measure estimate, show that $u > M^{k-1}$ in the predecessor of Q_j. Conclude that

$$|\{u > M^k\} \cap Q_{1/2}| \leq (1 - \mu)|\{u > M^{k-1}\} \cap Q_{1/2}|,$$

hence the claim.

The covering argument can also be accomplished using the Vitali lemma. The idea is to cover a universal fraction of $\{u > M^k\} \cap B_{1/2}$ by balls which just touch $\{u \leq M^k\}$ and apply the basic measure estimate to see that the sets $\{u > M^j\}$ decay by a fixed fraction each time j grows.



Exercise: Let $a_k = |\{u \leq M^k\} \cap B_{1/2}|/|B_{1/2}|$. Let F be the collection of balls centered in $\{u > M^k\} \cap B_{1/2}$ but touching $\{u \leq M^k\}$. Show that these balls cover $\{u > M^k\} \cap B_{1/2}$. Using the Vitali lemma, extract a disjoint subcollection

$$\{B_{r_i}\}$$

such that the 5-times dilations cover F. Using the basic measure estimate, show that

$$a_{k+1} \geq a_k + \frac{\mu}{10^n}(1 - a_k).$$

Letting $b_k = 1 - a_k$, this estimate becomes

$$b_{k+1} \leq \left(1 - \frac{\mu}{C(n)}\right) b_k,$$

proving the L^ϵ_W estimate.

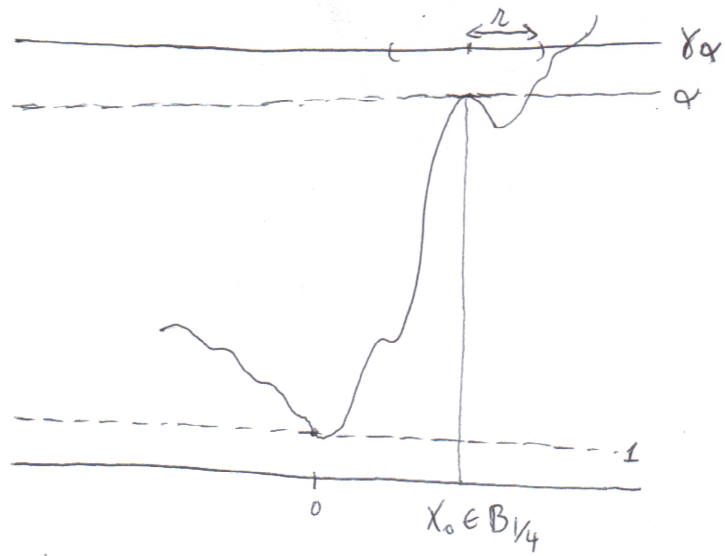
Exercise: Actually, I cheated a bit in the last exercise since some of the balls could be partly outside of $B_{1/2}$. To fix this, note that a universal fraction is still in $B_{1/2}$ and show that we can use a variant the basic measure lemma for this part.

9.3 Harnack Inequality

Let $u \in S(0)$ with $u \geq 0$ in B_2. The Harnack inequality is obtained by applying the L^ϵ_W estimate from both directions:

$$u|_{B_{1/4}} \leq Cu(0).$$

Assume that $u(0) = 1$. Intuitively, if u were too large at a point in $B_{1/4}$, then by applying the basic measure estimate upside-down u would be large on a set of large measure. This situation is ruled out by the L^ϵ_W estimate.



Choose $\gamma > 1$ s.t. $\gamma - M(\gamma - 1) = 1/2$.

Upside down measure estimate

$\Rightarrow |\{u \geq \frac{\alpha}{2}\} \cap B_{r/2}| \geq C r^n$

L^{ε}_{W} estimate

$\Rightarrow |\{u \geq \frac{\alpha}{2}\} \cap B_{r/2}| \leq C\alpha^{-\varepsilon}$

$\Rightarrow r \leq C\alpha^{-\varepsilon/n}$

Exercise: Suppose that $u(x_0) = \alpha$ for some $x_0 \in B_{1/4}$. Suppose that $u \leq \gamma\alpha$ in $B_r(x_0)$ for some $\gamma > 1$ universal to be chosen later. Using the basic measure estimate for $v = \gamma\alpha - u$, show that

$$|\{v \leq M\alpha(\gamma - 1)\} \cap B_{r/2}| \geq C(n)\mu r^n.$$

By choosing γ close enough to 1 that

$$\gamma - M(\gamma - 1) = \frac{1}{2},$$

this estimate becomes

$$cr^n \leq |\{u \geq \frac{\alpha}{2}\} \cap B_{r/2}|.$$

Exercise: Using the L^{ϵ}_{W} estimate, show that

$$|\{u \geq \frac{\alpha}{2}\} \cap B_{r/2}| \leq C\alpha^{-\epsilon}.$$

Conclude from this and the previous exercise that

$$r \leq C\alpha^{-\epsilon/n}.$$

Exercise: Using the previous exercises, show that for α large enough we could produce a sequence of points x_k converging to some $x_\infty \in B_{1/2}$ with $u(x_k) \geq \alpha\gamma^k$, contradicting boundedness of u.

Exercise: Prove the Harnack inequality as above for the nonhomogeneous equation. (Hint: We get the basic measure estimate for $\|f\|_{L^n}$ small. Dividing u by $u(0) + C\|f\|_{L^n}$ for C large, we reduce to this case).

9.4 Applications

Hölder Estimate: Let C be the constant from the Harnack inequality. Show the oscillation decay of $u \in S(0)$:

$$osc_{B_{1/2}} u \leq \left(\frac{1}{2}\right)^\alpha osc_{B_1} u$$

where $2^{-\alpha} = \frac{C}{C+1}$. Conclude that

$$\|u\|_{C^\alpha(B_{1/2})} \leq \tilde{C}\|u\|_{L^\infty(B_1)}.$$

Liouville Theorem: Prove that global bounded solutions $u \in S(0)$ are constant by applying the oscillation estimate.

Compactness: Let $\{u_k\}$ be an L^∞-bounded collection of solutions to the fully nonlinear equation $F_k(D^2u_k, x) = f(x)$ with F_k having the same ellipticity constants and converging uniformly to F on compact subsets in the matrix space. Show that a subsequence u_{k_j} converges uniformly to a solution of $F(D^2u, x) = f(x)$. (Hint: $u_k \in S(f - F_k(0, x))$ and the functions $f - F_k(0, x)$ have bounded L^n norm. Recall closedness.)

$C^{1,\alpha}$ **Estimate:** The difference of two C^2 solutions to $F(D^2u) = 0$ satisfies a linear elliptic equation (by balancing of eigenvalues). If F and u are smooth, we get the linearized equation by differentiating:

$$F_{ij}(D^2u)(u_e)_{ij} = 0.$$

Thus, we can expect to obtain $C^{1,\alpha}$ estimates for solutions to fully nonlinear equations. (We still require a way to regularize viscosity solutions. This is the content of the next section).

Hölder Estimate up to Boundary: Suppose that $u \in S(0)$ with $u|_{\partial B_1} = \phi \in C^\beta(\partial B_1)$. For simplicity assume that the ball has center e_n and that $\phi(0) = 0$.

Exercise: Show that $|x|^2 = 2x_n$ on the boundary of the ball. Conclude that

$$|\phi(x)| \leq C\|\phi\|_{C^\beta(\partial B_1)} x_n^{\beta/2}$$

for $x \in \partial B_1$.

Exercise: Consider the barriers

$$\pm\eta(x) = \pm\|\phi\|_{C^\beta(\partial B_1)} x_n^{\beta/2}.$$

Show that

$$M^+(D^2\eta) < 0.$$

Conclude that u is trapped by $\pm\eta$ in B_1 by the comparison principle with C^2 functions.

Exercise: Conclude from the previous exercises that

$$\|u\|_{C^\gamma(B_1)} \leq C\|\phi\|_{C^\beta(\partial B_1)}$$

where γ is the smaller of α from the Harnack inequality and $\frac{\beta}{2}$.

10 Jensen's Approximation

Reading: Chapter 5 of Caffarelli-Cabré.

10.1 Visualizing Jensen's Approximation

Jensen's approximation of a continuous function u on B_1 is a regularization that is natural in the viscosity solution setting. There are several ways to think of the approximations.

Upper Envelope of Downward-Opening Paraboids: The first way is to take each point on the graph of u, draw the paraboloid of opening $\frac{1}{\epsilon}$ with vertex at this point opening downward, and take the upper envelope of all such paraboloids. Show that this regularization is

$$u^\epsilon(x_0) = \sup_x (u(x) - \frac{1}{\epsilon}|x - x_0|^2).$$

Supremum of Slides of u: One can also think of the approximation as follows: Fix x_0 and slide u along the paraboloid $u(x_0) - \frac{1}{\epsilon}|x - x_0|^2$, and take the supremum at x_0 of all slides of u. Since a slide of u by y has the value

$$u(x_0 - y) - \frac{1}{\epsilon}|y|^2$$

at x_0, it is clear by taking the supremum that this interpretation and the previous interpretation are the same.

Exercise: Show that if we take a paraboloid of opening $\frac{1}{\epsilon}$ up centered at x_0 and slide it from above until it touches u, the height of the vertex is also the Jensen approximation.

We use paraboloids to generate the approximation because they have constant Hessian, and the Hessian is the object of greatest interest for second order equations. In other situations we may use different objects. For example, spheres are more natural for the minimal surface equation since spheres have constant mean curvature.

10.2 Key Properties

$C^{1,1}$ **by Below:** It is clear from the first interpretation that u^ϵ can be touched from below by a paraboloid of opening $\frac{1}{\epsilon}$ at every point. As a consequence, we have by Aleksandrov's theorem that the Jensen approximation is almost everywhere twice differentiable.

Convergence: Show as an exercise that u^ϵ decreases uniformly to u on compact subsets.

Subsolution Property: This is the most important. If $F(D^2u) \geq 0$ in the viscosity sense, then $F(D^2u^\epsilon) \geq 0$ in the viscosity sense. This is most easily shown using the slides-of-u interpretation. Suppose P touches u^ϵ by above at x_0. We have that

$$u^\epsilon(x_0) = u(x) - \frac{1}{\epsilon}|x - x_0|^2$$

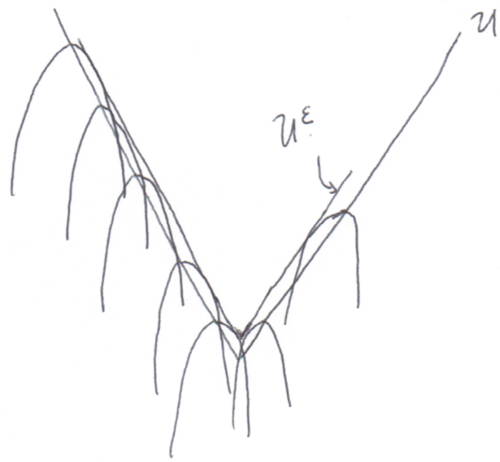
for some x. Sliding u^ϵ and P along the paraboloid with vertex x by $x - x_0$, we see that

$$u \leq \text{slide of } u^\epsilon \leq \text{slide of } P$$

so that the slide of P touches u by above at x. Hence,

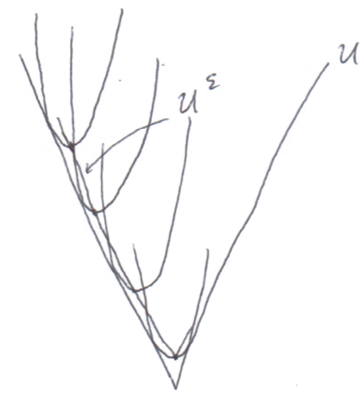
$$F(D^2P) \geq 0.$$

Exercise: Show why the subsolution property relies crucially on the translation invariance of $F(D^2u) = 0$. Why might this not work for $F(D^2u, x) = 0$? Does this still work for the equation $F(D^2u, Du) = 0$?

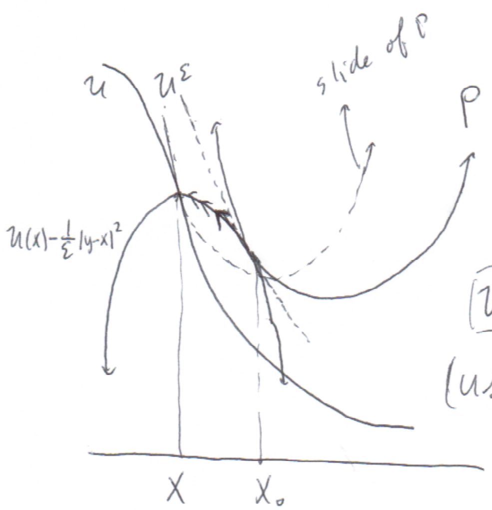


Sliding u/ Upper Envelope Picture.

(Easily gives u^ε $C^{1,1}$ by below)



Slide from above and take vertices picture.



Slide u^ε, P along $f(y) = u(x) - \frac{1}{\varepsilon}|x-y|^2$ until P touches u by above:

$u \leq$ slide of $u^\varepsilon \leq$ slide of P.

(uses translation invariance of $F(D^2u) = 0$).

10.3 Uniqueness

Theorem 13. *Suppose that $F(D^2u) = F(D^2v) = 0$ in the viscosity sense and that $u > v$ on ∂B_1. Then $u > v$ in B_1.*

Proof. Assume the conclusion is false. Take ϵ, δ small enough that

$$w = u_\epsilon - v^\epsilon - \delta|x|^2$$

is positive on ∂B_1 but negative somewhere in B_1. If w were twice differentiable at the minimum point x_0, we would have

$$D^2u_\epsilon(x_0) \geq D^2v^\epsilon(x_0) + 2I,$$

giving the contradiction $F(D^2u_\epsilon(x_0)) > 0$.

The idea is that we don't need twice differentiability at a local minimum of w. Since "changing by planes doesn't change the equation," we just need to find a point on the lower contact set $\{w = \Gamma_w\}$ where w is twice differentiable. Note that w is $C^{1,1}$ by above using the properties discussed in the previous section. Thus, Γ_w is $C^{1,1}$ on the contact set. Applying ABP, we see that

$$0 < \int_{w=\Gamma_w} \det(D^2\Gamma_w)$$

which gives that the contact set has positive measure. Recall that w is semiconvex, hence almost everywhere twice differentiable, completing the proof.

"changing by planes doesn't change eqn."

ABP $\Rightarrow$ $0 < \int_{\{\text{contact set}\}} \det D^2\Gamma_{u_\varepsilon - v^\varepsilon - \delta|x|^2} \Rightarrow |\{\text{contact set}\}| > 0$

so $\exists$ a pt. x_0 where all are $2\times$ differentiable

$\Rightarrow$ $D^2u_\varepsilon \geq D^2v^\varepsilon + 2\delta I$ at x_0

$\Rightarrow$ $F(D^2u_\varepsilon) > 0$, contradiction.
at x_0

□

Exercise: Notice that we only needed F elliptic in the last proof. For $F(D^2u, Du)$ show uniqueness for F uniformly elliptic and Lipschitz in Du. (Hint: Use the ABP-type estimate to show that we can find points where the largest eigenvalue μ of D^2u is $>> |Du|$.) Why don't we get uniqueness if F is only α-Hölder for $\alpha < 1$ in Du? Think about the relationship between this and non-uniqueness for ODEs, e.g. $f(x) = (x-a)^{\frac{1}{1-\alpha}}$ when $x > a$ and 0 for $x \leq a$ all solve $f'(x) = c(\alpha)f^{\alpha}$.

10.4 Applications

$C^{1,\alpha}$ **Interior Estimate**

Jensen's Approximation allows us to rigorously prove the $C^{1,\alpha}$ interior estimate for fully nonlinear equations $F(D^2u) = 0$.

Theorem 14. *Suppose $F(D^2u) = 0$ in the viscosity sense in B_1 and $F(0) = 0$. Then $u \in C^{1,\alpha}(B_{1/2})$ and*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

Formally, one differentiates the equation to obtain $F_{ij}(D^2u)(u_e)_{ij}$, and then apply the Krylov-Safonov Harnack inequality.

Proof. Since $F(0) = 0$ we have $u \in S(0)$. Dividing by $\|u\|_{L^\infty(B_1)}$ assume that

$$\|u\|_{L^\infty(B_1)} \leq 1.$$

Let $v_{h,\alpha}(x) = \frac{u(x+eh)-u(x)}{h^\alpha}$ where α is from the Harnack inequality. Then we have

$$\|v_{h,\alpha}\|_{L^\infty(B_{1/2})} \leq C$$

independently of h. By Jensen, $u(x+eh) - u(x) \in S(0)$, so we can apply the Harnack inequality to $v_{h,\alpha}$ for some β to obtain

$$\|v_{h,\alpha}\|_{C^\beta(B_{1/2})} \leq C$$

independently of h.

Exercise: Show that if $\alpha + \beta < 1$, the above implies that $u \in C^{\alpha+\beta}(B_{1/2})$ with

$$\|u\|_{C^{\alpha+\beta}(B_{1/2})} \leq C.$$

Show that if $\alpha + \beta > 1$, we have

$$\|u\|_{Lip(B_{1/2})} \leq C.$$

If $\alpha + \beta < 1$, repeat the above steps with $v_{h,\alpha+\beta}$ and iterate a universal number N times so that $\alpha + N\beta > 1$. Conclude after these steps the difference quotient estimate

$$\|D_h u\|_{C^\beta(B_{1/2})} \leq C,$$

giving $C^{1,\alpha}$ regularity and the desired estimate. □

Concave Equations

Suppose that $F(D^2u) = 0$ classically and that F is concave. Then u_{ee} are subsolutions to a linear equation. Formally, differentiate the equation twice to obtain

$$F_{ij}(D^2u)(u_{ee})_{ij} + F_{ij,kl}(D^2u)(u_e)_{ij}(u_e)_{kl} = 0$$

and apply concavity to the second term.

For the rigorous argument, consider the second order approximations

$$\Delta_h^2 u(x) = \frac{u(x+he) + u(x-he) - 2u(x)}{h^2}.$$

Exercise: Suppose that $F(D^2u) = F(D^2v) = 0$ classically and F is concave. Show that

$$F\left(\frac{1}{2}(D^2u + D^2v)\right) \geq 0$$

using only concavity. Conclude that $\Delta_h^2 u \in \underset{\sim}{S}(0)$.

Exercise: The above result is also true for viscosity solutions, with a little more work. Using a "changing by planes" ABP argument similar to the one above, show that the Jensen's approximation satisfies

$$F(\frac{1}{2}(D^2u^\epsilon + D^2v^\epsilon) \geq 0$$

in the viscosity sense. Conclude using Jensen that $\Delta_h^2 u \in \underset{\sim}{S}(0)$.

If $u \in C^2$, one concludes that u_{ee} are in $\underset{\sim}{S}(0)$ using closedness.

11 The Dirichlet Problem for Concave Equations

Reading: Chapters 6 and 9 of Caffarelli-Cabré.

11.1 Evans-Krylov $C^{2,\alpha}$ Interior Estimate

Theorem 15. *Assume $F(D^2u) = 0$ where $u \in C^2(B_1)$ and F is uniformly elliptic and concave. Then $u \in C^{2,\alpha}(B_{1/2})$ for some universal α and*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C\|D^2u\|_{C^{1,1}(B_1)}.$$

Proof. To prove a Hölder estimate on the second derivatives, we want to prove oscillation decay: there is some universal δ so that

$$diamD^2u(B_\delta) \leq \frac{1}{2}diamD^2u(B_1).$$

Assume that $diamD^2u(B_1) = 2$ by considering the operator $G(\cdot) = tF\left(\frac{1}{t}D^2\cdot\right)$, which has the same ellipticity constants.

The agenda is to show that if $diamD^2u(B_1) > 1$, we can remove a ball of universal size from the image of $D^2(B_1)$ so that what remains still covers $D^2u(B_{1/2})$. By repeating this procedure finitely many times, we are done. Cover $D^2u(B_1)$ by N balls of radius ϵ, to be chosen later.

Ellipticity: For $F(A) = F(B) = 0$, we know that $\|(A-B)^+\|$ is comparable to $\|A-B\|$. Quantitavely, there is a universal c_0 so that $\|(A-B)^+\| \geq c_0\|A-B\|$. Choose a subcollection $\{x_k\}_{k=1}^M$ of centers of the covering balls so that $B_k = B_{c_0/8}(x_k)$ cover $D^2u(B_1)$. Note that we can choose M universal. One of the B_k has universal nontrivial preimage in $B_{1/4}$, say

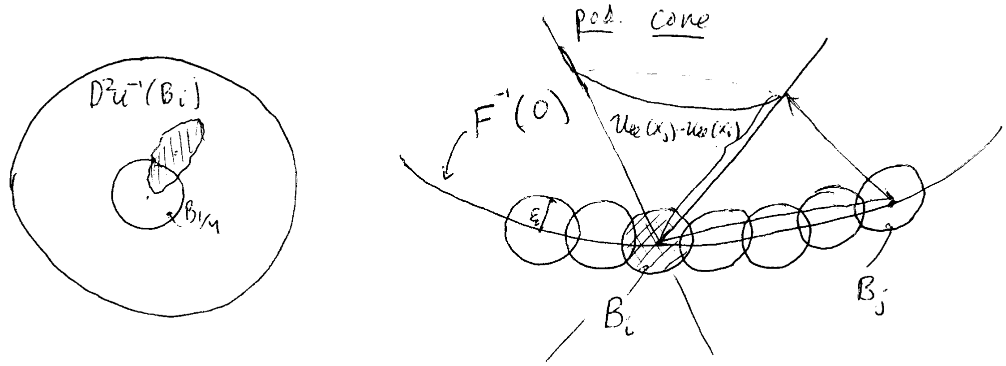
$$|D^2u^{-1}(B_i) \cap B_{1/4}| \geq \eta > 0.$$

Now, there is some $j \neq i$ so that the centers of B_i and B_j are far apart, i.e. $\|D^2u(x_i) - D^2u(x_j)\| \geq \frac{1}{4}$. Using ellipticity, we see that for some direction e, u_{ee} separates from its maximum by a universal amount at x_i:

$$u_{ee}(x_i) \leq u_{ee}(x_j) - \frac{c_0}{4}.$$

Since we chose the expanded balls to have radius $c_0/8$ we have nontrivial measure in $B_{1/4}$ where u_{ee} separates from its maximum K:

$$|\{u_{ee} < K - c_0/8\} \cap B_{1/4}| \geq \eta.$$



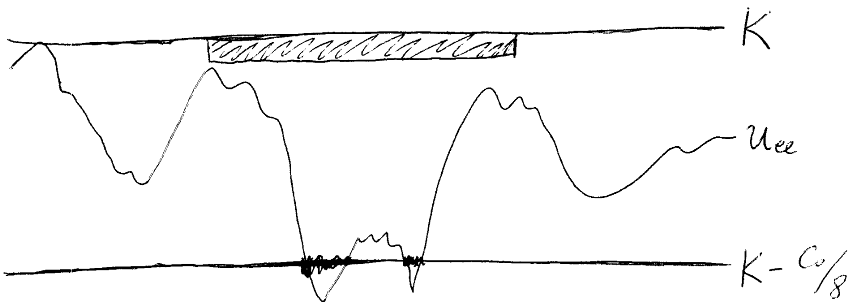
Fact that u satisfies elliptic eqn:

$$u_{ee}(x_i) \leq u_{ee}(x_j) - C_0 \| D^2u(x_j) - D^2u(x_i) \|$$
$$\leq K - \frac{C_0}{4} \quad (\text{uses } \operatorname{diam} D^2u(B_i) > 1)$$

$\Rightarrow$ for $\epsilon_0 < C_0/8$,

$$\frac{|\{u_{ee} < M - \frac{C_0}{8}\} \cap B_{1/4}|}{[\text{illegible}]} \geq \eta > 0 \text{ universal.}$$

Subsolution: Using concavity, we conclude the u_{ee} are subsolutions to some linear equation. The Harnack inequality implies that $u_{ee}|_{B_{1/2}} \leq K - \epsilon_0$ for some universal ϵ_0. By choosing $\epsilon \leq \epsilon_0/5$, we can remove the ball containing the maximum of u_{ee}.



Apply Krylov-Safonov for supersol'ns upside-down: (to $V \equiv K - u_{ee}$)

$$\eta \leqslant |\{V > c_0/8\} \cap B_{1/4}| \leqslant C \inf_{B_{1/2}} V$$

$\Longrightarrow u_{ee} \geqslant K - \varepsilon$ on $B_{1/2}$.

□

Some Philosophy: A rough way to think about the problem of finding $C^{2,\alpha}$ estimates for concave equations is the following. First recall that u_{ee} are subsolutions to the linearized equation. However, ellipticity from the original equation tells us roughly that we can write one u_{ii} as a negative sum of the other second derivatives, so that u_{ii} is both a subsolution and a supersolution. The Harnack inequality then gives oscillation decay.

Exercise: Was it really necessary that we had nontrivial measure of a preimage in $B_{1/4}$, or could we have taken a ball with universal nontrivial preimage in $B_{1-\delta}$? What happens to α as we take δ smaller?

Exercise: Suppose we have a collection of linear uniformly elliptic operators with constant coefficients:

$$L_k = a_k^{ij}\partial_i\partial_j.$$

Consider the Bellman equations

$$F(D^2u) = \inf_k L_k u = 0.$$

Show that this equation is concave and uniformly elliptic, and sketch the level set $F^{-1}(0)$ if we are working in 2 dimensions with the usual identification of $Sym(2)$ with $\mathbb{R}^3$. (Should be polygonal).

Exercise: Recall the weighted interior norms from Gilbarg-Trudinger:

$$[u]^*_{C^{k,\alpha}}(\Omega) = \sup_{x,y} d_{x,y}^{k+\alpha}\frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha}$$

where $d_{x,y}$ is the smaller distance of the distances from x and y to $\partial\Omega$. By rescaling the Evans-Krylov estimate show that

$$[u]^*_{C^{2,\alpha}(B_1)} \le C[u]^*_{C^{1,1}(B_1)}.$$

Recalling the interpolation inequality

$$[u]^*_{C^{1,1}(B_1)} \le C(\epsilon)\|u\|_{L^\infty(B_1)} + \epsilon[u]^*_{C^{2,\alpha}(B_1)},$$

show that

$$[u]*_{C^{2,\alpha}(B_1)} \le C\|u\|_{L^\infty(B_1)}.$$

Exercise: Read the first section of Chapter 9 and use the Bernstein technique to prove that for F smooth and concave, we have

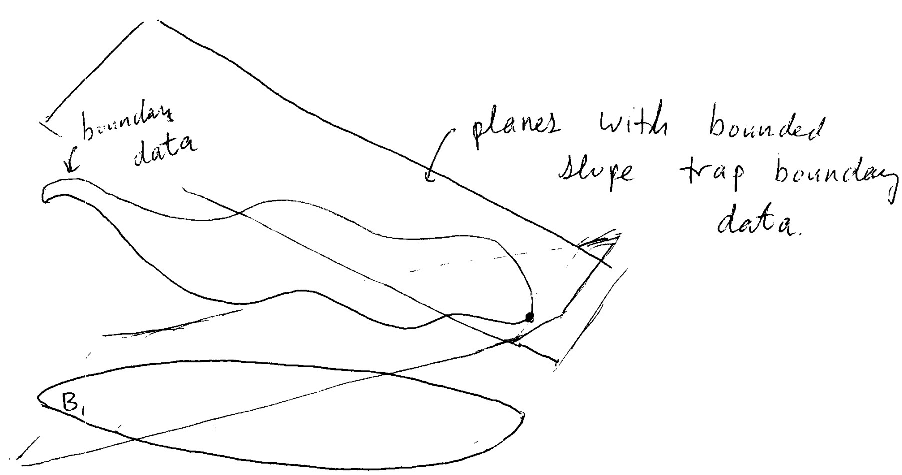
$$\|Du\|_{L^\infty(B_{1/2})}, \|D^2u\|_{L^\infty(B_{1/2})} \le C\|u\|_{L^\infty(B_1)}.$$

Roughly, for the Bernstein technique one cooks up a quantity which solves some elliptic PDE and applies the maximum principle. This is another technique to obtain the desired $C^{2,\alpha}$ apriori estimate.

11.2 Some Pictures for $C^{2,\alpha}$ up to the Boundary

Once we have $C^{2,\alpha}$ estimates up to the boundary (assuming nice boundary data) we can solve the Dirichlet problem via the method of continuity. In this last section we draw a few pictures giving intuition for how one obtains estimates up to the boundary. This is the "bread and butter" of elliptic PDE.

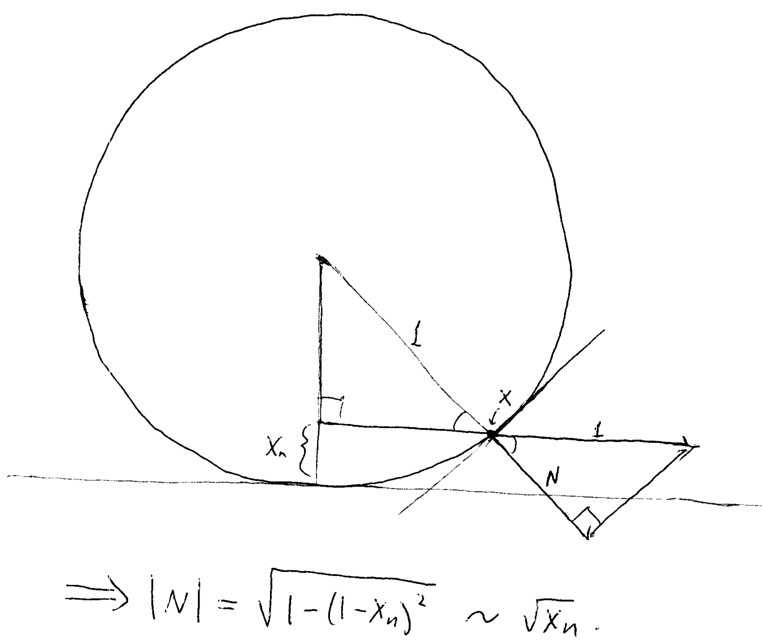
Gradient Estimate: The tangential derivatives are those of the boundary data. For the normal derivative, the intuition is that the boundary data is trapped by planes with slope controlled by the boundary data. Since u satisfies a linear elliptic PDE, we conclude using the maximum principle.



Second Derivative Estimate: The second tangential derivatives are controlled by derivatives along the boundary and the normal derivative, which we estimated above. The second normal derivative is controlled by the mixed partial normal and tangential derivatives by ellipticity, so we only need to estimate the mixed normal-tangential derivatives. Follow the same idea as above replacing u by a derivative of u along the boundary and trapping the derivative along the boundary by planes. However, the derivative along the boundary doesn't satisfy an obvious elliptic PDE. The key step is modifying by a large multiple of

$$\sum_{i=1}^{n-1} |u_i(x) - u_i(0)|^2,$$

which is a subsolution, and showing that we can still trap this by planes $\pm C x_n$ at 0. The idea is that we can decompose $\{u_1(x), ..., u_{n-1}(x)\}$ into its tangential and normal parts, at the boundary. The tangential part can be estimated by the boundary data while the normal part grows like $\sqrt{x_n}$.



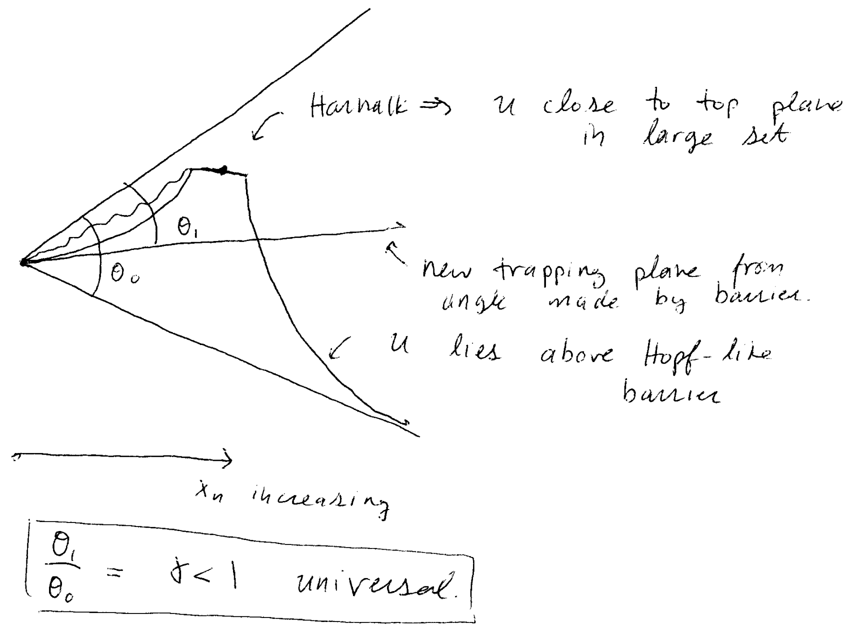
Hölder Estimate for Second Derivative at the Boundary: As an exercise reduce to the following "Boundary Harnack" estimate:

Theorem 16. *Suppose $a^{ij}(x)w_{ij} = 0$ in B_1^+ and $w = 0$ on $\Gamma = \{x_n = 0\}$ with a^{ij} uniformly elliptic but with no regularity assumptions. Then*

$$\|w_n\|_{C^\alpha(B_{1/2}(\mathbb{R}^{n-1})\times[0,\delta))} \leq C\|u\|_{L^\infty(B_1^+)}$$

for some α, δ, C universal.

The idea is the following: u is trapped by planes intersecting Γ. If u gets close to one of them, by the Harnack inequality it is close in a region around this point. We can then construct a Hopf-like barrier that intersects the other plane on Γ at a positive angle, showing that near Γ we have a better approximation of u by planes, and the angle improvement is universal. Iterating this procedure, we produce a sequence of planes with angle decreasing geometrically that trap u nearby Γ, which is $C^{1,\alpha}$ regularity in the x_n direction.



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