MINIMIZERS OF CONVEX FUNCTIONALS WITH SMALL DEGENERACY SET

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Abstract. We study the question whether Lipschitz minimizers of $\int F(\nabla u)\,dx$ in $\mathbb{R}^n$ are C^1 when F is strictly convex. Building on work of De Silva-Savin, we confirm the C^1 regularity when D^2F is positive and bounded away from finitely many points that lie in a 2-plane. We then construct a counterexample in $\mathbb{R}^4$, where F is strictly convex but D^2F degenerates on the intersection of a Simons cone with S^3. Finally we highlight a connection between the case $n = 3$ and a result of Alexandrov in classical differential geometry, and we make a conjecture about this case.

1. Introduction

In this paper we study the regularity of Lipschitz minimizers of

$$E(u) = \int_{B_1} F(\nabla u)\,dx \tag{1}$$

in $\mathbb{R}^n$, where $F : \mathbb{R}^n \to \mathbb{R}$ is convex. By Lipschitz minimizer we mean a function $u \in W^{1,\infty}(B_1)$ that satisfies $E(u+\varphi) \geq E(u)$ for all $\varphi \in C_0^1(B_1)$. It is straightforward to show that Lipschitz minimizers solve the Euler-Lagrange equation

$$\mathrm{div}(\nabla F(\nabla u)) = 0 \tag{2}$$

in the weak sense. Conversely, any Lipschitz weak solution of (2) is a minimizer of E by the convexity of F.

In the extreme case that the graph of F contains a line segment, minimizers are no better than Lipschitz by simple examples. In the other extreme that F is smooth and uniformly convex, De Giorgi and Nash proved that Lipschitz minimizers are smooth and solve the Euler-Lagrange equation $F_{ij}(\nabla u)u_{ij} = 0$ classically ([DG], [Na]). It remains largely open what happens in the intermediate case where F is strictly convex, but the eigenvalues of D^2F go to 0 or ∞ on some set D_F. Such functionals arise naturally in the study of anisotropic surface tensions [DMMN], traffic flow [CF], and statistical mechanics ([CKP], [KOS]).

In [DS] the authors raise the natural question:

(3) *Are Lipschitz minimizers in C^1 when F is strictly convex?*

They give evidence that the answer may be "yes," at least in two dimensions. In particular, they show that if $n = 2$ and D_F consists of finitely many points, then Lipschitz minimizers of E are C^1. In this paper we study this question in higher dimensions. We first confirm the C^1 regularity of Lipschitz minimizers when D_F is a finite set in some 2-plane. In particular, this covers the case that D_F consists of three points. We then show the answer to Question (3) is "no" in general, by constructing a singular Lipschitz minimizer in $\mathbb{R}^4$. In our example, F is in fact uniformly convex and C^1, but one eigenvalue of D^2F goes to ∞ on the intersection of a Simons cone with S^3. This leaves open the possibility that Lipschitz minimizers are C^1 in dimension $n \geq 3$ in the interesting case that D_F consists of finitely many points. To address this problem we connect it to a result of Alexandrov in the classical differential geometry of convex surfaces, and we propose a possible counterexample in $\mathbb{R}^3$ where D_F consists of four non-coplanar points.

Remark 1.1. Guided by the observation that the Legendre transform F^* of F solves $\text{div}(\nabla F(\nabla F^*)) = \text{div}(x) = n$, one could (more ambitiously) ask whether the minimizers are as regular as F^*. This is known in some special cases, e.g. for the p-Laplace case $F(x) = |x|^p$ when $p > 2$ and $n = 2$ (see [ATO] and also [IM], where it is shown that in this case minimizers are in fact *more* regular than F^*).

Remark 1.2. The case that D_F consists of a single point (e.g. p-Laplace) is well-studied (see [E], [Uh], [Ur]). The case that D_F is "large" is also understood: in [CF] the authors show that if D_F is convex and $F = 0$ on D_F, then for $x \in B_1$ the gradients $\nabla u(B_r(x))$ localize as $r \to 0$ either to a point outside D_F or to D_F.

Remark 1.3. One can show the existence of Lipschitz minimizers with additional hypotheses on the behavior of F at infinity. For example, if F has quadratic growth, then for $g \in H^1(B_1)$ the direct method gives the existence of a minimizer $u \in H^1(B_1)$ with $u - g \in H_0^1(B_1)$. If g is smooth enough ($C^{1,1}$ suffices) then u is Lipschitz by the comparison principle. Alternatively, if F is uniformly convex with bounded second derivatives at infinity, then u is locally Lipschitz. For a proof of this result, see [Ma2]. The local Lipschitz regularity of minimizers is in fact true under assumptions that allow for growth of D^2F at infinity (the so called $(p,\, q)$ growth conditions); see [Ma1], [Ma2], and the references therein. However, if the growth is sufficiently anisotropic, then there can be discontinuous and even unbounded minimizers (see [Gi], [Ma3]).

The paper is organized as follows. In Section 2 we give precise statements of our results, and we discuss a connection between the problem in dimension $n = 3$ and a result of Alexandrov. In Section 3 we prove the C^1 regularity result. In Section 4 we construct the counterexample. Finally, in the Appendix we record some technical results that we used to construct the counterexample.

2. Statements of Results

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^1 convex function, and let $D_F \subset \mathbb{R}^n$ be a compact set such that

$$F \in C^2(\mathbb{R}^n \backslash D_F), \quad D_F = \mathbb{R}^n \backslash \left(\cup_{k>1} \{k^{-1} I < D^2 F < kI\} \right).$$

Here and below, dependence on F means dependence on the sets

$$\mathcal{O}_k := \{k^{-1} I < D^2 F < kI\} \subset \mathbb{R}^n \backslash D_F$$

(in particular, the geometry of D_F), and on the moduli of continuity of $D^2 F$ in compact sets that exhaust $\mathbb{R}^n \backslash D_F$. Our first theorem is:

Theorem 2.1. *Let u be a Lipschitz solution of (2). If D_F is finite and is contained in a two-dimensional affine subspace of $\mathbb{R}^n$, then $u \in C^1(B_1)$, and the modulus of continuity of ∇u in $B_{1/2}$ depends only on on n, F, and $\|\nabla u\|_{L^\infty(B_1)}$.*

Remark 2.2. We conjecture that the assumption in Theorem 2.1 is optimal. That is, that there exists a singular minimizer in $\mathbb{R}^3$ where D_F consists of four non-coplanar points (see the discussion in Section 2.1).

The starting point of Theorem 2.1 is the well-known fact that convex functions of ∇u are sub-solutions to the linearized Euler-Lagrange equation. Using this fact we show that $\nabla u(B_r)$ localizes as $r \to 0$ either to a point outside D_F (in which case we are done), or to the convex hull of D_F. This was observed in [CF] in the case that D_F is a convex set and $F = 0$ on D_F, motivated by models of traffic congestion. The key observation in [DS] is that in two dimensions, certain slightly non-convex functions of ∇u are also sub-solutions to the linearized equation. If the convex hull of D_F is two-dimensional, we can use higher-dimensional versions of these functions to further localize the gradients to a point.

To state our second result we let $x = (x_1, x_2) \in \mathbb{R}^{2n}$ with $x_i \in \mathbb{R}^n$. We define

$$w(x) := \frac{1}{\sqrt{2}} \frac{|x_2|^2 - |x_1|^2}{|x|}. \tag{4}$$

Then w is a nontrivial one-homogeneous function on $\mathbb{R}^{2n}$ that is analytic outside of the origin. We show:

Theorem 2.3. *When $n \geq 2$, w is a minimizer of a functional of the form (1) with F uniformly convex and C^1, and $D_F = \{|x_1|^2 = |x_2|^2\} \cap \sqrt{2}\, S^{2n-1}$.*

Our approach to Theorem 2.3 is based on the observation that when $n \geq 2$, the gradient image $\Sigma_w := \nabla w(\mathbb{R}^{2n} \backslash \{0\}) = \nabla w(S^{2n-1})$ is a saddle-shaped hypersurface that is smooth away from a "cusp" singularity on $\{|x_1|^2 = |x_2|^2\} \cap \sqrt{2}\, S^{2n-1}$. This reflects that $D^2 w$ has positive and negative eigenvalues, and thus solves some elliptic equation. We then build the integrand F near Σ_w so that the Euler-Lagrange equation (2) is satisfied, and finally we make a global convex extension. In previous work with Savin we took a similar approach to construct singular minimizers of functionals with large degeneracy set in $\mathbb{R}^3$, where D_F consists of two disconnected convex sets with nonempty interior [MS].

2.1. **The Case $n = 3$ and Hyperbolic Hedgehogs.** To conclude the section we highlight a connection between our approach to Theorem 2.3 and classical differential geometry.

Natural candidates for singular minimizers are one-homogeneous functions with Hessians that have indefinite sign. Indeed, such functions are invariant under the rescalings that preserve (2), and they solve some elliptic PDE. It is useful to identify a one-homogeneous function u with its gradient image, a (possibly singular) hypersurface Σ_u. The function u is the support function of Σ_u, and the eigenvalues of $D^2 u$ on S^{n-1} are the principal radii of Σ_u. The set Σ_u is the parallel set a distance A in the direction of the *inward* unit normal from the convex body $\Sigma_{u+A|x|}$, where A is chosen large enough that $D^2 u + AI > 0$ on S^{n-1}. Such parallel surfaces to a convex body are known in the literature as "hedgehogs" (see e.g. [MM2]).

In dimension $n = 3$, a natural candidate for a singular minimizer thus corresponds to a hedgehog that is saddle-shaped away from its singularities, i.e. a parallel set a distance A in the inward direction from a convex surface with principal radii $r_1, r_2 > 0$ that satisfy $(r_1 - A)(r_2 - A) \leq 0$. Alexandrov originally conjectured that the only such convex surfaces in $\mathbb{R}^3$ are spheres. He proved his conjecture for analytic convex surfaces ([A1], [A2]). Thus, we cannot construct with our method a singular minimizer in $\mathbb{R}^3$ that is analytic outside of the origin (compare to Theorem 2.3). For C^2 surfaces, Alexandrov's conjecture remained open for a long time

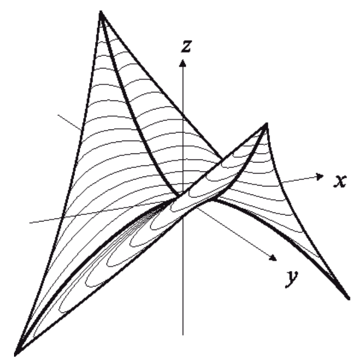

FIGURE 1. The hyperbolic hedgehog Σ_h from [MM1].

(with at least one incorrect proof). It was resolved in 2001 by a beautiful counterexample of Martinez-Maure ([MM1]). Martinez-Maure's hedgehog is built by gluing together four self-intersecting "cross caps" with figure-eight cross sections that shrink to cusps (see Figure 1). Motivated by this discussion and Theorem 2.1 we conjecture:

Conjecture 2.4. *The support function h of the hedgehog from* [MM1] *is a one-homogeneous singular minimizer of a functional of the type (1), where D_F consists exactly of the four cusps of Σ_h.*

This would show that the geometric conditions on D_F in Theorem 2.1 are optimal.

Remark 2.5. The surface Σ_h can be written as the union of two graphs, which makes writing the Euler-Lagrange equation on Σ_h relatively simple. Using this observation we can show that it is possible to construct F locally (in particular, in a small neighborhood of a cusp), with some tedious calculation. It seems challenging to construct F globally, but so far we do not see a fundamental obstruction. Our approach may benefit from calculations by a computer, as in [MM1].

Remark 2.6. The regularity of h in the example from ([MM1]) is C^2. Smooth counterexamples to Alexandrov's conjecture, with (a version of) Σ_h as a special case, were later constructed by Panina [P].

3. PROOF OF THEOREM 2.1

Choose M_0 large so that $D_F \subset B_{M_0}$, and let $M = \max\{M_0, \|u\|_{L^\infty(B_1)}\}$. By a standard approximation argument, to prove Theorem 2.1 it suffices to assume $u, F \in C^\infty$ and show that the modulus of continuity of ∇u in $B_{1/2}$ depends only on M, the sets $\{\mathcal{O}_k\}$, and the moduli of continuity of D^2F in the sets $\{B_M \cap \mathcal{O}_k\}$ (see e.g. [CF]).

3.1. **Preliminaries.** We record some important preliminary results. Our argument is based on applying the following estimate of De Giorgi (the "weak Harnack inequality") to various functions of ∇u:

Proposition 3.1. *Assume that $v \geq 0$ is in $H^1(B_1)$ and solves $\partial_i(a_{ij}(x)v_j) \geq 0$, with a_{ij} bounded measurable and $\lambda I \leq (a_{ij}) \leq \lambda^{-1}I$ for some $\lambda > 0$. Then for all $\mu > 0$, there exists $\nu(\mu, n, \lambda) > 0$ such that if*

$$\frac{|\{v > 0\} \cap B_1|}{|B_1|} \leq 1 - \mu$$

then

$$\sup_{B_{1/2}} v \leq (1 - \nu) \sup_{B_1} v.$$

To prove Proposition 3.1 apply the weak Harnack inequality for supersolutions (Theorem 8.18 in [GT]) to $\sup_{B_1} v - v$.

We now discuss the types of functions of ∇u that Proposition 3.1 applies to. We denote the linearized Euler-Lagrange operator by L_F. That is,

$$L_F(v) := \mathrm{div}(D^2F(\nabla u)\nabla v) = \partial_i(F_{ij}(\nabla u)v_j).$$

The key observation is that if η is slightly concave in only one direction, then $\eta(\nabla u)$ is a subsolution of L_F where ∇u avoids D_F. For $\Omega \subset \mathbb{R}^n$ let $\mathcal{N}_\delta(\Omega)$ denote the δ-neighborhood of Ω. We have:

Lemma 3.2. *Assume η is a smooth function in a neighborhood of $\nabla u(B_1)$. For any $\rho \in (0, 1)$, there exists $\lambda(\rho, F, M, n) > 0$ such that if $\nabla u(B_1) \cap \{\eta > 0\} \subset B_M \backslash \mathcal{N}_\rho(D_F)$, and in $\nabla u(B_1) \cap \{\eta > 0\}$ the eigenvalues $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_n$ of $D^2\eta$ satisfy $\gamma_2 > 0$ and $\gamma_1 \geq -\lambda\gamma_2$, then*

$$L_F(\eta_+(\nabla u)) \geq 0.$$

Here $\eta_+ := \max\{\eta, 0\}$.

Proof. Using that $L_F(u_k) = 0$ we compute

$$L_F(\eta(\nabla u)) = \mathrm{div}(D^2F(\nabla u)\nabla(\eta(\nabla u))) = F_{ij}u_{jk}\eta_{kl}u_{li}.$$

At a fixed point $x_0 \in \{\eta(\nabla u) > 0\}$ choose coordinates so that $\eta_{kl}(\nabla u(x_0)) = \gamma_k\delta_{kl}$. Summing over l we obtain

$$L_F(\eta(\nabla u))(x_0) = \sum_{k=1}^n \gamma_k \sum_{i,\,j=1}^n F_{ij}u_{ki}u_{kj}.$$

For some m large depending on ρ, F, M we have $\nabla u(B_1) \cap \{\eta > 0\} \subset \mathcal{O}_m$. Since $\gamma_1 \geq -\lambda\gamma_2$ in $\{\eta(\nabla u) > 0\}$ we conclude that

$$\gamma_2^{-1}L_F(\eta(\nabla u))(x_0) \geq m^{-1}\sum_{k=2}^n |\nabla u_k(x_0)|^2 - m\,\lambda|\nabla u_1(x_0)|^2.$$

If $L_F(\eta(\nabla u))(x_0) < 0$ then the above inequality gives

$$(\lambda^{-1}m^{-2} - 1)\sum_{(i,\,j)\neq(1,\,1)} u_{ij}^2(x_0) < u_{11}^2(x_0).$$

On the other hand, the equation $F_{11}u_{11} = -F_{ij}u_{ij}|_{(i,\,j)\neq(1,\,1)}$ gives

$$u_{11}^2 \leq C(n,\,m) \sum_{(i,\,j)\neq(1,\,1)} u_{ij}^2$$

in $\{\eta(\nabla u) > 0\}$. The previous two inequalities contradict each other for $\lambda(n,\,m)$ small. □

In order to apply Proposition 3.1 and Lemma 3.2, we need ∇u to be close to D_F in sets of positive measure. The alternative is that u nearly solves a non-degenerate equation. To handle this situation we will also use a "flatness implies regularity" result for u:

Proposition 3.3. *Assume that a_{ij} are smooth elliptic coefficients on $\mathbb{R}^n$ that satisfy $\lambda I \leq (a_{ij}) \leq \lambda^{-1} I$ in $B_\rho(p)$ for some fixed $\lambda,\,\rho > 0,\,p \in \mathbb{R}^n$. There exists $\epsilon > 0$ depending on $\rho,\,n,\,\lambda$ and the modulus of continuity of a_{ij} in $B_\rho(p)$ such that if $v \in C^\infty(B_1)$ solves $a_{ij}(\nabla v)v_{ij} = 0$ and*

$$\|v - l_p\|_{L^\infty(B_1)} \leq \epsilon$$

for some linear function l_p with $\nabla l_p = p$, then

$$\nabla v(B_{1/2}) \subset B_\rho(p).$$

Heuristically, $w := \epsilon^{-1}(v - l_p)$ solves $a_{ij}(p + \epsilon\nabla w)w_{ij} = 0$ which is nearly a constant-coefficient equation for ϵ small. The idea of Proposition 3.3 is due to Savin [S], who treated equations with degeneracy in the Hessian of v. For a proof of the proposition as stated (with degeneracy in the gradient of v) see [CF].

An easy consequence of Proposition 3.3 is:

Lemma 3.4. *For any $\rho \in (0,\,1)$ there exist $\epsilon_1,\,\mu_1(\rho,\,F,\,M,\,n) > 0$ such that if*

$$\frac{|\{\nabla u \in B_{\epsilon_1}(p)\} \cap B_r|}{|B_r|} \geq 1 - \mu_1$$

for some $p \in B_{2M} \backslash \mathcal{N}_{2\rho}(D_F)$ then

$$\nabla u(B_{r/2}) \subset B_\rho(p).$$

Proof. After taking $u \to r^{-1}u(rx)$ we may assume that $r = 1$. Since u solves $F_{ij}(\nabla u)u_{ij} = 0$, by Proposition 3.3 there exists $\epsilon_0 > 0$ depending on $\rho,\,F,\,M,\,n$ such that if

$$\|u - l_p\|_{L^\infty(B_1)} \leq \epsilon_0,$$

for some linear function l_p with $\nabla l_p = p$, then $\nabla u(B_{1/2}) \subset B_\rho(p)$. The above inequality holds by standard embeddings if we take e.g. $\epsilon_1 < c(n)\epsilon_0$ and take μ_1 small depending on $M,\,n,\,\epsilon_0$. □

Our approach to Theorem 2.1 is to first show that as $r \to 0$, the sets $\nabla u(B_r)$ localize to the convex hull of D_F, and then to show that if this set is two-dimensional, they localize to a point. We treat these two results separately in the following subsections, and then combine them.

3.2. **Localization to the Convex Hull.** Let K_F denote the convex hull of D_F. In this subsection we show:

Proposition 3.5. *For any $\rho > 0$, there exists $s(\rho, F, M, n) > 0$ such that either $\nabla u(B_s) \subset B_\rho(p)$ for some $p \in B_{2M} \backslash \mathcal{N}_{2\rho}(D_F)$, or $\nabla u(B_s) \subset \mathcal{N}_{4\rho}(K_F)$.*

In this subsection we call a constant universal if it depends only on ρ, M, F, n. Let β be a smooth uniformly convex function on $\mathbb{R}^n$ such that

$$\mathcal{N}_{3\rho}(K_F) \subset \{\beta \leq 0\} \subset \mathcal{N}_{4\rho}(K_F),$$

with $B_{2M} \subset \{\beta < \tilde{M}\}$ for some universal $\tilde{M}$. Let ϵ_1, $\mu_1 > 0$ be the universal constants from Lemma 3.4, corresponding to ρ.

Lemma 3.6. *There exists $\delta > 0$ universal such that if $\sup_{B_r} \beta(\nabla u) > 0$ and*

$$\frac{|\{\nabla u \in B_{\epsilon_1}(p)\} \cap B_r|}{|B_r|} < 1 - \mu_1$$

for all $p \in B_{2M} \backslash \mathcal{N}_{2\rho}(D_F)$, then

$$\sup_{B_{r/2}} \beta(\nabla u) \leq \sup_{B_r} \beta(\nabla u) - \delta.$$

Proof. After taking $u \to r^{-1}u(rx)$ we may assume that $r = 1$. Let

$$0 < t := \sup_{B_1} \beta(\nabla u) \leq \tilde{M}.$$

For any unit vector e let $s_e := \sup_{\{\beta < t\}}(p \cdot e)$. There is some universal $\delta_0 > 0$ (independent of e) such that for some $c_e \leq s_e - \delta_0$ we have

$$\mathrm{diam}(\{p \cdot e \geq c_e\} \cap \{\beta < t\}) < \epsilon_1$$

and that D^2F has universal ellipticity constant in $\{p \cdot e > c_e\}$. We denote the function $\nabla u \cdot e$ by u_e, and we note that $u_e \leq s_e$ in B_1 by the definition of s_e. By the hypotheses we may apply Proposition 3.1 to $v := (u_e - c_e)_+$ with $\mu = \mu_1$ to conclude that

$$u_e \leq (1 - \nu_1)s_e + \nu_1 c_e \leq s_e - \nu_1 \delta_0$$

in $B_{1/2}$, with $\nu_1 > 0$ universal. (Here we use that the coefficients of L_F have universal ellipticity constant in $\{v > 0\}$; we can replace the coefficients by e.g. δ_{ij} in $\{v = 0\}$ without changing the equation for v). Since

$$\cap_{e \in S^{n-1}} \{p \cdot e \leq s_e - \nu_1 \delta_0\} \subset \{\beta < t - \delta\}$$

for some universal $\delta > 0$ the proof is complete. □

Proof of Proposition 3.5. Apply the following algorithm for $k \geq 0$: if one of the hypotheses of Lemma 3.6 is not satisfied in $B_{2^{-k}}$, then stop. We either have $\nabla u(B_{2^{-k}}) \subset \{\beta \leq 0\} \subset \mathcal{N}_{4\rho}(K_F)$, or we can apply Lemma 3.4 to conclude $\nabla u(B_{2^{-k-1}}) \subset B_\rho(p)$ for some $p \in B_{2M} \backslash \mathcal{N}_{2\rho}(D_F)$. Otherwise, we apply Lemma 3.6. The algorithm terminates after at most k_0 steps with $\tilde{M} - k_0\delta \leq 0$. □

3.3. **Localization Beyond the Convex Hull.** In this subsection we show that if $\nabla u(B_1)$ is sufficiently close to a two-dimensional affine subspace, then as $r \to 0$ the gradients $\nabla u(B_r)$ localize to a connected component of D_F.

Let $(p,\, q) \in \mathbb{R}^n$ with $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^{n-2}$, and assume that $D_F \subset \{q = 0\}$. Let $8\rho_0$ be the smallest distance between a pair of points in D_F. In this subsection we call constants depending on ρ_0, F, M, n universal.

Proposition 3.7. *There exist $\sigma_0,\, r_0 > 0$ universal such that if $\nabla u(B_1) \subset \{|q| < \sigma_0\}$ then either $\nabla u(B_{r_0}) \subset B_{\rho_0}(p)$ for some $p \in B_{2M}\backslash \mathcal{N}_{2\rho_0}(D_F)$ or $\nabla u(B_{r_0}) \subset \mathcal{N}_{3\rho_0}(D_F)$.*

In particular, since ∇u is (qualitatively) continuous the set $\nabla u(B_{r_0})$ is connected, so it is contained in a ball with at most one point of D_F, and is a distance at least ρ_0 from the remaining points in D_F.

The idea is to localize the gradients using the level sets of non-convex functions of ∇u. Let $\epsilon_1,\, \mu_1 > 0$ be the (universal) constants from Lemma 3.4 with $\rho = \rho_0$. We assume by taking ϵ_1 smaller if necessary that $\epsilon_1 \leq \rho_0$. Let λ_1 be the constant from Lemma 3.2 with $\rho = \rho_0$. Finally, let ν_1 be the constant from Proposition 3.1 corresponding to $\mu = \mu_1$ and the ellipticity constants of D^2F in $B_{2M}\backslash \mathcal{N}_{\rho_0}(D_F)$. The following lemma says that when the gradient image is sufficiently close to $\{q = 0\}$, we can "chop" at its projection to $\{q = 0\}$ with circles (see Figure 2):

Lemma 3.8. *Let $(p_0,\, 0) \in B_{2M}\backslash \mathcal{N}_{2\rho_0}(D_F)$. There exist $\sigma_0,\, \delta_0 > 0$ universal such that if $\nabla u(B_r) \subset \{|q| < \sigma_0\} \cap \{|p - p_0| \geq \epsilon_1/4\}$ and*

$$\frac{|\{\nabla u \in B_{\epsilon_1}(p_0,\, 0)\} \cap B_r|}{|B_r|} < 1 - \mu_1,$$

then

$$\nabla u(B_{r/2}) \subset \{|p - p_0| \geq \epsilon_1/4 + \delta_0\}.$$

Proof. We may assume that $r = 1$ after a Lipschitz rescaling. Define

$$\eta_A(p,\, q) := e^{A^2|q|^2/2 - A|p|}, \quad \eta_{A,\, p_0} := \eta_A(p - p_0,\, q) - e^{-A\epsilon_1/2}.$$

In an appropriate system of coordinates we have in $\{|q| < A^{-3}\}$ that

$$(A^2\eta_A)^{-1} D^2\eta_A = \mathrm{diag}(-(A|p|)^{-1},\, 1,\, ...,\, 1) + O(A^{-2}),$$

and that $\{\eta_A > e^{-A\epsilon_1/2}\} \subset \{|p| < \epsilon_1/2 + A^{-5}\}$. Then by our first hypothesis, for A large universal and $\sigma_0 < A^{-3}$ we have $\nabla u(B_1) \cap \{\eta_{A,\, p_0} > 0\} \subset B_M \cap B_{\epsilon_1}(p_0,\, 0) \subset B_M\backslash \mathcal{N}_{\rho_0}(D_F)$, and that the eigenvalues $\gamma_1 \leq ... \leq \gamma_n$ of $D^2\eta_{A,\, p_0}$ satisfy $\gamma_2 > 0$ and $\gamma_1 > -\lambda_1\gamma_2$ in $\nabla u(B_1) \cap \{\eta_{A,\, p_0} > 0\}$. We conclude using Lemma 3.2 that the function $v_{p_0} := (\eta_{A,\, p_0})_+(\nabla u)$ satisfies $L_F(v_{p_0}) \geq 0$. By our second hypothesis we can apply Proposition 3.1 to v_{p_0}. In the extreme case that $\sigma_0 = 0$, Proposition 3.1 gives

$$\nabla u(B_{1/2}) \subset \{\eta_{A,\, p_0} < (1 - \nu_1)(e^{-A\epsilon_1/4} - e^{-A\epsilon_1/2})\} \subset \{|p - p_0| \geq \epsilon_1/4 + 2\delta_0\}$$

for some $\delta_0 > 0$ universal. By continuity we have the same inclusion with $2\delta_0$ replaced by δ_0 for sufficiently small $\sigma_0 < A^{-3}$, completing the proof. □

We can now prove Proposition 3.7.

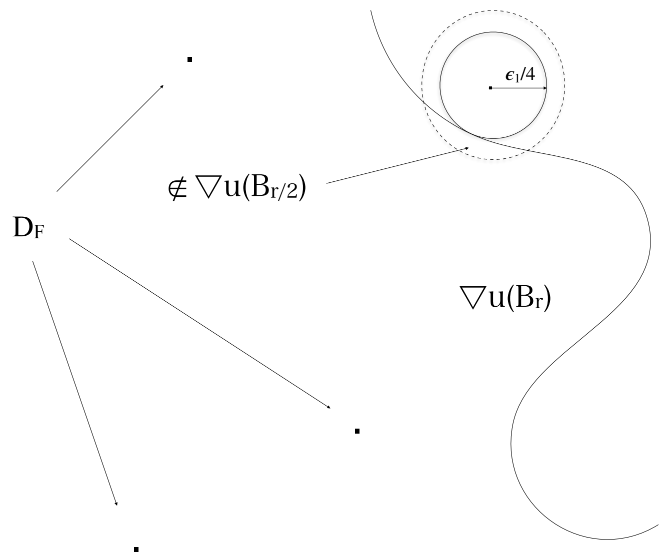

FIGURE 2. If $\nabla u(B_r)$ is "nearly $2D$" we can chop its projection to $\{q = 0\}$ with circles. The picture illustrates the case $n = 2$.

Proof of Proposition 3.7: Take σ_0 as in Lemma 3.8, and apply the following algorithm for $k \geq 0$: If

$$\frac{|\{\nabla u \in B_{\epsilon_1}(p_0, 0)\} \cap B_{2^{-k}}|}{|B_{2^{-k}}|} \geq 1 - \mu_1$$

for some $(p_0, 0) \in B_{2M} \backslash \mathcal{N}_{2\rho_0}(D_F)$, then stop. We have by Lemma 3.4 that $\nabla u(B_{2^{-k-1}}) \subset B_{\rho_0}(p_0, 0)$. If not, apply Lemma 3.8 to conclude that

$$\nabla u(B_{2^{-k-1}}) \subset \{|p - p_0| \geq \epsilon_1/4 + \delta_0\}$$

for all $(p_0, 0) \in B_{2M} \backslash \mathcal{N}_{2\rho_0}(D_F)$ such that $\nabla u(B_{2^{-k}}) \subset \{|p - p_0| \geq \epsilon_1/4\}$. If at this point we can conclude that $\nabla u(B_{2^{-k-1}}) \subset \mathcal{N}_{3\rho_0}(D_F)$, then stop.

To show that this algorithm terminates after a universal number of steps, we use a simple covering argument. Let $\mathcal{S}_k$ be the projections of the sets $\nabla u(B_{2^{-k}})$ to $\{q = 0\}$. Take a finite number of lines $\{l_i\}$ in $\{q = 0\}$ that avoid $\mathcal{N}_{2\rho_0}(D_F)$, whose $\epsilon_1/4$ neighborhoods cover $B_{2M} \backslash \mathcal{N}_{2\rho_0}(D_F) \cap \{q = 0\}$. For each l_i take a universal number $J + 1$ of two-dimensional balls $\{B_{ij} := B_{\epsilon_1/4}(p_{ij})\}_{j=1}^{J+1}$ in $\{q = 0\}$ that cover $B_{2M} \cap l_i \cap \{q = 0\}$, with centers $p_{ij} \in B_{2M} \cap l_i \cap \{q = 0\}$ on l_i such that $|p_{i,\,j+1} - p_{ij}| \leq \delta_0$ for $j = 1, ..., J$. Since $\mathcal{S}_0 \subset B_M$ we can arrange that $B_{i1} \cap \mathcal{S}_0 = \emptyset$ for all i. By induction, if the algorithm doesn't terminate after k steps, then $\mathcal{S}_k$ has empty intersection with the convex hull of B_{i1} and $B_{i,\,k+1}$, for each i. In particular, after J steps we have that $\mathcal{S}_J \subset \mathcal{N}_{2\rho_0}(D_F)$, and the proof is complete up to replacing σ_0 with $\min\{\sigma_0, \rho_0\}$. □

3.4. **Proof of Theorem 2.1.** We are now in position to prove Theorem 2.1. We call constants depending on F, M, n universal.

Proof. For any $\epsilon > 0$ we will show that there is some $\delta(\epsilon, F, M, n) > 0$ such that $\nabla u(B_\delta)$ is contained in a ball of radius ϵ.

Take σ_0 to be the constant from Proposition 3.7. Applying Proposition 3.5 with $\rho = \sigma_0/4$ we obtain $s_0 > 0$ universal such that either $\nabla u(B_{s_0}) \subset B_{\sigma_0/4}(p)$ for some $p \in B_{2M} \backslash \mathcal{N}_{\sigma_0/2}(D_F)$, or $\nabla u(B_{s_0}) \subset \mathcal{N}_{\sigma_0}(K_F)$. In the latter case, apply Proposition 3.7 to $s_0^{-1}u(s_0 x)$ to conclude for some $r_0 > 0$ universal that either $\nabla u(B_{r_0 s_0}) \subset B_{\rho_0}(p)$ for some $p \in B_{2M} \backslash \mathcal{N}_{2\rho_0}(D_F)$ or $\nabla u(B_{r_0 s_0}) \subset \mathcal{N}_{3\rho_0}(D_F)$.

In all cases, $\nabla u(B_{r_0 s_0})$ is contained in a ball $\mathcal{B}$ that has at most one point of D_F and is a positive universal distance from the remaining points of D_F. Thus, after restricting our attention to $\tilde{u} = (r_0 s_0)^{-1} u(r_0 s_0 x)$ we may assume that D_F contains at most one point (indeed, we can modify F outside of $\mathcal{B}$ without changing that $\tilde{u}$ is a minimizer). Applying Proposition 3.5 to $\tilde{u}$ with $\rho = \epsilon/4$ completes the proof. □

4. Proof of Theorem 2.3

In this section we construct the examples from Theorem 2.3. Here and below we let $k \geq 1$, and $q = (q_1, q_2)$, $y = (y_1, y_2) \in \mathbb{R}^{2k+2}$ with $q_i, y_i \in \mathbb{R}^{k+1}$, $i = 1, 2$. We will reduce the problem to making a certain one-dimensional construction using the symmetries of w.

4.1. Reduction to Two Dimensions.

We first reduce Theorem 2.3 to a problem in two dimensions. Let v be the one-homogeneous function on $\mathbb{R}^2$ given by

$$v(x_1, x_2) := \frac{1}{\sqrt{2}} \frac{x_2^2 - x_1^2}{|x|}.$$

We claim it suffices to construct a C^1, uniformly convex function $G(p_1, p_2)$ on $\mathbb{R}^2$ that is smooth away from $D_G = \sqrt{2}\, S^1 \cap \{p_1^2 = p_2^2\}$, such that G is invariant under reflection over the axes and over the lines $\{p_1 = \pm p_2\}$ (that is, $G(p_1, p_2) = G(-p_1, p_2) = G(p_2, p_1)$) and furthermore

$$\mathrm{tr}(D^2 G(\nabla v)\, D^2 v) + k\, \nabla G(\nabla v) \cdot \left(\frac{1}{x_1}, \frac{1}{x_2} \right) = 0 \tag{5}$$

for x in the positive quadrant. Indeed, if we manage to do this, note that by the symmetries of G and v, each term on the left is smooth away from $\{x_1^2 = x_2^2\}$, where ∇v maps to D_G. If we then take $F(q) = G(|q_1|, |q_2|)$ we obtain a C^1, uniformly convex function on $\mathbb{R}^{2k+2}$ that is smooth away from $D_F = \sqrt{2}\, S^{2k+1} \cap \{|q_1|^2 = |q_2|^2\}$. Using that $w(y) = v(|y_1|, |y_2|)$ we compute

$$\begin{aligned} \mathrm{tr}(D^2 F(\nabla w)\, D^2 w)(y) &= \mathrm{tr}(D^2 G(\nabla v)\, D^2 v)(|y_1|, |y_2|) \\ &\quad + k\, \nabla G(\nabla v(|y_1|, |y_2|)) \cdot \left(\frac{1}{|y_1|}, \frac{1}{|y_2|} \right) \\ &= 0 \end{aligned}$$

classically away from the cone $\{|y_1|^2 = |y_2|^2\}$. Here we used that $v_1 < 0$ and $v_2 > 0$ in the positive quadrant. It is not hard to show that the equation $\mathrm{div}(\nabla F(\nabla w)) = 0$ holds in the weak sense in B_1 by integrating away from a thin cone containing $\{|y_1|^2 = |y_2|^2\}$ and a small ball around the origin, using the C^1 regularity of F and the one-homogeneity of w, and taking a limit.

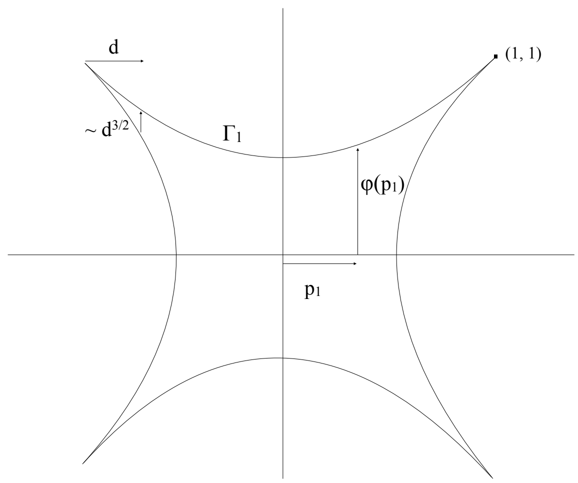

FIGURE 3. The set Σ_v consists of four congruent curves separating from the lines $p_2 = \pm p_1$ like $\text{dist}^{3/2}$.

4.2. **Reduction to One Dimension.** We now use that $\Sigma_v := \nabla v(S^1)$ is one-dimensional and an extension lemma to reduce our problem to one dimension. The set $\nabla v(S^1 \cap \{x_2 \geq |x_1|\})$ can be written as a graph $\Gamma_1 := \{(p_1, \varphi(p_1))\}$ with $p_1 \in [-1, 1]$, where $\varphi \in C^\infty(-1, 1) \cap C^1([-1, 1])$ is even, uniformly convex, and separates from the lines $p_2 = \pm p_1$ like $\text{dist.}^{3/2}$ at the endpoints. See the Appendix for a justification of these properties, as well as an expansion of φ near the endpoints. The set Σ_v consists of four rotations of Γ_1 by $\pi/2$ (see Figure 3). Let $S := \sqrt{2}\, S^1 \cap \{p_1^2 = p_2^2\}$, and let $\Sigma_0 := \Sigma_v \backslash S$, $\Gamma_0 := \Gamma_1 \backslash S$. We will use the following important extension lemma:

Lemma 4.1. *Assume that $g : \Sigma_v \to \mathbb{R}$ and $\mathbf{v} : \Sigma_v \to \mathbb{R}^2$ are smooth on Σ_0 and continuous on Σ_v, and satisfy the condition*

$$g(\tilde{p}) - g(p) - \mathbf{v}(p) \cdot (\tilde{p} - p) \geq \gamma |\tilde{p} - p|^2 \tag{6}$$

for some $\gamma > 0$ and all $\tilde{p}, p \in \Sigma_v$. Then there exists a C^1, uniformly convex function G on $\mathbb{R}^2$ with $G \in C^\infty(\mathbb{R}^2 \backslash S)$, such that $G = g$ and $\nabla G = \mathbf{v}$ on Σ_v.

We delay the proof of Lemma 4.1 to the Appendix, and proceed with the reduction. We claim that it suffices to construct even functions $f, h \in C^\infty(-1, 1) \cap C^1([-1, 1])$ such that

$$h = \frac{1}{1 + 2k} \left(\frac{f''}{\varphi''} + k \frac{f'}{\varphi'} \right) \tag{7}$$

on $(-1, 1)$, and furthermore the pair $(g, \mathbf{v})$ defined on Σ_v by

$$g(s, \varphi(s)) := f(s), \quad \mathbf{v}(s, \varphi(s)) := (f'(s) - h(s)\varphi'(s), h(s)) \tag{8}$$

on Γ_1 and extended by reflection over the lines $p_2 = \pm p_1$, satisfies the conditions of Lemma 4.1 for some $\gamma > 0$. Indeed, if this is accomplished, then the extension

G from Lemma 4.1 satisfies the Euler-Lagrange equation (5). To see this, let ν be the upward unit normal to Γ_0. By the one-homogeneity of v we have

$$x = \nu(\nabla v(x)) = \frac{(-\varphi',\, 1)}{\sqrt{1+\varphi'^2}}$$

for $x \in S^1 \cap \{x_2 > |x_1|\}$. Differentiating in x gives

$$D^2 v(x) = \frac{1}{\kappa}\tau \otimes \tau = -\frac{(1+\varphi'^2)^{3/2}}{\varphi''}\tau \otimes \tau$$

where τ is the unit tangent vector to Γ_0 at $\nabla v(x)$ (also the unit tangent to S^1 at x) and κ is the (signed) curvature of Γ_0. Finally, by (8) we have

$$G(s,\, \varphi(s)) = f(s), \quad (\nabla G \cdot e_2)(s,\, \varphi(s)) = h(s) \tag{9}$$

on Γ_0. Differentiating the first relation in (9) twice we obtain

$$\nabla G = (f' - h\varphi',\, h), \quad \tau^T \cdot D^2 G \cdot \tau = \frac{f'' - h\varphi''}{1+\varphi'^2}$$

on Γ_0. Putting these together gives the equivalence of (5) and (7). Finally, since $(G,\, \nabla G)$ has the desired symmetries on Σ_v, we can arrange that G has the desired symmetries globally by taking the average of its reflections.

Remark 4.2. One can compute the Euler-Lagrange equation (7) directly in $\mathbb{R}^{2k+2}$ using the geometry of $\Sigma_w := \nabla w(S^{2k+1})$, without much trouble. Using the one-homogeneity of w we see as above that the equation reduces to $\mathrm{tr}((II)^{-1}(q)\, D_T^2 F(q)) = 0$ on Σ_w, where II is the second fundamental form of Σ_w and T is the tangent hyperplane to Σ_w. Since Σ_w is obtained by taking rotations of Γ_1, it is tangent on one side to second order to a sphere of radius $\varphi\sqrt{1+\varphi'^2}$ and on the other side to second order to a sphere of radius $\frac{s}{\varphi'}\sqrt{1+\varphi'^2}$. It thus has one principal curvature $-\varphi''\,(1+\varphi'^2)^{-3/2}$, k principal curvatures $\frac{1}{\varphi\,\sqrt{1+\varphi'^2}}$ from rotating around the p_1 axis, and k principal curvatures $-\frac{\varphi'}{s\sqrt{1+\varphi'^2}}$ from rotating around the p_2 axis. The eigenvalues of D^2F corresponding to these directions are $\frac{f''-h\varphi''}{1+\varphi'^2}$, $\frac{h}{\varphi}$ and $\frac{f'-h\varphi'}{s}$, where the latter two come from rotations. Putting these together in the original Euler-Lagrange equation $\mathrm{tr}((II)^{-1}(q)\, D_T^2 F(q)) = 0$ we recover (7).

4.3. **The One-Dimensional Construction.** We now construct f and h. The Euler-Lagrange equation (7) determines h through our choice of f, so there is only one function f to construct. It is convenient to do this by choosing a positive even function $\eta \in C^\infty(-1,\, 1) \cap C([-1,\, 1])$, and then defining f by the relation

$$f''(s) := \eta(s)\varphi''(s),\ f'(0) = 0. \tag{10}$$

Fix $0 < \delta << 1$. We choose η satisfying the following conditions:

(i) η is even, concave, and $\eta(1) = 1/2$,
(ii) $\eta \geq \min\left\{1,\, \frac{1}{2}(1 + \delta^{-1/2}(1-s)^{1/2})\right\}$,
(iii) $\eta \equiv 1 + \mu$ on $[0,\, 1-\delta]$ for some $\mu > 0$,
(iv) $\int_0^1 \eta(s)\varphi''(s)\, ds = 1$.

We observe that by the first condition, the function η is decreasing on $(0,\, 1)$. Furthermore, since $\eta > 0$ we have by (10) that f is convex, and the last condition on

η implies that $f'(1) = 1$. Since $\int_0^1 \varphi''(s)\,ds = 1$, it is clear that $\mu \to 0$ as we take $\delta \to 0$. In particular, $f' \to \varphi'$ and $h \to \frac{1+k}{1+2k}$ as $\delta \to 0$. We will show below that for any such choice of η with δ sufficiently small, the pair $(g, \mathbf{v})$ defined by (8) satisfies the hypotheses of Lemma 4.1.

Continuity Condition: The condition that $\mathbf{v}$ is continuous on Σ_v and invariant under reflection over the diagonal is that

$$(f' - h\varphi',\, h)(1) \text{ is parallel to } (1,\, 1). \tag{11}$$

This follows from (7) and (10), using that $\eta(1) = 1/2$ and $f'(1) = \varphi'(1) = 1$.

Convexity Condition Along Top Graph. We check the convexity condition (6) on $\Gamma_1 \cap \{p_1 \geq 0\}$. Take $\tilde{p} = (y,\, \varphi(y))$ and $p = (x,\, \varphi(x))$ for some $x,\, y \in [0,\, 1]$. The quantity of interest is

$$\begin{aligned} g(y,\, \varphi(y)) &- g(x,\, \varphi(x)) - \mathbf{v}(x,\, \varphi(x)) \cdot (y - x,\, \varphi(y) - \varphi(x)) \\ &= [f(y) - f(x) - f'(x)(y - x)] - h(x)[\varphi(y) - \varphi(x) - \varphi'(x)(y - x)] \\ &= \int_x^y f''(x)(y - s)\,ds - h(x) \int_x^y \varphi''(s)(y - s)\,ds \\ &= \left(\int_x^y \varphi''(s)(y - s)\,ds \right) H(x,\, y), \end{aligned}$$

where

$$H(x,\, y) = \int_x^y (\eta - 1/2)(s) d\mu_y(s) - \frac{1}{1 + 2k}(\eta - 1/2)(x) - \frac{k}{1 + 2k} \frac{f' - \varphi'}{\varphi'}(x),$$

and $d\mu_y$ is the probability density

$$d\mu_y := \frac{\varphi''(s)(y - s)\,ds}{\int_x^y \varphi''(t)(y - t)\,dt}$$

on the interval from x to y. We claim that

$$H(x,\, y) \geq c_0 \max\{(1 - x)^{1/2},\, (1 - y)^{1/2}\} \tag{12}$$

for some $c_0 > 0$ independent of $x,\, y \in [0,\, 1]$. The desired inequality (6) then follows because

$$\frac{1}{(y - x)^2} \int_x^y \varphi''(s)(y - s)\,ds \geq c \min\{(1 - x)^{-1/2},\, (1 - y)^{-1/2}\}$$

for some $c > 0$, using that φ'' is positive, increasing on $[0,\, 1)$, and has the expansion $\varphi''(s) = \sqrt{\frac{2}{3}}(1 - s)^{-1/2} + O(1)$ near $s = 1$ (see Appendix). To show (12) we check several cases. Below, c_1, c_2 always denote universal positive constants which may change from line to line.

In the case that $x,\, y \leq 1 - \delta$ it is obvious that $H(x,\, y) \geq c_0 > 0$ since $\eta - 1/2 > 1/2$ is constant on $[0,\, 1 - \delta]$ and $f' - \varphi' = \mu\varphi'$ with μ small.

The next simplest case is that $y < x$ and $x > 1 - \delta$. Using that η is decreasing, that $f' \leq 1$, and the expansion of $\varphi'(s)$ near $s = 1$ (see Appendix) we have

$$H(x,\, y) \geq \frac{2}{3} \int_x^y (\eta - 1/2)(s) d\mu_y(s) - c_1\,(1 - x)^{1/2}.$$

It is elementary to check that the mass of $d\mu_y$ in the left half of the interval $[y, x]$ is at least $c_2 > 0$ independent of y using the properties of φ. Since η is decreasing and concave we conclude that

$$H(x, y) \geq \frac{1}{3}c_2\,(\eta - 1/2)(y) - c_1\,(1-x)^{1/2} \geq c_0(1-y)^{1/2}$$

using definition of η and that δ is small.

The most delicate case is that $x < y$ and $y > 1-\delta$. By the expansion of $\varphi(s)$ near $s = 1$ we may choose δ so small that $d\mu_y$ is decreasing on $[1-2\delta, y]$, independent of $y > 1-\delta$ (see Appendix). We first claim that

$$\int_x^y (\eta - 1/2)(s)d\mu_y(s) \geq \frac{1}{2}(\eta - 1/2)(x).$$

If $x < 1-2\delta$ then since most of the weight of $d\mu_y$ on $[x, y]$ is to the left of $1-\delta$ the inequality is obvious. When $x \geq 1-2\delta$, since both the weight $d\mu_y$ and η are decreasing on $[x, y]$, the average of $\eta - 1/2$ decreases if we redistribute the weight $d\mu_y$ evenly, and the concavity of η gives the result. We conclude that

$$H(x, y) \geq \frac{1}{6}(\eta - 1/2)(x) - \frac{f' - \varphi'}{2\varphi'}(x).$$

If $x \leq 1-\delta$ we have $f' - \varphi' = \mu\varphi'$ and $\eta - 1/2 > 1/2$ so $H(x, y) \geq c_0 > 0$. If $x > 1-\delta$ we argue as in the case $y < x$ that

$$H(x, y) \geq \frac{1}{12}\delta^{-1/2}(1-x)^{1/2} - c_1(1-x)^{1/2} > (1-x)^{1/2}$$

for δ small, and inequality (12) follows.

Global Convexity Condition. Here we show that the convexity condition (6) holds on all of Σ_v, using reflection symmetry and the case treated above. Let $p = (p_1, p_2)$ and $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$. We may assume by reflection symmetry that $p \in \Gamma_1$ and $p_1 \geq 0$, and we have already considered the case $\tilde{p}_2 \geq \tilde{p}_1 \geq 0$. To treat the remaining cases we claim that it suffices to prove the inequalities

$$\mathbf{v}(s, \varphi(s)) \cdot (1, 0) \geq c_0 s, \quad \mathbf{v}(s, \varphi(s)) \cdot (-1, 1) \geq c_0(\varphi(s) - s) \tag{13}$$

for some fixed $c_0 > 0$ and all $s \in [0, 1]$. Indeed, consider first the case that $\tilde{p}_1 \geq \tilde{p}_2 \geq 0$. Let $\tilde{q} = (\tilde{p}_2, \tilde{p}_1)$ be the reflection of $\tilde{p}$ over the diagonal. Then by reflection symmetry and the previous case we have

$$g(\tilde{p}) = g(\tilde{q}) \geq g(p) + \mathbf{v}(p) \cdot (\tilde{p} - p) + \mathbf{v}(p) \cdot (\tilde{q} - \tilde{p}) + \gamma|\tilde{q} - p|^2.$$

By the second inequality in (13), the sum of the last two terms is bounded from below by $c_0(\tilde{p}_1 - \tilde{p}_2)(p_2 - p_1) + \gamma|\tilde{q} - p|^2 \geq (\min\{c_0, \gamma\}/2)\,|\tilde{p} - p|^2$. The case $\tilde{p}_1 \leq 0$, $\tilde{p}_2 \geq 0$ is then treated similarly, taking instead $\tilde{q} = (-\tilde{p}_1, \tilde{p}_2)$ and using the first inequality in (13). For the remaining case case $\tilde{p}_2 \leq 0$ repeat again the argument with $\tilde{q} = (\tilde{p}_1, -\tilde{p}_2)$ and use that $\mathbf{v}(p) \cdot e_2 = h(p_1) > 1/4$.

To finish we confirm the inequalities in (13). For δ small it is straightforward to show that $h < 3/4$, so $f' - h\varphi' \geq c_0\varphi'$, and the first inequality in (13) follows.

For the second we compute

$$\begin{aligned}(1+2k)\mathbf{v}\cdot(-1,\,1) &= (1+2k)[(1+\varphi')h - f'] \\ &= (1+\varphi')\eta + [k(1+1/\varphi') - (1+2k)]f' \\ &= (1+\varphi')\eta + [k(1/\varphi' - 1) - 1]f' \\ &\geq (1+\varphi')\eta - f'.\end{aligned}$$

When $0 \leq s \leq 1-\delta$ this quantity is larger than 1 by the definition of η. For $s \geq 1-\delta$ we use the expansion $\varphi' = 1 - c_1\,(1-s)^{1/2} + O(1-s)$ (see Appendix) and that $f' \leq 1$ to get

$$\begin{aligned}(1+\varphi')\eta - f' &\geq (2 - c_1(1-s)^{1/2})\eta - 1 + O(1-s) \\ &= (2\eta - 1) - c_1\eta(1-s)^{1/2} + O(1-s) \\ &\geq (\delta^{-1/2} - 2c_1)(1-s)^{1/2} + O(1-s) \\ &\geq \frac{1}{2}\delta^{-1/2}(1-s)^{1/2}\end{aligned}$$

for δ small. We conclude that

$$\mathbf{v}\cdot(-1,\,1) \geq c_0 \min\{1,\, \delta^{-1/2}(1-s)^{1/2}\}.$$

Since $\varphi(s) - s \leq c_2(1-s)^{3/2}$ for some $c_2 > 0$, this confirms (13).

4.4. **Proof of Theorem 2.3.**

Proof. Choose η as in the previous subsection. Then the pair $(g,\,\mathbf{v})$ determined by η through the relations (10), (7), and (8) satisfies the hypotheses of Lemma 4.1. We showed above that the extension G then satisfies the Euler-Lagrange equation (5) and can be chosen symmetric over the axes and over the lines $p_1 = \pm p_2$, and that the result follows by taking $F(q) = G(|q_1|,\,|q_2|)$. □

Remark 4.3. If we choose $\eta - 1/2$ to be a multiple of $(1-s)^{1/2}$ near $s = 1$, then a straightforward computation shows that the $2k+1$ second derivatives of F in directions tangent to Σ_w are bounded, and the only second derivative of F that tends to ∞ is the one normal to Σ_w. The regularity of F near D_F is in fact $C^{1,\,\frac{1}{3}}$, by the computation that confirms the second inequality in (13).

5. Appendix

In the Appendix we record some properties of φ, and we prove the extension result Lemma 4.1.

5.1. **Properties of φ.**

We recall from [MS] that if we parametrize Γ_1 by the angle $\theta \in [\pi/4,\,3\pi/4]$ of its upward unit normal ν, then its curvature is given by $\kappa = \frac{\sqrt{2}}{3}\sec(2\theta)$. It follows easily that φ is smooth, even, and uniformly convex on $(-1,\,1)$, and φ'' is increasing on $[0,\,1)$. We recall also the expansion

$$\varphi\left(-1 + \frac{3}{2}\theta^2 + \theta^3 + O(\theta^4)\right) = 1 - \frac{3}{2}\theta^2 + \theta^3 + O(\theta^4)$$

from [MS]. By differentiating implicitly and using that φ is even we obtain near $s = 1$ the expansions

$$\varphi'(s) = 1 - 2\sqrt{\frac{2}{3}}(1-s)^{1/2} + O(1-s)$$

$$\varphi''(s) = \sqrt{\frac{2}{3}}(1-s)^{-1/2} + O(1)$$

$$\varphi'''(s) = \frac{1}{2}\sqrt{\frac{2}{3}}(1-s)^{-3/2} + O((1-s)^{-1}).$$

In particular, for $0 < s < y$ the derivative of the weight $\varphi''(s)(y-s)$ is bounded above by $\varphi'''(s)(1-s) - \varphi''(s) = -\frac{1}{2}\sqrt{\frac{2}{3}}(1-s)^{-1/2} + O(1) < 0$ for s close to 1.

5.2. **Proof of Extension Lemma.** We now prove Lemma 4.1. Our strategy is to first construct G in a set containing a neighborhood of every point on Σ_0. We then apply a global C^1 extension result to this local extension. To complete the construction we use a mollification and gluing procedure.

5.2.1. *Local Extension.*

Lemma 5.1. *There exists an open set $\mathcal{O}$ containing a neighborhood of each point on Σ_0 and a function $G_0 \in C^\infty(\mathcal{O})$ such that $G_0 = g$ and $\nabla G_0 = \mathbf{v}$ on Σ_0, and furthermore*

$$G_0(\tilde{p}) - G_0(p) - \nabla G_0(p) \cdot (\tilde{p} - p) \geq \frac{\gamma}{2}|\tilde{p} - p|^2 \tag{14}$$

for all p, $\tilde{p} \in \mathcal{O}$.

Proof. The squared distance function $d^2_{\Sigma_v}$ from Σ_v is smooth in a neighborhood of each point on Σ_0, as is the projection π_{Σ_v} to Σ_v. Let τ be a unit tangent vector field to Σ_v in a neighborhood of a point on Σ_0, and ν a unit normal vector field. Then $D^2(d^2_{\Sigma_v}/2)$ projects in the normal direction ν on Σ_0. See e.g. [AS] for proofs of these properties.

Let $A > 0$ be a smooth function on Σ_0 to be chosen, and define

$$G_0(x) := g(\pi_{\Sigma_v}(x)) + \mathbf{v}(\pi_{\Sigma_v}(x)) \cdot (x - \pi_{\Sigma_v}(x)) + A(\pi_{\Sigma_v}(x))d^2_{\Sigma_v}.$$

It is elementary to check using (6) that $\nabla_{\Sigma_0} g$ is the tangential component of $\mathbf{v}$ on Σ_0, and as a consequence that $G_0 = g$, $\nabla G_0 = \mathbf{v}$ on Σ_0. Furthermore, it follows from (6) that $(G_0)_{\tau\tau} \geq 2\gamma$ on Σ_0. Since $D^2(A(\pi_{\Sigma_v}(x))d^2_{\Sigma_v})$ is $2A$ times the matrix that projects in the direction ν on Σ_0, we have that $(G_0)_{\nu\nu} = 2A$ on Σ_0, and that $(G_0)_{\tau\nu}$ on Σ_0 depends only on g, $\mathbf{v}$, and the geometry of Σ_0 (in particular, not on A). By choosing $A(p)$ sufficiently large depending on γ and these quantities (and perhaps going to ∞ as $p \to S$), we have that

$$D^2 G_0 \geq \frac{3}{2}\gamma I \tag{15}$$

on Σ_0. In particular, (14) holds in a small neighborhood of each point on Σ_0.

Now, for δ, $\sigma > 0$ let S_δ be the closed δ-neighborhood of S, and let Σ_σ be the open σ-neighborhood of Σ_v. By (6), (15), and continuity, for each $\delta > 0$ there exists $\sigma(\delta) > 0$ small and an open set O_δ containing $\Sigma_{\sigma(\delta)} \backslash S_\delta$ and a neighborhood of each point in Σ_0, such that (14) holds for all p, $\tilde{p} \in O_\delta$ such that at least one of p, $\tilde{p}$ is in $\Sigma_{\sigma(\delta)} \backslash S_\delta$. For $\delta = 1/k$ we may choose $\sigma(1/k)$ and $O_{1/k}$ such that $\Sigma_{\sigma(1/k)} \backslash S_{1/k} \subset O_{1/k} \subset O_{1/(k-1)}$ for $k > 1$. Taking

$$\mathcal{O} = \cup_{k>1}(\Sigma_{\sigma(1/k)} \backslash S_{1/k})$$

completes the proof. ∎

5.2.2. *Global C^1 Extension.*

Lemma 5.2. *There exists a C^1 convex function G_1 on $\mathbb{R}^2$ such that $G_1 = G_0$ on an open set $\mathcal{U} \subset \mathcal{O}$ that contains a neighborhood of each point on Σ_0, such that $D^2 G_1 \geq \frac{\gamma}{2} I$ on $\mathbb{R}^2$.*

Proof. Let K be a compact set containing Σ_v and a neighborhood of each point on Σ_0, such that $K \backslash S \subset \mathcal{O}$, and G_0, ∇G_0 are continuous up to S in K. Let $H_0 := G_0 - \frac{\gamma}{4}|x|^2$. On S, define $H_0 = g - \frac{\gamma}{2}$, $\nabla H_0 = \mathbf{v} - \frac{\gamma}{2} x$. Then by (14) we have

$$H_0(\tilde{p}) - H_0(p) - \nabla H_0(p) \cdot (\tilde{p} - p) \geq \frac{\gamma}{4} |\tilde{p} - p|^2$$

for all p, $\tilde{p} \in K$. We may thus apply Theorem 1.10 from [AM] to obtain a global C^1, convex function H_1 on $\mathbb{R}^2$ such that $H_1 = H_0$, $\nabla H_1 = \nabla H_0$ on K. To finish take $G_1 = H_1 + \frac{\gamma}{4}|x|^2$. □

5.2.3. *Smoothing.*

Lemma 5.3. *There exists a convex function $G \in C^1(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \backslash S)$ such that $G = G_1$ in a neighborhood of each point on Σ_0, and $D^2 G \geq \frac{\gamma}{4} I$ on $\mathbb{R}^2$. In particular, $D_G = S$, and $(G, \nabla G) = (g, \mathbf{v})$ on Σ_v.*

Proof. We begin with a simple observation. Let F be a C^1 convex function on $\mathbb{R}^n$ with $D^2 F \geq \mu I$. Let ρ_ϵ be a standard mollifier, and let $F_\epsilon := \rho_\epsilon * F$. Then if $\varphi \in C_0^\infty(B_R)$ with $0 \leq \varphi \leq 1$ we have

$$\begin{aligned} \|(\varphi F_\epsilon + (1-\varphi)F) - F\|_{C^1(\mathbb{R}^n)} &= \|\varphi(F_\epsilon - F)\|_{C^1(\mathbb{R}^n)} \\ &\leq C(\varphi)\|F_\epsilon - F\|_{C^1(B_R)} \end{aligned}$$

and

$$\begin{aligned} D^2(\varphi F_\epsilon + (1-\varphi)F) &= \varphi D^2 F_\epsilon + (1-\varphi) D^2 F + D^2\varphi\,(F_\epsilon - F) \\ &\quad + \nabla\varphi \otimes (\nabla F_\epsilon - \nabla F) + (\nabla F_\epsilon - \nabla F) \otimes \nabla \varphi \\ &\geq \left(\mu - C(\varphi)\|F_\epsilon - F\|_{C^1(B_R)}\right) I. \end{aligned}$$

Since F is C^1, by taking $\epsilon(\varphi)$ small we have that $\varphi F_\epsilon + (1-\varphi)F$ is as close as we like to F in $C^1(\mathbb{R}^n)$, and we have a lower bound for $D^2(\varphi F_\epsilon + (1-\varphi)F)$ that is as close as we like to μI. We will apply this observation to a sequence of mollifications of G_1.

Let $\{B_{r_i}(x_i)\}_{i=1}^\infty$ be a Whitney covering of $\mathbb{R}^2 \backslash \Sigma_v$. That is, for

$$r(x) := \min\{1, d_{\Sigma_v}(x)\}/20,$$

we have $r_i = r(x_i)$, the balls $B_{r_i}(x_i)$ are disjoint, $\cup_i B_{5r_i}(x_i) = \mathbb{R}^2 \backslash \Sigma_v$, and for each $x \in \mathbb{R}^2 \backslash \Sigma_v$ the ball $B_{10r(x)}(x)$ intersects at most $(129)^2$ of the $\{B_{10r_i}(x_i)\}$. For the existence of such a covering see for example [EG].

Now take $\varphi_i \in C_0^\infty(B_{10r_i}(x_i))$ such that $\varphi_i = 1$ in $B_{5r_i}(x_i)$, $0 \leq \varphi \leq 1$. For $j \geq 1$ we define G_{j+1} inductively as follows: If $B_{10r_j}(x_j) \subset \mathcal{U}$ then let $G_{j+1} = G_j$. If not, then let

$$G_{j+1} = \varphi_j(\rho_{\epsilon_j} * G_j) + (1 - \varphi_j) G_j,$$

with ϵ_j chosen so small that

$$\|G_{j+1} - G_j\|_{C^1(\mathbb{R}^2)} \leq 2^{-j} \tag{16}$$

and

$$D^2G_{j+1} \geq \left(\frac{\gamma}{2} - \frac{\gamma}{4}\sum_{i=1}^{j} 2^{-i}\right) I. \tag{17}$$

By the covering properties, for any $x \in \mathbb{R}^2 \backslash \Sigma_v$, we have that G_j are smooth and remain constant in $B_{10r(x)}(x)$ for all j sufficiently large. In addition, for any point on Σ_0, every $B_{10r_j}(x_j)$ that intersects a small neighborhood of this point is contained in $\mathcal{U}$, so $G_j = G_1$ in a small neighborhood of every point in Σ_0. We conclude using (16) and (17) that $G := \lim_{j\to\infty} G_j$ is C^1, smooth on $\mathbb{R}^2 \backslash S$, agrees with G_1 in a neighborhood of every point on Σ_0, and satisfies

$$D^2G \geq \left(\frac{\gamma}{2} - \frac{\gamma}{4}\sum_{i=1}^{\infty} 2^{-i}\right) I = \frac{\gamma}{4} I.$$

□

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References

[A1] Alexandrov, A. D. On the curvature of surfaces. *Vestnik Leningrad. Univ. 21* **19** (1966), 5-11.

[A2] Alexandrov, A. D. On uniqueness theorem for closed surfaces. *Doklady Akad. Nauk. SSSR* **22** (1939), 99-102.

[AS] Ambrosio, L.; Soner, H. M. Level set approach to mean curvature flow in arbitrary codimension. *J. Differential Geom.* **43** (1996), 693-737.

[ATO] Araújo, D. J.; Teixeira, E. V.; Urbano, J. M. A proof of the $C^{p'}$-regularity conjecture in the plane. *Adv. Math.* **316** (2017), 541-553

[AM] Azagra, D.; Mudarra, C. Whitney extension theorems for convex functions of the classes C^1 and $C^{1,\omega}$. *Proc. London Math. Soc.* **3** (2017), 133 -158.

[CKP] Cohn, H.; Kenyon, R.; Propp, J. A variational principle for domino tilings. *J. Amer. Math. Soc.* **14** (2001), no. 2, 297-346.

[CF] Colombo, M.; Figalli, A. Regularity results for very degenerate elliptic equations. *J. Math. Pures Appl.* (9) **101** (2014), no. 1, 94-117.

[DG] De Giorgi, E.: Sulla differenziabilità e l'analicità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino cl. Sci. Fis. Fat. Nat.* 3, 25-43 (1957).

[DS] De Silva, D.; Savin O. Minimizers of convex functionals arising in random surfaces. *Duke Math. J.* **151**, no. 3 (2010), 487-532.

[DMMN] Delgadino, M. G.; Maggi, F.; Mihaila, C.; Neumayer, N. Bubbling with L^2-almost constant mean curvature and an Alexandrov-type theorem for crystals. *Arch. Ration. Mech. Anal.* **230** (2018), no. 3, 1131-1177.

[E] Evans, L. C. A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. *J. Differential Equations* **45** (1982), no. 3, 356-373.

[EG] Evans, L. C.; Gariepy, R. F. *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[Gi] Giaquinta, M. Growth conditions and regularity, a counterexample. *Manuscripta Math.* **59** (1987), 245-248.

[GT] Gilbarg, D.; Trudinger, N. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[IM] Iwaniec, T.; Manfredi, J. Regularity of p-harmonic functions on the plane. *Rev. Mat. Iberoam.* **5** (1989), 1-19.

[KOS] Kenyon, R.; Okounkov, A.; Sheffield, S. Dimers and amoebae. *Ann. of Math.* (2) 163 (2006), no. 3, 1029-1056.

[Ma1] Marcellini, P. Regularity and existence of solutions of elliptic equations with (p, q)-growth conditions. *J. Differential Equations* **90** (1991), 1-30.

[Ma2] Marcellini, P. Regularity for some scalar variational problems under general growth conditions. *J. Optim. Theory Appl.* **90** (1996), 161-181.

[Ma3] Marcellini, P. Regularity under general and p, q-growth conditions. *Discrete Contin. Dyn. Syst. Ser. S* (2019), to appear.

[MM1] Martinez-Maure, Y. Contre-exemple à une caractérisation conjecturée de la sphère. *C. R. Acad. Sci. Paris* **332** (2001), 41-44.

[MM2] Martinez-Maure, Y. New notion of index for hedgehogs of $\mathbb{R}^3$ and applications. *European J. Combin.* **31** (2010), 1037-1049.

[MS] Mooney, C.; Savin, O. Some singular minimizers in low dimensions in the calculus of variations. *Arch. Ration. Mech. Anal.* **221** (2016), 1-22.

[Mo] Morrey, C. B. *Multiple Integrals in the Calculus of Variations.* Springer-Verlag, Heidelberg, NY (1966).

[Na] Nash, J. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* 80, 931-954 (1958).

[P] Panina, G. New counterexamples to A. D. Alexandrov's hypothesis. *Adv. Geom.* **5** (2005), 301-317.

[S] Savin, O. Small perturbation solutions to elliptic equations. *Comm. Partial Differential Equations* **32** (2007), no. 4-6, 557-578.

[Uh] Uhlenbeck, K. Regularity for a class of non-linear elliptic systems. *Acta Math.* **138** (1977), 219-240.

[Ur] Ural'tseva, N. Degenerate quasilinear elliptic systems. *Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov* **7** (1968), 184-222.

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